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Stability and Unstability of the Standing Wave to Euler Equations

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Abstract. In this paper, we first discuss the well-posedness of linearizing equations, and then study the stability and unstability of the 3-D compressible Euler Equation, by analysing the existence of saddle point. In addition, we give the existence of local solutions of the compressible Euler equation.

AMS subject classifications: 35Q31, 35L04, 35C07, 35B35

Key words: Euler Equations, Lie Group, well-posedness, saddle point.

1 Introduction

In this paper we consider the compressible Euler system

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho v) = 0, \\ \partial_t (\rho v) + \nabla_x \cdot (\rho v \otimes v) + \nabla_x p = 0, \\ \partial_t \left[\rho \left(e + \frac{1}{2}|v|^2 \right) \right] + \nabla_x \cdot \left[\rho v \left(e + \frac{1}{2}|v|^2 \right) \right] + \nabla_x \cdot (pv) = 0, \end{cases} \tag{1.1}$$

with the equation of state

$$p = \rho RT = \frac{2}{3}\rho e, \tag{1.2}$$

where (ρ, v, T) represent the macroscopic density, velocity, temperature, respectively. Compressible Euler system (1.1) is a fundamental example of a system of hyperbolic conservation laws. The three equations in the compressible Euler system (1.1) express

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respectively conservation of mass, momentum and energy. It is well-known [1] that the system (1.1) is strictly hyperbolic if the density is bounded below from zero: $\rho>0$.

Althought the study of the compressible Euler Equation has a long history, stability still be one open problem. Recalling that, namely when the system is strictly hyperbolic everywhere, one can use the theory of symmetric hyperbolic systems developed by Friedrichs-Lax-Kato to construct smooth solutions; for instance, see Majda [2]. The breakdown of classical solutions was demonstrated by Sideris [3]. Sideris [3] gave a sufficient conditions for nonglobal existence of C^1 solutions when $\rho^0(x)>0$, where $\rho^0(x)>0$ is the initial density. The nonexistence of C^1 solutions in [3] is related to shock formation. When inf $\rho^0(x)=0$, Makino [4] proved the nonglobal existence of regular solutions by assuming the initial data $(\rho^0(x),v^0(x))$ to have compact support, where $v^0(x)$ is the initial velocity. In [5], the authors discovered the Lax pairs of the two-dimensional and three-dimensional Euler equations. In [6], the authors excluded the possibility of δ function or bigger blowup for spherically symmetric or non-spherically symmetric gas by the total potential energy control and Fourier transformation. Li [7] obtained two types of special solutions to Euler equations with spherical symmetry. The authors in [8] present some explicit self-similar blow-up solutions and some other solutions of the incompressible three-dimensional Navier-Stokes equations. These solutions indicate that in C^∞ the solution of Navier-Stokes equations does not always tend to a solution of Euler equations.

The purpose of this article is to establish the stability and unstability of the standing Wave to the compressible Euler equations. After linearization about the uniform equilibrium $\rho=T=1$, $v=0$ of the compressible Euler system and a suitable choice of units, the fluid fluctuations (σ,u,θ) satisfy

$$\begin{cases} \partial_t\sigma+\nabla_x\cdot u=0, \\ \partial_t u+\nabla_x(\sigma+\theta)=0, \\ \partial_t\theta+\dfrac{2}{3}\nabla_x\cdot u=0. \end{cases} \tag{1.3}$$

For discussing the stability and unstability of (1.1), we first study the stability of the linearizing equations (1.3): referring to the ideas of reference [9], by using the Lie Group method, we get some explicit solutions of (1.3), then we give the proof that the well-posedness of those explicit solutions. Therefor we prove the stability and unstability of the 3-D compressible Euler Equation, by analysing the existence of saddle point.

Besides the method of this paper, there are other methods to study nonlinear problems, such as [10–14]. Xing Lü in [10,12], derive bright and dark envelope solutions for a generalized mixed nonlinear Schrödinger models, by virtue of the corresponding solitary wave solutions for the generalized stationary Gardner equations. Xing Lü, in [11], solve the three coupled higher order nonlinear Schrödinger equations with the achievement of N-soliton solution formula, by means of Darboux transformation. Xing Lü, in [11], directly construct a bilinear Bäcklund transformation of a (2+1)-dimensional Korteweg-de Vries-like model, based on a so-called quadrilinear representation. In [11], the Gerdjikov-CIvanov envelope solitons are derived and discussed, under suitable hypothesis for the

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current velocity.

2 Explicit solutions of linearizing equations (1.3)

Let $F=(\sigma,u_1,u_2,u_3,\theta)^T$ denote the transpose of vector $(\sigma,u_1,u_2,u_3,\theta)$, then we transformed Eqs. (1.3) as follows:

$$F_t=-AF, \tag{2.1}$$

where A is one partial differential operator as follow:

$$A=\begin{pmatrix} 0 & \partial_{x_1} & \partial_{x_2} & \partial_{x_3} & 0 \\ \partial_{x_1} & 0 & 0 & 0 & \partial_{x_1} \\ \partial_{x_2} & 0 & 0 & 0 & \partial_{x_2} \\ \partial_{x_3} & 0 & 0 & 0 & \partial_{x_3} \\ 0 & \frac{2}{3}\partial_{x_1} & \frac{2}{3}\partial_{x_2} & \frac{2}{3}\partial_{x_3} & 0 \end{pmatrix}.$$

In terms of $F=F(x_1,x_2,x_3,t)$, we can fix

$$F(x_1,x_2,x_3,t)=G(a_1x_1+a_2x_2+a_3x_3+bt)=G(\eta).$$

Now put it into Eqs. (2.1) and merge similar terms to derive new equations

$$\begin{pmatrix} b & a_1 & a_2 & a_3 & 0 \\ a_1 & b & 0 & 0 & a_1 \\ a_2 & 0 & b & 0 & a_2 \\ a_3 & 0 & 0 & b & a_3 \\ 0 & \frac{2}{3}a_1 & \frac{2}{3}a_2 & \frac{2}{3}a_3 & b \end{pmatrix} G_\eta=0.$$

After taking elementary transformation to the coefficient matrix of the equations above, we get

$$\begin{pmatrix} -\frac{2}{3}b & 0 & 0 & 0 & b \\ 0 & b & 0 & 0 & \frac{5}{2}a_1 \\ 0 & 0 & b & 0 & \frac{5}{2}a_2 \\ 0 & 0 & 0 & b & \frac{5}{2}a_3 \\ 0 & 0 & 0 & 0 & \frac{3}{2}b-\frac{5}{2b}(a_1^2+a_2^2+a_3^2) \end{pmatrix} G_\eta=0.$$

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Considering this case when $b\neq 0$ and $3b^2=5(a_1^2+a_2^2+a_3^2)$, there has a conclusion that for each function $\theta=\theta(a_1x_1+a_2x_2+a_3x_3+bt)$, function

$$(\sigma,u_1,u_2,u_3,\theta)=\left(\frac{3}{2}\theta+c_1,-\frac{5a_1}{2b}\theta+c_2,-\frac{5a_2}{2b}\theta+c_3,-\frac{5a_3}{2b}\theta+c_4,\theta\right)$$

is the solution of Eq. (1.3) where c_i $(i=1,2,3,4)$ are arbitrary constants.

3 Calculations of Lie point symmetry groups

Normalizing $R\equiv 1$ in the equation of state (1.2), we can write the inviscid flow equations (1.1) in variables ρ, v, T as follows:

$$\begin{cases}\partial_t\rho+(v\cdot\nabla)\rho+\rho\nabla_x\cdot v=0,\\ \rho\partial_t v+\rho(v\cdot\nabla)v+\rho\nabla T+T\nabla\rho=0,\\ \partial_t T+(v\cdot\nabla)T+\dfrac{2}{3}T\nabla_x\cdot v=0.\end{cases}\tag{3.1}$$

In this section, the Lie group method is used to study the analytic solutions of Eqs. (1.3) and (3.1).

According to the method of determining the infinitesimal generator of nonlinear partial differential equation, we take the infinitesimal generator of equation as follows: Let

$$v=\sum_{i=1}^{p}\xi^i(x,u)\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{q}\phi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open subset $M\subset X\times U$. The n-th prolongation of v is the vector field

$$pr^{(n)}v=v+\sum_{\alpha=1}^{p}\sum_{J}\phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha},$$

defined on the corresponding jet space $M^{(n)}\subset X\times U^{(n)}$, the second summation being over all multi-indices $J=(j_1,j_2,\cdots,j_k)$, $1\leq j_k\leq p$, $1\leq k\leq n$, and the coefficient functions $\phi_\alpha^J(x,u^{(n)})$ of $pr^{(n)}v$ are given by the following formula

$$\phi_\alpha^J(x,u^{(n)})=D_J\left(\phi_\alpha-\sum_{i=1}^{p}\xi^iu_i^a\right)+\sum_{i=1}^{p}\xi^iu_{J,i}^a,$$

where $u_i^a=\frac{\partial u^\alpha}{\partial x^i}$ and $pr^{(1)}v$ (in this section $x^4=t$).

