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# TRANSIENT ANALYSIS OF PERMANENT CUSTOMERS IN A SINGLE-SERVER QUEUE WITH MIXED TRAFFIC

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Single-server queueing systems with mixed traffic are a flexible modeling tool that have been used to analyze polling systems and database striping strategies. They also have applications in manufacturing and telecommunication systems. Ordinary (open) customers arrive according to a Poisson process, receive service, and leave the system. Permanent (closed) customers remain in the system, reentering the queue after receiving service. This work examines the transient behavior of the permanent customers. The cycle times and output/departure process for the permanent customers are described.

## 1. INTRODUCTION

Many service systems, such as manufacturing systems, computer systems, and communication networks, have traffic that is mixed, consisting of both ordinary (open) and permanent (closed) customers. Ordinary customers enter from outside the system and, eventually, leave it. Permanent customers never leave the system. Instead

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they reenter previously visited queues as they circulate or cycle in the system. General equilibrium results for networks of Markovian queues with mixed traffic have existed for some time [2]. More recently, a heavy-traffic approximation has been developed for a two-station mixed-queueing network with applications to production systems employing a combination of make-to-order and make-to-stock operations [15]. Equilibrium results have also been obtained for an $M/G/1$ queue with mixed traffic [4] and have been used to analyze polling systems [5] and database striping strategies [14]. However, little work related to the transient behavior of mixed-traffic queueing models appears to have been done. Transient results are plentiful for open queueing systems. Those related to this work include output-process [8,18] and sojourn-time results [10]. Transient results for closed systems include [3,6,11].

This work examines the transient behavior of permanent customers in a single-server queue with mixed traffic, as depicted in Figure 1. (Note that we use transient to refer to nonstationary rather than time dependent.) The results have previously been used to analyze transient behavior in telecommunication networks following communication link failures [13]. The server alternates between periods of serving ordinary customers and permanent customers. This alternating process is used as the basis for analyzing the system. Initially, a mixed-traffic system with one permanent customer is examined. For this system, a cycle is described as an "ordinary" service period followed by a "permanent" service period. Length information is obtained for cycle $i$. By adding the cycle lengths, the time of the $n$th departure of the permanent customer can be found. These results are presented in Section 2. In Section 3, the alternating process is used to obtain results for a system containing $M$ permanent customers. Interestingly, the $M$-customer system can be viewed as the "interleaving" of $M$ single-permanent-customer systems. Finally, conclusions are presented in Section 4.

## 2. SINGLE PERMANENT CUSTOMER

Consider a single-server queue with one permanent customer and immediate feedback. The permanent customer always remains in the system. After completing ser-

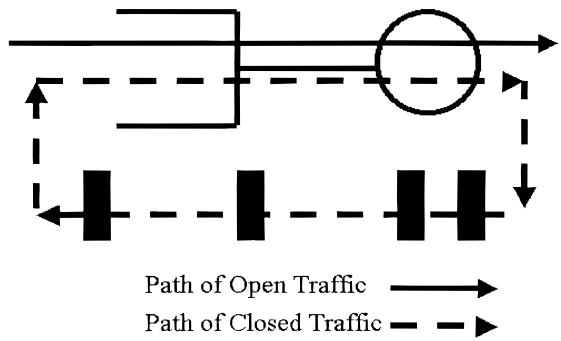



**FIGURE 1.** Single-server queue with multiple permanent customers.

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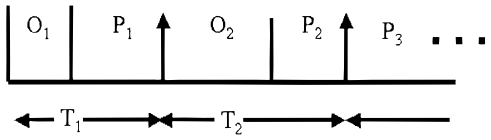



**FIGURE 2.** Service process with one permanent customer.

vice, it immediately feeds back to reenter the queue. As a result, the server is never idle. A cycle is defined to be the time from one arrival of the permanent customer to the next. Each cycle consists of an "ordinary" period, during which the ordinary customers are served, and a "permanent" period, during which the permanent customer is served. (Note that the ordinary period will have zero length if no ordinary customers arrive before the permanent customer completes service, feeds back, and reenters the queue.) The service process then has the form of an alternating process as shown in Figure 2. This figure also illustrates the departure process for the permanent customer. The following definitions are used:

$$
\begin{aligned}
O_i &= \text{length of the } i\text{th ordinary period}\\
P_i &= \text{length of the } i\text{th permanent period}\\
T_i &= O_i + P_i = \text{length of the } i\text{th cycle}\\
N_i &= \text{number of ordinary customers arriving during the } i\text{th cycle}\\
F_{O_i}(t) &= \text{cumulative distribution function for } O_i\\
F_{P_i}(t) &= \text{cumulative distribution function for } P_i\\
F_{T_i}(t) &= \text{cumulative distribution function for } T_i\\
F_{N_i}(n) &= \text{cumulative distribution function for } N_i\\
O_i^*(s) &= \text{Laplace–Stieltjes transform (LST) of } F_{O_i}(t)\\
P_i^*(s) &= \text{Laplace–Stieltjes transform of } F_{P_i}(t)\\
T_i^*(s) &= \text{Laplace–Stieltjes transform of } F_{T_i}(t)\\
\mathfrak{F}_{N_i}(z) &= \text{probability generating function (pgf) of } F_{N_i}(n).
\end{aligned}
$$

The cycle time of the permanent customer is analyzed next.

