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Ion acoustic traveling waves

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Models for traveling waves in multi-fluid plasmas give essential insight into fully nonlinear wave structures in plasmas, not readily available from either numerical simulations or from weakly nonlinear wave theories. We illustrate these ideas using one of the simplest models of an electron–proton multi-fluid plasma for the case where there is no magnetic field or a constant normal magnetic field present. We show that the traveling waves can be reduced to a single first-order differential equation governing the dynamics. We also show that the equations admit a multi-symplectic Hamiltonian formulation in which both the space and time variables can act as the evolution variable. An integral equation useful for calculating adiabatic, electrostatic solitary wave signatures for multi-fluid plasmas with arbitrary mass ratios is presented. The integral equation arises naturally from a fluid dynamics approach for a two fluid plasma, with a given mass ratio of the two species (e.g. the plasma could be an electron–proton or an electron–positron plasma). Besides its intrinsic interest, the integral equation solution provides a useful analytical test for numerical codes that include a proton–electron mass ratio as a fundamental constant, such as for particle in cell (PIC) codes. The integral equation is used to delineate the physical characteristics of ion acoustic traveling waves consisting of hot electron and cold proton fluids.

1. Introduction

Electrostatic solitary waves (ESWs) have been observed simultaneously with reflected suprathermal ions at collisionless shocks, e.g. by the geotail spacecraft at Earth's bow shock (Shin et al. 2008). Such waves are observed to be of the order of 2–7 Debye lengths and amplitudes of ~ 100 mV/m, by the WIND spacecraft (Bale et al. (1998)).

Quasi-perpendicular shock models in which suprathermal ions gain energy in the motional electric field upon reflection from an electrostatic shock potential (ESSP) have had some success as a possible dissipation mechanism for super-critical collisionless shocks, and as an explanation for some observations. For example, reflected pick-up ions can explain the cooler than expected solar wind observed by Voyager 2 downstream of the heliospheric termination shock (Zank et al. 1996, 2010; Richardson et al. 2008; Burrows et al. 2010; Oka et al. 2011). Zank et al. (1996, 2010) point out the importance of the cross shock electrostatic potential at the solar wind termination shock and at traveling interplanetary shocks, and the acceleration of pick-up ions by the shock surfing mechanism. The dissipation mechanism for the

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solar wind termination shock is due in large part to the interaction of pick-up ions with the shock since they carry most of the momentum flux of the solar wind (Zank et al. 1996, 2010). Oka et al. (2011) studied in detail the dissipation mechanism and pick-up ion distribution at the solar wind termination shock by using PIC simulations.

Observations indicate that electrostatic solitary waves reflect particles and are fundamental components of shocks (Zank et al. 1996; Wilson et al. 2007; Shin et al. 2008). Self-consistent models of the solar wind termination shock based in part on a fluid dynamics approach needs to incorporate the effects of reflected particles such as shock surfing pick-up ions. These models cannot make the typical assumption of charge neutrality since electrostatic structures arise from charge separation (e.g. Tidman and Krall 1971). To show the relevance of our solutions to simulations that use an artificial ratio of the electron to proton mass (e.g. Oka et al. 2010, 2011), we discuss the electron to proton mass ratio dependence of the amplitude of ion acoustic solitary waves, and find that the amplitude of the wave increases with increasing m_e/m_p. We also find that the wave amplitude increases with the ion acoustic Mach number, and the width of the solitons decreases with increasing Mach number, which may be important in understanding the dissipation mechanism at quasi-perpendicular shocks (Zank et al. 1996; Lipatov and Zank 1999; Burrows et al. 2010; Oka et al. 2010).

There are more complicated models of ion acoustic waves than the model adopted in the present paper (see e.g. Baluku et al. (2010) who investigated ion acoustic solitary waves in a plasma with both cool and hot electrons). We will not use these more complicated models in the present paper.

McKenzie and co-workers investigated a variety of two fluid models of fully nonlinear traveling waves in space plasmas (e.g. McKenzie et al. 2004). These solutions encompass both cases in which the charge neutrality assumption is a valid approximation (e.g. for whistler oscillitons), and also for other cases where the charge neutrality assumption is not a good approximation (e.g. for ion acoustic traveling waves where the charge separation electric field is essential in describing the wave structure). Verheest et al. (2004a) discuss the charge neutrality assumption for whistler oscillitons. Dubinin et al. (2007) carried out extensive data analysis on the coherent whistler emissions in the magneto-sphere Cluster observations, and found strong evidence of whistler oscillitons in the data. Dubinov (2007a,b) studied periodic space charge waves. In the limit as the wave spatial period goes to infinity, these solutions reduce to the ion acoustic solitons studied by McKenzie (2002). Dubinov (2007a,b) did not use the same parameters as McKenzie (2002), but the underlying model is the same as that used by McKenzie (2002). McKenzie (2002) did not consider periodic traveling waves with a finite wave period.

Webb et al. (2005, 2007, 2008) developed a Hamiltonian formulation for nonlinear traveling whistler waves in quasi-charge neutral plasmas. Sauer et al. (2001, 2002), Sauer et al. (2003), Dubinin et al. (2003), Dubinin et al. (2007) and McKenzie et al. (2004) studied whistler oscillitons. Webb et al. (2007, 2008) showed that the traveling waves in this model are described by two different but equivalent Hamiltonian formulations. In the first formulation, the Hamiltonian is the total conserved, longitudinal x-momentum integral of the system, in which the energy flux integral $\varepsilon = const.$ is a constraint and for which d/dx is the Hamiltonian evolution operator. In the second Hamiltonian approach, the Hamiltonian is the energy flux integral ε, in which the x-momentum integral $P_x = const.$ is a constraint. In the latter case, the Hamiltonian evolution operator is the advective Lagrangian time derivative operator $d/d\tau = u_x d/dx$. These dual variational principles are analogous

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to the dual or multi-symplectic variational principles obtained by *Bridges* (1992) in studies of traveling water waves (see also Bridges 1997a,b; Bridges et al. 2005). McKenzie et al. (2006) cast the spatial evolution equations for solitary traveling waves in a Hall current plasma in Hamiltonian form, in which the energy flux integral ε is the Hamiltonian and the longitudinal momentum flux integral $P_x = const.$ acts as a constraint. Mace et al. (2007) derived conservation laws for traveling waves in multi-fluid plasmas using Bernoulli type theorems and generalized vorticities for the different plasma species. Hydon (2005) develops the general theory for multi-symplectic Hamiltonian systems, and Cotter et al. (2007) show that multi-symplectic equations can be derived for fluid systems based on Clebsch variables which act as canonically conjugate momenta, and in which the Clebsch variables are Lagrange multipliers for the constraints in the variational principle. Bridges and Reich (2001, 2006), Marsden et al. (1998) and Marsden and Shkoller (1999) use multi-symplectic systems in numerical finite difference schemes for Hamiltonian wave equations (see also Brio et al. (2010) for a discussion of multi-symplectic difference schemes).

In the present paper we develop a multi-symplectic description for ion acoustic traveling waves in an electron–proton, or electron–positron plasma, in which there is a non-trivial electric field induced by charge separation. We also present a detailed description of ion acoustic traveling waves in order to illustrate the different types of solution that are possible (see McKenzie 2002, 2003; McKenzie and Doyle 2003; McKenzie et al. 2004 for a related analysis of fully nonlinear, ion acoustic traveling waves). In Sec. 2, we present the basic equations of the model. In order to keep the analysis simple we consider only the case of plasmas in which there is no magnetic field present, or a constant parallel magnetic field is present in the traveling wave. In Sec. 3, we discuss the dispersion equation for linear waves in a two-fluid, electron–ion plasma. In Sec. 4, we develop the dual variational principles for traveling ion acoustic waves. Section 5 presents examples of the traveling ion acoustic waves, illustrating the physics and including a discussion of their possible applications to both spacecraft observations, and simulations. Section 6 concludes with a summary and discussion.

2. Basic equations

In this section we formulate the traveling wave solution in an electron–proton two-fluid model as a Hamiltonian system. The analysis is based on the multi-fluid equations for an electron–proton fluid. To simplify the analysis we consider only the case where there is a charge separation electric field, but there is no magnetic field. The electron and proton mass continuity equations for the system are:

$$\begin{aligned} &\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\mathbf{u}_e) = 0, \\ &\frac{\partial n_p}{\partial t} + \nabla\cdot(n_p\mathbf{u}_p) = 0, \end{aligned} \tag{2.1}$$

where n_e and n_p are the electron and proton fluid number densities, and $\mathbf{u}_e$ and $\mathbf{u}_p$ are the electron and proton fluid velocities. Poisson's equation for the system is:

$$\varepsilon_0\nabla\cdot\mathbf{E} = e(n_p - n_e), \tag{2.2}$$

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where $\mathbf{E}$ is the electric field. The momentum equations for the system can be written in the form:

$$\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e\cdot\nabla\mathbf{u}_e = -\frac{1}{m_e n_e}\nabla p_e - \frac{e}{m_e}\mathbf{E},$$
$$\frac{\partial \mathbf{u}_p}{\partial t} + \mathbf{u}_p\cdot\nabla\mathbf{u}_p = -\frac{1}{m_p n_p}\nabla p_p + \frac{e}{m_p}\mathbf{E}. \tag{2.3}$$

To complete the equation system, we assume a polytropic equation of state for the electron and proton fluids, i.e. $p_a = p_{a0}(s_a)n_a^{\gamma_a}$ where s_a is the entropy ($a = e, p$). For simplicity, we assume that the entropies s_a are constants throughout the flow (this restriction can be lifted if necessary).