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3.1 Lie point symmetry Groups of linearizing equations (1.3)

We transformed Eqs. (1.3) as follows:

$$\begin{cases} \sigma_t+u_{1x_1}+u_{2x_2}+u_{3x_3}=0, \\ u_{1t}+\sigma_{x_1}+\theta_{x_1}=0, \\ u_{2t}+\sigma_{x_2}+\theta_{x_2}=0, \\ u_{3t}+\sigma_{x_3}+\theta_{x_3}=0, \\ \theta_t+\dfrac{2}{3}(u_{1x_1}+u_{2x_2}+u_{3x_3})=0. \end{cases} \tag{3.2}$$

The most general Lie symmetry group admitted by Eqs. (3.2) is realized by the vector field of first-order differential operators of the form

$$v=\xi_1\frac{\partial}{\partial x_1}+\xi_2\frac{\partial}{\partial x_2}+\xi_3\frac{\partial}{\partial x_3}+\xi_4\frac{\partial}{\partial t}+\phi_1\frac{\partial}{\partial\sigma}+\phi_2\frac{\partial}{\partial u_1}+\phi_3\frac{\partial}{\partial u_2}+\phi_4\frac{\partial}{\partial u_3}+\phi_5\frac{\partial}{\partial\theta},$$

where coefficient functions ξ_i $(i=1,2,3,4)$ and ϕ_α $(\alpha=1,2,3,4,5)$ are arbitrary smooth functions of x_1, x_2, x_3, t, σ, u_1, u_2, u_3, θ. And the first prolongation of v as

$$\begin{aligned} pr^{(1)}v=v&+\phi_1^{x_1}\frac{\partial}{\partial\sigma_{x_1}}+\phi_1^{x_2}\frac{\partial}{\partial\sigma_{x_2}}+\phi_1^{x_3}\frac{\partial}{\partial\sigma_{x_3}}+\phi_1^{t}\frac{\partial}{\partial\sigma_{t}}+\phi_2^{x_1}\frac{\partial}{\partial u_{1x_1}} \\ &+\phi_2^{x_2}\frac{\partial}{\partial u_{1x_2}}+\phi_2^{x_3}\frac{\partial}{\partial u_{1x_3}}+\phi_2^{t}\frac{\partial}{\partial u_{1t}}+\phi_3^{x_1}\frac{\partial}{\partial u_{2x_1}}+\phi_3^{x_2}\frac{\partial}{\partial u_{2x_2}} \\ &+\phi_3^{x_3}\frac{\partial}{\partial u_{2x_3}}+\phi_3^{t}\frac{\partial}{\partial u_{2t}}+\phi_4^{x_1}\frac{\partial}{\partial u_{3x_1}}+\phi_4^{x_2}\frac{\partial}{\partial u_{3x_2}}+\phi_4^{x_3}\frac{\partial}{\partial u_{3x_3}} \\ &+\phi_4^{t}\frac{\partial}{\partial u_{3t}}+\phi_5^{x_1}\frac{\partial}{\partial\theta_{x_1}}+\phi_5^{x_2}\frac{\partial}{\partial\theta_{x_2}}+\phi_5^{x_3}\frac{\partial}{\partial\theta_{x_3}}+\phi_5^{t}\frac{\partial}{\partial\theta_{t}}. \end{aligned}$$

Applying $pr^{(1)}v$ to (3.2), we find

$$\begin{cases} \phi_2^t+\phi_1^{x_1}+\phi_5^{x_1}=0, \\ \phi_3^t+\phi_1^{x_2}+\phi_5^{x_2}=0, \\ \phi_4^t+\phi_1^{x_3}+\phi_5^{x_3}=0, \\ \phi_1^t+\phi_2^{x_1}+\phi_3^{x_2}+\phi_4^{x_3}=0, \\ \phi_5^t+\dfrac{2}{3}(\phi_2^{x_1}+\phi_3^{x_2}+\phi_4^{x_3})=0. \end{cases}$$

Then merge similar terms coefficient, we have

$$\begin{cases} \phi_{1,t}+\phi_{2,x_1}+\phi_{3,x_2}+\phi_{4,x_3}=0, \quad \dfrac{3}{2}\phi_{5,t}+\phi_{2,x_1}+\phi_{3,x_2}+\phi_{4,x_3}=0, \\ \phi_{1,u_1}=\phi_{1,u_2}=\phi_{1,u_3}=\phi_{1,\theta}=0, \quad \phi_{2,\sigma}=\phi_{2,u_2}=\phi_{2,u_3}=\phi_{2,\theta}=0, \\ \phi_{3,\sigma}=\phi_{3,u_1}=\phi_{3,u_3}=\phi_{3,\theta}=0, \quad \phi_{4,\sigma}=\phi_{4,u_1}=\phi_{4,u_2}=\phi_{4,\theta}=0, \\ \xi_{1,x_1}=\xi_{2,x_2}=\xi_{3,x_3}=\xi_{4,t}=\phi_{1,\sigma}=\phi_{2,u_1}=\phi_{3,u_2}=\phi_{4,u_3}=\phi_{5,\theta}, \\ \phi_{2,t}+\phi_{1,x_1}+\phi_{5,x_1}=0, \quad \phi_{3,t}+\phi_{1,x_2}+\phi_{5,x_2}=0, \quad \phi_{4,t}+\phi_{1,x_3}+\phi_{5,x_3}=0, \\ \xi_{1,x_2}=\xi_{1,x_3}=\xi_{1,t}=\xi_{2,x_1}=\xi_{2,x_3}=\xi_{2,t}=\xi_{3,x_1}=\xi_{3,x_2}=\xi_{3,t}=\xi_{4,x_1}=\xi_{4,x_2}=\xi_{4,x_3}=0. \end{cases}$$

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Finally, solving the above differential equations, we conclude that the most general infinitesimal symmetry of the Eqs. (1.3) has coefficient functions of the form

$$\begin{cases} \xi_1 = c_1 x_1 + c_2, & \xi_2 = c_1 x_2 + c_3, \\ \xi_3 = c_1 x_3 + c_4, & \xi_4 = c_1 t + c_5, \\ \phi_1 = f_1 + c_1 \sigma, & \phi_2 = f_2 + c_1 u_1, \\ \phi_3 = f_3 + c_1 u_2, & \phi_4 = f_4 + c_1 u_3, \\ \phi_5 = f_5 + c_1 \theta, \end{cases}$$

where c_i $(i=1,2,3,4,5)$ are arbitrary constants and f_i $(i=1,2,3,4,5)$ are arbitrary functions of x_1, x_2, x_3, t, which satisfy the following equations

$$\begin{cases} f_{1t} = \dfrac{3}{2} f_{5t}, \\ f_{2t} + f_{1x_1} + f_{5x_1} = 0, \\ f_{3t} + f_{1x_2} + f_{5x_2} = 0, \\ f_{4t} + f_{1x_3} + f_{5x_3} = 0, \\ f_{5t} + \dfrac{2}{3}(f_{2x_1} + f_{3x_2} + f_{4x_3}) = 0. \end{cases}$$

Soving the equations as following: as

$$\begin{cases} f_1 = \dfrac{3}{2} f_5 + g(x_1, x_2, x_3), \\ f_{2tx_1} + g_{x_1 x_1} + \dfrac{5}{2} f_{5x_1 x_1} = 0, \\ f_{3tx_2} + g_{x_2 x_2} + \dfrac{5}{2} f_{5x_2 x_2} = 0, \\ f_{4tx_3} + g_{x_3 x_3} + \dfrac{5}{2} f_{5x_3 x_3} = 0, \\ f_{5tt} + \dfrac{2}{3}(f_{2tx_1} + f_{3tx_2} + f_{4tx_3}) = 0, \end{cases}$$

so

$$\left(f_5 + \frac{2}{5} g\right)_{tt} = \frac{5}{3} \Delta \left(f_5 + \frac{2}{5} g\right),$$

than we have

$$\begin{cases} f_1 = \dfrac{3}{2} f_5 + g(x_1, x_2, x_3), \\ f_2 = -t g_{x_1} - \dfrac{5}{2} \displaystyle\int_0^t f_{5x_1} dt, \\ f_3 = -t g_{x_2} - \dfrac{5}{2} \displaystyle\int_0^t f_{5x_2} dt, \\ f_4 = -t g_{x_3} - \dfrac{5}{2} \displaystyle\int_0^t f_{5x_3} dt, \\ f_5 = -\dfrac{2}{5} g + \dfrac{1}{4\pi t^2} \displaystyle\int_{\partial B\left(x, \sqrt{\frac{5}{3}}\right)} [\phi(y) + D\phi(y).(y-x) + t\psi(y)] dS(y). \end{cases}$$

