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Asymptotic treatment of the three-wave nonlinear parametric interaction

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Abstract. A novel integral representation suitable for the study of asymptotic ($L/L_0 \gg 1$, L being the plasma length and L_0 the basic gain length) stability of the inhomogeneous three-wave interaction is formulated and applied to the special case of uniform gain and ramp wavenumber mismatch. The absolute instability condition is $\kappa L_0 < \pi/8$ (κ being the mismatch gradient), with the wave conversion performed in a zone of width $4/\kappa L_0$ located at the boundary of wave incidence.

1. Introduction

In this contribution we present a novel representation of the three-wave nonlinear coupling problem in an inhomogeneous plasma, convenient for the study of strong excitation conditions (long plasma and/or strong pump). The coupling problem discussed herein can represent various laser–plasma interactions, such as stimulated Raman or Brillouin scattering (e.g. Kruer 1988). Our starting point are the nonlinearly coupled spatio-temporal equations for slowly varying amplitudes, which have a wide range of applications in optics as well as laser–plasmas. They were used, for example, for the study of saturation of Brillouin back-scatter (Cairns 1982).

Here, rather than proceeding via the standard (Rosenbluth 1972; Pesme et al. 1973; White et al. 1974; Nicholson and Kaufman 1974; Dubois et al. 1974; Chambers and Bers 1977) complex second-order differential equation, we prefer to treat a pair of coupled equations for the total phase Φ and its spatial integral $\eta = \int \cos\Phi \, d\xi$. An equation for the phase invariant $\sin\Phi \sin 2\eta$ then leads us to an integral equation whose phase is stationary at the wavenumber matching point. It will become clear, in this representation, that the inhomogeneous interaction is a nonlinear diffractive phenomenon, with significant interference occurring between emission originating from the endpoints, and that produced resonantly, if and only if the phase modes are symmetric with respect to the center of the gain region. Otherwise, asymmetric modes associated with product waves, which are, respectively, up-shifted and down-shifted in frequency with respect to the original assumption, can form a proper interaction-zone centered around the point of best phase matching. In the special case of uniform gain and a ramp mismatch, the asymptotic ($L/L_0 \gg 1$, with L the plasma length and L_0 the basic gain length) absolute instability condition is $\kappa L_0^2 < 8/\pi$, where κ is the mismatch gradient, and the asymptotic frequency shift is $\Omega = 2L\rho/\pi L_0$ where ρ is the growth rate of the absolute instability in an infinite homogeneous plasma. The plasma tends to perform the wave conversion in a zone of width $4\kappa L_0$ located at the irradiated boundary of the uniform gain region.

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2. Analysis of the coupled mode equations

To start with, we will determine the time-asymptotic equilibria of the inhomogeneous coupled mode equations (Cairns 1982):

$$\frac{\partial E_1}{\partial t} + c_1 \frac{\partial E_1}{\partial x} = -M_1 E_2 E_3 \exp[iP(x)] \tag{1a}$$

$$\frac{\partial E_2^*}{\partial t} - c_2 \frac{\partial E_2^*}{\partial x} = M_2 E_1^* E_3^* \exp[iP(x)] \tag{1b}$$

$$\frac{\partial E_3^*}{\partial t} + c_3 \frac{\partial E_3^*}{\partial x} = M_3 E_1^* E_2^* \exp[iP(x)] \tag{1c}$$

which describe the slow space–time modulations of the complex wave amplitudes $E_\alpha, \alpha = 1, 2, 3$. c_α are the absolute values of the group velocities $v_\alpha, M_\alpha > 0$ are coupling constants, and

$$P(x) = \int_0^x \Delta k(x')\, dx', \quad \Delta k = k_2 = k_3 - k_1 \tag{2}$$

is the WKBJ (Wentzel-Kramers-Brillouin-Jeffries) wavenumber-mismatch phase-integral (Rosenbluth 1972; Pesme et al. 1973; White et al. 1974; Nicholson and Kaufman 1974; Dubois et al. 1974; Chambers and Bers 1977). We consider the boundary value problem of a steady pump wave 1 incident on the plasma slab at $x = 0$ (low-density region) and emerging somewhat depleted at $x = L$. For the oppositely propagating product waves it is assumed that $E_3(0, t) = E_2(L, t) = 0$ so that decay and an associated nontrivial time-asymptotic solution of (1) is possible only above the absolute instability threshold. Since the incident pump amplitude is assumed steady, $E_1(0, t) \equiv E_{10} =$ constant it is clear that the time-asymptotic moduli $|E_\alpha|, \alpha = 1, 2, 3$, are also steady. Furthermore, it is not difficult to deduce that the equilibrium of the pump amplitude is also steady, whereas the amplitudes of the product waves can, in general, develop as $\exp(\pm i\Omega t)$, respectively. The time-asymptotic equilibria are, therefore,