Poisson's equation (2.2) is related to the charge conservation law:

$$\frac{\partial \rho_q}{\partial t} + \nabla\cdot\mathbf{J} = 0, \tag{2.4}$$

where

$$\rho_q = e(n_p - n_e), \quad \mathbf{J} = e(n_p\mathbf{u}_p - n_e\mathbf{u}_e), \tag{2.5}$$

define the charge density ρ_q and the electric current density $\mathbf{J}$. The electric current $\mathbf{J}$ is related to Ampere's law:

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2.6}$$

where $\mathbf{H} = \mathbf{B}/\mu_0$ is the magnetic field strength, $\mathbf{B}$ is the magnetic field induction and $\mathbf{D} = \varepsilon_0\mathbf{E}$ is the electric field displacement. Taking the divergence of Ampere's equation (2.6) gives the charge conservation law (2.4) where we identify the charge density $\rho_q = \nabla\cdot\mathbf{D}$ with the divergence of the electric field displacement $\mathbf{D}$. The latter equation is equivalent to Poisson's equation (2.2).

3. Dispersion equation

For the electron–ion, two fluid plasma model (2.1)–(2.5) without magnetic field effects, the perturbed equations governing linear waves have the form:

$$\frac{\partial \delta n_i}{\partial t} + \frac{\partial}{\partial x}(n_0\delta u_i) = 0,$$
$$\frac{\partial \delta n_e}{\partial t} + \frac{\partial}{\partial x}(n_0\delta u_e) = 0,$$
$$n_0 m_i\frac{\partial \delta u_i}{\partial t} = e n_0\delta E - \frac{\gamma_i p_i}{n_0}\frac{\partial \delta n_i}{\partial x},$$
$$n_0 m_e\frac{\partial \delta u_e}{\partial t} = -e n_0\delta E - \frac{\gamma_e p_e}{n_0}\frac{\partial \delta n_e}{\partial x},$$
$$\frac{\partial \delta E}{\partial x} = 4\pi e\,(\delta n_i - \delta n_e)\,, \tag{3.1}$$

where $\delta\psi$ denotes the perturbation of the physical quantity ψ. Assuming perturbations of the form $\delta\psi \propto \exp(ikx - i\omega t)$, the system (3.1) reduces to the algebraic equation:

$$k^2\left(1 - \frac{\omega_{pi}^2}{\omega^2 - k^2c_i^2} - \frac{\omega_{pe}^2}{\omega^2 - k^2c_e^2}\right)\delta E = 0. \tag{3.2}$$

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Thus, the dispersion equation for the system is:

$$D(k,\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 c_i^2} - \frac{\omega_{pe}^2}{\omega^2 - k^2 c_e^2} = 0, \tag{3.3}$$

where

$$c_i = \left(\frac{\gamma_i p_i}{n_0 m_i}\right)^{1/2}, \quad c_e = \left(\frac{\gamma_e p_e}{n_0 m_e}\right)^{1/2}, \tag{3.4}$$

are the adiabatic sound speeds for the ions (c_i) and electrons (c_e). In (3.1) we assume adiabatic equations of state for the electron and proton fluids, with adiabatic indices γ_i and γ_e. The quantities

$$\omega_{pi} = \left(\frac{4\pi n e^2}{m_i}\right)^{1/2} \quad \text{and} \quad \omega_{pe} = \left(\frac{4\pi n e^2}{m_e}\right)^{1/2}, \tag{3.5}$$

are the ion and electron plasma frequencies respectively.

The general structure of the dispersion equation (3.3) may be deduced by writing (3.3) in the form:

$$k^2 = \frac{\omega_{pi}^2}{\lambda^2 - c_i^2} + \frac{\omega_{pe}^2}{\lambda^2 - c_e^2} \quad \text{where} \quad \lambda = \frac{\omega}{k}, \tag{3.6}$$

is the phase speed. A sketch of k^2 versus λ (assuming $c_e > c_i$) reveals that the roots for the phase speed $\lambda = \omega/k$ are located in the ranges (a) $c_i^2 < \lambda^2 < \zeta^2$ and (b) $\lambda^2 > c_e^2$ where

$$\zeta^2 = \frac{c_i^2 \omega_{pe}^2 + c_e^2 \omega_{pi}^2}{\omega_{pe}^2 + \omega_{pi}^2}. \tag{3.7}$$

The solutions (a) correspond to the ion acoustic branch of the dispersion equation, and range (b) corresponds to the Langmuir wave branch. The quantity ζ is the generalized ion acoustic speed. As $k \to \infty$, $\lambda \to c_i$ for branch (a) and $\lambda \to c_e$ for branch (b), i.e. the characteristic wave speeds at short length scales ($k \to \infty$) correspond to electron and ion sound waves.

Alternatively, ω^2 from (3.3) satisfies the biquadratic equation:

$$\omega^4 - \alpha\omega^2 + \beta = 0, \tag{3.8}$$

where

$$\begin{aligned} \alpha &= k^2 c_e^2 + k^2 c_i^2 + \omega_{pi}^2 + \omega_{pe}^2, \\ \beta &= k^4 c_i^2 c_e^2 + k^2\left(\omega_{pi}^2 c_e^2 + \omega_{pe}^2 c_i^2\right). \end{aligned} \tag{3.9}$$

The solutions of (3.8) are

$$\omega_{\pm}^2 = \frac{1}{2}\left(\alpha \pm \sqrt{\Delta}\right), \quad \Delta = \alpha^2 - 4\beta. \tag{3.10}$$

The discriminant Δ reduces to:

$$\Delta = \left(\omega_{pe}^2 + \omega_{pi}^2\right)^2 + 2k^2\left(\omega_{pe}^2 - \omega_{pi}^2\right)\left(c_e^2 - c_i^2\right) + k^4\left(c_e^2 - c_i^2\right)^2. \tag{3.11}$$

The ω_+^2 root corresponds to the Langmuir wave branch (b) and ω_-^2 corresponds to the ion acoustic wave branch (a).

3.1. *Long wavelength approximation*

It is instructive to investigate the dispersion equation (3.8) solutions in the limit as $k \to 0$. As $k \to 0$ the approximate solution of the dispersion equation for the ion

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acoustic wave to $O(k^4)$ is:

$$\omega = kc_{ia} - \gamma k^3, \tag{3.12}$$

where

$$c_{ia} \equiv \zeta = \left(\frac{c_i^2 \omega_{pe}^2 + c_e^2 \omega_{pi}^2}{\omega_{pe}^2 + \omega_{pi}^2} \right)^{1/2}, \tag{3.13}$$

$$\gamma = \frac{\omega_{pe}^2 \omega_{pi}^2}{\left(\omega_{pe}^2 + \omega_{pi}^2\right)^3} \frac{(c_e^2 - c_i^2)^2}{2c_{ia}}. \tag{3.14}$$

Here c_{ia} is the generalized ion acoustic wave speed (in the limit $\omega_{pe}^2 \gg \omega_{pi}^2$ and $c_i = 0$ the ion acoustic speed $c_{ia} = [\gamma_e p_e/(nm_i)]^{1/2}$), and γ gives the dispersion of the wave. Using the association $\partial/\partial t \to -i\omega$ and $\partial/\partial x \to ik$ for perturbations $\delta\psi \propto \exp(ikx - i\omega t)$, (3.12) gives the linearized Korteweg de Vries (KdV) equation:

$$\left(\frac{\partial}{\partial t} + c_{ia} \frac{\partial}{\partial x} + \gamma \frac{\partial^3}{\partial x^3} \right) \delta\psi = 0, \tag{3.15}$$

for long-wavelength ion acoustic waves. The original derivation of the KdV equation describing ion acoustic solitons was derived by Washimi and Taniuti (1966). Their derivation assumed isothermal hot electrons and cold ions. Moslem (1999, 2000) derived the KdV equation for ion acoustic waves, including negative ions, positrons and other particle species. They assumed adiabatic equations of state for the different species, with adiabatic index of $\gamma = 3$, and included a beam plasma component. Other authors developed kinetic plasma models for the ion acoustic wave including the effects of Landau damping. In particular, Ott and Sudan (1969) developed a nonlinear theory for a nonlinear ion acoustic wave. They obtained a KdV equation, with a Cauchy Principal value integral term representing the effects of Landau damping.

Dimensional analysis of (3.12)–(3.15) gives the characteristic length and time scales:

$$L = \sqrt{\frac{\gamma}{c_{ia}}}, \quad T = \frac{L}{c_{ia}}, \tag{3.16}$$

for the waves. In the cold ion gas limit ($c_i = 0$) , and for an electron–proton plasma with $\omega_{pe}^2 \gg \omega_{pi}^2$, the length and time scales (3.16) reduce to:

$$L = \frac{c_e}{\omega_{pe}}, \quad T = \frac{1}{\omega_{pi}} \quad \text{and} \quad c_{ia} \approx \left(\frac{\gamma_e p_e}{nm_i} \right)^{1/2}. \tag{3.17}$$

It is interesting to note that $\gamma = 0$ if $c_e = c_i$. In this case the ion acoustic wave is non-dispersive (see Sec. 3.1.1; note however, for a hot plasma Landau damping is important and a plasma kinetic treatment is required).