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In particular,

$$\begin{cases} f_1=\dfrac{3}{2}f_5+g(x_1,x_2,x_3), \\ f_2=-tg_{x_1}-\dfrac{5}{2}\displaystyle\int_0^t f_{5x_1}dt, \\ f_3=-tg_{x_2}-\dfrac{5}{2}\displaystyle\int_0^t f_{5x_2}dt, \\ f_4=-tg_{x_3}-\dfrac{5}{2}\displaystyle\int_0^t f_{5x_3}dt, \\ f_5=-\dfrac{2}{5}g+f\Big(ax_1+bx_2+cx_3+\sqrt{\dfrac{5}{2}(a^2+b^2+c^2)}t\Big), \end{cases}$$

for any continuous derivable $g(x_1,x_2,x_3)$ and $f(y)$.

Thus the Lie algebra of infinitesimal symmetries of the Eqs. (1.3) is spanned by the five vector fields

$$\begin{aligned} &\nu_1=\partial_{x_1},\quad \nu_2=\partial_{x_2},\quad \nu_3=\partial_{x_3},\quad \nu_4=\partial_t, \\ &\nu_5=x_1\partial_{x_1}+x_2\partial_{x_2}+x_3\partial_{x_3}+t\partial_t+\sigma\partial_\sigma+u_1\partial_{u_1}+u_2\partial_{u_2}+u_3\partial_{u_3}+\theta\partial_\theta, \end{aligned}$$

and the five infinite-dimensional subalgebras

$$\nu_6=f_1\partial_\sigma,\quad \nu_7=f_2\partial_{u_1},\quad \nu_8=f_3\partial_{u_2},\quad \nu_9=f_4\partial_{u_3},\quad \nu_{10}=f_3\partial_\theta.$$

The one-parameter groups G_i $(i=1,2,\cdots,10)$ generated by the ν_i are given in the following table:

$$\begin{aligned} &G_1:(x_1+\varepsilon,x_2,x_3,t,\sigma,u_1,u_2,u_3,\theta), \\ &G_2:(x_1,x_2+\varepsilon,x_3,t,\sigma,u_1,u_2,u_3,\theta), \\ &G_3:(x_1,x_2,x_3+\varepsilon,t,\sigma,u_1,u_2,u_3,\theta), \\ &G_4:(x_1,x_2,x_3,t+\varepsilon,\sigma,u_1,u_2,u_3,\theta), \\ &G_5:(e^\varepsilon x_1,e^\varepsilon x_2,e^\varepsilon x_3,e^\varepsilon t,e^\varepsilon\sigma,e^\varepsilon u_1,e^\varepsilon u_2,e^\varepsilon u_3,e^\varepsilon\theta), \\ &G_6:(x_1,x_2,x_3,t,\sigma+\varepsilon f_1,u_1,u_2,u_3,\theta), \\ &G_7:(x_1,x_2,x_3,t,\sigma,u_1+\varepsilon f_2,u_2,u_3,\theta), \\ &G_8:(x_1,x_2,x_3,t,\sigma,u_1,u_2+\varepsilon f_3,u_3,\theta), \\ &G_9:(x_1,x_2,x_3,t,\sigma,u_1,u_2,u_3+\varepsilon f_4,\theta), \\ &G_{10}:(x_1,x_2,x_3,t,\sigma,u_1,u_2,u_3,\theta+\varepsilon f_5), \end{aligned}$$

where ε is an arbitrary constant. We say that if

$$F=(\sigma(x_1,x_2,x_3,t),u_1(x_1,x_2,x_3,t),u_2(x_1,x_2,x_3,t),u_3(x_1,x_2,x_3,t),\theta(x_1,x_2,x_3,t))$$

is an arbitrary classical solution of Eqs. (1.3), so is the flowing

$$\begin{aligned} F^{(1)}=(&\sigma(x_1-\varepsilon,x_2,x_3,t),u_1(x_1-\varepsilon,x_2,x_3,t),u_2(x_1-\varepsilon,x_2,x_3,t), \\ &u_3(x_1-\varepsilon,x_2,x_3,t),\theta(x_1-\varepsilon,x_2,x_3,t)), \end{aligned}$$

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$$
\begin{aligned}
F^{(2)} &= (\sigma(x_1,x_2-\varepsilon,x_3,t),u_1(x_1,x_2-\varepsilon,x_3,t),u_2(x_1,x_2-\varepsilon,x_3,t),\\
&\quad u_3(x_1,x_2-\varepsilon,x_3,t),\theta(x_1,x_2-\varepsilon,x_3,t)),\\
F^{(3)} &= (\sigma(x_1,x_2,x_3-\varepsilon,t),u_1(x_1,x_2,x_3-\varepsilon,t),u_2(x_1,x_2,x_3-\varepsilon,t),\\
&\quad u_3(x_1,x_2,x_3-\varepsilon,t),\theta(x_1,x_2,x_3-\varepsilon,t)),\\
F^{(4)} &= (\sigma(x_1,x_2,x_3,t-\varepsilon),u_1(x_1,x_2,x_3,t-\varepsilon),u_2(x_1,x_2,x_3,t-\varepsilon),\\
&\quad u_3(x_1,x_2,x_3,t-\varepsilon),\theta(x_1,x_2,x_3,t-\varepsilon)),\\
F^{(5)} &= e^{\varepsilon}(\sigma(e^{-\varepsilon}(x_1,x_2,x_3,t)),u_1(e^{-\varepsilon}(x_1,x_2,x_3,t)),u_2(e^{-\varepsilon}(x_1,x_2,x_3,t)),\\
&\quad u_3(e^{-\varepsilon}(x_1,x_2,x_3,t)),\theta(e^{-\varepsilon}(x_1,x_2,x_3,t))),\\
F^{(6)} &= (\sigma(x_1,x_2,x_3,t)+\varepsilon f_1(x_1,x_2,x_3,t),u_1(x_1,x_2,x_3,t),u_2(x_1,x_2,x_3,t),\\
&\quad u_3(x_1,x_2,x_3,t),\theta(x_1,x_2,x_3,t)),\\
F^{(7)} &= (\sigma(x_1,x_2,x_3,t),u_1(x_1,x_2,x_3,t)+\varepsilon f_2(x_1,x_2,x_3,t),u_2(x_1,x_2,x_3,t),\\
&\quad u_3(x_1,x_2,x_3,t),\theta(x_1,x_2,x_3,t)),\\
F^{(8)} &= (\sigma(x_1,x_2,x_3,t),u_1(x_1,x_2,x_3,t),u_2(x_1,x_2,x_3,t)+\varepsilon f_3(x_1,x_2,x_3,t),\\
&\quad u_3(x_1,x_2,x_3,t),\theta(x_1,x_2,x_3,t)),\\
F^{(9)} &= (\sigma(x_1,x_2,x_3,t),u_1(x_1,x_2,x_3,t),u_2(x_1,x_2,x_3,t),\\
&\quad u_3(x_1,x_2,x_3,t)+\varepsilon f_4(x_1,x_2,x_3,t),\theta(x_1,x_2,x_3,t)),\\
F^{(10)} &= (\sigma(x_1,x_2,x_3,t),u_1(x_1,x_2,x_3,t),u_2(x_1,x_2,x_3,t),\\
&\quad u_3(x_1,x_2,x_3,t),\theta(x_1,x_2,x_3,t)+\varepsilon f_5(x_1,x_2,x_3,t)).
\end{aligned}
$$

3.2 Lie point symmetry Groups of compressible Euler equations (3.1)

Now we transformed Eqs. (3.1) as follows:

$$
\begin{cases}
\rho_t+v_1\rho_{x_1}+v_2\rho_{x_2}+v_3\rho_{x_3}+\rho(v_{1x_1}+v_{2x_2}+v_{3x_3})=0,\\
\rho v_{1t}+\rho(v_1v_{1x_1}+v_2v_{1x_2}+v_3v_{1x_3})+\rho T_{x_1}+T\rho_{x_1}=0,\\
\rho v_{2t}+\rho(v_1v_{2x_1}+v_2v_{2x_2}+v_3v_{2x_3})+\rho T_{x_2}+T\rho_{x_2}=0,\\
\rho v_{3t}+\rho(v_1v_{3x_1}+v_2v_{3x_2}+v_3v_{3x_3})+\rho T_{x_3}+T\rho_{x_3}=0,\\
T_t+v_1T_{x_1}+v_2T_{x_2}+v_3T_{x_3}+\dfrac{3}{2}T(v_{1x_1}+v_{2x_2}+v_{3x_3})=0.
\end{cases}
\tag{3.3}
$$

