Robotica (2013) volume 31, pp. 1037–1050. © Cambridge University Press 2013
doi:10.1017/S0263574713000271

Cyclic pursuit in a multi-agent robotic system with double-integrator dynamics under linear interactions

Balaji R. Sharma†, Subramanian Ramakrishnan‡* and Manish Kumar§

†*Cooperative Distributed Systems Laboratory, School of Dynamic Systems, University of Cincinnati, Cincinnati, OH 45220, USA*
E-mail: sharmabr@mail.uc.edu
‡*Center for Nonlinear Dynamics and Control, Villanova University, Villanova, PA 19085, USA*
§*Cooperative Distributed Systems Laboratory, Department of Mechanical, Industrial and Manufacturing Engineering, University of Toledo, Toledo, OH 43606, USA*
E-mail: Manish.Kumar2@utoledo.edu

(Accepted March 13, 2013. First published online: April 23, 2013)

SUMMARY
We investigate the controlled realization of a stable circular pursuit model in a multi-agent robotic system described by double-integrator dynamics with homogeneous controller gains. The dynamic convergence of the system starting from a randomly chosen, non-overlapping initial configuration to a sustained, stable pursuit configuration satisfying velocity matching and uniform inter-agent separation is demonstrated using the proposed control framework. The cyclic pursuit configuration emerges from local, linear, inter-agent interactions and is shown to be robust under stochastic perturbations of small and moderate intensities. The stability criterion discussed in this work is independent of the number of agents, permitting dynamic addition/deletion of agents without affecting overall system stability. Experimental results that validate the key theoretical results are also presented. Potential applications of the results obtained include cooperative perimeter tracking and resource distribution applications such as border patrol and wildfire monitoring.

KEYWORDS: Multi-robot systems; Control of robotic systems; Robot dynamics; Aerial robotics; Novel applications of robotics.

1. Introduction

The problem of cyclic pursuit has long captured the attention of mathematicians. One of the earliest known characterizations of cyclic pursuit is the now-famous "three-bug" problem, with early investigations conducted by H. Brocard[1,2] in response to a question by E. Lucas[3] in 1877. Some of the early research, documented in refs. [4–6], involved evaluating the trajectories of three bugs starting from the vertices of an equilateral triangle. Each bug was directed to follow the bug ahead of it in cyclic succession to itself, with all the bugs moving at constant speeds. It was demonstrated analytically that the agents mutually captured each other within a specific period of time and after having covered a specific distance, and that the trajectory leading to mutual capture could be represented by logarithmic spirals.[4] It was further shown that in such a scenario, the bugs would meet simultaneously at a point that is now referred to as the Brocard point of the triangle. The problem was subsequently extended to the case of n-bugs, initiated at the vertices of an n-sided regular polygon (n-gon), and it was shown[5] that the trajectories of the bugs would be simultaneously concurrent even when the constraint of uniform speeds for all the bugs was relaxed. The case of the bugs being initiated at the vertices of a non-degenerate, non-equilateral triangle was considered in ref. [6], and it was shown for this unsymmetrical three-bug problem with the bugs traveling at different speeds, that the capture would still occur at a point within the initial triangle, and that it would be mutual. Further analytical investigation into cyclic pursuit using a rotating frame of reference, and the cases of non-mutual captures in n-bug problems are discussed in ref. [7]. The reader is directed to[6,7] for accounts of some of the early investigations of the n-bug problem and its variants.

Developing cyclic pursuit models is expected to have potentially significant applications in multi-agent cooperative or swarm robotics such as networks of ground robots or unmanned aerial vehicles. For instance, in the case of perimeter tracking in applications such as wildfire monitoring or surveillance using unmanned systems, it may be necessary for the agents comprising the system to swiftly and adaptively achieve stable, cyclic pursuit configurations. In addition, the effort reported in this paper is aligned with extensive research being pursued toward agent coordination[8] and consensus control,[9,10] heterogeneous agent segregation,[11] aggregation and formation control[12] in multi-agent systems. Distributed control laws such as those presented here form critical precursors to developing more complex and more general distributed control laws for cooperative multi-agent systems. Integrating and implementing cyclic pursuit strategies in such multi-agent systems for perimeter tracking applications could potentially lead to significant advances in the optimal utilization of agents and resources. Having discussed the practical

* Corresponding author. E-mail: subramanian.ramakrishnan@villanova.edu

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

applications involving cyclic pursuit, we now turn to a brief survey of the more recent literature allied with the problem.

The "three-bug" problem with the imposition of an additional single non-holonomic constraint on each bug, and the trajectories followed by these identical kinematic unicycles, was analyzed in refs. [13, 14]. It was demonstrated that the agents converged to generalized polygons of pursuit at equilibrium, and the stability of these equilibrium polygons for cyclic pursuit was discussed in detail therein. Linear and non-linear kinematic cyclic pursuit models have been investigated and documented in refs. [15–17]. The case of asynchronous cyclic pursuit was studied in ref. [18], and convergence of agents to a point under such a pursuit model was demonstrated. A three-dimensional cyclic pursuit with target-enclosing objectives was discussed in ref. [19], with the stability of formation discussed by invoking the notion of $\mathcal{D}$-stability.[20] We note that in ref. [19], while the authors demonstrated convergence to a well-distributed spatial formation around a target to be enclosed, the convergence of inter-agent speeds to a common value during pursuit was not discussed. The use of a dynamic control model for convergence of a multi-agent system in three dimensions to circular and the Archimedean spiral formations was discussed in detail in refs. [21, 22]. Notably, this control model relied on sharing position *and* velocity information with neighboring agents in the realization of stable formations. Single and double-integrator pursuit models for agents in a three-dimensional space in a distributed control framework were also discussed in ref. [23]; however, the motion of agents while in formation was governed by the motion of leader agents. The motion in two dimensions of a string of vehicles connected in a unidirectional ring topology, based on local interactions, was discussed in ref. [24]. In addition to the assessment of the stability of the system, the influence of a partly functioning agent on the overall dynamics of the system was presented in the above-mentioned paper, thus incorporating critical considerations into real-world implementation of such pursuit models.

It is pertinent to note that much of the published work on cyclic pursuit control models employ and evaluate single-integrator kinematic models to represent the agent dynamics. In this paper, we focus on double-integrator dynamics, which represent force–acceleration models, and are typically applicable to a broad range of vehicles. Double-integrator models have been studied and experimentally implemented for a broad range of robotic applications. For example, authors in ref. [25] evaluated and experimentally implemented a distributed containment control algorithm, based on double-integrator dynamics, in a multi-agent system with multiple stationary or dynamic leaders. An experimental implementation of a flocking algorithm was demonstrated in ref. [26], with the use of control forces and torques to achieve flocking behavior in ground robots based on double-integrator dynamics. A double-integrator control model was also presented in ref. [27], utilizing both *a priori* path data and real-time sensor information for achieving autonomous navigation in dynamic environments in robotic systems. The use of double-integrator control models for aerial and ground robots for real-time active cooperative tracking of dynamic targets was discussed in detail in ref. [28].

For the control model presented in this work, interactions are restricted to between each agent and the nearest agent ahead of it along the curve in the direction of pursuit. A key result of the present effort is that such a control model governing local interactions between agents leads to a global configuration where the velocity of each agent converges to a constant magnitude. It is consequently shown that the inter-agent distances also converge to a common constant value, leading to cyclic pursuit among agents with uniform spatial distribution around the pursuit curve. The advantages of using such a control model, such as its efficacy in the case of randomly chosen initial configuration of agents and guaranteed convergence to stable configuration, without inter-agent collisions, are demonstrated through analysis, numerical simulations and experiments involving mobile robots.

Furthermore, the conditions for stability of the pursuit model are discussed based on an evaluation of the eigenvalues of the system matrix. Specifically, it is shown that the poles lie on a Cassini curve defined by the gains of the control model, thus placing a very tight bound on the location of the poles in the complex plane. We reiterate our finding that this stability criterion, for a set of control gain values, remains invariant under the addition or deletion of agents and consequent variations in the pole locations in the complex plane. In other words, the characteristics of the Cassini curve that defines the bounds on the eigenvalues remain independent of the number of agents involved in pursuit and we believe that this has important implications for practical implementation of cyclic pursuit models.

While parallels can be drawn between the approach and results presented in ref. [24] and the present effort, importantly, in the current work, a tighter bound on the system poles is presented for stability that is independent of the number of agents participating in the pursuit. Therefore, our results allow for dynamic addition and deletion of agents without affecting the pursuit stability. It is also critical to note that for the system, the pole placement in the complex plane is governed by the geometrical characteristics of a class of closed curves called the Cassini curves. The parameters defining the Cassini curves are the gains of the control model chosen for pursuit. Therefore, from a design point of view, the framework employed in our approach allows one to draw conclusions about the system stability characteristics based on a straightforward graphical inspection of the pole locations.

Given that noise invariably plays a role in real-world applications, the validity of our results in the presence of stochastic forces is an important question. Hence, we demonstrate, via numerical simulations, the robustness of the cyclic pursuit configurations achieved by the control framework presented in this paper to stochastic perturbations of the dynamics resulting from white Gaussian noise of minimal to moderate intensity.