A similar analysis of the Langmuir wave branch in (3.8)–(3.10) for ω_+^2 gives:

$$\omega_+^2 = \omega_{pe}^2 + \omega_{pi}^2 + k^2 a_2 + k^4 a_4, \tag{3.18}$$

where

$$a_2 = \frac{c_e^2 \omega_{pe}^2 + c_i^2 \omega_{pi}^2}{\omega_{pe}^2 + \omega_{pi}^2}, \tag{3.19}$$

$$a_4 = \frac{\omega_{pe}^2 \omega_{pi}^2 (c_e^2 - c_i^2)^2}{(\omega_{pe}^2 + \omega_{pi}^2)^3}. \tag{3.20}$$

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Using the association $\partial_t = -i\omega$ and $\partial_x = ik$, and dropping the $O(k^4)$ term in (3.18) we obtain the Klein–Gordon equation:

$$\left(\frac{\partial^2}{\partial t^2} - a_2\frac{\partial^2}{\partial x^2} + \left(\omega_{pe}^2 + \omega_{pi}^2\right)\right)\delta\psi = 0, \tag{3.21}$$

for linear wave perturbations. An alternative first-order wave equation follows from (3.18) by noting:

$$\omega_+ = \bar{\omega}\left[1 + \frac{k^2 a_2}{2\bar{\omega}^2} + O(k^4)\right], \tag{3.22}$$

where $\bar{\omega} = (\omega_{pe}^2 + \omega_{pi}^2)^{1/2}$. The wave equation corresponding to (3.22) is the linear Schroedinger equation:

$$\left[i\frac{\partial}{\partial t} + \kappa\frac{\partial^2}{\partial x^2} - \bar{\omega}\right]\delta\psi = 0, \tag{3.23}$$

where

$$\kappa = \frac{c_e^2\omega_{pe}^2 + c_i^2\omega_{pi}^2}{2\bar{\omega}^3}, \quad \bar{\omega} = \left(\omega_{pe}^2 + \omega_{pi}^2\right)^{1/2}. \tag{3.24}$$

3.1.1. *Special case* $c_e = c_i$.

In this case, the solutions for $\omega_\pm^2$ in (3.10) reduce to:

$$\omega_- = \pm kc_e \quad \text{and} \quad \omega_+^2 = k^2c_e^2 + \omega_{pe}^2 + \omega_{pi}^2. \tag{3.25}$$

Thus, the ion acoustic wave in this limit (i.e. $\omega = \omega_- = \pm kc_e$) reduces to a non-dispersive sound wave satisfying either of the equations:

$$\left(\frac{\partial}{\partial t} + c_e\frac{\partial}{\partial x}\right)\delta\psi = 0 \quad \text{or} \quad \left(\frac{\partial}{\partial t} - c_e\frac{\partial}{\partial x}\right)\delta\psi = 0 \tag{3.26}$$

corresponding to the forward and backward sound waves respectively.

Similarly, the Langmuir wave branch solution reduces to the dispersion equation:

$$\omega_+^2 = k^2c_e^2 + \omega_{pe}^2 + \omega_{pi}^2. \tag{3.27}$$

The corresponding wave equation is the Klein–Gordon equation:

$$\left(\frac{\partial^2}{\partial t^2} - c_e^2\frac{\partial^2}{\partial x^2} + \left(\omega_{pe}^2 + \omega_{pi}^2\right)\right)\delta\psi = 0. \tag{3.28}$$

Equation (3.28) shows that wave dispersion for the ion acoustic wave is intrinsically linked to the difference $c_e^2 - c_i^2$ of the squares of the electron and ion sound speeds. The dispersionless limit $c_e \to c_i$ for the nonlinear version of the linear KdV equation (3.15) is not straightforward. The dispersionless limit of the KdV equation has been investigated by Levermore (1988).

3.1.2. *Short wavelength limit.*

At short wavelengths ($k \to \infty$) the dispersion equation solutions (3.10) reduce to:

$$\omega_+^2 = k^2c_e^2 + \omega_{pe}^2 + \frac{\omega_{pe}^2\omega_{pi}^2}{(c_e^2 - c_i^2)}\frac{1}{k^2} + O\left(\frac{1}{k^4}\right), \tag{3.29}$$

$$\omega_-^2 = k^2c_i^2 + \omega_{pi}^2 - \frac{\omega_{pe}^2\omega_{pi}^2}{(c_e^2 - c_i^2)}\frac{1}{k^2} + O\left(\frac{1}{k^4}\right). \tag{3.30}$$

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It is straightforward to write down partial differential equations (or integral equations) corresponding to the dispersion equation expansions (3.29) and (3.30) using the Fourier correspondence: $i\partial/\partial t \to \omega$ and $i\partial/\partial x \to -k$.

Examples of fully nonlinear traveling waves of the ion acoustic or Langmuir wave type are investigated in Sec. 5. In the next section, we describe a multi-symplectic Hamiltonian formulation of the nonlinear traveling waves for the ion acoustic and Langmuir traveling waves.

4. Multi-symplectic traveling waves

The multi-fluid plasma system described by (2.1)–(2.5) is a Hamiltonian system (e.g. Spencer 1982; Spencer and Kaufman 1982; Holm and Kupershmidt 1983). The non-canonical Poisson bracket for the two fluid electron–proton two fluid plasma described by (2.1)–(2.5) can be used to write the evolution equations for the system in terms of the non-canonical Eulerian physical variables using the Poisson bracket. We use a direct approach to write the traveling wave solutions in a canonical Hamiltonian form.

For traveling wave solutions of (2.1)–(2.5), we look for solutions of the form $\psi^\alpha = \psi^\alpha(\xi)$ where the physical variables ψ^α depend only on the traveling wave variable $\xi = x - \lambda t$ where λ is the speed of the traveling wave. Substitution of the ansatz $\psi^\alpha = \psi^\alpha(\xi)$ into the governing equations (2.1)–(2.5) gives the system of ordinary differential equations:

$$\frac{d}{d\xi}(n_e u_e) = \frac{d}{d\xi}(n_p u_p) = 0, \tag{4.1}$$

$$u_e \frac{du_e}{d\xi} = -\frac{1}{m_e n_e}\frac{dp_e}{d\xi} - \frac{e}{m_e}E_x, \tag{4.2}$$

$$u_p \frac{du_p}{d\xi} = -\frac{1}{m_p n_p}\frac{dp_p}{d\xi} + \frac{e}{m_p}E_x, \tag{4.3}$$

$$\frac{d}{d\xi}J = \frac{d}{d\xi}[e(n_p u_p - n_e u_e)] = 0, \tag{4.4}$$

$$\varepsilon_0 \frac{dE_x}{d\xi} = e(n_p - n_e), \tag{4.5}$$

Here u_e and u_p now refer to the x-component of the fluid velocity in the traveling waves frame, and E_x is the x-component of the electric field.

Integration of the number density continuity equations and the current continuity equation (4.3) gives the integrals:

$$n_p u_p = n_e u_e = j, \tag{4.6}$$

where j is the common integration constant for the particle fluxes, that ensures that the electric current J is zero. Multiplying (4.2) by $m_e n_e$ and (4.3) by $m_p n_p$, adding the two resultant equations, and using Poisson's equation to integrate the electric field term $eE_x(n_p - n_e)$ gives the x—momentum integral for the system as:

$$j(m_e u_e + m_p u_p) + p_e + p_p - \varepsilon_0 \frac{E_x^2}{2} = P_x, \tag{4.7}$$

where P_x is the total x-momentum integration constant. Similarly multiplying (4.2) by $m_e n_e u_e$ and (4.3) by $m_p n_p u_p$, adding the two equations, and integrating the resultant equation with respect to x gives the total energy flux conservation equations for the

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system in the form:

$$j\left(\frac{1}{2}m_e u_e^2+\frac{\gamma_e p_e}{(\gamma_e-1)n_e}\right)+j\left(\frac{1}{2}m_p u_p^2+\frac{\gamma_p p_p}{(\gamma_p-1)n_p}\right)=\varepsilon. \tag{4.8}$$

In the integration of (4.8) we assumed polytropic equations of state for the electron and proton gases, pressures, i.e.

$$p_e=p_{e0}\left(\frac{n_e}{n_{e0}}\right)^{\gamma_e} \quad \text{and} \quad p_p=p_{p0}\left(\frac{n_p}{n_{p0}}\right)^{\gamma_p}. \tag{4.9}$$

Using (4.5) and the equations of state (4.9) allows the electron and proton $x-$momentum equations to be written in the forms:

$$\frac{(u_e^2-c_e^2)}{u_e}\frac{du_e}{d\xi}=-\frac{e}{m_e}E_x, \quad \text{and} \quad \frac{(u_p^2-c_p^2)}{u_p}\frac{du_p}{d\xi}=\frac{e}{m_p}E_x, \tag{4.10}$$

where $c_e^2=\gamma_e p_e/(n_e m_e)$ and $c_p^2=\gamma_p p_p/(n_p m_p)$ are the squares of the electron and proton sound speeds respectively. Equation (4.10) can also be written in the form:

$$\frac{d\varepsilon_e}{d\xi}=-\frac{e}{m_e}E_x, \quad \text{and} \quad \frac{d\varepsilon_p}{d\xi}=\frac{e}{m_p}E_x, \tag{4.11}$$

where

$$\varepsilon_e=\frac{1}{2}u_e^2+\frac{c_e^2}{\gamma_e-1} \quad \text{and} \quad \varepsilon_p=\frac{1}{2}u_p^2+\frac{c_p^2}{\gamma_p-1}, \tag{4.12}$$

are the kinetic and enthalpy contributions to the normalized energy fluxes of the electron and proton gases. Multiplying the first equation (4.11) by jm_e and the second equation (4.11) by jm_p and integrating with respect to x gives the total energy equation (4.8).

4.1. *Variational principles*

PROPOSITION 4.1. *The electron–proton multi-fluid traveling waves described by (4.1)–(4.12) are described by the stationary point conditions for the variational functional:*

$$\mathscr{A}=\int L d\xi, \tag{4.13}$$

where the Lagrangian density L is:

$$L=E_x\frac{d\tilde{\varepsilon}_e}{d\xi}-[\Pi_x(u_e,u_p,E_x)-P_x]+\lambda\left(\tilde{\varepsilon}_e+\tilde{\varepsilon}_p-\tilde{\varepsilon}\right). \tag{4.14}$$

Here

$$\tilde{\varepsilon}_e=\frac{\varepsilon_0 m_e}{e}\varepsilon_e, \quad \tilde{\varepsilon}_p=\frac{\varepsilon_0 m_p}{e}\varepsilon_p, \quad \tilde{\varepsilon}=\frac{\varepsilon_0}{je}\varepsilon, \tag{4.15}$$

are the normalized electron and proton energy fluxes defined in (4.12), and ε is the total energy integration constant in (4.8), $j=n_e u_e=n_p u_p$ and $\Pi_x(u_e,u_p,E_x)=P_x$ is the x-momentum integral (4.7). The Lagrange multiplier λ ensures that the energy conservation integral (4.8) is satisfied.