The most general Lie symmetry group admitted by Eqs. (3.3) is realized by the vector field of first-order differential operators of the form

$$
\nu=\xi_1\frac{\partial}{\partial x_1}+\xi_2\frac{\partial}{\partial x_2}+\xi_3\frac{\partial}{\partial x_3}+\xi_4\frac{\partial}{\partial t}+\phi_1\frac{\partial}{\partial\rho}+\phi_2\frac{\partial}{\partial v_1}+\phi_3\frac{\partial}{\partial v_2}+\phi_4\frac{\partial}{\partial v_3}+\phi_5\frac{\partial}{\partial T},
$$

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where coefficient functions ξ_i $(i=1,2,3,4)$ and ϕ_α $(\alpha=1,2,3,4,5)$ are arbitrary smooth functions of x_1, x_2, x_3, t, ρ, v_1, v_2, v_3, T. And the first prolongation of ν as

$$\begin{aligned}pr^{(1)}\nu=\nu&+\phi_1^{x_1}\frac{\partial}{\partial\rho_{x_1}}+\phi_1^{x_2}\frac{\partial}{\partial\rho_{x_2}}+\phi_1^{x_3}\frac{\partial}{\partial\rho_{x_3}}+\phi_1^{t}\frac{\partial}{\partial\rho_{t}}+\phi_2^{x_1}\frac{\partial}{\partial v_{1x_1}}\\&+\phi_2^{x_2}\frac{\partial}{\partial v_{1x_2}}+\phi_2^{x_3}\frac{\partial}{\partial v_{1x_3}}+\phi_2^{t}\frac{\partial}{\partial v_{1t}}+\phi_3^{x_1}\frac{\partial}{\partial v_{2x_1}}+\phi_3^{x_2}\frac{\partial}{\partial v_{2x_2}}\\&+\phi_3^{x_3}\frac{\partial}{\partial v_{2x_3}}+\phi_3^{t}\frac{\partial}{\partial v_{2t}}+\phi_4^{x_1}\frac{\partial}{\partial v_{3x_1}}+\phi_4^{x_2}\frac{\partial}{\partial v_{3x_2}}+\phi_4^{x_3}\frac{\partial}{\partial v_{3x_3}}\\&+\phi_4^{t}\frac{\partial}{\partial v_{3t}}+\phi_5^{x_1}\frac{\partial}{\partial T_{x_1}}+\phi_5^{x_2}\frac{\partial}{\partial T_{x_2}}+\phi_5^{x_3}\frac{\partial}{\partial T_{x_3}}+\phi_5^{t}\frac{\partial}{\partial T_{t}}.\end{aligned}$$

Applying $pr^{(1)}\nu$ to (3.3), we find

$$\begin{aligned}&\phi_1^t+v_1\phi_1^{x_1}+v_2\phi_1^{x_2}+v_3\phi_1^{x_3}+\rho_{x_1}\phi_2+\rho_{x_2}\phi_3+\rho_{x_3}\phi_4\\&\quad+\rho(\phi_2^{x_1}+\phi_3^{x_2}+\phi_4^{x_3})+(v_{1x_1}+v_{2x_2}+v_{3x_3})\phi_1=0,\\&\rho\phi_2^t+v_{1t}\phi_1+\rho(v_1\phi_2^{x_1}+v_2\phi_2^{x_2}+v_3\phi_2^{x_3}+v_{1x_1}\phi_2+v_{1x_2}\phi_3+v_{1x_3}\phi_4)\\&\quad+(v_1v_{1x_1}+v_2v_{1x_2}+v_3v_{1x_3})\phi_1+\rho\phi_5^{x_1}+T_{x_1}\phi_1+T\phi_1^{x_1}+\rho_{x_1}\phi_5=0,\\&\rho\phi_3^t+v_{2t}\phi_1+\rho(v_1\phi_3^{x_1}+v_2\phi_3^{x_2}+v_3\phi_3^{x_3}+v_{2x_1}\phi_2+v_{2x_2}\phi_3+v_{2x_3}\phi_4)\\&\quad+(v_1v_{2x_1}+v_2v_{2x_2}+v_3v_{2x_3})\phi_1+\rho\phi_5^{x_2}+T_{x_2}\phi_1+T\phi_1^{x_2}+\rho_{x_2}\phi_5=0,\\&\rho\phi_4^t+v_{3t}\phi_1+\rho(v_1\phi_4^{x_1}+v_2\phi_4^{x_2}+v_3\phi_4^{x_3}+v_{3x_1}\phi_2+v_{3x_2}\phi_3+v_{3x_3}\phi_4)\\&\quad+(v_1v_{3x_1}+v_2v_{3x_2}+v_3v_{3x_3})\phi_1+\rho\phi_5^{x_3}+T_{x_3}\phi_1+T\phi_1^{x_3}+\rho_{x_3}\phi_5=0,\\&\phi_5^t+v_1\phi_5^{x_1}+v_2\phi_5^{x_2}+v_3\phi_5^{x_3}+T_{x_1}\phi_2+T_{x_2}\phi_3+T_{x_3}\phi_4\\&\quad+\frac{3}{2}T(\phi_2^{x_1}+\phi_3^{x_2}+\phi_4^{x_3})+\frac{3}{2}(v_{1x_1}+v_{2x_2}+v_{3x_3})\phi_5=0.\end{aligned}$$

Then merge similar terms coefficient, we have

$$\begin{aligned}&\phi_1=\phi_5=0,\\&\phi_2=\xi_{1,t},\quad \phi_3=\xi_{4,t},\quad \phi_4=\xi_{3,t},\quad \phi_{1,\rho}=\xi_{4,t},\\&\phi_{2,v_1}=\xi_{1,x_1},\quad \phi_{3,v_1}=\xi_{2,x_1},\quad \phi_{4,v_1}=\xi_{3,x_1},\\&\phi_{2,v_2}=\xi_{1,x_2},\quad \phi_{3,v_2}=\xi_{2,x_2},\quad \phi_{4,v_2}=\xi_{3,x_2},\\&\phi_{2,v_3}=\xi_{1,x_3},\quad \phi_{3,v_3}=\xi_{2,x_3},\quad \phi_{4,v_3}=\xi_{3,x_3},\\&\xi_{1,x_1}=\xi_{1,x_2}=\xi_{1,x_3}=\xi_{1,\rho}=\xi_{1,v_1}=\xi_{1,v_2}=\xi_{1,v_3}=\xi_{1,T}=0,\\&\xi_{2,x_1}=\xi_{2,x_2}=\xi_{2,x_3}=\xi_{2,\rho}=\xi_{2,v_1}=\xi_{2,v_2}=\xi_{2,v_3}=\xi_{2,T}=0,\\&\xi_{3,x_1}=\xi_{3,x_2}=\xi_{3,x_3}=\xi_{3,\rho}=\xi_{3,v_1}=\xi_{3,v_2}=\xi_{3,v_3}=\xi_{3,T}=0,\\&\xi_{4,x_1}=\xi_{4,x_2}=\xi_{4,x_3}=\xi_{4,\rho}=\xi_{4,v_1}=\xi_{4,v_2}=\xi_{4,v_3}=\xi_{4,T}=0,\\&\phi_{2,t}=\phi_{2,\rho}=\phi_{2,T}=\phi_{3,t}=\phi_{3,\rho}=\phi_{3,T}=\phi_{4,t}=\phi_{4,\rho}=\phi_{4,T}=0.\end{aligned}$$

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Finally, solving the above differential equations, we conclude that the most general infinitesimal symmetry of the Eqs. (3.1) has coefficient functions of the form

$$\xi_1=c_2t+c_5,\quad \xi_2=c_3t+c_6,\quad \xi_3=c_4t+c_7,\quad \xi_4=c_1,$$
$$\phi_1=0,\quad \phi_2=c_2,\quad \phi_3=c_3,\quad \phi_4=c_4,\quad \phi_5=0,$$

where c_i $(i=1,2,3,4,5,6,7)$ are arbitrary constants. Thus the Lie algebra of infinitesimal symmetries of the Eqs. (3.1) is spanned by the seven vector fields

$$\begin{aligned}
&v_1=\partial_t,\\
&v_2=t\partial_{x_1}+\partial_{v_1},\\
&v_3=t\partial_{x_2}+\partial_{v_2},\\
&v_4=t\partial_{x_3}+\partial_{v_3},\\
&v_5=\partial_{x_1},\\
&v_6=\partial_{x_2},\\
&v_7=\partial_{x_3}.
\end{aligned}$$