Finally, we validate our theoretical and numerical results by presenting experimental results obtained from implementing the control framework on a system of mobile robots. The cyclic pursuit control model is experimentally

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

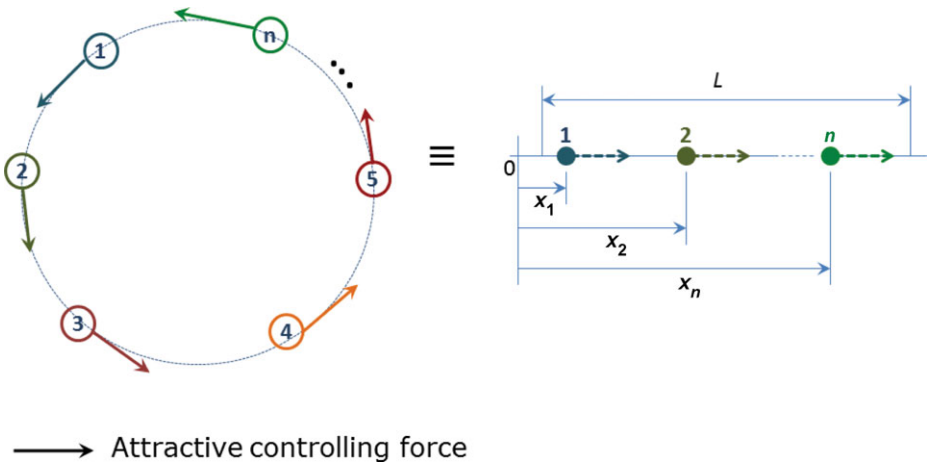


Fig. 1. (Colour online) Schematic of forces on agents.

implemented using a three-agent ground robotic system aided by a vision-based motion tracking system. The validity of the control laws for stable cyclic pursuit is thus experimentally demonstrated. The pursuit behavior is observed to be robust to sources of noise unavoidable in experimental implementations, such as induced by the use of a vision-based tracking system and actuation errors in physical robots.

The rest of the paper is organized as follows. Section 2 formulates the problems with descriptions of the agent dynamics and control laws, and underlying assumptions therein. Section 3 presents a detailed analysis of issues relating to the stability of the multi-agent system under controller action. Simulation results validating results from the analytical studies are presented in Section 4. The experimental results are presented in Section 5. The paper concludes in Section 6, summarizing the results from the present work and with a word on future directions for research.

2. Problem and Control Law Formulation

We consider a multi-agent system comprising n agents, with $n > 2$. Since the motion of the agents is constrained to a circular perimeter, they can be assumed to be moving in a one-dimensional space, as represented in Fig. 1. The agents follow the double-integrator dynamics given by

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{v}_i, \\ \dot{\mathbf{v}}_i &= \mathbf{u}_i; \quad i = 1 \ldots n, \end{aligned} \tag{1}$$

where $\mathbf{x}_i$ is the one-dimensional position coordinate for agent i, and $\mathbf{v}_i$ is the corresponding one-dimensional velocity. In addition, keeping in mind the objectives of this study, the following general assumptions are made regarding the agent dynamics:

1. All agents are identical in their physical properties, and their motion is governed by point mass dynamics in one-dimensional real space $\mathbf{R}^1$.
2. Each agent is capable of identifying and communicating with its leading agent along the pursuit curve at all times and at all points along the curve.
3. The motion of the agents, initiated at unique points around the perimeter of the circle, is synchronous with no time delays, i.e., the agents move simultaneously, and are aware of the positions of their neighboring agents at all times.

We now present the cyclic pursuit controller that acts on agent i and accounts for interactions between neighboring agents, with the variable $\mathbf{x}$ representing position along the curve as

$$\begin{aligned} \mathbf{u}_i &= -k_{cp}\,(\mathbf{x}_i - \mathbf{x}_{i+1}) - k_v \mathbf{v}_i, \qquad i = 1, 2, \ldots, n-1. \\ \mathbf{u}_n &= -k_{cp}\,(\mathbf{x}_n - \mathbf{x}_1 - L) - k_v \mathbf{v}_n; \end{aligned} \tag{2}$$

Note that k_{cp} and k_v are the controller gains, and L represents the overall length of the closed pursuit curve, or the perimeter of the pursuit circle in this case.

We note that in Eq. (2), the term $-k_{cp}(\mathbf{x}_i - \mathbf{x}_j)$ represents the linear attractive component of the forcing function acting on the ith agent under the influence of the jth agent ahead of it along the curve. Evidently, the influence of this force is quite pronounced at moderate to large distances, while the opposite is true at small distances of separation between the agents. The circulant matrix $\mathbf{C}$ represents the agent interactions for this force, and is given by

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & \ldots & 0 & 1 \end{bmatrix} = circ\,([1, -1, \ldots, \ldots, 0]) \in \mathbf{R}^{n \times n}.$$

The linear forces acting on each agent are representatively shown in Fig. 1. The control vector for the overall system can, therefore, be rewritten as

$$\mathbf{U} = -k_{cp}\mathbf{C}\mathbf{X} - k_{cp}\mathbf{L} - k_v\mathbf{V}. \tag{3}$$

where $\mathbf{X} = (\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_n)$, $\mathbf{V} = (\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n)$, and $\mathbf{U} = (\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_n)$ are the stacked position, velocity, and control vectors, respectively, and $\mathbf{L} = (0\,0\ \ldots\ -L)$.

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

The system dynamics under the influence of this control action can be written for an n-agent system in state-space form as

$$\dot{X} = AX + B, \tag{4}$$

where $X = \left(\mathbf{x}_2 - \mathbf{x}_1 \ldots \mathbf{x}_1 - \mathbf{x}_n \; \dot{\mathbf{x}}_2 - \dot{\mathbf{x}}_1 \ldots \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_n\right)^T$, $A = \left[\begin{array}{c|c} [0]_{nxn} & [I]_{nxn} \\ \hline -k_{cp}.[\mathbf{C}]_{nxn} & -k_v.[I]_{nxn} \end{array}\right]$, and $B = (0\ 0 \ldots -k_{cp}L)^T$. $[I]_{nxn}$ represents an identity matrix of size n, while $[0]_{nxn}$ represents a null matrix of dimension $n \times n$. Section 3 discusses the stability of the linear system given by Eq. (4).

3. Analysis of the Control Law

In this section, we investigate analytically the stability of the multi-agent system in which dynamics of each agent is given by Eq. (1) and each agent experiences control given by Eq. (2). We study the convergence properties of the entire system to a stable constant velocity configuration in cyclic pursuit. In pertinence to the results presented further in this section, we begin with a brief discussion of a class of curves called the Cassini curves.

A Cassini curve (or Cassini oval) is a quartic curve, described as the locus of a point such that the product of its distances from two focal points is constant. The curve could be either a single closed curve or the union of two ovals depending on the curve parameters. Further details on Cassini curves and their characteristics can be found in refs. [29, 30].

Cassini curves are typically discussed in the context of eigenvalue localization (see, for instance, ref. [31]). They often find mention alongside the relatively more popular Gershgorin discs that arise in the context of placing bounds on eigenvalues in the complex plane. An example of the use of Gershgorin discs can be found in ref. [16]. However, it is generally established that the ovals of Cassini provide a much tighter inclusion of eigenvalues than Gershgorin discs.[29] The ovals of Cassini are found to be very pertinent to the localization of eigenvalues for the cyclic pursuit system dynamics presented in this work, and subsequently, in establishing the stability of the system, as will be demonstrated in what follows.

Theorem 1. *Consider a system of n identical agents whose dynamics are governed by the double-integrator model given by Eq. (1) and the control laws given by Eq. (2). All the roots of the characteristic equation corresponding to the dynamic system represented by Eq. (4), for any number of agents, lie on a Cassini curve given by the expression* $[(x + \frac{k_v}{2})^2 - y^2 + (k_{cp} - \frac{k_v^2}{4})]^2 + 4y^2(x + \frac{k_v}{2})^2 - k_{cp}^2 = 0$.