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Proof. The stationary point conditions $\delta\mathscr{A}/\delta\lambda = 0$, $\delta\mathscr{A}/\delta u_p = 0$, $\delta\mathscr{A}/\delta u_e = 0$ give the equations:

$$\mathscr{A}_\lambda = \tilde{\varepsilon}_e + \tilde{\varepsilon}_p - \tilde{\varepsilon} = 0, \tag{4.16}$$

$$\mathscr{A}_{u_p} = m_p \left(1 - \frac{c_p^2}{u_p^2}\right)(\lambda u_p - j) = 0, \tag{4.17}$$

$$\mathscr{A}_{u_e} = m_e \left(1 - \frac{c_e^2}{u_e^2}\right)\left(-\frac{\varepsilon_0}{e}\frac{dE_x}{d\xi} - (n_e - \lambda)\right) = 0, \tag{4.18}$$

where we use the notation $\mathscr{A}_\psi \equiv \delta\mathscr{A}/\delta\psi$ for the variational derivative of $\mathscr{A}$ with respect to the variable ψ. Equation (4.16) is equivalent to the energy integral (4.8). Equation (4.17) determines the Lagrange multiplier λ, i.e.

$$\lambda = \frac{j}{u_p} = n_p. \tag{4.19}$$

Using $\lambda = n_p$ in (4.18) gives the Poisson equation (4.5), i.e.

$$\varepsilon_0 \frac{dE_x}{d\xi} = e(n_p - n_e). \tag{4.20}$$

The variational equation

$$\mathscr{A}_{E_x} = \frac{\varepsilon_0 m_e}{e}\left(\frac{d\varepsilon_e}{d\xi} + \frac{e}{m_e}E_x\right) = 0, \tag{4.21}$$

gives the electron momentum equation in (4.11). The remaining equations of the system follow from the above variational equations. For example, differentiation of the energy integral (4.16) with respect to ξ, coupled with the electron momentum equation (4.21), gives the proton momentum equation in (4.11). The total momentum equation (4.7) follows by the integration of a suitable combination of the electron and proton momentum equations as indicated in (4.12) et seq. This completes the proof. □

Proposition 4.2. *The electron energy equation in (4.11) and Poisson's equation (4.5) can be written in the Hamiltonian form:*

$$\frac{d\tilde{\varepsilon}_e}{d\xi} = \frac{\partial P_x}{\partial E_x}, \tag{4.22}$$

$$\frac{dE_x}{d\xi} = -\frac{\partial P_x}{\partial \tilde{\varepsilon}_e}, \tag{4.23}$$

where the partial derivatives of the total momentum function P_x are taken keeping the total energy flux ε in (4.8) constant, and

$$\tilde{\varepsilon}_e = \frac{\varepsilon_0 m_e}{e}\varepsilon_e \equiv \frac{\varepsilon_0 m_e}{e}\left[\frac{1}{2}u_e^2 + \frac{c_e^2}{\gamma_e - 1}\right],$$

$$\tilde{\varepsilon}_p = \frac{\varepsilon_0 m_p}{e}\varepsilon_p \equiv \frac{\varepsilon_0 m_p}{e}\left[\frac{1}{2}u_p^2 + \frac{c_p^2}{\gamma_p - 1}\right], \tag{4.24}$$

are re-normalized energy fluxes of the electron and proton gases.

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Proof. To prove (4.22) note from (4.11) and (4.7) that

$$\frac{d\tilde{\varepsilon}_e}{d\xi} = -\varepsilon_0 E_x \quad \text{and} \quad \frac{\partial P_x}{\partial E_x} = -\varepsilon_0 E_x. \tag{4.25}$$

These two equations imply (4.22).

To prove (4.23) note that as the total energy flux ε is held constant, that $u_p = u_p(u_e)$ is a function of u_e. Hence

$$\frac{\partial P_x}{\partial \tilde{\varepsilon}_e} = \frac{\partial P_x}{\partial u_e} \Big/ \frac{d\tilde{\varepsilon}_e}{du_e}, \quad \frac{\partial P_x}{\partial u_e} = \left(\frac{\partial P_x}{\partial u_e}\right)_{u_p} + \frac{\partial P_x}{\partial u_p}\frac{du_p}{du_e}. \tag{4.26}$$

Separately evaluating the different terms in (4.26) we obtain:

$$\begin{gathered}
\frac{d\tilde{\varepsilon}_e}{du_e} = \frac{\varepsilon_0 m_e}{e}\frac{d\varepsilon_e}{du_e} = \frac{\varepsilon_0 m_e}{e}\frac{(u_e^2 - c_e^2)}{u_e}, \\
\left(\frac{\partial P_x}{\partial u_e}\right)_{u_p} = \frac{j m_e (u_e^2 - c_e^2)}{u_e^2}, \quad \frac{\partial P_x}{\partial u_p} = \frac{j m_p (u_p^2 - c_p^2)}{u_p^2}, \\
\frac{du_p}{du_e} = -\frac{m_e u_p (u_e^2 - c_e^2)}{m_p u_e (u_p^2 - c_p^2)}, \quad \frac{\partial P_x}{\partial u_e} = \frac{j m_e (u_e^2 - c_e^2)}{u_e^2}\left(1 - \frac{n_p}{n_e}\right).
\end{gathered} \tag{4.27}$$

Using the results (4.27) in (4.26) gives

$$\frac{\partial P_x}{\partial \tilde{\varepsilon}_e} = \frac{e(n_e - n_p)}{\varepsilon_0} = -\frac{dE_x}{d\xi}. \tag{4.28}$$

This completes the proof. □

PROPOSITION 4.3. *The electron–proton multi-fluid traveling waves (4.1)–(4.12) can be obtained from the stationary point conditions for the variational functional:*

$$\mathscr{A} = \int L_2 \, d\tau, \tag{4.29}$$

where the Lagrangian density L_2 has the form:

$$L_2 = E_x \frac{d\tilde{\varepsilon}_e}{d\tau} + \left[E(u_e, u_p) - \varepsilon\right] + \lambda \left[\Pi_x(u_e, u_p, E_x) - P_x\right]. \tag{4.30}$$

In (4.30)

$$E(u_e, u_p) = \varepsilon, \quad \Pi_x(u_e, u_p, E_x) = P_x, \tag{4.31}$$

are the total energy and momentum conservation laws (4.8) and (4.7) respectively, and $\tilde{\varepsilon}_a = \varepsilon_0(m_a \varepsilon_a)/e$ $(a = e, p)$. The Lagrange multiplier λ in (4.30) ensures that the momentum conservation equation (4.7) is satisfied. The parameter

$$\tau = \int_{\xi_0}^{\xi} \frac{d\xi}{u_p}, \tag{4.32}$$

is the time traveled by the proton fluid from some fiducial point ξ_0 to the position ξ in the wave frame. Thus,

$$\frac{d\tilde{\varepsilon}_e}{d\tau} = u_p \frac{d\tilde{\varepsilon}_e}{d\xi}. \tag{4.33}$$

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Proof. Evaluating $\delta\mathscr{A}/\delta u_p$, $\delta\mathscr{A}/\delta E_x$ and $\delta\mathscr{A}/\delta u_e$ gives the equations:

$$\mathscr{A}_{u_p} = jm_p\left(1-\frac{c_p^2}{u_p^2}\right)(\lambda+u_p) = 0, \tag{4.34}$$

$$\mathscr{A}_{E_x} = \frac{d\tilde{\varepsilon}_e}{d\tau} - \lambda(\varepsilon_0 E_x) = 0, \tag{4.35}$$

$$\mathscr{A}_{u_e} = -\left(\frac{\varepsilon_0 m_e}{e}\right)u_e\left(1-\frac{c_e^2}{u_e^2}\right)\left[\frac{dE_x}{d\tau} - u_p\frac{e}{\varepsilon_0}(n_p-n_e)\right] = 0. \tag{4.36}$$

Solving (4.34) for λ gives:

$$\lambda = -u_p. \tag{4.37}$$

Equations (4.35) and (4.36) reduce to:

$$\frac{d\tilde{\varepsilon}_e}{d\tau} = -u_p(\varepsilon_0 E_x) \quad \text{or} \quad \frac{d\tilde{\varepsilon}_e}{d\xi} = -(\varepsilon_0 E_x), \tag{4.38}$$

$$\frac{dE_x}{d\tau} = u_p\frac{e}{\varepsilon_0}(n_p-n_e) \quad \text{or} \quad \frac{dE_x}{d\xi} = \frac{e}{\varepsilon_0}(n_p-n_e). \tag{4.39}$$

Thus, (4.35)–(4.36) or (4.38)–(4.39) are equivalent to the electron momentum equation in (4.11) and to Poisson's equation (4.5). The equation $\delta\mathscr{A}/\delta\lambda = 0$ gives the total x-momentum conservation law (4.7). The above equations and the total momentum conservation equation (4.7) imply the two-fluid equations (4.1)–(4.11) for the traveling waves. This completes the proof. □

Remark 1. The present formulation of the action for the traveling waves clearly differs from that in Proposition 4.1, where the evolution operator is the spatial operator $d/d\xi$ which is different from the time evolution operator $d/d\tau = u_p d/d\xi$ in the present formulation.