### 2.1. Cycle Time for *M/M/*1 System

Ordinary customers arrive according to a Poisson process with rate $\lambda$. Service times are independent and both types of customers draw their service times from the same exponential distribution with rate $\mu$. As a result, the lengths of the permanent periods, the $P_i$'s, are independent and identically distributed (i.i.d.), so the subscript may be dropped; then,

$$F_P(t) = F_{P_i}(t) = 1 - e^{-\mu t}, \quad t \geq 0 \text{ for all } i,$$

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and

$$P^*(s) = P_i^*(s) = \frac{\mu}{s+\mu} \quad \text{for all } i.$$

Cycle $i$ begins as the permanent customer departs the server and reenters the queue for the $i$th time. The permanent customer then waits while the ordinary customers it finds in the system, if any, receive service. The length of this wait is $O_i$, which can be zero. After the ordinary customers are gone, the permanent customer receives service, then departs the server and reenters the queue, starting cycle $i+1$. Note that the number of ordinary customers left in the queue by the permanent customer at the end of cycle $i$ is the number it finds at the start of cycle $i+1$. If it looks back just before leaving the server, the permanent customer sees all the ordinary customers that have arrived since it last entered the system. Because the last arrival of the permanent customer started cycle $i$, this is $N_i$. When the permanent customer reenters the queue, starting cycle $i+1$, it finds the $N_i$ ordinary customers there. Because the time to serve these customers is $O_{i+1}$, the length of cycle $i+1$ depends on the number of ordinary arrivals during cycle $i$, which depends on the length of cycle $i$.

More formally, begin with the distribution of the number of customers found by the arriving permanent customer at the start of cycle $i$ or, equivalently, the number of ordinary customers arriving in cycle $i-1$. Using this distribution and the Law of Total Probability,

$$O_i^*(s) = \sum_{j=0}^{\infty} [O_i^*(s) | j \text{ arrivals in cycle } (i-1)] P\{j \text{ arrivals in cycle } (i-1)\}.$$

The time to serve $j$ customers is the sum of their individual service times and its distribution is the $j$-fold convolution of their individual service-time distributions. Because the service times are all independent and drawn from the same distribution, the LST of the convolution is given by the LST of the service-time distribution raised to the $j$th power; then,

$$O_i^*(s) = \sum_{j=0}^{\infty} \left( \frac{\mu}{s+\mu} \right)^j P\{j \text{ arrivals in cycle } (i-1)\}. \tag{2.1}$$

Note that if there were no arrivals in cycle $i-1$, $O_i$ will have zero length and $O_i^*(s) = 1 = (\mu/(s+\mu))^0$.

The length of cycle $i$ is

$$T_i = O_i + P.$$

Because the service time of the permanent customer is independent of everything and the ordinary customers making up $O_i$ clear the system before the permanent customer begins service, $O_i$ is independent of $P$ and

$$T_i^*(s) = O_i^*(s) P^*(s). \tag{2.2}$$

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The following theorem gives the pgf of the number of Poisson arrivals during a random length of time.

THEOREM 1: *Let $T^*(s)$ denote the Laplace–Stieltjes transform of a length of time $T$. Let $\mathfrak{F}_N(z)$ denote the probability generating function of the number of events from a Poisson process having rate $\lambda$ that occurs during $T$; then,*

$$\mathfrak{F}_N(z) = T^*(\lambda[1 - z]). \tag{2.3}$$

PROOF: See [7, Thm. 4.2]. ■

Applying this result to (2.2) gives the pgf of the distribution for the number of ordinary customer arrivals in cycle $i$ as

$$\mathfrak{F}_{N_i}(z) = T_i^*(\lambda[1 - z]).$$

Inverting this pgf gives the distribution to use as a starting point for analyzing cycle $i + 1$.

This approach is now applied to analyze the transient behavior of the case in which the permanent customer is initially in the queue behind a geometric (starting from 0) number of ordinary customers.