$$E_1 = A_1(x) \exp[i\varphi_1(x)], \quad E_{2,3} = A_{2,3}(x) \exp[i\varphi_{2,3}(x) \pm i\Omega t] \tag{3}$$

where the frequency shift Ω is as yet unspecified.

Upon substitution of (3) into (1), and making use of the normalizations

$$\xi = x/L_0, \quad \Lambda = L/L_0, \quad L_0 = 1/E_{10}(M_2 M_3/c_2 c_3)^{1/2} \tag{4}$$

$$a_\alpha = L_0 L_\alpha (M_\beta M_\gamma / c_\beta c_\gamma)^{1/2} \tag{5}$$

where L_0 is the basic gain length and α, β, γ are permuted in cyclic order over 1, 2, 3, we obtain

$$a_1' = -a_2 a_3 \cos\varphi, \quad a_2' = -a_1 a_3 \cos\varphi, \quad a_3' = a_1 a_2 \cos\varphi \tag{6a}$$

$$\varphi' = f(\xi) - [\ln(a_1 a_2 a_3)]' \tan\varphi \tag{6b}$$

where the prime means differentiation with respect to ξ,

$$f(\xi) = \Delta k L_0 - Q, \quad Q = \Omega L_0 (1/c_2 + 1/c_3) \tag{7}$$

and

$$\Phi = \int_0^\xi f(\tau)\, d\tau + \varphi_2 + \varphi_3 - \varphi_1. \tag{8}$$

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The formal solution of (6) are the Jacobian elliptic functions

$$a_1 = dn(\eta, \kappa), \quad a_2 = kcn(\eta, \kappa), \quad a_3 = ksn(\eta, \kappa) \tag{9}$$

where $k = a_{20}$ is the modulus and $\eta = \int_0^\xi \cos\varphi(\tau)\, dt$. It follows that the condition $\eta(\Lambda) = K(k)$, K being the elliptic integral of the first kind, determines the anomalous (action) reflectivity k^2 of the fundamental mode. For $k \to 0$ we thus obtain $\eta(\Lambda) = \pi/2$, the most general form of the absolute instability criterion. To examine the threshold, use is made of this condition in an associated boundary value problem formulated as follows. Upon substituting (6) into (7) and performing the limit $k \to 0$, we obtain the basic threshold equation

$$\varphi' = f(\xi) - 2\sin\varphi \operatorname{cotg} 2\eta. \tag{10}$$

$$\varphi(0) = \varphi(\Lambda) = \eta(0), \quad \eta(\Lambda) = \frac{\pi}{2}.$$