Proposition 4.4. *The electron momentum equation (4.11) and Poisson's equation (4.5) can be written in the Hamiltonian form:*

$$\frac{d\tilde{\varepsilon}_e}{d\tau} = -\frac{\partial E}{\partial E_x}, \tag{4.40}$$

$$\frac{dE_x}{d\tau} = \frac{\partial E}{\partial \tilde{\varepsilon}_e}, \tag{4.41}$$

where

$$\tau = \int^{\xi}\frac{d\xi}{u_p}, \tag{4.42}$$

is the evolution variable and $E(u_p, u_e) = \varepsilon$ is the energy integral (4.8). The partial derivatives of $E(u_e, u_p)$ are taken keeping the momentum integral $\Pi_x(u_e, u_p, E_x)$ constant where $\Pi_x(u_e, u_p, E_x) = P_x$ is the momentum integral (4.7).

Proof. Since Π_x is constant, then the momentum integral $\Pi_x(u_e, u_p, E_x) = P_x$ can be solved for $u_p = u_p(u_e, E_x)$ as a function of u_e and E_x. To compute $\partial E/\partial E_x$ in (4.40) we note

$$\left(\frac{\partial E}{\partial E_x}\right)_{u_e} = \frac{\partial E}{\partial u_p}\frac{\partial u_p}{\partial E_x}. \tag{4.43}$$

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Using the results:

$$\frac{\partial u_p}{\partial E_x} = \frac{\varepsilon_0 E_x u_p}{n_p m_p (u_p^2 - c_p^2)}, \quad \left(\frac{\partial E}{\partial u_p}\right)_{u_e} = n_p m_p (u_p^2 - c_p^2), \tag{4.44}$$

(4.43) reduces to:

$$\frac{\partial E}{\partial E_x} = u_p \varepsilon_0 E_x. \tag{4.45}$$

The electron momentum equation (4.11) is:

$$\frac{d\varepsilon_e}{d\xi} = -\frac{e}{m_e} E_x. \tag{4.46}$$

Equations (4.45) and (4.46) imply Hamilton's equation (4.40).

To compute $\partial E/\partial \tilde{\varepsilon}_e$ we note:

$$\frac{\partial E}{\partial \tilde{\varepsilon}_e} = \frac{\partial E}{\partial u_e} \Big/ \frac{d\tilde{\varepsilon}_e}{du_e}, \quad \left(\frac{\partial E}{\partial u_e}\right)_{E_x} = \left(\frac{\partial E}{\partial u_e}\right)_{u_p} + \frac{\partial E}{\partial u_p}\left(\frac{\partial u_p}{\partial u_e}\right)_{E_x}. \tag{4.47}$$

Separately computing the terms in (4.47) we obtain:

$$\begin{aligned}
&\frac{d\tilde{\varepsilon}_e}{du_e} = \frac{\varepsilon_0 m_e}{e}\frac{d\varepsilon_e}{du_e} = \frac{\varepsilon_0 m_e}{e}\frac{(u_e^2 - c_e^2)}{u_e}, \quad \left(\frac{\partial E}{\partial u_e}\right)_{u_p} = \frac{j m_e u_e (u_e^2 - c_e^2)}{u_e^2}, \\
&\frac{\partial E}{\partial u_p} = \frac{j m_p u_p (u_p^2 - c_p^2)}{u_p^2}, \quad \frac{du_p}{du_e} = -\frac{m_e u_p (u_e^2 - c_e^2)}{m_p u_e (u_p^2 - c_p^2)}, \\
&\frac{\partial E}{\partial u_e} = \frac{j m_e (u_e^2 - c_e^2)}{u_e}\left(1 - \frac{n_e}{n_p}\right).
\end{aligned} \tag{4.48}$$

Using (4.48) in (4.47) gives:

$$\frac{\partial E}{\partial \tilde{\varepsilon}_e} = u_p \left[\frac{e}{\varepsilon_0}(n_p - n_e)\right] \equiv u_p \frac{dE_x}{d\xi} = \frac{dE_x}{d\tau}, \tag{4.49}$$

which establishes the Hamiltonian equation (4.41) for $dE_x/d\tau$. This completes the proof. □

5. Integral form of steady-state solitary wave signatures in a multi-fluid plasma

We consider a steady-state, multi-fluid plasma system, viewed in the wave frame, where the bulk velocity of each plasma species is $\hat{x}u_0$ as $x \to -\infty$ and the wave form appears stationary so that $\partial/\partial t \to 0$. Thus the momentum equations and Poisson's equation are

$$\frac{d}{dx}\left(p_j + m_j n_j u_j^2\right) = q_j n_j E_x; \tag{5.1}$$

$$\frac{dE_x}{dx} = 4\pi \sum_j q_j n_j. \tag{5.2}$$

By adopting the McKenzie approach of writing dimensionless velocities as $u_j \to u_j/u_0$ and using the continuity

$$n_j u_j = n_{j0} u_0$$

and adiabatic energy

$$p_j u_j^{\gamma_j} = p_{j0} u_0^{\gamma_j}$$

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Charge	Mach number	Potential	Velocity from IP
Ions ($q > 0$)	$M_i > 1$	$\phi > 0$	decelerates
	$M_i > 1$	$\phi < 0$	accelerates
	$M_i < 1$	$\phi > 0$	accelerates
	$M_i < 1$	$\phi < 0$	decelerates
Electrons ($q < 0$)	$M_e > 1$	$\phi > 0$	accelerates
	$M_e > 1$	$\phi < 0$	decelerates
	$M_e < 1$	$\phi > 0$	decelerates
	$M_e < 1$	$\phi < 0$	accelerates

TABLE 1. Velocity behavior at the initial point (IP), acceleration or deceleration as determined from (5.6), for ions and electron species. $\phi > 0$ indicates a potential hill whereas $\phi < 0$ indicates a potential well.

relations, the momentum equation (5.1) can be written as

$$\frac{d}{dx}\left[\frac{u_j^2}{2} + \frac{u_j^{1-\gamma_j}}{(\gamma_j - 1)M_j^2}\right] = \frac{q_j E_x}{m_j u_0^2}, \tag{5.3}$$

where the 0-subscript denotes a constant upstream state and

$$M_j = \frac{u_0}{c_{j0}}, \quad c_{j0}^2 = \frac{\gamma_j p_{j0}}{m_j n_{j0}},$$

is the jth species sound speed Mach number. Integrating (5.1), using (5.2), yields

$$E_x = \pm\sqrt{8\pi u_0^2 \sum_j m_j n_{j0} P(u_j)}, \tag{5.4}$$

$$P(u_j) = (u_j - 1) + \frac{1}{\gamma_j M_j^2}(u^{-\gamma_j} - 1), \tag{5.5}$$

which gives the electric field in terms of conservation of total momentum and where the generalized momentum functions P_j are composed of two terms: the first associated with the dynamic pressure and the second with the thermal pressure of the jth species. Introducing the electrostatic potential $d\phi/dx = -E_x$ and again integrating the momentum equation (5.3) yields conservation of energy

$$\frac{m_j u_0^2}{q_j}\varepsilon(u_j) = \frac{m_i u_0^2}{q_i}\varepsilon(u_i) = -\phi, \tag{5.6}$$

$$\varepsilon(u_j) = \frac{u_j^2 - 1}{2} + \frac{(u_j^{1-\gamma_j} - 1)}{(\gamma_j - 1)M_j^2}, \tag{5.7}$$

where the first equation (5.6) expresses the energy proportionality between the jth and ith species, and (5.7) defines the Bernoulli energy function ε_j, composed of two terms: the first associated with dynamic energy and the second with the enthalpy of the species. The relations (5.4) through (5.7) yield the wave amplitude and necessary conditions for a solitary wave to exist. As summarized in Table 1, the energy proportionality relations (5.6) and (5.7) can be employed to determine whether a fluid species will accelerate or decelerate from the initial point (McKenzie 2002; Verheest et al. 2004b).

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The functions P_j and ε_j can be used to calculate solitary wave signatures. Using (5.6) and (5.7), the velocity of any plasma species i can be expressed as a function of the jth species in the form $u_i = u_i(u_j)$. Thus all the ith momentum functions can be written in terms of the jth species velocity, $P_i = P(u_i(u_j))$, so that the electric field can also be expressed as a function of the same single species velocity: $E_x = E_x(u_j)$. The above procedure can always (in principle) be carried out, although it is sometimes more convenient to do so numerically.

We write the structure equation (5.3) as

$$x = \int_{u_c}^{u_j} \frac{u\left(1 - \dfrac{1}{M_j^2 u^{\gamma_j+1}}\right) \mathrm{d}u}{q_j E_x / m_j u_0^2}, \tag{5.8}$$

where u_c is the velocity of species j at the center of the wave where $x \equiv 0$. Note that since $E_x = E_x(u_j)$, equation (5.8) can be used directly to integrate solitary wave signatures for a plasma system composed of any number of different fluid species. Thus equations (5.4) through (5.7) along with the structure equation (5.8) form a complete description for solitary waves in an adiabatic, multi-fluid plasma. However, the determination of existence conditions and integration of the structure equation is not generally straightforward and should be treated carefully on a case-by-case basis.

5.1. *Integration of a plasma composed of cold protons and hot electrons*

In this sub-section we illustrate the utility of the closed system of multi-fluid conservation laws by considering a two-fluid plasma composed of cold (highly supersonic, $T_p = 0$) protons and hot (subsonic) electrons. The utility of the structure equation (5.8) is exploited to integrate exact solutions, revealing properties particularly important for particle reflection – the width and amplitude of the wave, both of which are found to depend critically on the Mach number of the incident flow.