*2.1.1. Geometric number of ordinary customers.* Assume that the permanent customer has just completed service and the number of ordinary customers found in the system is given by the equilibrium occupancy distribution for an $M/M/1$ queue [12]

$$P\{j \text{ ordinary customers in system}\} = (1 - \rho)\rho^j, \quad j = 0,1,\ldots, 0 < \rho < 1. \tag{2.4}$$

Substituting into (2.1) yields

$$\begin{aligned} O_1^*(s) &= \sum_{j=0}^{\infty} \left(\frac{\mu}{s + \mu}\right)^j (1 - \rho)\rho^j \\ &= (1 - \rho) \sum_{j=0}^{\infty} \left(\frac{\lambda}{s + \mu}\right)^j \\ &= (1 - \rho) \frac{1}{1 - \lambda/(s + \mu)} \\ &= \frac{(1 - \rho)(s + \mu)}{s + \mu - \lambda}. \end{aligned} \tag{2.5}$$

The occupancy distribution of (2.4) is geometric. Thus, if the permanent customer finds at least one ordinary customer, it must wait a geometric number of exponential periods, which is itself an exponential length of time. If the permanent customer finds the system empty, there is no delay. The LST of the mixture [9, p. 53] of these two distributions is given by (2.5).

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Now, using (2.2), the LST of the length of the first cycle is

$$\begin{aligned} T_1^*(s) &= O_1^*(s)P^*(s) \\ &= \frac{(1-\rho)(s+\mu)}{s+\mu-\lambda}\,\frac{\mu}{s+\mu} \\ &= \frac{\mu-\lambda}{s+\mu-\lambda}. \end{aligned} \tag{2.6}$$

This is the LST of an exponential distribution with rate $\mu - \lambda$. To explain, the total number of customers served in the cycle is one more than the number of ordinary customers served. This eliminates the possibility of a zero-length cycle, so there is always a geometric number of exponential service times, which is exponential. This result implies that the sojourn time of the permanent customer in the mixed-traffic system is equivalent to a service time in a system with the ordinary customers removed and the server slowed by the arrival rate for ordinary customers. This is the approach used to account for interfering traffic in the flow-control models of [1,16,17].

Applying (2.3),

$$\tilde{\delta}_{N_1}(z) = \frac{\mu-\lambda}{\mu-\lambda z} = (1-\rho)\left(\frac{1}{1-\rho z}\right).$$

Inverting this yields

$$P\{j \text{ arrivals in cycle } i\} = (1-\rho)\rho^j, \quad j = 0,1,\ldots, 0 < \rho < 1. \tag{2.7}$$

The distribution of the number of customers left in the system by the permanent customer is the same as the distribution of the number it found. The system is in equilibrium. It is important to note that although the future cycles are identically distributed, they are not independent because the number of arrivals in a cycle affects the length of the next cycle.

In the next case examined, the permanent customer begins with no ordinary customers in the system.

*2.1.2. System starting empty.* Assume that the permanent customer has just completed service and finds the system empty of ordinary customers when it feeds back and reenters the system to start cycle 1. The following theorem gives results for cycle *i*.

THEOREM 2: *For a mixed-traffic M/M/1 queue having one permanent customer with the system initially empty of ordinary customers*,

$$O_i^*(s) = \frac{(1-\rho)(s+\mu)}{(1-\rho^i)s+\mu-\lambda}, \quad i = 1,2,\ldots,$$

$$T_i^*(s) = \frac{\mu-\lambda}{(1-\rho^i)s+\mu-\lambda}, \quad i = 1,2,\ldots,$$

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*and*

$$P\{j \text{ arrivals in cycle } i\} = \left(\frac{1-\rho}{1-\rho^{i+1}}\right)\left(\frac{\rho(1-\rho^i)}{1-\rho^{i+1}}\right)^j,$$
$$i = 1,2,\ldots, j = 0,1,\ldots, 0 < \rho < 1.$$

PROOF: As the first step in an inductive proof, let $i = 1$. Because there are initially no ordinary customers, $O_1$ has length zero and

$$O_1^*(s) = 1, \tag{2.8}$$

$$T_1^*(s) = \frac{\mu}{s+\mu}, \tag{2.9}$$

$$\begin{aligned}\mathfrak{F}_{N_i}(z) &= \frac{\mu}{\lambda[1-z]+\mu} \\ &= \frac{1}{1+\rho-\rho z} \\ &= \frac{1}{1+\rho}\,\frac{1}{1-[\rho/(1+\rho)]z} \\ &= \frac{1}{1+\rho}\sum_{j=0}^{\infty}\left(\frac{\rho}{1+\rho}\,z\right)^j.\end{aligned}$$

Inverting $\mathfrak{F}_{N_1}(z)$ yields

$$P\{j \text{ arrivals in cycle } 1\} = \left(\frac{1}{1+\rho}\right)\left(\frac{\rho}{1+\rho}\right)^j, \quad j = 0,1,\ldots. \tag{2.10}$$

Thus, it can be seen that the case of $i = 1$ is proved.