Let us now introduce the functions

$$Y(\xi) = \sin 2\eta \sin\varphi, \quad G(\xi) = \cos 2\eta \tag{11}$$

and multiply (10) by $\sin 2\eta \cos\varphi$. This produces

$$Y'(\xi) = f(\xi) \sin 2\eta \cos\varphi, \tag{12}$$

which differentiated with respect to ξ yields

$$Y'' - (\ln f)' Y' + f^2 Y = 2fG \tag{13}$$

$$Y(0) = Y'(0) = Y(\Lambda) = Y'(\Lambda) = 0.$$

To proceed further we now introduce the variable

$$F(\xi) = \int_0^\xi f(\tau)\, d\tau \tag{14}$$

producing

$$Y(\xi) = 2[\sin F C(0, \xi) - \cos F S(0, \xi)] \tag{15}$$

where the functions C and S are defined by

$$C(0, \xi) + iS(0, \xi) = \int_0^\xi G(\tau) \exp[iF(\tau)]\, d\tau. \tag{16}$$

Furthermore,

$$Y'(\xi) = 2f[\cos F C(0, \xi) + \sin F S(0, \xi)], \tag{17}$$

with the result that

$$\sin^2 2\eta = 4[C^2(0, \xi) + S^2(0, \xi)]. \tag{18}$$

To complement this integral equation we note that (17) is equivalent to

$$G'(\xi) = -4 \int_0^\xi G(t) \cos[F(\xi) - F(t)]\, dt \tag{19}$$

which, differentiated twice, yields

$$G''' - (\ln f)' G'' + (4 + f^2) G' - 4(\ln f)' G = 0$$

$$G(0) = 1, \quad G'(0) = 0, \quad G''(0) = -4. \tag{20}$$

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As a consequence, in the first place, of the relatively simple nature of the function G, which satisfies $G(0)=1$, $G(\Lambda)=1$, and which is definite in sign on either side of the phase resonance point ξ_{R}, and, in the second place, of the oscillatory character of $\sin F$ and $\cos F$, the asymptotic ($\Lambda \gg 1$) behaviour of the integrals C and S is determined by the form of G at the stationary point of F, i.e. at the phase resonance $f=0$. In order to obtain $G(\xi_{\mathrm{R}})$ we observe that (20) at $f=0$ is equivalent to

$$H' - (f'/f)H = 0, \quad H = G'' + 4G, \tag{21}$$

yielding the result

$$G(f \cong 0) = A\cos 2\xi + B\sin 2\xi. \tag{22}$$

In conclusion, (18) complemented by (20) permits the discussion of physically significant regions of the inhomogeneous plasma, i.e. the phase resonance and the endpoints regions.

To permit comparison with the work of other authors (Rosenbluth 1972; Pesme et al. 1973; White et al. 1974; Nicholson and Kaufman 1974; Dubois et al. 1974; Chambers and Bers 1977), we now consider the case of a ramped mismatch $f(\xi) = \alpha(\Lambda/2-\xi)-Q, \alpha=\kappa L_0^2$, where we, as have others, suppose uniform gain and place the assumed resonance at the center of the gain region. Let us recognize two cases.

Case (i)

The frequency of product waves is unshifted (Rosenbluth 1972; Pesme et al. 1973; White et al. 1974), $Q=0$. In this case $G(\xi)$ is odd with respect to $\xi=\xi_{\mathrm{R}}=\Lambda/2$, and thus $G(\xi)=\xi$ around $\xi=\Lambda/2$. Hence (see Theorem 13.2 of Olver (1974)) for large Λ the contribution of the endpoints 0 and Λ is asymptotically equivalent to that of the stationary point $\xi_{\mathrm{R}}=\Lambda/2$. Consequently, waves propagating inward from 0 and Λ can interfere strongly with the signal produced resonantly (Pesme 1974). To formulate these ideas quantitatively, we rewrite (18) in terms of the variable f, for which

$$f_0 \equiv f(0) = \alpha\Lambda/2 - Q, \quad f_\Lambda = -\alpha\Lambda/2 - Q, \quad f(\xi_R) = 0 \tag{23}$$

producing at $\xi=\xi_{\mathrm{R}}$,

$$\alpha^2 = 4|I(f_0)|^2, \quad I(f_0) = \int_0^{\alpha\Lambda/2} G(t)\exp(it^2/2\alpha)\,dt. \tag{24}$$

Changing variables to $\tau = t\alpha\Lambda/2$ introduces the large parameter $\lambda=\alpha\Lambda^2/8$ and we obtain (Olver 1974)

$$I(f_0) = (2i/\Lambda)(\alpha\Lambda\gamma/2 - \cos\lambda - i\sin\lambda) \tag{25}$$

where $\gamma = G'(0) = (4/\alpha^2)\mathrm{Re}\,I(f_0)$, as follows from (19). Hence, (24) yields the equation

$$\alpha^4 + [(8/\lambda)\sin 2\lambda - (16/\Lambda^2)]\alpha^2 - (16\sin\lambda/\Lambda)^2 = 0 \tag{26}$$

whose asymptotic root is $\alpha^2 = (16/\Lambda)|\sin(\alpha\Lambda^2/8)|$.