On using the normalizations,

$$\begin{aligned} &N = n_p/n_0, \quad && n = n_e/n_0, \quad && U = u_p/u_0, \quad && u = u_e/u_0, \\ &E_x \to E_x/E_0, \quad && x \to x/\lambda_D, \quad && E_0 = \frac{kT_{e0}}{e\lambda_D}, \quad && \lambda_D^2 = \frac{kT_{e0}}{4\pi n_0 e^2}, \\ &\alpha_0 = m_e/m_p, \quad && \alpha_1^2 = \frac{\alpha_0}{\gamma M_e^2}, \end{aligned} \tag{5.9}$$

equation (5.8) becomes

$$x = \int_{u_c}^{u} f_{\pm}(u)du; \tag{5.10}$$

$$f_{\pm}(u) = \frac{\left(\dfrac{1}{M_e^2 u^{\gamma+1}} - 1\right)\alpha_0 u}{\pm\sqrt{2\alpha_1^2\left(U - 1 + \alpha_0(u-1) + \alpha_1^2(u^{-\gamma} - 1)\right)}},$$

where, in view of (5.6) and (5.7),

$$U = \sqrt{1 + \alpha_0(1-u^2) + \frac{2\gamma\alpha_1^2}{\gamma - 1}(1 - u^{1-\gamma})} \tag{5.11}$$

and u_c is the electron velocity at the center of the solitary wave where $x = 0$. This expression explicitly retains the electron to proton mass ratio α_0.

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The ion acoustic Mach number M_{ia} is defined by the equation:

$$M_{ia}^2 = \frac{u_0^2}{c_{ia}^2} \quad \text{where} \quad c_{ia}^2 = \frac{c_i^2 m_i + c_e^2 m_e}{m_i + m_e}. \tag{5.12}$$

Expression (5.12) for c_{ia}^2 is equivalent to the result (3.13) for c_{ia}^2. It is also useful to define the Mach number M_{ep}^2 used by McKenzie (2002):

$$M_{ep}^2 = \frac{u_0^2}{c_{ep}^2} \quad \text{where} \quad c_{ep}^2 = \frac{\gamma_e p_e}{n m_p} \equiv \frac{\gamma_e k_B T_e}{m_p}. \tag{5.13}$$

The relationship between M_{ia}^2 and M_{ep}^2 is:

$$M_{ia}^2 = (1 + \alpha_0) M_{ep}^2. \tag{5.14}$$

Note if $\alpha_0 = m_e/m_p << 1$ then $M_{ia}^2 \approx M_{ep}^2$.

The Mach number regime over which solitary wave solutions exist can be determined by the methods used by McKenzie (2002) and Verheest et al. (2004b). The ion acoustic Mach number M_{ia} is restricted to the range:

$$1 < M_{ia}^2 < M_{max}^2. \tag{5.15}$$

The lower limit $M_{ia} = 1$ in (5.15) corresponds to a weakly nonlinear ion acoustic wave, i.e. to a linear ion acoustic wave. For a weakly nonlinear, long wavelength ion acoustic wave to have spatial wave growth and decay for the wave envelope like $\exp(\pm\kappa x)$, requires $M_{ia}^2 > 1$ (see below). Note that in a dispersive soliton or a solitary wave the nonlinearity is balanced by the dispersion. To derive the condition $M_{ia} > 1$ we consider solutions of the linear dispersion equation (3.3) in conjunction with the traveling wave condition $\omega = k u_0$ where u_0 is the speed of the weak traveling wave (cf. McKenzie et al. 2004). Thus, we look for solutions of the equations:

$$k^2 = \frac{\omega_{pi}^2}{\lambda^2 - c_i^2} + \frac{\omega_{pe}^2}{\lambda^2 - c_e^2} \quad \text{where} \quad \lambda = \frac{\omega}{k} = u_0. \tag{5.16}$$

For the sake of simplicity we assume $c_e^2 > c_i^2$ (i.e. hot electrons and cooler ions). It is straightforward to prove in this case that

$$c_i^2 < c_{ia}^2 < c_e^2, \tag{5.17}$$

where the ion acoustic speed c_{ia} is given by (5.12). Eliminating reference to ω in (5.16) we obtain:

$$k^2 = \frac{(\omega_{pi}^2 + \omega_{pe}^2)(u_0^2 - c_{ia}^2)}{(u_0^2 - c_i^2)(u_0^2 - c_e^2)}. \tag{5.18}$$

A sketch of k^2 versus u_0^2 reveals that if $c_{ia}^2 < u_0^2 < c_e^2$ then $k^2 < 0$ (i.e. the dispersion equation has pure imaginary solutions for $k = i\kappa$). Thus $k^2 < 0$ if:

$$1 < M_{ia}^2 < \frac{c_e^2}{c_{ia}^2} \equiv \frac{c_e^2(1 + \alpha_0)}{(c_i^2 + c_e^2 \alpha_0)}. \tag{5.19}$$

Hence $M_{ia}^2 > 1$ is required for spatial wave growth and decay as occurs for a soliton envelope. The upper limit on the Mach number M_{ia}^2 requires the use of the fully nonlinear equations of the system. The determination of M_{max} is discussed below.

The upper bound, M_{max}, in (5.15) comes from requiring $E_x = 0$ at an equilibrium point of the flow (assumed to occur at the center of the wave). For a compressive

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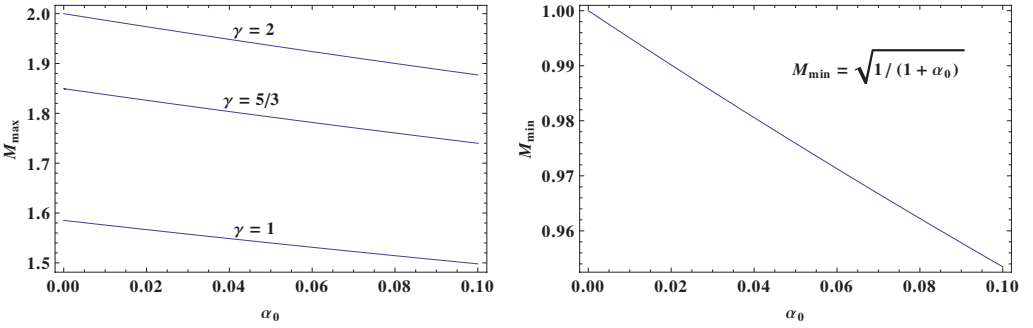


FIGURE 1. (Colour online) Electron–ion Mach number M_{ep} (see (5.13)) range as a function of electron to ion mass ratio, for which solitary wave solutions exist, for several choices of the electron adiabatic gas index γ.

ion acoustic solitary wave, the equilibrium point $u_p = u_{eq}$ is such that $u_{eq} > u_{sonic}$ where $u_{sonic} = u_0 M_p^{-2/(\gamma_p+1)}$ is the value of u_p at the proton sonic point. Thus, the basic equations to determine $(M_{ep})_{max} = M_c$ are:

$$\begin{aligned} E_x^2 &= \frac{2u_0^2 n_0}{\varepsilon_0}\left[m_e P_e(\bar{u}_e) + m_p P_p(\bar{u}_p)\right] = 0, \\ & m_e \varepsilon_e(\bar{u}_e) + m_p \varepsilon_p(\bar{u}_p) = 0, \\ u_p &= u_{sonic} = u_0 M_p^{-2/(\gamma_p+1)}, \end{aligned} \tag{5.20}$$

where $\bar{u}_j = u_j/u_0$ $(j = e, p)$, and

$$\begin{aligned} P_j(\bar{u}_j) &= \bar{u}_j - 1 + \frac{1}{\gamma_j M_j^2}\left(\bar{u}_j^{-\gamma_j} - 1\right), \quad (j = e, p), \\ \varepsilon_j(\bar{u}_j) &= \frac{1}{2}\left(\bar{u}_j^2 - 1\right) + \frac{1}{(\gamma_j - 1)M_j^2}\left(\bar{u}_j^{-(\gamma_j+1)} - 1\right), \quad (j = e, p), \end{aligned} \tag{5.21}$$

are the momentum and energy functions for species j $(j = e, p)$.

For arbitrary γ_e, M_{max} can be calculated implicitly as done for the curves corresponding to $\gamma_e = 5/3$ and $\gamma_e = 1$ of Fig. 1.

For the case of a cold proton gas ($M_p \to \infty$ or $p_p = 0$) and $\gamma_e = 2$, and a finite electron mass ($\alpha \neq 0$), (5.20)–(5.21) admit analytical solutions for $\bar{u}_e$ and M_{ep} (McKenzie (2002) also obtained the analytical solution for $\gamma_e = 2$ and $\alpha_0 = 0$).