For the inductive step, assume that

$$P\{j \text{ arrivals in cycle } i-1\} = \left(\frac{1-\rho}{1-\rho^i}\right)\left(\frac{\rho(1-\rho^{i-1})}{1-\rho^i}\right)^j,$$
$$i = 1,2,\ldots, j = 0,1,\ldots.$$

Then, using (2.1) and (2.2) as earlier,

$$\begin{aligned}O_i^*(s) &= \sum_{j=0}^{\infty}\left(\frac{\mu}{s+\mu}\right)^j\left(\frac{1-\rho}{1-\rho^i}\right)\left(\frac{\rho(1-\rho^{i-1})}{1-\rho^i}\right)^j \\ &= \left(\frac{1-\rho}{1-\rho^i}\right)\sum_{j=0}^{\infty}\left[\left(\frac{\mu}{s+\mu}\right)\left(\frac{\rho-\rho^i}{1-\rho^i}\right)\right]^j \\ &= \left(\frac{1-\rho}{1-\rho^i}\right)\left(1-\left(\frac{\mu}{s+\mu}\right)\left(\frac{\rho-\rho^i}{1-\rho^i}\right)\right)^{-1} \\ &= \frac{(1-\rho)(s+\mu)}{(1-\rho^i)(s+\mu)-\mu(\rho-\rho^i)} \\ &= \frac{(1-\rho)(s+\mu)}{(1-\rho^i)s+\mu-\lambda}\end{aligned}$$

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and

$$T_i^*(s) = \frac{\mu - \lambda}{(1-\rho^i)s + \mu - \lambda}.$$

Applying (2.3) yields

$$\begin{aligned}\tilde{\mathcal{S}}_{N_i}(z) &= \frac{\mu - \lambda}{(1-\rho^i)\lambda[1-z] + \mu - \lambda} \\ &= \frac{\mu - \lambda}{\mu - \lambda\rho^i - \lambda(1-\rho^i)z} \\ &= \frac{1-\rho}{1-\rho^{i+1} - \rho(1-\rho^i)z} \\ &= \left(\frac{1-\rho}{1-\rho^{i+1}}\right)\left(1 - \frac{\rho(1-\rho^i)}{1-\rho^{i+1}}\,z\right)^{-1}\end{aligned}$$

from which

$$P\{j \text{ arrivals in cycle } i\} = \left(\frac{1-\rho}{1-\rho^{i+1}}\right)\left(\frac{\rho(1-\rho^i)}{1-\rho^{i+1}}\right)^j, \quad i = 1,2,\ldots, j = 0,1,\ldots. \ \blacksquare$$

Note that as $i \to \infty$, these results reduce to those of the equilibrium system, as one would expect. The output process for the permanent customer is now examined.

### 2.2. Output Process for *M/G/*1 System

Let $S_n$ be the time of the $n$th departure of the permanent customer. The distribution of $S_n$ and its Laplace–Stieltjes transform (LST) will be denoted by $F_{S_n}(t)$ and $S_n^*(s)$, respectively. Theorem 3 gives results for the output process.

THEOREM 3: *For the mixed-traffic $M/G/1$ queue with a single permanent customer that is initially in the queue behind $j$ ordinary customers, the Laplace–Stieltjes transform of the distribution of the time of the $n$th departure of the permanent customer is*

$$S_n^*(s) = (Y_n^*(s))^j \prod_{i=1}^{n} X_i^*(s),$$

*where $P^*(s)$ is the LST of the service-time distribution for the permanent customer and $C^*(s)$ is the LST of the service-time distribution for the ordinary customers,*

$$\begin{aligned}&X_1^*(s) = P^*(s), \qquad Y_1^*(s) = C^*(s), \\ &X_k^*(s) = P^*(s + \lambda(1 - Y_{k-1}^*(s))), \quad k = 2,3,\ldots, \\ &Y_k^*(s) = C^*(s + \lambda(1 - Y_{k-1}^*(s))), \quad k = 2,3,\ldots.\end{aligned}$$

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PROOF: The time of the $n$th departure is the sum of the first $n$ cycle times, so

$$S_n = \sum_{i=1}^{n} T_i$$

and

$$S_n^*(s) = E[e^{-s(\sum_{i=1}^{n} T_i)}].$$

Conditioning on the lengths of the first $n-1$ cycles,

$$\begin{aligned} S_n^*(s) &= \int_0^\infty \cdots \int_0^\infty E[e^{-s(\sum_{i=1}^{n} T_i)} | T_1 = t_1, \ldots, T_{n-1} = t_{n-1}]\, dF_{T_1,\ldots,T_{n-1}}(t_1, \ldots, t_{n-1}) \\ &= \int_0^\infty \cdots \int_0^\infty E[e^{-s(T_n + \sum_{i=1}^{n-1} t_i)} | T_1 = t_1, \ldots, T_{n-1} = t_{n-1}]\, dF_{T_{n-1}|T_{n-2}}(t_{n-1}|t_{n-2}) \\ &\quad \cdots dF_{T_1}(t_1) \\ &= \int_0^\infty \cdots \int_0^\infty e^{-st_1} \cdots e^{-st_{n-1}} E[e^{-sT_n} | T_1 = t_1, \ldots, T_{n-1} = t_{n-1}]\, dF_{T_{n-1}|T_{n-2}}(t_{n-1}|t_{n-2}) \\ &\quad \cdots dF_{T_1}(t_1). \end{aligned} \tag{2.11}$$