The one-parameter groups G_i $(i=1,2,\cdots,7)$ generated by the v_i are given in the following table:

$$\begin{aligned}
&G_1:(x_1,x_2,x_3,t+\varepsilon,\rho,v_1,v_2,v_3,T),\\
&G_2:(x_1+\varepsilon t,x_2,x_3,t,\rho,v_1+\varepsilon,v_2,v_3,T),\\
&G_3:(x_1,x_2+\varepsilon t,x_3,t,\rho,v_1,v_2+\varepsilon,v_3,T),\\
&G_4:(x_1,x_2,x_3+\varepsilon t,t,\rho,v_1,v_2,v_3+\varepsilon,T),\\
&G_5:(x_1+\varepsilon,x_2,x_3,t,\rho,v_1,v_2,v_3,T),\\
&G_6:(x_1,x_2+\varepsilon,x_3,t,\rho,v_1,v_2,v_3,T),\\
&G_7:(x_1,x_2,x_3+\varepsilon,t,\rho,v_1,v_2,v_3,T),
\end{aligned}$$

where ε is an arbitrary constant. We say that if

$$F=(\rho(x_1,x_2,x_3,t),v_1(x_1,x_2,x_3,t),v_2(x_1,x_2,x_3,t),v_3(x_1,x_2,x_3,t),T(x_1,x_2,x_3,t))$$

is an arbitrary classical solution of Eqs. (3.1), so is the flowing

$$\begin{aligned}
F^{(1)}=&(\rho(x_1,x_2,x_3,t-\varepsilon),v_1(x_1,x_2,x_3,t-\varepsilon),v_2(x_1,x_2,x_3,t-\varepsilon),\\
&v_3(x_1,x_2,x_3,t-\varepsilon),T(x_1,x_2,x_3,t-\varepsilon)),\\
F^{(2)}=&(\rho(x_1-\varepsilon t,x_2,x_3,t),v_1(x_1-\varepsilon t,x_2,x_3,t)+\varepsilon,v_2(x_1-\varepsilon t,x_2,x_3,t),\\
&v_3(x_1-\varepsilon t,x_2,x_3,t),T(x_1-\varepsilon t,x_2,x_3,t)),\\
F^{(3)}=&(\rho(x_1,x_2-\varepsilon t,x_3,t),v_1(x_1,x_2-\varepsilon t,x_3,t),v_2(x_1,x_2-\varepsilon t,x_3,t)+\varepsilon,\\
&v_3(x_1,x_2-\varepsilon t,x_3,t),T(x_1,x_2-\varepsilon t,x_3,t)),
\end{aligned}$$

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$$
\begin{aligned}
F^{(4)} &= (\rho(x_1,x_2,x_3-\varepsilon t,t),v_1(x_1,x_2,x_3-\varepsilon t,t)+\varepsilon,v_2(x_1,x_2,x_3-\varepsilon t,t),\\
&\qquad v_3(x_1,x_2,x_3-\varepsilon t,t)+\varepsilon,T(x_1,x_2,x_3-\varepsilon t,t)),\\
F^{(5)} &= (\rho(x_1-\varepsilon,x_2,x_3,t),v_1(x_1-\varepsilon,x_2,x_3,t),v_2(x_1-\varepsilon,x_2,x_3,t),\\
&\qquad v_3(x_1-\varepsilon,x_2,x_3,t),T(x_1-\varepsilon,x_2,x_3,t)),\\
F^{(6)} &= (\rho(x_1,x_2-\varepsilon,x_3,t),v_1(x_1,x_2-\varepsilon,x_3,t),v_2(x_1,x_2-\varepsilon,x_3,t),\\
&\qquad v_3(x_1,x_2-\varepsilon,x_3,t),T(x_1,x_2-\varepsilon,x_3,t)),\\
F^{(7)} &= (\rho(x_1,x_2,x_3-\varepsilon,t),v_1(x_1,x_2,x_3-\varepsilon,t),v_2(x_1,x_2,x_3-\varepsilon,t),\\
&\qquad v_3(x_1,x_2,x_3-\varepsilon,t),T(x_1,x_2,x_3-\varepsilon,t)).
\end{aligned}
$$

4 Well-posedness of linearizing equations (1.3)

In this section, we discuss the stability and uniqueness of the initial boundary value problem for the equations below, respectively, in $L^2(\Omega)$ and $H^m(\Omega)$:

$$
\begin{cases}
\partial_t\sigma+\nabla_x\cdot u=0, & \sigma(x,0)=\sigma^0(x), \quad \sigma(x,t)|_{x\in\partial\Omega}=\phi(x,t),\\
\partial_t u+\nabla_x(\sigma+\theta)=0, & u(x,0)=u^0(x), \quad u(x,t)|_{x\in\partial\Omega}=\varphi(x,t),\\
\partial_t\theta+\dfrac{2}{3}\nabla_x\cdot u=0, & \theta(x,0)=\theta^0(x), \quad \theta(x,t)|_{x\in\partial\Omega}=\psi(x,t).
\end{cases}
\tag{4.1}
$$

4.1 Uniqueness

We as always assume that $\Omega\subset R^3$ is a bounded, open set. Suppose also $\sigma^0(x)$, $u^0(x)$, $\theta^0(x)$, $\phi(x,t)$, $\varphi(x,t)$ and $\psi(x,t)$ are given.

Theorem 4.1 (Uniqueness in $L^2(\Omega)$)**.** *Assume*

$$\sigma^0(x),u^0(x),\theta^0(x)\in L^2(\Omega),$$

and

$$\phi(x,t),\varphi(x,t),\psi(x,t)\in L^2(0,\tau;L^2(\Omega)).$$

Then the initial boundary value problem (4.1) *has a unique solution* $(\sigma,u,\theta)\in L^2(0,\tau;L^2(\Omega))$.

Proof. Due to $F(x,t)=F(\theta(x,t))$, supposing $F_1(\theta_1(x,t))$ and $F_2(\theta_2(x,t))$ are two solutions of the same initial boundary value problem (3.1). Let $\overline{\theta}(x,t)=\theta_1(x,t)-\theta_2(x,t)$ and $\overline{F}(x,t))=\overline{F}(\overline{\theta}(x,t))=F_1(\theta_1(x,t))-F_2(\theta_2(x,t))$, then we have

$$\overline{F}^0(x)=0, \quad \overline{F}_t=-A\overline{F}, \quad \overline{F}(x,t)|_{x\in\partial\Omega}=0,$$

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where $\overline{F}^0(x)=\overline{F}(x,0)$. Multiplying the second equation above by $\overline{F}(x,t)$ and integrating over Ω, we obtain

$$\frac{d}{dt}\|\overline{F}\|^2_{L^2(\Omega)}=-\frac{1}{2}\int_\Omega(\overline{\theta}^2)_t dx.$$

Integrate from 0 to τ with respect to t and than change the variable τ to t:

$$\|\overline{F}\|^2_{L^2(\Omega)}+\frac{1}{2}\|\overline{\theta}\|^2_{L^2(\Omega)}=\|\overline{F}^0\|^2_{L^2(\Omega)}+\frac{1}{2}\|\overline{\theta}^0\|^2_{L^2(\Omega)}=0. \tag{4.2}$$

Then we have

$$\theta_1(x,t)=\theta_2(x,t),F_1(\theta_1(x,t))=F_2(\theta_2(x,t)).$$

This completes the proof. □

Next we consider equations

$$F_{t,x_i}=-(AF)_{x_i}. \tag{4.3}$$

Such equations can be formally derived from the Eqs. (2.1) by differentiating Eqs. (2.1) with respect to x_i $(i=1,2,3)$ (take t as x_4).

Similarly, we have the following result.

Theorem 4.2 (Uniqueness in $H^1(\Omega)$)**.** *Assume*

$$\sigma^0(x),u^0(x),\theta^0(x)\in H^1(\Omega),$$

and

$$\phi(x,t),\varphi(x,t),\psi(x,t)\in H^1(0,\tau;H^1(\Omega)).$$

Then the solution (σ,u,θ) *of the initial boundary value problem* (4.1) *is unique in* $H^1(0,\tau;H^1(\Omega))$.