In the above case, (5.20) reduces to:

$$\bar{u}_p - 1 + \frac{1}{\gamma_e M_{ep}^2}\left(\bar{u}_e^{-\gamma_e} - 1\right) + \alpha_0(\bar{u}_e - 1) = 0, \tag{5.22}$$

$$\alpha_0 \frac{1}{2}\left(\bar{u}_e^2 - 1\right) + \frac{1}{(\gamma_e - 1)M_{ep}^2}\left(\bar{u}_e^{1-\gamma_e} - 1\right) + \frac{1}{2}\left(\bar{u}_p^2 - 1\right) = 0, \tag{5.23}$$

$$\bar{u}_p = 0 \quad \text{(proton fluid sonic point condition)}. \tag{5.24}$$

For $\gamma_e = 2$, (5.22)–(5.23) have the root

$$\bar{u}_e = \frac{3}{2\tilde{M}^2 + 1} \quad \text{where} \quad \tilde{M}^2 = (1 + \alpha_0)M_{ep}^2, \tag{5.25}$$

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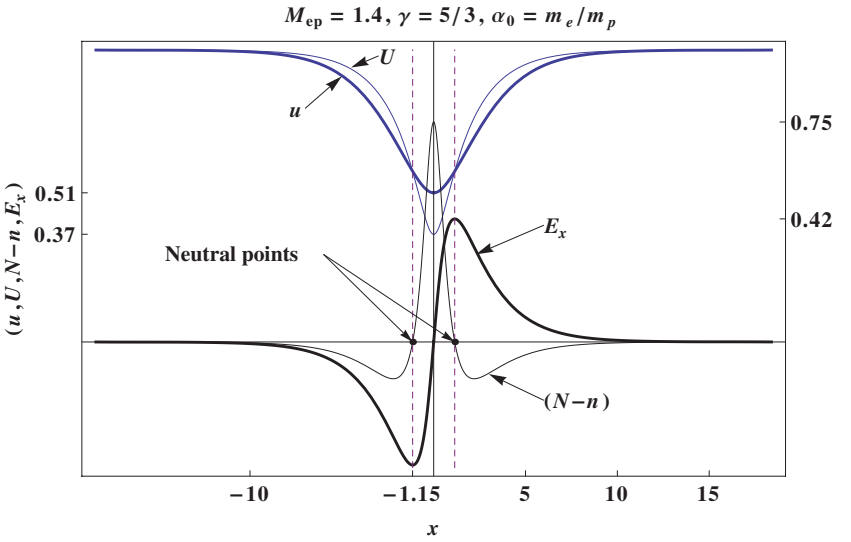


FIGURE 2. (Colour online) Typical ion acoustic solitary wave signature.

and $\tilde{M}^2$ satisfies the equation:

$$\left(2\tilde{M}^2+1\right)^3 - 9\left(2\tilde{M}^2+1\right)^2 + \frac{54\alpha_0}{(\alpha_0+1)}\tilde{M}^2 = 0. \tag{5.26}$$

Equation (5.26) can be written in the form:

$$\left(\tilde{M}^2-1\right)^3 - b\left(\tilde{M}^2-1\right) - b = 0 \quad \text{where} \quad b = \frac{27}{4(1+\alpha_0)}. \tag{5.27}$$

Using standard formulae for the solution of a cubic equation (Abramowitz and Stegun 1965, formula 3.8.2, p. 17), (5.27) has one real root for $\theta = \tilde{M}^2 - 1$. This real root gives $\tilde{M}^2 \equiv M^2_{max}$:

$$M^2_{\max} = 1 + s_1 + s_2 \quad \text{where} \quad s_{1,2} = \frac{3}{2}\left[\frac{1}{(1+\alpha_0)}\left(1 \pm \left(\frac{\alpha_0}{(1+\alpha_0)}\right)^{1/2}\right)\right]^{1/3}. \tag{5.28}$$

For the massless electron limit $\alpha_0 \to 0$ (5.28) gives $M_{\max} = 2$ which agrees with the result of McKenzie (2002).

Figure 2 illustrates a typical solitary wave signature resulting from the integration of (5.10), the (f_+) kernel of which is plotted in Fig. 3. Figure 2 reveals several properties of ion acoustic solitons. Both fluids are decelerated in the first part of the wave, consistent with a potential hill (see Table 1), with the protons running ahead of the electrons, up to the neutral point where the electrons overtake the protons and the electric field reaches an extrema. At the center of the wave, where velocities are minimum, $E_x = 0$ and the charge density reaches a maximum. As suggested by Fig. 4, for zero temperature protons the charge density and slope of the electric field at the wave center actually tend to infinity as $M_{ep} \to M_{\max}$. The wave behavior described by Fig. 2 is related to the formation of a cross-shock potential in that the protons, which due to their mass, are able to penetrate deeper into the wave then the electrons, creating charge imbalance and subsequent electro-static forces which act to restore charge neutrality and prevent either species from 'running away' in the flow. The charge imbalance (which here

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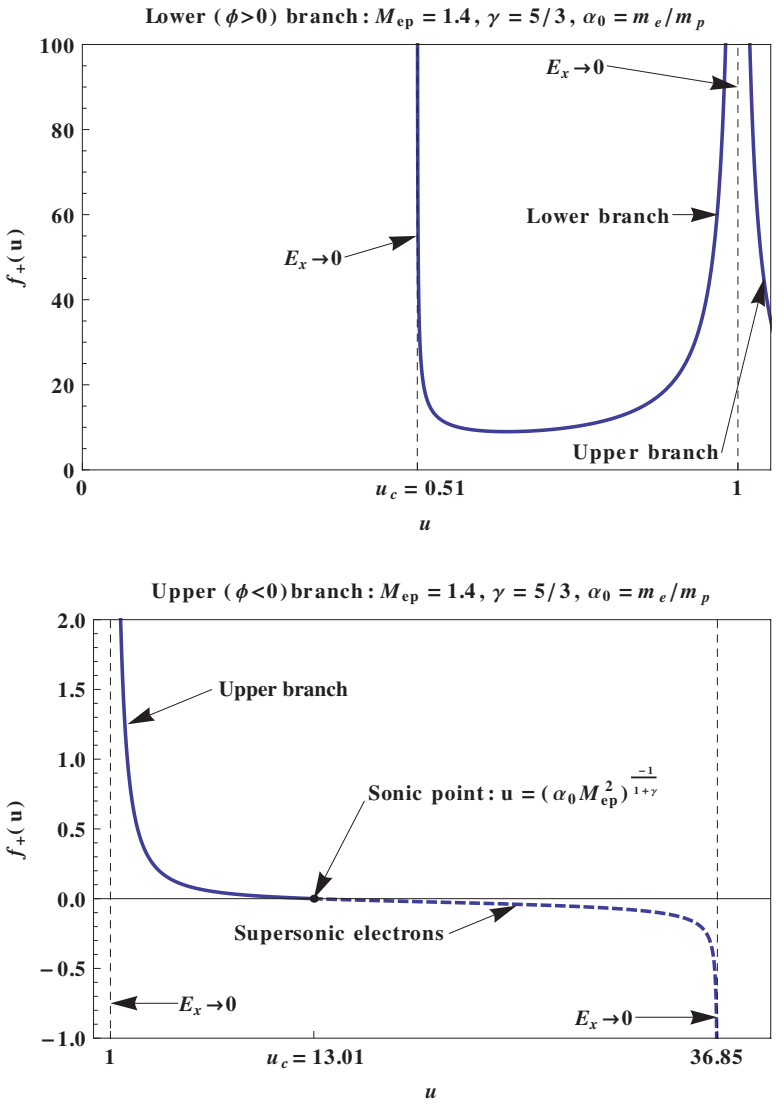


FIGURE 3. (Colour online) The kernel of the structure equation (5.10) plotted as a function of electron velocity (where the plus-sign has been selected). The vertical separatrix at $x = 1$ separate the solution space into lower (physical) and upper (unphysical) branches.

tends to infinity at the critical Mach number owing to the zero proton pressure) that occurs throughout the wave is worth emphasizing in that it underlines the care required when imposing conditions of quasi-charge neutrality in plasma-fluid models.

It can be shown (Verheest et al. 2004b) that, solitary waves, associated with potential wells, which result from integrating the $f_\pm$ upper-branch (see Fig. 3) are unphysical. To see this, note that the area under the curve is positive to the left and negative to the right of the electron sonic point, which means that the solution curve becomes double valued there. The most physically plausible way to find a single valued, upper-branch solution $u = u(x)$, then, is to place u_c at the sonic point and patch together solution branches that satisfy the original ODEs and boundary conditions. However, as can

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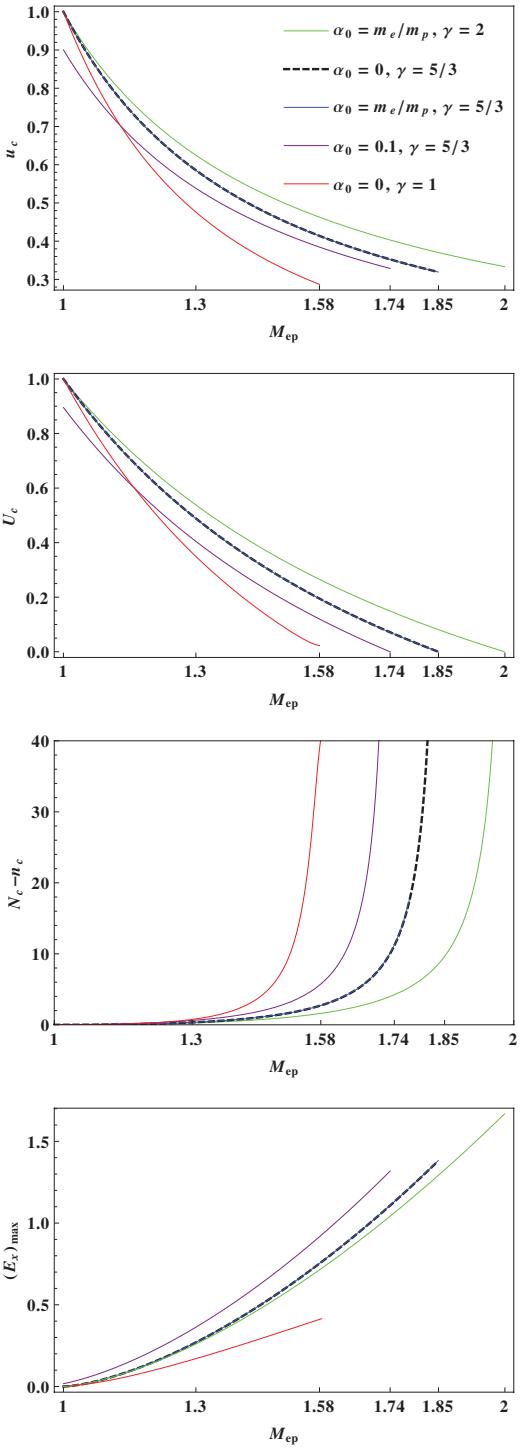


FIGURE 4. (Colour online) Wave amplitude as a function of Mach number M_{ep}, determined by setting $E_x = 0$ (at the center of the wave) in (5.4) through (5.7) to obtain the maximum values of U and u, and by noting that the electric field reaches a maximum at the charge neutral points where $u = U$.