Proof. The general form of the characteristic equation for an n-agent system in terms of complex variable s, given the system matrix of the form A, can be obtained as

$$\left(s^2 + k_v s + k_{cp}\right)^n - {k_{cp}}^n = 0. \tag{5}$$

Equation (5) can now be rewritten in the form

$$\left(\frac{s^2 + k_v s + k_{cp}}{k_{cp}}\right)^n = 1. \tag{6}$$

Each of the n roots of unity is known to lie on a unit circle centered at the origin in the complex plane. The kth root of unity can, therefore, be represented by $c_k = a_k + jb_k$, such that

$${a_k}^2 + {b_k}^2 = 1. \tag{7}$$

Now, the $2n$ roots of the characteristic polynomial given by Eq. (6) can be obtained as the solutions of the following quadratic equation for each of the n roots of unity

$$s^2 + k_v s + k_{cp} = k_{cp}\left(a_k + jb_k\right). \tag{8}$$

Therefore, corresponding to each of the n roots of unity, two roots of the characteristic polynomial for the system are obtained as

$$s = -\frac{k_v}{2} \pm \frac{1}{2}\sqrt{{k_v}^2 - 4k_{cp}\left(1 - a_k - jb_k\right)}. \tag{9}$$

Since $s = x + jy$, it follows from Eq. (9) that

$$x + jy = -\frac{k_v}{2} \pm \frac{1}{2}\sqrt{{k_v}^2 - 4k_{cp}\left(1 - a_k - jb_k\right)}. \tag{10}$$

Therefore,

$$\left(x + jy + \frac{k_v}{2}\right)^2 = \frac{1}{4}\left[{k_v}^2 - 4k_{cp}\left(1 - a_k\right)\right] + j\left[k_{cp}b_k\right]. \tag{11}$$

Comparison of the real and imaginary components on either side of Eq. (11) gives

$$a_k = \frac{1}{k_{cp}}\left[\left(x + \frac{k_v}{2}\right)^2 - y^2 + \left(k_{cp} - \frac{k_v^2}{4}\right)\right], \tag{12}$$

and

$$b_k = \frac{1}{k_{cp}}\left[(2y)\left(x + \frac{k_v}{2}\right)\right]. \tag{13}$$

Imposing the constraint given by Eq. (7), we then obtain

$$\left[\left(x + \frac{k_v}{2}\right)^2 - y^2 + \left(k_{cp} - \frac{k_v^2}{4}\right)\right]^2 + 4y^2\left(x + \frac{k_v}{2}\right)^2 - k_{cp}^2 = 0. \tag{14}$$

Equation (14) represents a Cassini curve symmetric about the axes defined by $x = -\frac{k_v}{2}$ and $y = 0$. Furthermore, it can be observed that this curve always passes through the points represented by $s = (0, 0)$ and $s = (-k_v, 0)$. □

Of critical importance is the observation that the eigenvalues of the system lie on the Cassini curve given by

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

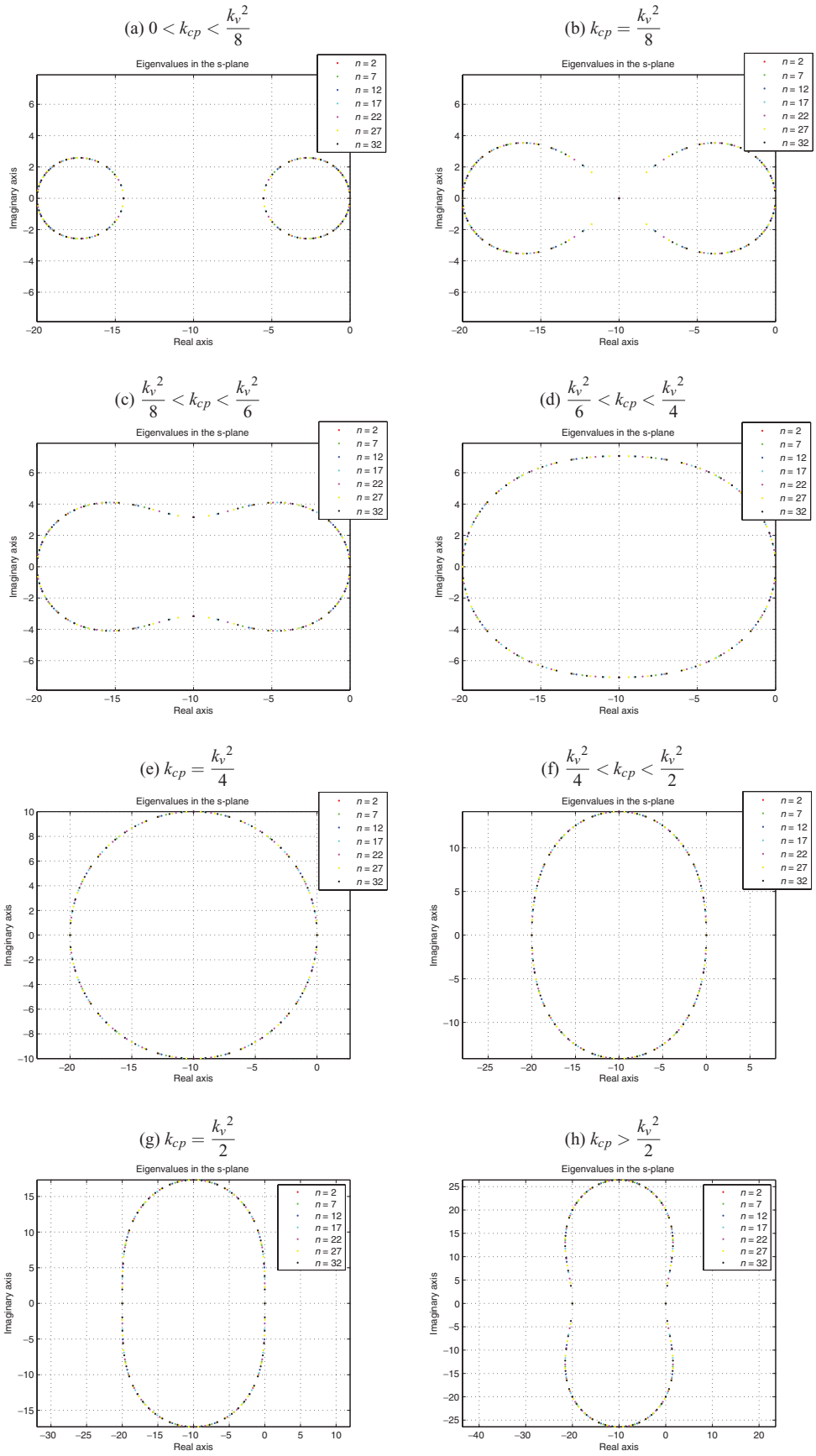


Fig. 2. Effect of k_{cp} and k_v on the location of eigenvalues in the complex plane.

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

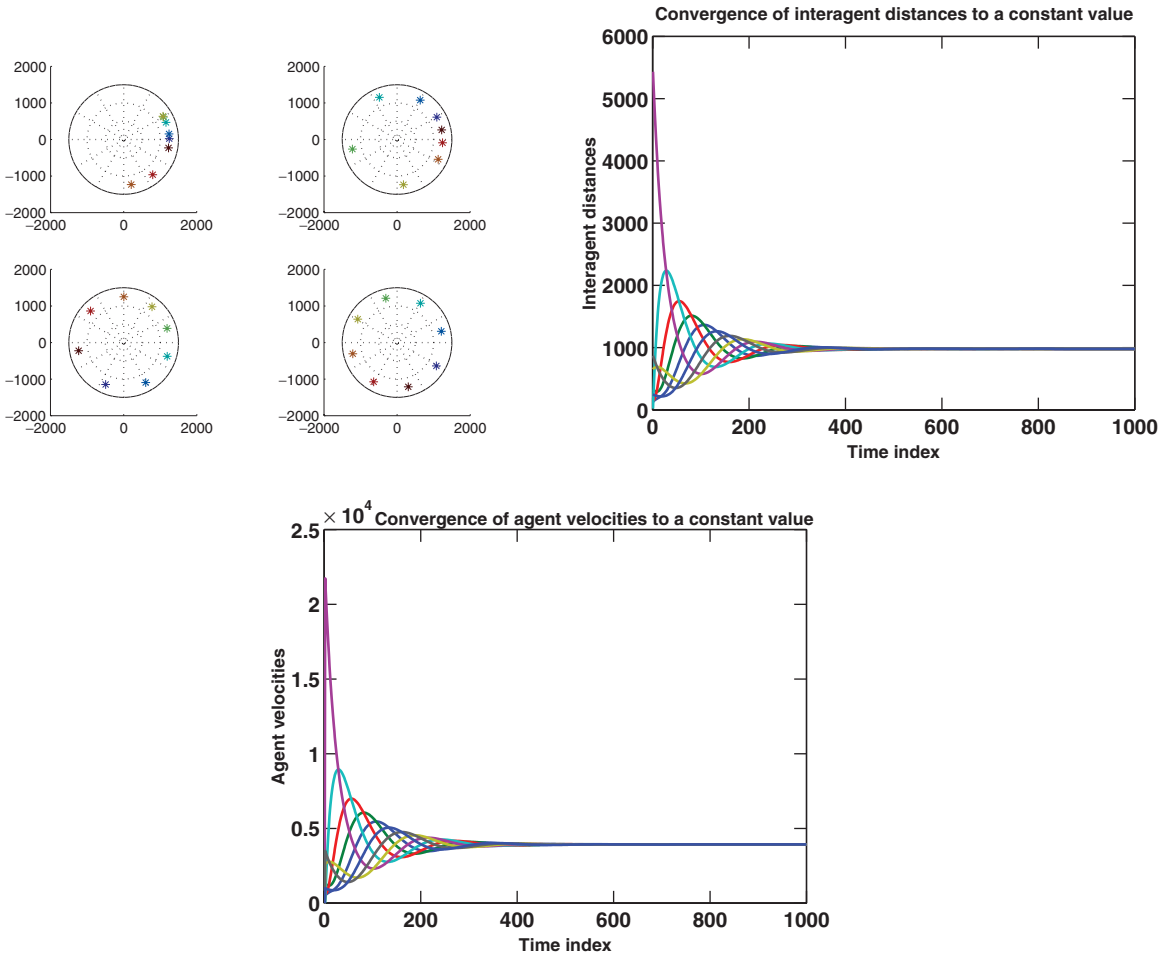


Fig. 3. (Colour online) Cyclic pursuit for Case 1: (a) System configurations after (i) 0 timesteps, (ii) 50 timesteps, (iii) 150 timesteps, and (iv) 1000 timesteps configuration; (b) Distances between neighboring agents (an agent and its leading agent) during pursuit; (c) Agent velocities during pursuit.