Proof. Supposing $F_1(\theta_1(x,t))$ and $F_2(\theta_2(x,t))$ are two solutions of the same the initial boundary value problem (3.1). Let $\overline{\theta}(x,t)=\theta_1(x,t)-\theta_2(x,t)$ and $\overline{F}(x,t))=\overline{F}(\overline{\theta}(x,t))=F_1(\theta_1(x,t))-F_2(\theta_2(x,t))$, then we have

$$\overline{F}^0_{x_i}(x)=0,\quad \overline{F}_{t,x_i}=-(A\overline{F})_{x_i},\quad \overline{F}_{x_i}(x,t)|_{x\in\partial\Omega}=0.$$

Multiplying the second equation above by $\overline{F}_{x_i}(x,t)$ and integrating over Ω, we obtain

$$\frac{d}{dt}\|\overline{F}_{x_i}\|^2_{L^2(\Omega)}=-\frac{1}{2}\int_\Omega(\overline{\theta}^2_{x_i})_t dx.$$

Integrate from 0 to τ with respect to t and than change the variable τ to t:

$$\|\overline{F}_{x_i}\|^2_{L^2(\Omega)}+\frac{1}{2}\|\overline{\theta}_{x_i}\|^2_{L^2(\Omega)}=\|\overline{F}^0_{x_i}\|^2_{L^2(\Omega)}+\frac{1}{2}\|\overline{\theta}^0_{x_i}\|^2_{L^2(\Omega)}=0.$$

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Then we have

$$\theta_1(x,t)=\theta_2(x,t), F_1(\theta_1(x,t))=F_2(\theta_2(x,t)).$$

This completes the proof. □

Similarly, we can prove the uniqueness of the initial boundary value problem (4.1) in $H^m(\Omega)$. The proof procedure is omitted.

4.2 Analysis of stability

Let $\tilde{\sigma}$, $\tilde{u}$, and $\tilde{\theta}$ denote the solution pair of a slight disturbance with the corresponding initial value $\tilde{\sigma}^0$, $\tilde{u}^0$, and $\tilde{\theta}^0$ respectively. Let $f=\sigma-\tilde{\sigma}$ be the difference of σ and $\tilde{\sigma}$, with initial value $f^0(x)\to 0$ in the sense of $L^2(\Omega)$ and boundary value $\sigma\to\tilde{\sigma}$ in the sense of $L^1(\partial\Omega)$. Let $g=u-\tilde{u}$ be the difference of u and $\tilde{u}$, with initial value $g^0(x)\to 0$ in the sense of $L^2(\Omega)$ and boundary value $u\to\tilde{u}$ in the sense of $L^\infty(\partial\Omega)$. Let $h=\theta-\tilde{\theta}$ be the difference of θ and $\tilde{\theta}$, with initial value $h^0(x)\to 0$ in the sense of $L^2(\Omega)$ and boundary value $\theta\to\tilde{\theta}$ in the sense of $L^1(\partial\Omega)$. Then, subtracting one equation from the other, we obtain

$$\begin{cases} \partial_t f+\nabla_x\cdot g=0, & f(x,)=f^0(x), & f(x,t)|_{x\in\partial\Omega}=\tilde{\phi}(x,t), \\ \partial_t g+\nabla_x(f+h)=0, & g(x,0)=g^0(x), & g(x,t)|_{x\in\partial\Omega}=\tilde{\varphi}(x,t), \\ \partial_t h+\dfrac{2}{3}\nabla_x\cdot g=0, & h(x,0)=h^0(x), & h(x,t)|_{x\in\partial\Omega}=\tilde{\psi}(x,t). \end{cases} \tag{4.4}$$

Similarly, we transformed equations above as follows:

$$\tilde{F}_t=-A\tilde{F}, \tag{4.5}$$

where A is the partial differential operator given in Section 2 and $\tilde{F}=(f,g,h)^T$. Multiplying Eq. (4.5) by $\tilde{F}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{F}\|^2_{L^2(\Omega)}=-(A\tilde{F},\tilde{F}),$$

where we use $(\cdot,\cdot)$ to denote the standard L^2 inner product.

Using the integration by parts formula and $\frac{3}{2}\partial_t h+\nabla_x\cdot g=0$,

$$\begin{aligned}(A\tilde{F},\tilde{F})&=\frac{1}{4}\frac{d}{dt}\|h\|^2_{L^2(\Omega)}+\int_{\partial\Omega}(f+h)g\cdot n ds\\ &\le\frac{1}{4}\frac{d}{dt}\|h\|^2_{L^2(\Omega)}+\|f+h\|_{L^1(\partial\Omega)}\|g\|_{L^\infty(\partial\Omega)},\end{aligned}$$

where n stands for the outward unit normal to Ω.

Since $\sigma\to\tilde{\sigma}$ in the sense of $L^1(\partial\Omega)$, $u\to\tilde{u}$ in the sense of $L^\infty(\partial\Omega)$ and $\theta\to\tilde{\theta}$ in the sense of $L^1(\partial\Omega)$, we can make, for every given ε ($\varepsilon>0$),

$$\|f+h\|_{L^1(\partial\Omega)}\|g\|_{L^\infty(\partial\Omega)}\le\varepsilon.$$

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Combining the formulas above, we have

$$\frac{d}{dt}\|\tilde{F}\|^2_{L^2(\Omega)} \leq -\frac{1}{2}\frac{d}{dt}\|h\|^2_{L^2(\Omega)} - \varepsilon.$$

Integrate from 0 to τ with respect to t and than change the variable τ to t:

$$\|\tilde{F}\|^2_{L^2(\Omega)} + \frac{1}{2}\|h\|^2_{L^2(\Omega)} \leq \|\tilde{F}^0\|^2_{L^2(\Omega)} + \frac{1}{2}\|h^0\|^2_{L^2(\Omega)} + \varepsilon t \to 0,$$

as $f^0(x) \to 0$, $g^0(x) \to 0$ and $g^0(x) \to 0$ in the sense of $L^2(\Omega)$. So we reach the stability of the solution in finite time.

5 Existence for local solution

In this section we discuss the Existence of the local solution to 3-D Compressible Euler equation (3.1).

Lemma 5.1 (see [15]). *Consider the compressible Euler system* (3.1) *with initial value*

$$\rho^0 = 1 + \delta\sigma^0, \quad v^0 = \delta u^0, \quad T^0 = 1 + \delta\theta^0, \tag{5.1}$$

for any given $(\sigma^0, u^0, \theta^0) \in H^s$ *with* $s > 5/2$. *Choose* $\delta_1 > 0$ *so that for any* $0 < \delta \leq \delta_1$, *the positivity of* ρ^0 *and* T^0 *is guaranteed. Then, for each* $0 < \delta \leq \delta_1$, *there is a family of classical solutions* $(\rho^\delta, v^\delta, T^\delta) \in C([0,\tau^\delta]; H^s) \cap C^1([0,\tau^\delta]; H^{s-1})$ *of the Euler system* (3.1) *such that* $\rho^\delta > 0$, $T^\delta > 0$, *and the following estimates hold:*

$$\|(\rho^\delta, v^\delta, T^\delta) - (\rho_0, v_0, T_0)\|_{C([0,\tau^\delta]; H^s) \cap C^1([0,\tau^\delta]; H^{s-1})} \leq C.$$

Here the constant C *is independent of* δ, *depending only on the* H^s*-norm of* $(\sigma^0, u^0, \theta^0)$.

Remark 5.1. 1. The Lemma 5.1 presents a classical result that giving a small perturbation of amplitude δ from the constant state, there exist a family of classical solutions of the compressible Euler system (3.1) and the family of classical solutions is stable.

2. $(\sigma^0, u^0, \theta^0) \in H^s$ means $[(\rho^0, v^0, T^0) - (1,0,1)] \in H^s$.