In (Λ,α) space the individual lobes of this function map onto closed regions whose vertices lie at $\Lambda_N = [(2N+1)\pi]^{2/3}$ and decay as $\Lambda^{-1/2}$. The individual branches of the stability regions converge to the Λ-axis as Λ^{-1}. In conclusion, the $Q=0$ modes cannot support absolute instability for large Λ (Rosenbluth 1972; Pesme et al. 1973; White et al. 1974).

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Case (ii)

A completely different picture emerges if we allow for a frequency shift, $Q \neq 0$ (Dubois et al. 1974; Chambers and Bers 1977). The corresponding modes are now asymmetric so that G is, in general, not zero at $f = 0$. In consequence (see Theorem 13.1 of Olver (1974)), the contribution of the endpoints 0 and Λ is dominated by that of the resonant point (which is now shifted). The disproportion between the resonant and endpoint contributions increases, of course, with the degree of asymmetry, i.e. with Q, and will thus be maximized for the largest admissible eigenvalue Q. The wave coupling process in the resonant zone then proceeds undisturbed by signals arriving from other points on the gradient. This describes a typically homogeneous situation and hence the concept of an interaction zone which could be treated as a homogeneous slab can be introduced. Let us first point out that α no longer declines with Λ, but rather saturates at a level dependent on Q, in consequence of the fact that the asymptotic behaviour of C and S is now given by that of the Fresnel integrals (Olver 1974; Born and Wolf 1970). To establish the spectrum of Q we thus write

$$C^2(f_0, 0) + S^2(f_0, 0) = C^2(0, f_\Lambda) + C^2(0, f_\Lambda) \tag{27}$$

obtained from (18) on realizing that $\sin 2\eta \cong 1$ at $f = 0$, and $\sin 2\eta = 0$ at $f = f_\Lambda$. As $\Lambda \to \infty$. Equation (27) becomes

$$C^2(\bar{f}_0) + S^2(\bar{f}_0) = \tfrac{1}{2}, \quad \bar{f}_0 = f_0/(\alpha\pi)^{1/2} \tag{28}$$

where $C + iS = \int_0^t \exp(i\pi\tau^2/2)\, d\tau$. Hence, the spectrum of all $\bar{f}_0$ is given by the intersections of Cornu's spiral (Born and Wolf 1970) with a circle of radius $\frac{1}{2}$ drawn from the origin $C = S = 0$. The smallest $\bar{f}_0$ thus determined is $\bar{f}_{0\,\min} = 1/\sqrt{2}$. With regard to the width, Δ_{int}, of the interaction zone, we substitute for G from (22) into (16) to find that C and S are the linear combinations of integrals of the type

$$J^{(\pm)} = \int_{f_0}^{f} \exp[i(t \pm 2)^2/2\alpha]\, dt. \tag{29}$$

Asymptotically, there is no contribution to $J^{(\pm)}$ outside the interval $(-2, +2)$ centered around $f = 0$. Since the function $\sin^2 2\eta = 4a_2^2 a_3^2$ measures the intensity of the product waves, (18) immediately suggests that $\Delta_{\text{int}}^{(f)} = 4$, which in physical x-space gives $\Delta_{\text{int}}^{(x)} = 4\kappa L_0$. Applying the homogeneous threshold condition $\Delta_{\text{int}}^{(x)} = L_0\pi/2$ immediately gives the absolute stability condition for large Λ : $\alpha = \kappa L_0^2 = 8/\pi$. Thus, it seems, the instability can withstand a mismatch gradient 2.5 times greater than that suggested in (Dubois et al. 1974). Furthermore, given that $f_{0\,\min} = 2$, the asymptotic frequency shift is $Q = 4\Lambda/\pi - 2$, and the best-matched point lies at $x = 2/kL_0$. The placing of the resonance is obviously due to the assumed uniform gain at the irradiated boundary.

3. Summary

The new representation of the inhomogeneous wave coupling problem presented here reduces the three-wave coupling problem to the discussion of the effect of the two physically significant regions on the gradient: namely the phase resonance and the boundary regions, which are the sources of secondary emission. The analogy to the effects of obstacles in diffraction phenomena is clear (Born and Wolf 1970).

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This new representation may prove useful in situations more realistic than is the case of ramped mismatch and uniform gain.

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