Eq. (14) *independent of the number of agents in the system*, n. Importantly, this fact permits the scalability of agents in the system for a given set of controller gains without affecting the system stability. Furthermore, dynamic addition and deletion of agents during the cyclic pursuit process may be achieved without compromising the stability characteristics of the system.

Having thus established bounds on the eigenvalues of the multi-agent system, we now consider their influence on the stability of the system.

Theorem 2. *Consider a system of n identical agents whose motions are governed by the double-integrator dynamic model given by Eq. (1) and the cyclic pursuit control laws given in Eq. (2). The cyclic pursuit control model results in an asymptotically stable configuration, for $k_{cp} < \frac{k_v{}^2}{2}$. Additionally, the agents asymptotically converge to a configuration with identical velocities and constant and identical inter-agent separations.*

Proof. The asymptotic stability of the system is governed by the eigenvalues of the system matrix A. Theorem 1 demonstrates that the eigenvalues of the system matrix A, represented by the roots of the characteristic polynomial (Eq. (5)), lie on the boundary of a Casssini curve represented by Eq. (14). The properties of the curve in the complex plane are governed by the gains k_{cp} and k_v.

To obtain bounds on the gain values resulting in stable configurations, we now determine the conditions for which the eigenvalues, lying on the Cassini curve described in Eq. (14), fall in open right-half of the complex plane. Since the Cassini curve is a closed curve, the extreme points along the real-axis, i.e., the points corresponding to largest and smallest real components, correspond to points on the curve where the slope of the curve tends to infinity.

Now, differentiating Eq. (14) with respect to x, the expression for the slope of the curve can be obtained as

$$\frac{dy}{dx} = \frac{\left(x+\frac{k_v}{2}\right)\left[\left(x+\frac{k_v}{2}\right)^2 + y^2 + k_{cp} - \frac{k_v^2}{4}\right]}{y\left[-\left(x+\frac{k_v}{2}\right)^2 - y^2 + k_{cp} - \frac{k_v^2}{4}\right]}. \tag{15}$$

Considering finite values of x and y, the condition for slope of the curve to tend to infinity, or $\frac{dy}{dx} \to \infty$, holds true for one of the following two cases

1. $y = 0$, or
2. $-\left(x+\frac{k_v}{2}\right)^2 - y^2 + k_{cp} - \frac{k_v^2}{4} = 0.$

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

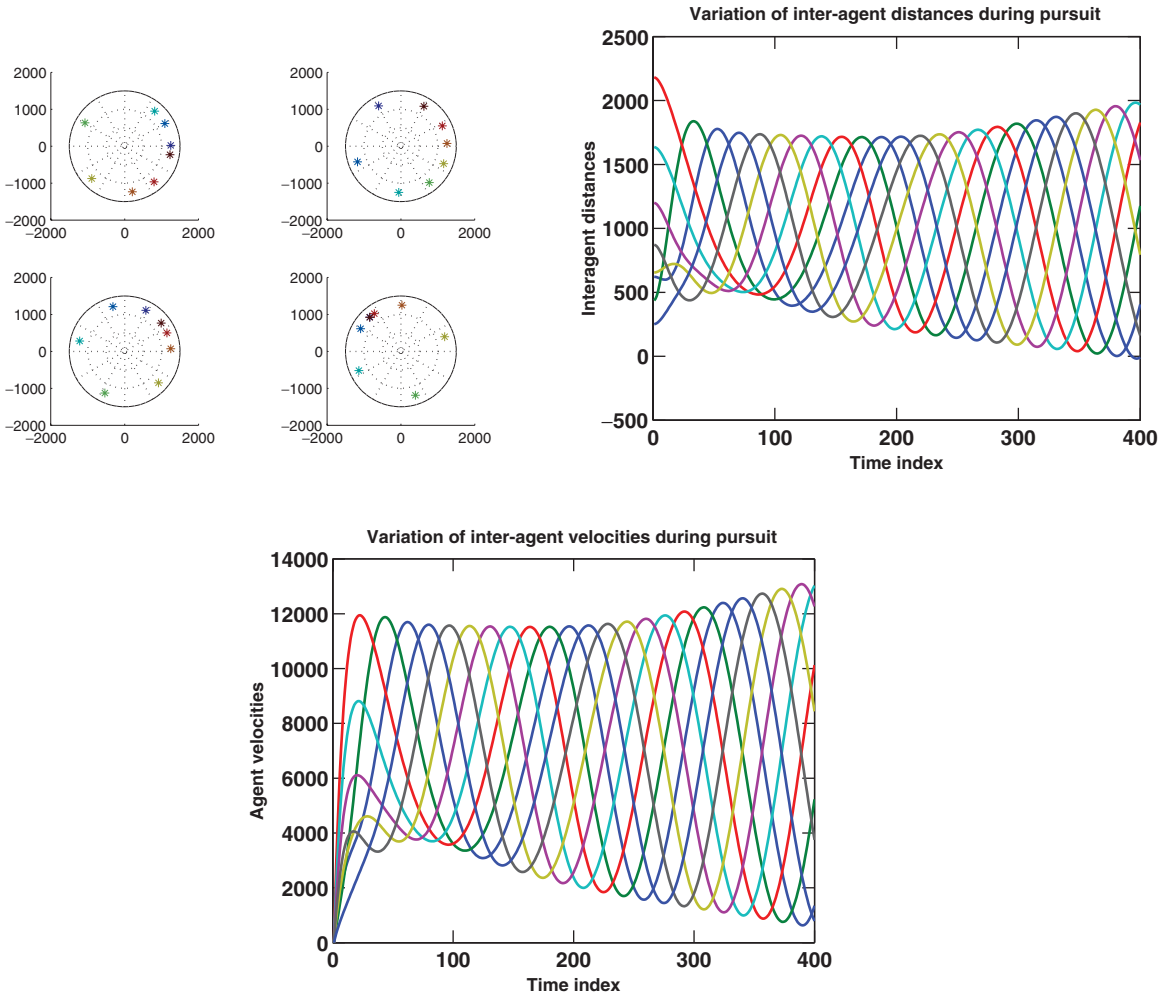


Fig. 4. (Colour online) Cyclic pursuit for Case 2: (a) System configurations after (i) 0 timesteps, (ii) 50 timesteps, (iii) 150 timesteps, and (iv) 400 timesteps configuration; (b) Distances between neighboring agents (an agent and its leading agent) during pursuit; (c) Agent velocities during pursuit.

Substituting $y = 0$ in Eq. (14) yields the points $(0, 0)$ and $(-k_v, 0)$ in the complex plane. As mentioned earlier, the curve passes through these points for all positive values of k_{cp} and k_v.

The second case implies that the extreme points on the Cassini curve on the real axis, corresponding to a slope of infinity, lie on a circle centered at $(\frac{-k_v}{2}, 0)$ with a radius of $\sqrt{(k_{cp} - \frac{k_v^2}{4})}$ (for $k_{cp} > \frac{k_v^2}{4}$, $x, y \in \mathbf{R}$). The equation of the circle is given by

$$\left(x + \frac{k_v}{2}\right)^2 + y^2 = k_{cp} - \frac{k_v^2}{4}. \tag{16}$$

It is essential for stability that $x < 0$. Therefore, the limiting case corresponding to which the system becomes unstable, from Eq. (16), is obtained as

$$k_{cp} - \frac{k_v^2}{4} \geq \frac{k_v^2}{4} \Rightarrow k_{cp} \geq \frac{k_v^2}{2}.$$

Therefore, for the multi-agent system to be stable, the limiting condition is given by

$$k_{cp} < \frac{k_v^2}{2}. \tag{17}$$

The condition given by Eq. (17) ensures that the eigenvalues of the system matrix A remain in the left-half of the complex plane, except one eigenvalue, which lies at the origin. As in refs. [13] and [16], taking into account the non-vanishing eigenvalues, we observe that they all fall in the open left-half of the complex plane for $k_{cp} < \frac{k_v^2}{2}$. Under this constraint, the system represented by Eq. (4) will therefore be asymptotically stable, i.e., $X_{t\to\infty} = constant$. Consequently, $\mathbf{v}_{i+1} - \mathbf{v}_i \to constant$ $(\forall i < n)$, and $\mathbf{v}_n - \mathbf{v}_1 \to constant$. Since the inter-agent distances, being a part of the stacked state vector for the system, also converge to a constant value, we have $\mathbf{x}_{i+1} - \mathbf{x}_i \to c_i \Rightarrow \mathbf{v}_{i+1} - \mathbf{v}_i \to 0$ $(\forall i < n)$, and $\mathbf{x}_1 + L - \mathbf{x}_n \to c_n \Rightarrow \mathbf{v}_1 - \mathbf{v}_n \to 0$, where the terms $c_1, \ldots, c_n$ are constants. The velocities of the individual agents thus converge to a uniform value among agents. Hence, since $\mathbf{v}_i = \mathbf{v}_j$ $(\forall i, j \neq i)$, it is clear that $\mathbf{u}_i = \dot{\mathbf{v}}_i$ will be uniform among all agents, i.e., $\mathbf{u}_i = \mathbf{u}_j$. Now, referring to Eq. (2), for uniform $\mathbf{v}_i$ and $\mathbf{u}_i$ $(\forall i)$, we observe