By the lemma above, we have the following theorem:

Theorem 5.1. *If* $[(\rho^0, v^0, T^0) - (1,0,1)] \in H^s$ *with* $s > 5/2$ *and* $\rho^0 > 0$, $T^0 > 0$, *then Euler equation* (3.1) *with initial value* (ρ^0, v^0, T^0) *has a classical solution* $(\rho, v, T) \in C([0,\tau^\delta]; H^s) \cap C^1([0,\tau^\delta]; H^{s-1})$ *satisfying*

$$\|(\rho, v, T) - (1,0,1)\|_{C([0,\tau^\delta]; H^s) \cap C^1([0,\tau^\delta]; H^{s-1})} \leq C.$$

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6 The uniqueness and stability of the solutions to the nonlinear equations

In this section, we discuss the stability and uniqueness of the initial boundary value problem below:

$$\begin{cases} \partial_t\rho+(v\cdot\nabla)\rho+\rho\nabla_x\cdot v=0, \\ \rho(x,0)=\rho^0(x), \quad \rho(x,t)|_{x\in\partial\Omega}=\phi(x,t), \\ \rho\partial_t v+\rho(v\cdot\nabla)v+\rho\nabla T+T\nabla\rho=0, \\ v(x,0)=v^0(x), \quad v(x,t)|_{x\in\partial\Omega}=\varphi(x,t), \\ \partial_t T+(v\cdot\nabla)T+\dfrac{2}{3}T\nabla_x\cdot v=0, \\ T(x,0)=T^0(x), \quad T(x,t)|_{x\in\partial\Omega}=\psi(x,t). \end{cases} \tag{6.1}$$

As before, $\Omega\subset R^3$ is a bounded, open set. Suppose also $\rho^0(x)$, $v^0(x)$, $T^0(x)$, $\phi(x,t)$, $\varphi(x,t)$ and (x,t) are given.

6.1 The uniqueness of the nonlinear equations

Theorem 6.1. *Assume*

$$\rho^0(x),v^0(x),T^0(x)\in L^2(\Omega)$$

and

$$\phi(x,t),\varphi(x,t),\psi(x,t)\in L^2(0,\tau;L^2(\Omega)).$$

Then the initial boundary value problem (6.1) *has a unique solution* $(\rho,v,T)\in L^2(0,\tau;L^2(\Omega))$.

Remark 6.1. ρ, T represent the macroscopic density and temperature respectively, so it is reasonable to assume that ρ,T have a positive lower bound and a positive upper bound. Considering the situation that the fluid volume remains unchanged, therefore ρ is proportional to T. Might as well fix $T=M\rho$, where M is an positive constant. In this case, we must get $\nabla_x\cdot v=0$, namely, the fluid boundary velocity is zero, which is reasonable in the actual situation.

Proof. Supposing (ρ_1,v_1,T_1) are solutions constructed in Section 3. From Section 3, it is reasonable to assume that (ρ_1,v_1,T_1) are uniformly bounded. Supposing (ρ_2,v_2,T_2) are arbitrary solutions of the initial boundary value problem (6.1). Let their difference be $(\overline{\rho},\overline{v},\overline{T})=(\rho_1,v_1,T_1)-(\rho_2,v_2,T_2)$. Then, subtracting the equations from each other in (6.1), we have

$$\begin{cases} \partial_t\overline{\rho}+v_1\cdot\nabla\rho_1-v_2\cdot\nabla\rho_2=0, \\ \overline{\rho}(x,0)=0, \quad \overline{\rho}(x,t)|_{x\in\partial\Omega}=0, \\ \partial_t\overline{v}+v_1\cdot\nabla v_1-v_2\cdot\nabla v_2+2M\nabla\overline{\rho}=0, \\ \overline{v}(x,0)=0, \quad \overline{v}(x,t)|_{x\in\partial\Omega}=0. \end{cases} \tag{6.2}$$

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Multiplying the second equation of (6.2) by $\overline{v}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\overline{v}\|^2_{L^2(\Omega)}+\int_\Omega (v_1\cdot\nabla)v_1\cdot\overline{v}dx-\int_\Omega (v_2\cdot\nabla)v_2\cdot\overline{v}dx+2M\int_\Omega \nabla\overline{\rho}\cdot\overline{v}dx=0.$$

Calculate the second and third term on the left:

$$\begin{aligned}&\int_\Omega (v_2\cdot\nabla)v_2\cdot\overline{v}dx-\int_\Omega (v_1\cdot\nabla)v_1\cdot\overline{v}dx\\=&\int_\Omega (v_2\cdot\nabla v_1-v_2\cdot\overline{v}-v_1\cdot\nabla v_1)\cdot\overline{v}dx\\=&\int_\Omega (-\overline{v}\cdot\nabla v_1-v_1\cdot\nabla\overline{v}+\overline{v}\cdot\nabla\overline{v})\cdot\overline{v}dx.\end{aligned}$$

Thus, the formula of partial integration implies

$$\begin{aligned}\int_\Omega v_1\cdot\nabla\overline{v}\cdot\overline{v}dx=\int_{\partial\Omega}\frac{1}{2}|\overline{v}|^2v_1\cdot nds-\int_\Omega\frac{1}{2}|\overline{v}|^2\nabla\cdot v_1dx=0,\\\int_\Omega \overline{v}\cdot\nabla\overline{v}\cdot\overline{v}dx=\int_{\partial\Omega}\frac{1}{2}|\overline{v}|^2\overline{v}\cdot nds-\int_\Omega\frac{1}{2}|\overline{v}|^2\nabla\cdot\overline{v}dx=0,\end{aligned}$$

and Hölder inequality yields the bound

$$\begin{aligned}&\int_\Omega -\overline{v}\cdot\nabla v_1\cdot\overline{v}dx\\\le&\|\nabla v_1\|_{L^\infty(\Omega)}\|\overline{v}\|_{L^2(\Omega)}\|\overline{v}\|_{L^2(\Omega)}\\\le&c\|\overline{v}\|^2_{L^2(\Omega)}.\end{aligned}$$

Calculate the last term on the left by using the formula of partial integration:

$$\int_\Omega\nabla\overline{\rho}\cdot\overline{v}dx=\int_{\partial\Omega}\overline{\rho v}\cdot nds-\int_\Omega\overline{\rho}\nabla\cdot\overline{v}dx=0.$$

We finally combine the formulas above, to discover

$$\frac{d}{dt}\|\overline{v}\|^2_{L^2(\Omega)}\le c\|\overline{v}\|^2_{L^2(\Omega)}.$$

In view of the Gronwall inequality, we deduce

$$\|\overline{v}\|^2_{L^2(\Omega)}\le e^{ct}\|\overline{v}(x,0)\|^2_{L^2(\Omega)}=0,$$

and thus $v_1=v_2$. Multiplying the first equation of (6.2) by $\overline{\rho}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\overline{\rho}\|^2_{L^2(\Omega)}+\int_\Omega\overline{\rho}v_1\cdot\nabla\rho_1dx-\int_\Omega\overline{\rho}v_2\cdot\nabla\rho_2dx=0.$$

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Calculate the second and third term on the left of the resulting expression by using the formula of partial integration:

$$\begin{aligned}&\int_\Omega \overline{\rho} v_1 \cdot \nabla \rho_1 dx - \int_\Omega \overline{\rho} v_2 \cdot \nabla \rho_2 dx\\ =&\int_\Omega \overline{\rho} v_1 \cdot \nabla \overline{\rho} dx = \int_{\partial\Omega} \frac{1}{2}|\overline{\rho}|^2 v_1 \cdot n ds - \int_\Omega \frac{1}{2}|\overline{\rho}|^2 \nabla \cdot v_1 dx\\ =&0.\end{aligned}$$

Returning now to the resulting expression, we find $\|\overline{\rho}\|_{L^2(\Omega)} = 0$, and thus $\rho_1 = \rho_2$. □

6.2 The stability of the nonlinear equations

By the Theorem 5.1, giving a small perturbation to Eq. (3.1), the solutions of equation still exists. Therefore, modelled on the proof process of uniqueness, stability can be proved.

Let $(\tilde{\rho}, \tilde{v}, \tilde{T})$ denote the solution pair of a slight disturbance with the corresponding initial value. Fix $(f, g, h) = (\tilde{\rho}, \tilde{v}, \tilde{T}) - (\rho, v, T)$, satisfying the following conditions

1. boundary value $f \to 0$ in the sense of $L^2(\partial\Omega)$,
2. initial value $g^0 \to 0$ in the sense of $L^2(\Omega)$ and boundary value $g \to 0$ in the sense of $L^2(\partial\Omega) \cap L^\infty(\partial\Omega)$.