Because the length of cycle $n$, $T_n$, depends only on the length of cycle $n-1$, $T_{n-1}$ (and the number of ordinary customers that arrive during cycle $n-1$),

$$E[e^{-sT_n} | T_1 = t_1, \ldots, T_{n-1} = t_{n-1}] = E[e^{-sT_n} | T_{n-1} = t_{n-1}].$$

Given that $T_{n-1} = t_{n-1}$, the number of ordinary customer arrivals in cycle $n-1$ is distributed Poisson with rate $\lambda t_{n-1}$. The LST of the time to serve these ordinary customers is

$$O^*_{n|T_{n-1}=t_{n-1}}(s) = E[e^{-sO_n} | T_{n-1} = t_{n-1}]$$

and can be obtained by conditioning on the number of ordinary customer arrivals in cycle $T_{n-1}$,

$$O^*_{n|T_{n-1}=t_{n-1}}(s) = \sum_{i=0}^{\infty} E[e^{-sO_n} | N_{n-1} = i] P\{N_{n-1} = i\}.$$

Because there are $i$ independent service times,

$$\begin{aligned} O^*_{n|T_{n-1}=t_{n-1}}(s) &= \sum_{i=0}^{\infty} (C^*(s))^i \frac{(\lambda t_{n-1})^i}{i!} e^{-\lambda t_{n-1}} \\ &= e^{-\lambda t_{n-1}} \sum_{i=1}^{\infty} \frac{(\lambda t_{n-1} C^*(s))^i}{i!} \\ &= e^{-\lambda t_{n-1}(1 - C^*(s))}. \end{aligned}$$

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Recall, $T_n = O_n + P$, so

$$E[e^{-sT_n}|T_{n-1} = t_{n-1}] = P^*(s)e^{-\lambda t_{n-1}(1-C^*(s))}. \tag{2.12}$$

In general,

$$E[e^{-sT_k}|T_{k-1} = t_{k-1}] = P^*(s)e^{-\lambda t_{k-1}(1-C^*(s))}. \tag{2.13}$$

Substituting (2.12) in (2.11) and rearranging yields

$$S_n^*(s) = P^*(s)\int_0^\infty e^{-st_1}\cdots\int_0^\infty e^{-st_{n-1}}e^{-\lambda t_{n-1}(1-C^*(s))}\,dF_{T_{n-1}|T_{n-2}}(t_{n-1}|t_{n-2})$$
$$\cdots\, dF_{T_1}(t_1). \tag{2.14}$$

Note that

$$T^*_{n-1|T_{n-2}}(s+\lambda(1-C^*(s))) = \int_0^\infty e^{-st_{n-1}}e^{-\lambda t_{n-1}(1-C^*(s))}\,dF_{T_{n-1}|T_{n-2}}(t_{n-1}|t_{n-2}). \tag{2.15}$$

Let $X_1^*(s) = P^*(s)$ and $Y_1^*(s) = C^*(s)$ and define

$$X_k^*(s) = P^*(s+\lambda(1-Y_{k-1}^*(s))), \quad k = 2,3,\ldots,$$
$$Y_k^*(s) = C^*(s+\lambda(1-Y_{k-1}^*(s))), \quad k = 2,3,\ldots. \tag{2.16}$$

Substituting (2.15) into (2.14) results in

$$S_n^*(s) = X_1^*(s)\int_0^\infty e^{-st_1}\cdots\int_0^\infty e^{-st_{n-2}}T^*_{n-1|T_{n-2}}(s+\lambda(1-Y_1^*(s)))\,dF_{T_{n-2}|T_{n-3}}(t_{n-2}|t_{n-3})$$
$$\cdots\, dF_{T_1}(t_1). \tag{2.17}$$

The process can now be repeated for the next cycle. Using (2.13) and (2.16),

$$T^*_{n-1|T_{n-2}}(s+\lambda(1-Y_1^*(s))) = P^*(s)e^{-\lambda t_{n-2}(1-C^*(s))}\big|_{s=s+\lambda(1-Y_1^*(s))}$$
$$= X_2^*(s)e^{-\lambda t_{n-2}(1-Y_2^*(s))}.$$

Then, (2.17) becomes

$$S_n^*(s) = X_1^*(s)\int_0^\infty e^{-st_1}\cdots\int_0^\infty e^{-st_{n-2}}X_2^*(s)e^{-\lambda t_{n-2}(1-Y_2^*(s))}\,dF_{T_{n-2}|T_{n-3}}(t_{n-2}|t_{n-3})$$
$$\cdots\, dF_{T_1}(t_1)$$
$$= X_1^*(s)X_2^*(s)\int_0^\infty e^{-st_1}\cdots\int_0^\infty e^{-st_{n-3}}T^*_{n-2|T_{n-3}}(s+\lambda(1-Y_2^*(s)))$$
$$\times\, dF_{T_{n-3}|T_{n-4}}(t_{n-3}|t_{n-4})\cdots dF_{T_1}(t_1). \tag{2.19}$$