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

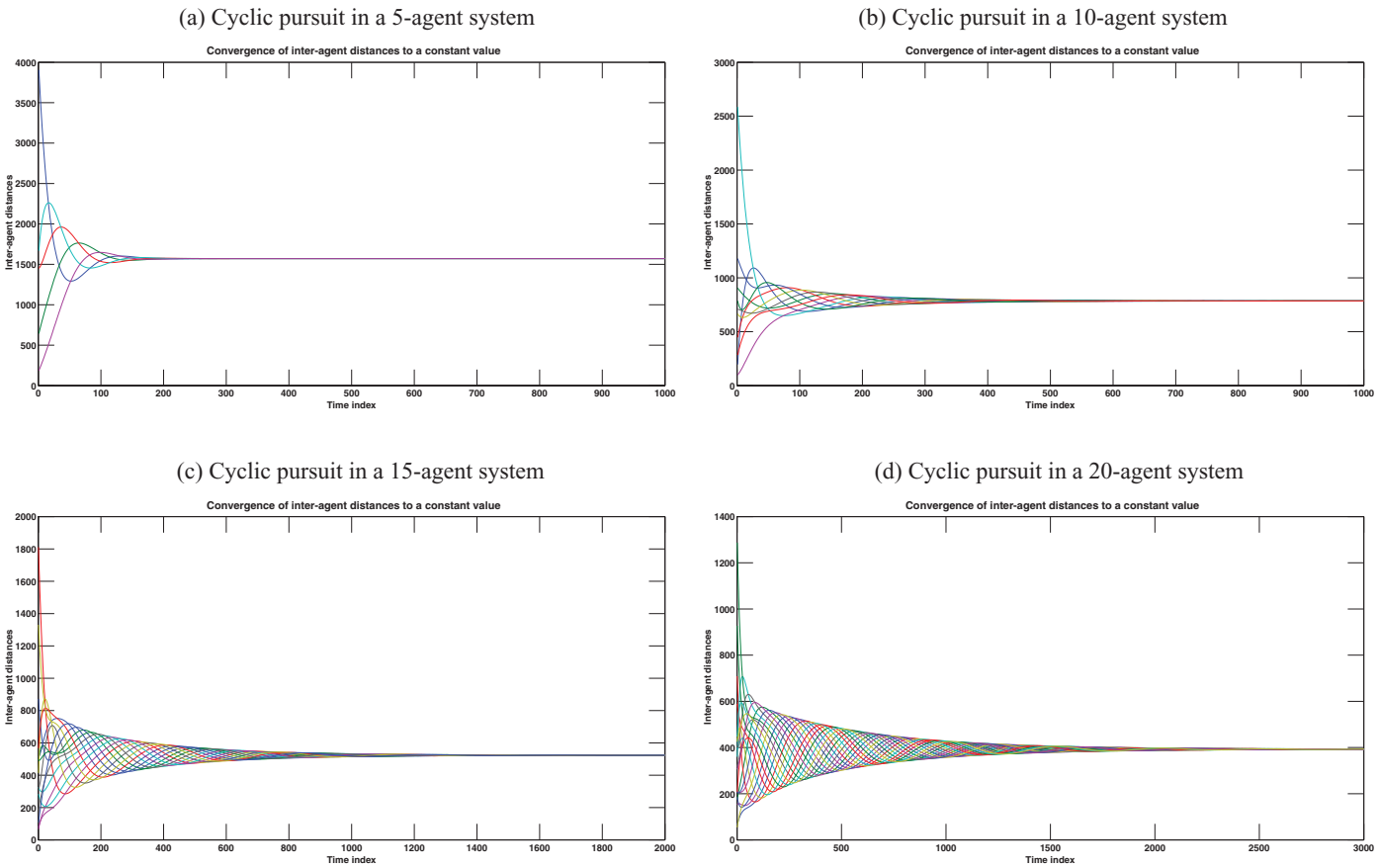


Fig. 5. (Colour online) Cyclic pursuit for different number of agents in pursuit.

that inter-agent distances also converge to uniform values. Hence, considering that the pursuit curve is a closed curve of length L, the agents would asymptotically converge to a cyclic pursuit configuration with inter-agent distances given by L/n. Thus, the agents attain a cyclic pursuit configuration with uniform agent velocities, and uniform and constant inter-agent distances. □

Figures 2(a)–(h) facilitate a graphical interpretation of the effect of variation in parameters k_{cp} and k_v on the attributes of the curve in the complex plane. Each of the figures shows the eigenvalues for a certain set of parameters for different values of n, and as expected from Theorem 1 all eigenvalues lie on the same curve. These reinforce the analytical exposition on the limiting conditions for stability of the multi-agent system. For $k_{cp} > \frac{k_v^2}{2}$, the poles spill over into the right-half of the complex plane, as graphically depicted in Fig. 2(h), and is indicative of instability. The analyses are reinforced through simulation results described in the next section.

4. Numerical Simulation Results

Results of numerical simulations carried out in order to validate the analytical cyclic pursuit control model are presented in this section. First, the emergence of stable and unstable configurations based on choices of parameter values as dictated by the theoretical results of the previous sections are demonstrated. The second set of results establish that stable configurations are achieved irrespective of the number of agents involved in the pursuit. The third set of results show convergence to stable pursuit configurations even when the agents start from locations that are randomly distributed along the perimeter of the pursuit trajectory. The fourth set of results indicate the robustness of the control framework under the influence of stochastic forces characterized by white Gaussian noise of varying intensity. The final set of numerical results lead to the important conclusion that, under the condition that the control gains are homogeneous and chosen from the stable region, the control framework allows for the dynamic addition and deletion of agents without affecting the stability of the overall system.

4.1. Stable and unstable configurations

Simulations were conducted for a system with eight identical agents initiated at randomly chosen points along the perimeter of a circle for two cases: Case 1: for $k_{cp} = 400$, and $k_v = 100$ that evolves into a stable configuration; and Case 2: for $k_{cp} = 70$, and $k_v = 10$, which evolves into an unstable configuration. For Case 1, it was observed that the agents resolved themselves into a spatially well-distributed configuration around the perimeter over time under the influence of the cyclic forces, eventually converging to constant-velocity, constant-inter-agent distance dynamics. Numerical results pertaining to the system response for Case 1 are presented in Figs. 3(a)–(c). Results corresponding to Case 2 are presented in Figs. 4(a)–(c).