Subtracting one equation from another, we have

$$\begin{cases} \partial_t f + \tilde{v} \cdot \nabla \tilde{\rho} - v \cdot \nabla \rho = 0, \\ f(x,0) = f^0(x), \quad f(x,t)|_{x \in \partial\Omega} = \tilde{\phi}(x,t), \\ \partial_t g + \tilde{v} \cdot \nabla \tilde{v} - v \cdot \nabla v + 2M \nabla f = 0, \\ g(x,0) = g^0(x), \quad g(x,t)|_{x \in \partial\Omega} = \tilde{\varphi}(x,t), \end{cases} \tag{6.3}$$

for $\forall t \in (0, T]$. Multiplying the second equation of (6.3) by g and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|g\|^2_{L^2(\Omega)} + \int_\Omega \tilde{v} \cdot \nabla \tilde{v} \cdot g dx - \int_\Omega v \cdot \nabla v \cdot g dx + 2M \int_\Omega \nabla f \cdot g dx = 0.$$

Calculate the second and third term on the left:

$$\begin{aligned}&\int_\Omega \tilde{v} \cdot \nabla \tilde{v} \cdot g dx - \int_\Omega v \cdot \nabla v \cdot g dx\\ =&\int_\Omega (\tilde{v} \cdot \nabla v + \tilde{v} \cdot \nabla g - v \cdot \nabla v) \cdot g dx\\ =&\int_\Omega (g \cdot \nabla v + g \cdot \nabla g + v \cdot \nabla g) \cdot g dx.\end{aligned}$$

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The formula of partial integration implies

$$\int_\Omega v\cdot\nabla g\cdot g dx=\int_{\partial\Omega}\frac{1}{2}|g|^2 v\cdot n ds-\int_\Omega\frac{1}{2}|g|^2\nabla\cdot v dx=\int_{\partial\Omega}\frac{1}{2}|g|^2 v\cdot n ds,$$
$$\int_\Omega g\cdot\nabla g\cdot g dx=\int_{\partial\Omega}\frac{1}{2}|g|^2 g\cdot n ds-\int_\Omega\frac{1}{2}|g|^2\nabla\cdot g dx=\int_{\partial\Omega}\frac{1}{2}|g|^2 g\cdot n ds.$$

Hölder inequality and $g\to 0$ in the sense of $L^2(\partial\Omega)\cap L^\infty(\partial\Omega)$ yield the bound

$$\int_{\partial\Omega}|g|^2 v\cdot n ds\le\|v\|_{L^\infty(\partial\Omega)}\|g\|_{L^2(\partial\Omega)}\|g\|_{L^2(\partial\Omega)}\to 0,$$
$$\int_{\partial\Omega}|g|^2 g\cdot n ds\le\|g\|_{L^\infty(\partial\Omega)}\|g\|_{L^2(\partial\Omega)}\|g\|_{L^2(\partial\Omega)}\to 0,$$
$$\int_\Omega g\cdot\nabla v\cdot g dx\le\|\nabla v\|_{L^\infty(\Omega)}\|g\|^2_{L^2(\Omega)}\le c\|g\|^2_{L^2(\Omega)}.$$

Estimating the last term on the left, we have

$$\int_\Omega\nabla f\cdot g dx=\int_{\partial\Omega}fg\cdot n ds-\int_\Omega f\nabla\cdot g dx=\int_{\partial\Omega}fg\cdot n ds\le\|f\|_{L^2(\partial\Omega)}\|g\|_{L^2(\partial\Omega)}\to 0.$$

Here we used the formula of partial integration, Hölder inequality and $f,g\to 0$ in the sense of $L^2(\partial\Omega)$. We finally combine the formulas above, to discover

$$\frac{d}{dt}\|g\|^2_{L^2(\Omega)}\le c\|g\|^2_{L^2(\Omega)}.$$

In view of the Gronwall inequality and $g^0\to 0$ in the sense of $L^2(\Omega)$, we deduce, for every $t\in(0,T]$,

$$\|g\|^2_{L^2(\Omega)}\le e^{ct}\|g(x,0)\|^2_{L^2(\Omega)}\to 0,$$

and thus $\tilde{v}\to v$. Multiplying the first equation of (6.3) by g and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(\Omega)}+\int_\Omega f\tilde{v}\cdot\nabla\tilde{\rho}dx-\int_\Omega fv\cdot\nabla\rho dx=0.$$

Calculate the second and third term on the left of the resulting expression by using the formula of partial integration

$$\int_\Omega f\tilde{v}\cdot\nabla\tilde{\rho}dx-\int_\Omega fv\cdot\nabla\rho dx=\int_\Omega fv\cdot\nabla f dx$$
$$=\int_{\partial\Omega}\frac{1}{2}|f|^2 v\cdot n ds-\int_\Omega\frac{1}{2}|f|^2\nabla\cdot v dx=\frac{1}{2}\int_{\partial\Omega}|f|^2 v\cdot n ds.$$

Thus, Hölder inequality and $f\to 0$ in the sense of $L^2(\Omega)$ yield the bound

$$\int_{\partial\Omega}|f|^2 v\cdot n ds\le\|v\|_{L^\infty(\partial\Omega)}\|f\|_{L^2(\partial\Omega)}\|f\|_{L^2(\partial\Omega)}\to 0.$$

Returning now to the resulting expression, we find $\|f\|_{L^2(\Omega)}\to 0$, and thus $\tilde{\rho}\to\rho$. So we reach the stability of the solutions in finite time.

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7 Existence of the saddle point solution

In this section, we discuss the eigenvalue problem for the equation below:

$$\begin{cases} \partial_t\sigma+\lambda\nabla_x\cdot u=0, \\ \partial_t u+\lambda\nabla_x(\sigma+\theta)=0, \\ \partial_t\theta+\dfrac{2}{3}\lambda\nabla_x\cdot u=0. \end{cases} \tag{7.1}$$

Similarly to the Section 2, we transform Eq. (7.1) as follows:

$$F_t=-\lambda AF \tag{7.2}$$

and fix

$$F(x_1,x_2,x_3,t)=G(a_1x_1+a_2x_2+a_3x_3+bt)=G(\eta).$$

Now put it into Eqs. (7.1) and merge similar terms to derive new equation

$$\begin{pmatrix} \dfrac{b}{\lambda} & a_1 & a_2 & a_3 & 0 \\ a_1 & \dfrac{b}{\lambda} & 0 & 0 & a_1 \\ a_2 & 0 & \dfrac{b}{\lambda} & 0 & a_2 \\ a_3 & 0 & 0 & \dfrac{b}{\lambda} & a_3 \\ 0 & \dfrac{2}{3}a_1 & \dfrac{2}{3}a_2 & \dfrac{2}{3}a_3 & \dfrac{b}{\lambda} \end{pmatrix} G_\eta=0. \tag{7.3}$$

Taking elementary transformation to the coefficient matrix of Eq. (7.3):

$$\begin{pmatrix} -\dfrac{2}{3}\dfrac{b}{\lambda} & 0 & 0 & 0 & \dfrac{b}{\lambda} \\ 0 & \dfrac{b}{\lambda} & 0 & 0 & \dfrac{5}{2}a_1 \\ 0 & 0 & \dfrac{b}{\lambda} & 0 & \dfrac{5}{2}a_2 \\ 0 & 0 & 0 & \dfrac{b}{\lambda} & \dfrac{5}{2}a_3 \\ 0 & 0 & 0 & 0 & \dfrac{3}{2}\dfrac{b}{\lambda}-\dfrac{5}{2}\dfrac{\lambda}{b}(a_1^2+a_2^2+a_3^2) \end{pmatrix} G_\eta=0.$$

Solving the equation, we get

$$\lambda_1=\frac{-\sqrt{3}b}{\sqrt{5(a_1^2+a_2^2+a_3^2)}},$$

$$\lambda_2=\frac{\sqrt{3}b}{\sqrt{5(a_1^2+a_2^2+a_3^2)}}.$$

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Remark 7.1. 1. When $b \neq 0$ and $a_1 a_2 a_3 \neq 0$, $\lambda_1 \lambda_2 < 0$.

2. This presents a classical result that the compressible Euler system (7.1) is not stable.

8 Conclusions

In another paper, authors study the numerical solution of the Euler equation and get the conclusion that the central difference scheme for the Euler equation is convergent under certain initial boundary value conditions. But this conclusion can not provide evidence for the stability.

In order to confirm the result of numerical simulation, we discuss the stability of the solutions to the Euler equation in a finite time period.

Firstly, we use the Lie Group to establish quite a wide range of two types exact solution. After traveling wave analysis, we obtain a class of exact solutions of the linearization equation (1.3) in Section 2. Then, in in Section 3, the classical Lie group method is used to study the linearization equation and the compressible Euler Equation respectively: using a first-order prolongation operator, we obtain group-invariant solution of equations, and establish relationships between the new solution and the original solution.

Secondly, in Section 4, we discuss the stability and uniqueness of the linearized equation with arbitrary initial boundary value respectively in different space, namely the well-posedness analysis.

Thirdly, in Section 5, generalize the existence theorem of local solutions to the Compressible Euler Equation. Afterwards, we prove the uniqueness and stability of the solutions to the Compressible Euler Equation in a finite time period in Section 6.

Finally, through the analysis of the characteristic equation, we get the instability of the compressible Euler Equation. It is concluded that the result (the stability of the solution of the Euler equation in the finite time interval) cannot be extended to the infinite time.

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