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Repeating this process eventually yields

$$S_n^*(s) = X_1^*(s)X_2^*(s) \cdots X_{n-1}^*(s)T_1^*(s + \lambda(1 - Y_{n-1}^*(s))). \quad \textbf{(2.20)}$$

If there are initially $j$ ordinary customers in the queue, $T_1^*(s) = (C^*(s))^j P^*(s)$, then

$$S_n^*(s) = (Y_n^*(s))^j \prod_{i=1}^{n} X_i^*(s). \quad \textbf{(2.21)}$$

■

Using this theorem, results for $j = 0$ and identically distributed exponential service times for the permanent and ordinary customers are

$$S_1^*(s) = \frac{\mu}{s + \mu},$$

$$S_2^*(s) = \frac{\mu}{s^2 + (2\mu + \lambda)s + \mu^2},$$

$$S_3^*(s) = \frac{\mu}{s^3 + (3\mu + 2\lambda)s^2 + (3\mu^2 + 2\lambda\mu + \lambda^2)s + \mu^3}.$$

THEOREM 4: *For the mixed-traffic $M/G/1$ queue where the permanent and ordinary customers have the same mean service time with a single permanent customer that is initially in the queue behind a geometric number of ordinary customers given by*

$$P\{j \text{ ordinary customers in system}\} = (1 - \rho)\rho^j, \quad j = 0,1,\ldots, 0 < \rho < 1, \quad \textbf{(2.22)}$$

*the Laplace–Stieltjes transform of the distribution of the time of the nth departure of the permanent customer is*

$$S_n^*(s) = \frac{\rho}{1 - \rho} \frac{\prod_{i=1}^{n} X_i^*(s)}{1 - (1 - \rho)X_n^*(s)},$$

*where $P^*(s)$ is the LST of the service-time distribution,*

$$X_1^*(s) = P^*(s)$$

*and*

$$X_k^*(s) = P^*(s + \lambda(1 - X_{k-1}^*(s))), \quad k = 2,3,\ldots.$$

PROOF: For the case in which the single permanent customer is initially behind a geometric (starting from 0) number of ordinary customers, (2.20) still holds, but the

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form of $T_1^*(s)$ is different. Using (2.4) to condition on the number of ordinary customers initially in the queue ahead of the permanent customer yields

$$O_1^*(s) = \sum_{i=0}^{\infty} (P^*(s))^i (1-\rho)^{i-1}\rho$$

$$= \frac{\rho}{1-\rho} \sum_{i=0}^{\infty} ((1-\rho)P^*(s))^i$$

$$= \frac{\rho}{1-\rho} \frac{1}{1-(1-\rho)P^*(s)}.$$

Since $T_1 = O_1 + P$,

$$T_1^*(s) = \frac{\rho}{1-\rho} \frac{P^*(s)}{1-(1-\rho)P^*(s)}.$$

Substituting into (2.20) yields

$$S_n^*(s) = \frac{\rho}{1-\rho} \frac{\prod_{i=1}^{n} X_i^*(s)}{1-(1-\rho)X_n^*(s)}. \tag{2.23}$$

■

## 3. MULTIPLE PERMANENT CUSTOMERS

Consider the single-server queue with multiple permanent customers depicted in Figure 1. Let the number of permanent customers be $M$ and label them $P^1, P^2, \ldots, P^M$ such that customer $P^1$ precedes $P^2$ who, in turn, precedes $P^3$, and so on. A cycle is now the time between successive arrivals of the same permanent customer. Let $T_i^k$ be the length of the $i$th cycle for permanent customer $k$. The service process now consists of a sequence of subcycles, each consisting of an ordinary period followed by a permanent-customer service time. Let $W_i^k$ be the length of the $i$th subcycle for permanent customer $k$. The structure is illustrated in Figure 3.

For ease of illustration, consider the behavior of subcycle $i$ for permanent customer $P^k$ with $1 < i$ and $1 < k < M$. The subcycle begins as permanent customer