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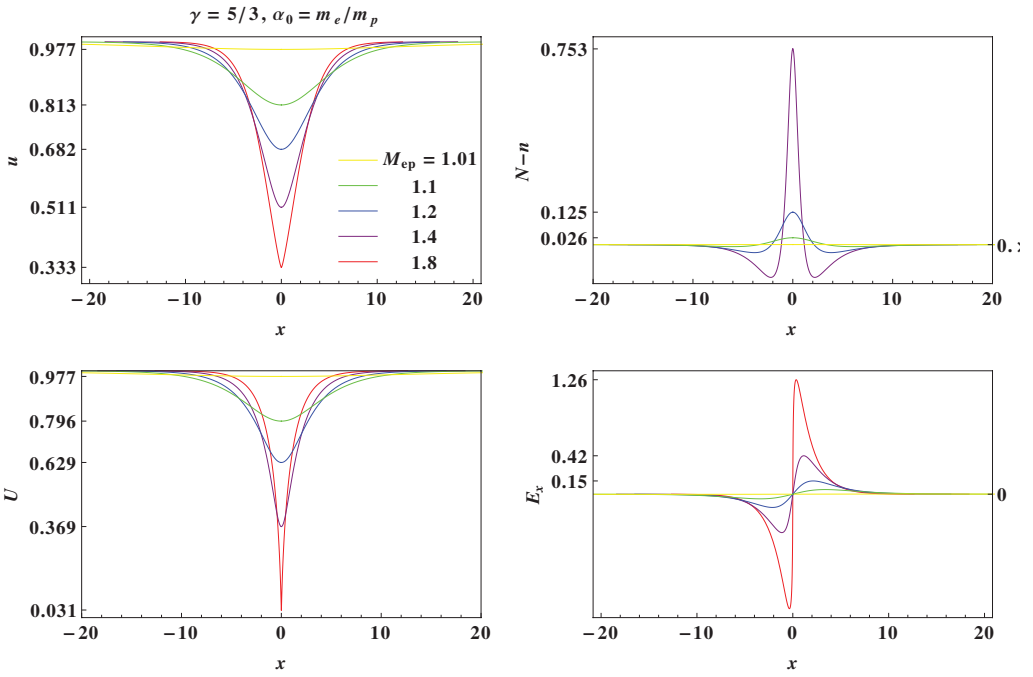


FIGURE 5. (Colour online) Ion acoustic solitary wave signatures for $\gamma \equiv \gamma_e = 5/3$ and $\alpha_0 = 0.0005446$, and Mach numbers M_{ep} ranging from 1.01 to 1.8.

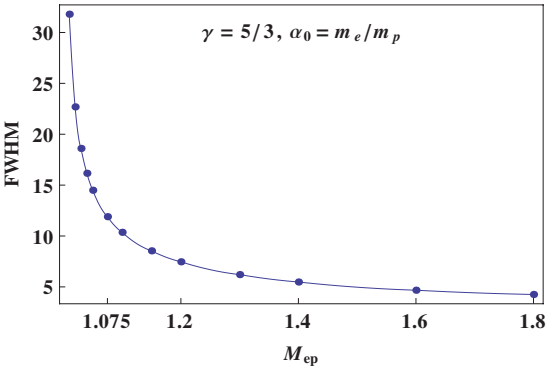


FIGURE 6. (Colour online) Full width at half minimum (FWHM) of the electron velocity for solitary waves as characterized by Fig. 2. On choosing a Mach number, M_{ep}, arbitrary soliton structures can be hand sketched, by finding the corresponding wave amplitudes from Fig. 4, using Fig. 2 to note the typical wave structure, and finding the width of the wave using this figure.

be shown by evaluating $dE_x^2/du_i = 0$, E_x has extrema at sonic points and neutral points which contradicts the requirement that $E_x = 0$ at the wave center.

Several solitary wave profiles are plotted in Fig. 5 for the case $\gamma = 5/3$ and $\alpha_0 = 0.0005446$ and for Mach numbers ranging from 1.01 to 1.8. This figure illustrates graphically the nonlinear wave steepening and amplification that occurs with increasing Mach number. Note that near the critical Mach number (here $M_{ep} = 1.84886$) the flow becomes completely 'choked' as the gradients of the plasma parameters approach infinity. The charge density curve $(N - n)$ for the near critical Mach number ($M_{ep} = 1.8$) case has been purposely omitted from the plots since

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its amplitude (of about 30 in normalized units) is so large that its inclusion would obscure the fine structure of the other charge density curves.

An additional property of solitary waves (perhaps not clearly visible in Fig. 5) is that they become more narrow with increasing Mach number. To see this, the full width at half minimum (FWHM) of the electron velocity is plotted verses Mach number in Fig. 6. Evidently, the wave amplitude increases (Fig. 4) and its structure narrows (Fig. 6), both with increasing Mach number. The, not surprising, implication of this behavior is that waves associated with faster flow should be better able to act as particle reflectors.

6. Conclusion

In this paper we have investigated the physical characteristics of ion acoustic and Langmuir-acoustic traveling waves in a multi-fluid plasma. We first determined the dispersion equation for the system (Sec. 3), which shows that there are two main branches, namely the ion acoustic and the Langmuir wave branch. The long wavelength dispersion equation for the ion acoustic branch results in the linearized KdV equation, whereas the long wavelength limit of the Langmuir wave branch results in a Klein–Gordon equation for linear waves. At short wavelengths, the basic wave modes are the electron acoustic and ion acoustic waves. These results are well known. Our main aim was to describe how the linear dispersion equation contains important information describing the traveling waves. In particular, far upstream the traveling wave is essentially a linear wave and the intersection of the dispersion equation $D(\omega, k) = 0$ with the traveling wave condition $\omega = kV$ (V is the traveling wave speed), yields the wave number of the wave far upstream.

The main results of the paper are presented in Secs 4 and 5. In Sec. 4, we establish the multi-symplectic Hamiltonian structure for the traveling waves for an electron–ion two fluid plasma. We expect that the same multi-symplectic structure will also apply for multi-fluid plasmas that have a Hamiltonian structure governed by a canonical or non-canonical Poisson bracket (e.g. Spencer 1982; Spencer and Kaufman 1982; Holm and Kupershmidt 1983). In the first formulation the Hamiltonian is identified as the total conserved longitudinal x-momentum of the system, in which the total energy flux integral acts as a constraint, and for which $d/d\xi$ is the Hamiltonian evolution operator ($\xi = x - Vt$ is the traveling wave variable and V is the velocity of the wave). In this formulation, the canonical variables are $(\tilde{\varepsilon}_e, E_x)$ where

$$\tilde{\varepsilon}_e = \frac{\varepsilon_0 m_e}{e}\left(\frac{1}{2}u_e^2 + \frac{c_e^2}{\gamma_e - 1}\right), \tag{6.1}$$

is the normalized electron energy flux and E_x is the electric field intensity in the x-direction.

In the second Hamiltonian formulation the total energy flux integral

$$\varepsilon = j\left(\frac{1}{2}m_e u_e^2 + \frac{\gamma_e p_e}{(\gamma_e - 1)n_e}\right) + j\left(\frac{1}{2}m_p u_p^2 + \frac{\gamma_p p_p}{(\gamma_p - 1)n_p}\right), \tag{6.2}$$

is the Hamiltonian, and the total x-momentum integral

$$P_x = j(m_e u_u + m_p u_p) + p_e + p_p - \frac{\varepsilon_0}{2}E_x^2, \tag{6.3}$$

is held constant as a constraint. Here $j = n_e u_e = n_p u_p$ is the constant number density flux for both the protons and the electrons. In the latter Hamiltonian formulation,

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the Hamiltonian evolution operator is:

$$\frac{d}{d\tau} = u_p \frac{d}{d\xi}, \tag{6.4}$$

where $\xi = x - Vt$ is the traveling wave variable. The canonical variables in this formulation are $(E_x, \tilde{\varepsilon}_e)$.

Section 5 presented examples of solitary traveling wave solutions. We used the McKenzie (2002) fluid dynamics approach to write the momentum equation in integral form. This form is especially convenient for calculating solitary wave signatures since, for a plasma system composed of any number and type of plasma species, it can be written as an integral over a single variable. By examining the special case of completely cold protons and hot (subsonic) electrons we have demonstrated how the fundamental integrals of the plasma system can be used to find the solitary wave existence conditions, the overall wave amplitude and the behavior (acceleration or deceleration) from the initial point of a given species.

The fluid dynamics expressions of conservation of energy and momentum were used to describe ion acoustic solitons, including (1) the range of Mach numbers over which ion acoustic solitons can exist as a function of the electron to proton mass ratio α_0 for several values of adiabatic gas index, showing that the range is slightly left-shifted and reduced for increasing α_0 and the upper bound increases with increasing γ, (2) the fluid velocity behavior for each species, acceleration or deceleration from the initial point as summarized in Table 1 and (3) the wave amplitude, Fig. 4, which indicates a very large charge imbalance at the wave center occurring for Mach numbers approaching the upper bound $M_{\max}$. The structure equation in integral form was employed to calculate solitary wave forms for a range of Mach numbers yielding the characteristic wave structure (Fig. 2) and scale width – expressed in terms of FWHM (Fig. 6). Our approach yields a rather complete description of ion acoustic solitary waves for an arbitrary proton–electron mass ratio, which can be readily employed to calculate solitary wave structures to a high degree of numerical accuracy.

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