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

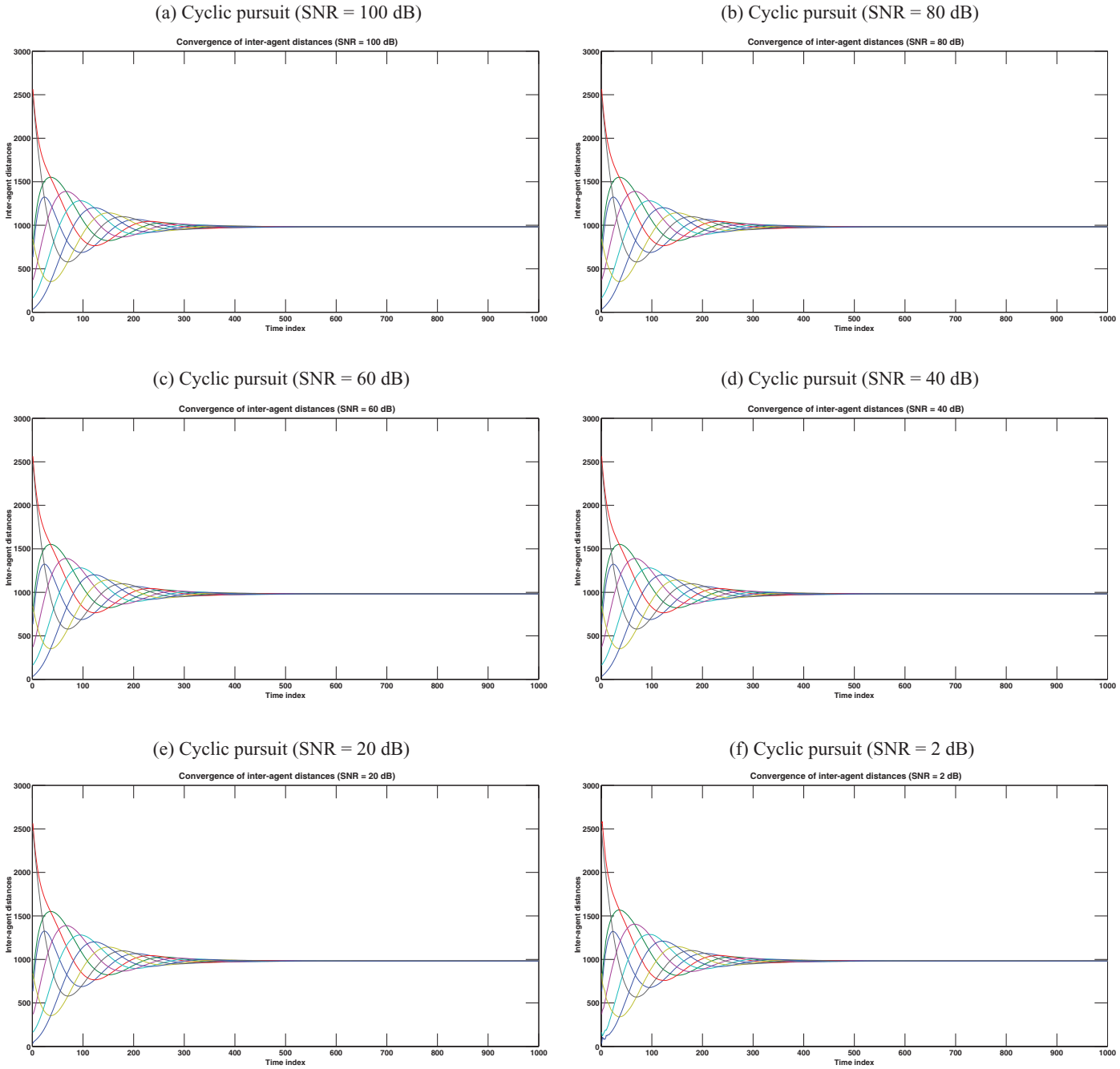


Fig. 6. (Colour online) Cyclic pursuit implementation under varying noise levels.

4.2. Number of agents

The pursuit characteristics and convergence of agents to stable, well-distributed pursuit configurations with equal inter-agent separations along the perimeter occur independent of the number of agents involved in the pursuit. This is demonstrated in Figs. 5(a)–(d) with results from pursuit implementation among 5, 10, 15, and 20 agents, respectively. While it appears that the rate of convergence is strongly influenced by the number of agents in the system, it is also dependent on the initial conditions of the agents along the perimeter. Hence, an evaluation of the influence of the number of agents in pursuit on convergence rate, in isolation from the initial conditions of the agents, is not possible. Results for simulations presented further in this work correspond to control gains chosen from the stable region for the gains, specifically with $k_1 = 400$ and $k_v = 100$.

4.3. Random initial conditions

The convergence to equal inter-agent separations in stable pursuit occurs independent of any non-overlapping initial conditions. This is demonstrated with the help of 50 simulation runs performed on eight agents with the initial positions of the agents chosen from uniformly distributed random locations along the perimeter. Convergence criteria for inter-agent separations is defined to be within 0.2% around the expected steady-state value for the given perimeter length and number of agents, corresponding to which the rate of convergence, which is defined as the time (in seconds) for

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

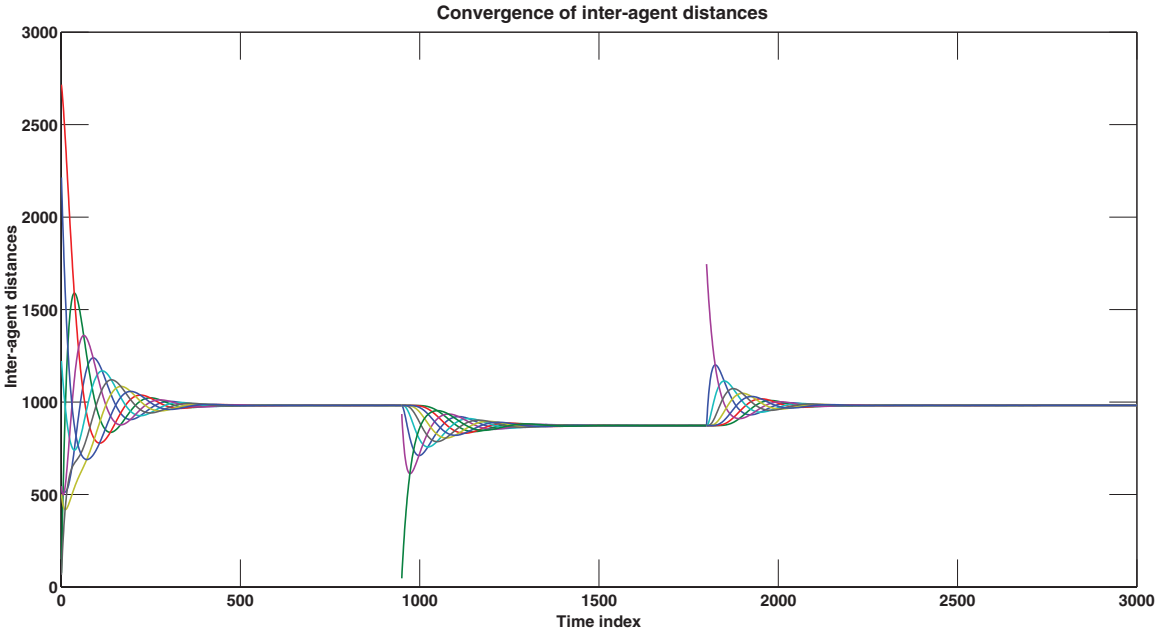


Fig. 7. (Colour online) Cyclic pursuit with dynamic addition and deletion of agents.

convergence, is obtained numerically. Presented in Table I are statistical results from the simulations for an eight-agent system averaged over 50 simulation runs with randomly chosen initial positions of the agents along the perimeter.

4.4. Stochastic effects

The control model is further shown to be robust to the presence of moderately high levels of additive white Gaussian noise. The eight-agent system is subjected to increasing noise levels, and it is observed that stable pursuit occurs without marked deviation from expected convergence attributes, even when the noise is high, as reflected by SNR of 2 dB. The SNR (in decibels) is defined in Eq. (18), where the terms P and A represent the power and amplitude, respectively, of the signal or noise.

$$\mathrm{SNR}_{\mathrm{dB}} = 10 \cdot log_{10}\left(\frac{P_{\mathrm{signal}}}{P_{\mathrm{noise}}}\right) = 20 \cdot log_{10}\left(\frac{A_{\mathrm{signal}}}{A_{\mathrm{noise}}}\right). \tag{18}$$

Note that the inter-agent distances at steady state for an eight-agent system for pursuit along a curve of radius 1250 units (as is chosen for the simulations presented in this work) are expected to converge to $\frac{2\pi * 1250}{8} \sim= 981.75$ units. This is

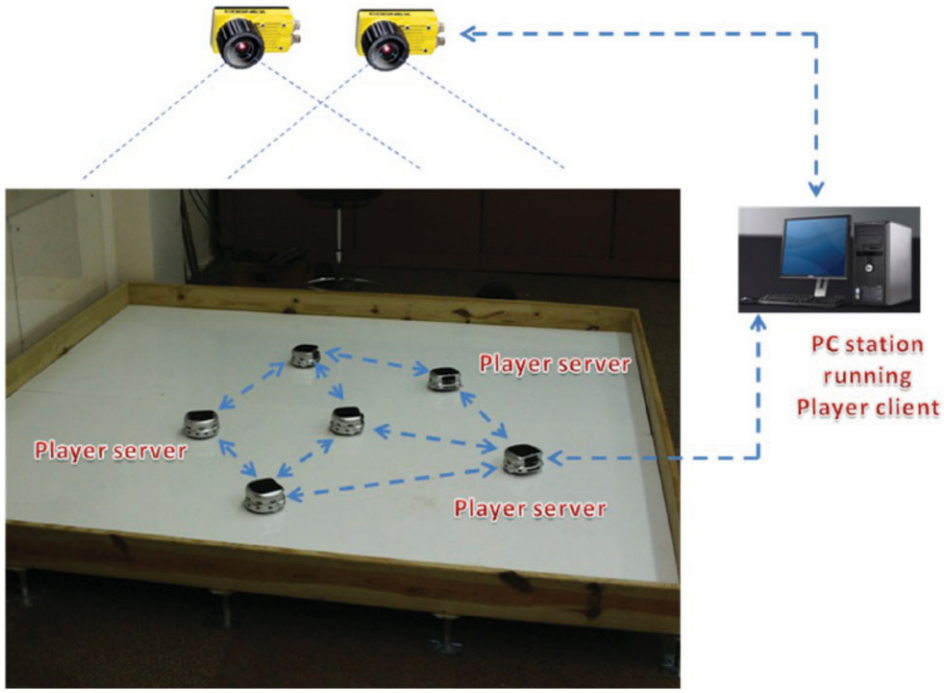


Fig. 8. (Colour online) UC test bed configuration.