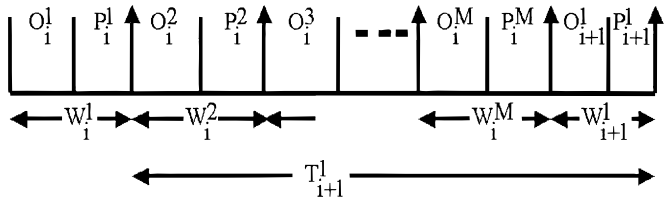



**FIGURE 3.** Service process with multiple permanent customers.

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$P^{k-1}$ completes service and a period of service for ordinary customers begins. The length of this period is $O_i^k$, which can be zero. After the ordinary customers are gone, permanent customer $P^k$ receives service and then reenters the queue, which starts subcycle $i$ for permanent customer $P^{k+1}$. As it reenters, $P^k$ finds a number of ordinary customers between itself and $P^{k-1}$. These are the ordinary customers arriving to the system after $P^{k-1}$ reentered the queue and before $P^k$ does (i.e., the ordinary customers arriving during subcycle $i$ for $P^k$). Given the LST of the distribution of $W_i^k$ (i.e., the length of the $i$th subcycle for customer $P^k$), Theorem 1 can be used to determine the distribution of the number of ordinary arrivals. These arrivals are the customers who are served during the ordinary customer service period of subcycle $i+1$ for permanent customer $P^k$ during subcycle $i+1$ (i.e., $O_{i+1}^k$).

Thus, the behavior of the subcycles for permanent customer $P^k$ is independent of the subcycles for the other permanent customers and depend only on the number of ordinary customer arrivals during the previous subcycle for permanent customer $P^k$; that is,

$$P\{W_i^k \le w | W_j^l\} = P\{W_i^k \le w\} \quad \text{for any } l = 1,\ldots,M, j = 1,\ldots, j \ne i. \qquad \textbf{(3.1)}$$

As a result, the behavior of the subcycles for permanent customer $P^k$ looks exactly like the behavior of a single-permanent-customer system and the sequence of subcycles for the system with $M$ permanent customers looks like an interleaving of cycles from $M$ single-customer systems.

### 3.1. Cycle Times for an *M/M/*1 System

The following theorem gives the LST for the permanent-customer cycle times when the $M/M/1$ system is initially empty of ordinary customers and when the permanent customers are adjacent to each other at the end of the queue behind a geometric (starting from 0) number of ordinary customers.

THEOREM 5: *For the $M/M/1$ system containing $M$ permanent customers using the same service-time distribution for permanent and ordinary customers, which is initially empty of ordinary customers, the LST of the length of the ith cycle of customer $P^k$ is*

$$T_i^{k*}(s) = \left(\frac{\mu-\lambda}{(1-\rho^{i-1})s+\mu-\lambda}\right)^{M-k}\left(\frac{\mu-\lambda}{(1-\rho^{i})s+\mu-\lambda}\right)^{k}, \quad i = 1,2,\ldots.$$

*The LST of the length of the ith cycle of customer $P^k$ when $M$ permanent customers are initially adjacent to each other at the end of the queue behind a geometric (starting from 0) number of ordinary customers is*

$$T_i^{k*}(s) = \frac{\mu-\lambda}{s+\mu-\lambda}\left(\frac{\mu-\lambda}{(1-\rho^{i-1})s+\mu-\lambda}\right)^{M-k}\left(\frac{\mu-\lambda}{(1-\rho^{i})s+\mu-\lambda}\right)^{k-1},$$
$$i = 1,2,\ldots.$$

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PROOF: Assume the system is initially empty of ordinary customers as the first permanent customer, $P^1$, begins service. The permanent customers are adjacent to each other in the queue, so that there are no ordinary customers between them. As with the single-permanent-customer system, starting empty, the LST of the distribution of the length of the first ordinary period is given by (2.8) and the LST of the distribution of the length of the first subcycle is given by (2.9). In addition, the distribution of the number of ordinary customer arrivals before $P^1$ reenters the queue is given by (2.10). The service times of these arrivals, and only these arrivals, along with the service time of customer $P^1$ determine the length of the second subcycle for permanent customer $P^1$. Thus, the future behavior of the subcycles for permanent customer $P^1$ is that of a system with one permanent customer starting empty, as described by Theorem 2.

Similarly, consider the remaining permanent customers, $P^k, k = 2,3,\ldots,M$. Because they are initially adjacent, there are no ordinary customers between them to be served, so their subcycles have the same probabilistic behavior as the subcycles for permanent customer $P^1$. For $k = 1,2,3,\ldots,M$,

$$O_i^{k*}(s) = \frac{(1-\rho)(s+\mu)}{(1-\rho^i)s+\mu-\lambda}, \quad i = 1,2,\ldots,$$

$$W_i^{k*}(s) = \frac{\mu-\lambda}{(1-\rho^i)s+\mu-\lambda}, \quad i = 1,2,\ldots, \tag{3.2}$$

and

$$P\{j \text{ arrivals in subcycle } k \text{ of cycle } i\} = \left(\frac{1-\rho}{1-\rho^{i+1}}\right)\left(\frac{\rho(1-\rho^i)}{1-\rho^{i+1}}\right)^j,$$

$$i = 1,2,\ldots, j = 0,1,\ldots.$$

The length of the first cycle for permanent customer $P^k$ is equal to the sum of the lengths of the first subcycle for each of the preceding $k-1$ permanent customers plus its own subcycle length. Then,

$$T_1^k = \sum_{j=1}^{k} W_1^j, \quad k = 1,2,\ldots,M.$$

Because these subcycles are independent,

$$T_1^{k*}(s) = \prod_{j=1}^{k} W_1^{j*}(s), \quad k = 1,2,\ldots,M.$$

The permanent customers are initially adjacent and the system is initially empty of ordinary customers, so the subcycle lengths are just the service times for each customer, which are i.i.d.; thus,

$$T_1^{k*}(s) = \left(\frac{\mu}{s+\mu}\right)^k = \left(\frac{\mu-\lambda}{(1-\rho)s+\mu-\lambda}\right)^k, \quad k = 1,2,\ldots,M.$$