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

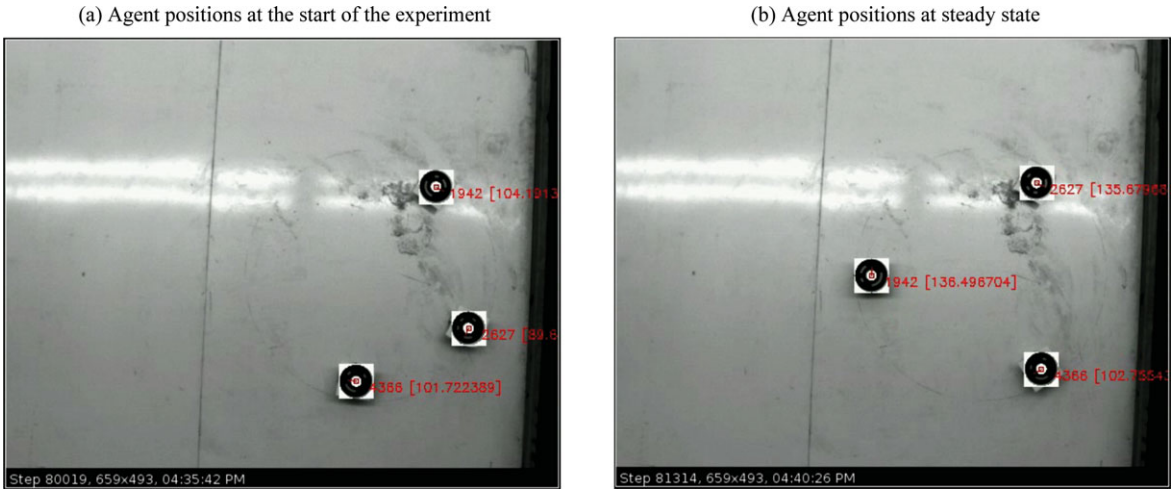


Fig. 9. (Colour online) Agent positions during the experiment.

Table I. Inter-agent separations at steady-state and rate of convergence (after 50 runs).

Agent Pair	Inter-agent separation		Rate of convergence	
	Avg.	S.D.	Avg.	S. D.
1-2	981.7467	0.0111	6.3356	0.5585
2-3	981.7442	0.0104	6.3256	0.6019
3-4	981.7479	0.0104	6.3322	0.5455
4-5	981.7488	0.0103	6.3230	0.5525
5-6	981.7496	0.0111	6.3308	0.6164
6-7	981.7483	0.0100	6.3346	0.5701
7-8	981.7481	0.0100	6.3314	0.5341
8-1	981.7481	0.0087	6.3148	0.6194

observed to be true even in the presence of moderately high noise levels as would correspond to a SNR of 2 dB. These results can be observed in Figs. 6(a)–(f).

4.5. Dynamic addition and deletion of agents

The control model and the analytical discussion presented for the control model are independent of the number of agents in the system, and allow for dynamic addition and deletion of agents without affecting the stability of the overall system, as long as the (homogeneous) control gains are chosen from the stable region. This is demonstrated in the simulation results presented in Fig. 7. Initially, the pursuit controller is implemented in an eight-agent system, and the agents converge under the influence of the control laws to stable pursuit at steady state. At around the 950th time iteration, an additional agent is randomly deployed on the perimeter, and it can be observed that, under the influence of purely the local control laws, the agents adapt to the addition and reconverge to a globally stable pursuit configuration with equal inter-agent separations. The distance between agents at steady state remains a function of the perimeter length and the number of agents in deployment. Furthermore, around the 1800th time iteration, the agent is removed from the perimeter, and the agents react to the deletion by reconverging to the steady-state configuration corresponding to the original eight-agent system.

5. Experimental Results

The control model is, additionally, implemented on a three-agent system in an experimental test bed available at the Cooperative Distributed Systems Laboratory at the University of Cincinnati. The UC test bed is based on Khepera III wheeled mobile robots,[32] each with an embedded XScale processor, infrared and ultrasonic sensor arrays, and capable of communicating over a wireless network. The Khepera III robot has a diameter of 130 mm and a height of 70 mm. The open source Player/Stage software is used in order to communicate with and control the robots. The Player server is installed on each Khepera robot and provides access to each sensor via TCP/IP protocol to interact with the robots. Controller development and implementation are achieved using MATLAB R2011a, particularly the Instrument Control Toolbox therein. An overhead camera, the Prosilica GC650 model, is connected over an ethernet port to a tracking system installed on a Linux-based PC. SwisTrack,[33] an open source software application, is used for real-time tracking of the robots on a two-dimensional test bed. Unique patterns installed on each robot act as markers for the camera and the tracking software to uniquely identify each robot's position on the test bed, with the information being relayed in real-time to the PC. Control actuations and information on respective neighboring agents are sent in real-time to each agent from the remote PC over a local Wi-Fi network. An illustration of UC's test bed is depicted in Fig. 8.

For this work, three Khepera robots are deployed at randomly chosen locations along a circular perimeter and at relatively close proximity to one another, to demonstrate the control action. The control gains for the experiments are chosen from the region for stable pursuit, specifically $k_1 = 0.01$ and $k_v = 0.5$. The controller input in terms of accelerations are computed as a function of agent velocities and distance of separation from its leading agent along the curve as described in Eqs. (1) and (2), and the corresponding velocity terms act as control inputs to the robots. The controller is implemented in a one-dimensional sense, with the robots constrained to move along a circular perimeter at a rate governed by the control input

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

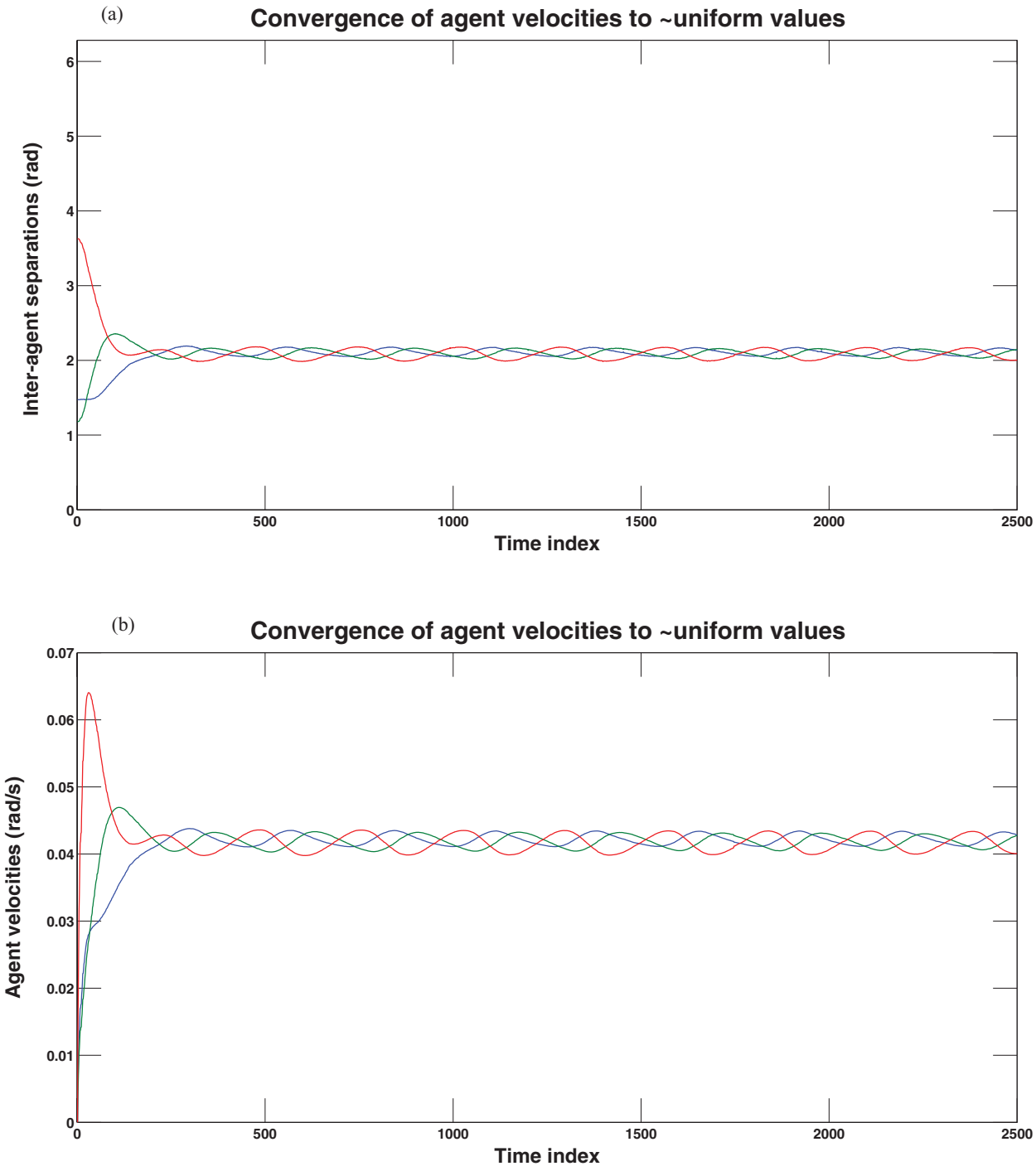


Fig. 10. (Colour online) Results from the experimental implementation on a three-agent robotic system.

actuations. As can be observed from Figs. 9(a)–(b), and in agreement with the simulation results presented earlier, the agents converge toward a spatially well-distributed pursuit configuration, with inter-agent separations and agent velocities converging to uniform values globally based on the distributed local controller action. These results on convergence attributes are shown, respectively, in Figs. 10(a)–(b), for inter-agent separations and agent velocities, over the length of the experiment. Small oscillations observed in the graphs are attributed to errors in vision sensing, actuation errors, and time delays in communication over TCP/IP.

6. Conclusions and Future Work

A novel cyclic pursuit strategy has been implemented for a multi-agent system constrained to move in a circle. The control laws are governed by directed local interactions between neighboring agents and it is demonstrated through analysis, numerical simulations, and experiments using mobile robots that the multi-agent system, under the influence of the controller, converges to a stable configuration characterized by constant and identical agent speeds and uniform distances of separation between agents. The effectiveness of such a control law toward achieving convergence to stability from a randomly chosen,

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

non-overlapping initial configuration of agents, is discussed and highlighted. In addition, it is shown that the stable pursuit configuration is achieved irrespective of the number of agents in the system. Moreover, it is also shown that the robust convergence to the aforementioned configuration occurs even in the presence of stochastic forces characterized by white Gaussian noise of small to moderate intensities. Furthermore, it is shown that dynamic addition and deletion of agents does not alter the overall system stability as long as the controller gains chosen correspond to the stable region.

In the light of the results presented in this paper, it is of interest to ask whether stable pursuit configurations may be achieved on closed curves in three-dimensional physical space. One observes that, since the control framework is dependent exclusively on knowledge (measurement) of distances (arc lengths) along curves, the framework applies equally well to any closed curve in three-dimensional space. In other words, since any curve in three-dimensional space may be parameterized using a single variable, the problem of pursuit along a closed curve in space remains a one-dimensional problem.

The experimental efforts reported in this paper may be generalized to the case of many more robots and other scenarios of increasing complexity. Ultimately, it would be of interest to pursue the implementation of the results reported in this paper in applications such as cooperative perimeter tracking for wildfire monitoring and border patrol. Of interest in this context would be optimization of the proposed control law with respect to various performance evaluation metrics, which would depend on the particular application considered. Additionally, from the analytical perspective, it is of interest to investigate cyclic pursuit models using non-linear interaction potentials such as the Lennard–Jones and the Morse potential functions. Among others, the possible existence of correlations between the type of potential and rate of convergence to stable configurations is an important open question.

Acknowledgment

The authors gratefully acknowledge support from the National Science Foundation via grant no. EFRI-1024608.

References

1. H. Brocard, *Nouvo Corres. Math.* **3**(280) (1877).
2. H. Brocard, *Nouvo Corres. Math.* **6**(211) (1880).
3. E. Lucas, *Nouvo Corres. Math.* **3**(175) (1877).
4. H. Bernhart, "Polygons of pursuit," *Scr. Math.* **24**, 23–50 (1959).
5. A. Watton and D. W. Kydon, "Analytical aspects of the *n*-bug problem," *Am. J. Phys.* **37**, 220–221 (1969).
6. M. S. Klamkin and D. J. Newman, "Cyclic pursuit or the 'three bugs problem'," *Am. Math. Mon.* **78**(6), 631–639 (1971).
7. F. Behroozi and R. Gagnon, "Cyclic pursuit in a plane," *J. Math. Phys.* **20**(11), 2212–2216 (1979).
8. H. Yamaguchi, "A distributed motion coordination strategy for multiple nonholonomic mobile robots in cooperative hunting operations," *Robot. Auton. Syst.* **43**(4), 257–282 (2003).
9. W. Ren, R. W. Beard and E. Atkins, "Information consensus in multivehicle cooperative control: Collective group behavior through local interaction," *IEEE Control Syst. Mag.* **27**(2), 71–82 (2007).
10. M. A. Joordens and M. Jamshidi, "Consensus control for a system of underwater swarm robots," *IEEE Syst. J.* **4**(1), 65–73 (2010).
11. M. Kumar, D. P. Garg and V. Kumar, "Segregation of heterogeneous units in a swarm of robotic agents," *IEEE Trans. Autom. Control* **55**(3), 743–748 (2010).
12. V. Gazi, B. Fidan, Y. S. Hanay and M. I. Koksal, "Aggregation, foraging, and formation control of swarms with non-holonomic agents using potential functions and sliding mode techniques," *Turk. J. Electr. Eng.* **15**(2), 149–168 (2007).
13. J. A. Marshall, M. E. Broucke and B. A. Francis, "Formations of vehicles in cyclic pursuit," *IEEE Trans. Autom. Control* **49**(11), 1963–1974 (2004).
14. J. A. Marshall, M. E. Broucke and B. A. Francis, "Pursuit formations of unicycles," *Automatica* **42**, 3–12 (2006).
15. A. Sinha and D. Ghose, "Control of multiagent systems using linear cyclic pursuit with heterogenous controller gains," *J. Dyn. Syst. Meas. Control* **129**, 742–749 (2007).
16. A. Sinha and D. Ghose, "Generalization of linear cyclic pursuit with application to rendezvous of multiple autonomous agents," *IEEE Trans. Autom. Control* **51**(11), 1819–1824 (2006).
17. A. Sinha and D. Ghose, "Generalization of nonlinear cyclic pursuit," *Automatica* **43**, 1954–1960 (2007).
18. A. Samiloglu, V. Gazi and B. Koku, "Asynchronous cyclic pursuit," *Lecture Notes in Computer Science* **4095**, 667–678 (2006).
19. T. Kim and S. Hara, "Stabilization of Multi-Agent Dynamical Systems for Cyclic Pursuit Behavior," *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico (Dec. 9–11, 2008) pp. 4370–4375.
20. R. Pavani, "D-stability from a Numerical Point of View," *SIAM Conference on Applied Algebra*, Williamsburg, VA, USA (Jul. 15–19, 2003).
21. J. L. Ramirez, M. Pavone, E. Frazzoli and D. W. Miller, "Distributed Control of Spacecraft Formation via Cyclic Pursuit: Theory and Experiments," *Proceedings of the American Control Conference*, St. Louis, MO, USA (Jun. 10–12, 2009) pp. 4811–4817.
22. J. L. Ramirez, M. Pavone, E. Frazzoli and D. W. Miller, "Distributed control of spacecraft formations via cyclic pursuit: Theory and experiments," *J. Guid. Control Dyn.* **33**(5), 1655–1669 (2010).
23. W. Ding, G. Yan and Z. Lin, "Collective motions and formations under pursuit strategies on directed acyclic graphs," *Automatica* **46**(1), 174–181, 2010.
24. J. A. Rogge and D. Aeyels, "Vehicle platoons through ring coupling," *IEEE Trans. Autom. Control* **53**(6), 1370–1377 (2008).
25. Y. Cao, D. Stuart, W. Ren and Z. Meng, "Distributed Containment Control for Double-Integrator Dynamics: Algorithms and Experiments," *Proceedings of the American Control Conference*, Baltimore, MD, USA (Jun. 30–Jul. 2, 2010) pp. 3830–3835.
26. A. Regmi, R. Sandoval, R. Byrne, H. Tanner and C. Abdallah, "Experimental implementation of flocking algorithms in wheeled mobile robots," *Proc. Am. Control Conf.* **2**, 4917–4922 (2005).
27. V. Delsart and T. Fraichard, "Reactive Trajectory Deformation to Navigate Dynamic Environments," *Proceedings of the European Robotics Symposium*, Prague, Czech Republic (Mar. 26–28, 2008) pp. 233–241.
28. F. Morbidi, C. Ray and G. Mariottin, "Cooperative Active Target Tracking for Heterogeneous Robots with Application to Gait Monitoring," *IEEE/RSJ International Conference on Intelligent Robots and Systems*, San Francisco, CA, USA (Sep. 25–30, 2011) pp. 3608–3613.
29. R. S. Varga, "Gershgorin disks, Brauer ovals of Cassini (a vindication) and Brualdi sets," *Information* **4**(2), 171–178 (2001).

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press

30. V. G. Melnikov, "About root-clustering in sophisticated regions," *Latest Trends on Systems* **1**, 297–300 (2010).
31. A. de Paor, "The root locus method: Famous curves, control designs and non-control applications," *Int. J. Electr. Eng. Educ.* **37**(4), 344–356 (2000).
32. F. Mondada, E. Franzi and A. Guignard, "The Development of Khepera," *Experiments with the Mini-Robot Khepera, Proceedings of the First International Khepera Workshop*, HNI-Verlagsschriftenreihe, Heinz Nixdorf Institut, Paderborn, Germany (1999) pp. 7–14.
33. T. Lochmatter, P. Roduit, C. Cianci, N. Correll, J. Jacot and A. Martinoli, "Swistrack - A Flexible Open Source Tracking Software for Multi-Agent Systems," *Intelligent Robots and Systems, 2008 (IROS 2008). IEEE/RSJ International Conference on*, Nice, France (Sep. 22–26, 2008) pp. 4004–4010.

https://doi.org/10.1017/S0263574713000271 Published online by Cambridge University Press