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The $i$th cycle for customer $P^k$ is the time between the $(i-1)$st and $i$th departures of customer $P^k$. Following the $(i-1)$st departure of customer $P^k$ are the $(i-1)$st departures of customers $P^{k+1}, P^{k+1}, \ldots, P^M$ and the $i$th departures of customers $P^1, P^2, \ldots, P^{k-1}$. Each departure has a subcycle associated with it. The sum of the lengths of these plus the length of the $i$th subcycle for customer $P^k$ is the length of the $i$th cycle for customer $P^k$. Then,

$$T_i^k = \sum_{j=k+1}^{M} W_{i-1}^j + \sum_{j=1}^{k} W_i^j, \quad i = 2,3,\ldots.$$

Using the independence of subcycles for different customers yields

$$T_i^{k*}(s) = \prod_{j=k+1}^{M} W_{i-1}^{j*}(s) \prod_{j=1}^{k} W_i^{j*}(s), \quad i = 2,3,\ldots.$$

Because the $i$th subcycles for all the customers are identically distributed, substituting from (3.2) results in

$$T_i^{k*}(s) = \left(\frac{\mu-\lambda}{(1-\rho^{i-1})s+\mu-\lambda}\right)^{M-k}\left(\frac{\mu-\lambda}{(1-\rho^{i})s+\mu-\lambda}\right)^{k}, \quad i = 2,3,\ldots.$$

If the permanent customers are adjacent to each other at the end of the queue behind a geometric (starting from 0) number of ordinary customers, all the subcycles for permanent customer $P^1$ are in equilibrium as described by (2.5)–(2.7). Then,

$$W_i^{1*}(s) = \frac{\mu-\lambda}{s+\mu-\lambda}, \quad i = 1,2,\ldots$$

and

$$T_1^{k*}(s) = \frac{\mu-\lambda}{s+\mu-\lambda}\left(\frac{\mu}{s+\mu}\right)^{k-1} = \frac{\mu-\lambda}{s+\mu-\lambda}\left(\frac{\mu-\lambda}{(1-\rho)s+\mu-\lambda}\right)^{k-1},$$
$$k = 1,2,\ldots,M,$$
$$T_i^{k*}(s) = \frac{\mu-\lambda}{s+\mu-\lambda}\left(\frac{\mu-\lambda}{(1-\rho^{i-1})s+\mu-\lambda}\right)^{M-k}\left(\frac{\mu-\lambda}{(1-\rho^{i})s+\mu-\lambda}\right)^{k-1},$$
$$i = 2,3,\ldots. \quad \blacksquare$$

### 3.2. Output Process for *M/G/*1 System

Let $S_n$ be the time of the $n$th departure of a permanent customer. The distribution of $S_n$ and its Laplace–Stieltjes transform are denoted by $F_{S_n}(t)$ and $S_n^*(s)$, respectively. Without loss of generality, again assume the permanent customers are ordered and numbered as $P^1, P^2, \ldots, P^M$. The permanent customers never leave the system, so the $n$th departure of a permanent customer is the $i$th departure of some permanent customer, say $P^k$. Note that $i = \lceil n/M \rceil$ and $k = n - \lfloor n/M \rfloor * M$. For example, if $M = 5$, the

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eighth departure of a permanent customer is the second departure of permanent customer $P^3$.

THEOREM 6: *For the M/G/1 system containing M permanent customers with permanent customer j initially in the queue behind $c_j$ ordinary customers, the LST of the time of the nth departure of a permanent customer is*

$$S_n^*(s) = (Y_i^*(s))^{\sum_{j=1}^{k} c_j} (Y_{i-1}^*(s))^{\sum_{j=k-1}^{M} c_j} (X_i^*(s))^k \left( \prod_{l=1}^{i-1} X_l^*(s) \right)^M, \tag{3.3}$$

*where $i = \lceil n/M \rceil$ and $k = n - \lfloor n/M \rfloor * M$, $P^*(s)$ is the LST of the service-time distribution, $X_1^*(s) = P^*(s)$, $Y_1^*(s) = C^*(s)$, and*

$$\begin{aligned} X_k^*(s) &= P^*(s + \lambda(1 - Y_{k-1}^*(s))), \quad k = 2,3,\ldots, \\ Y_k^*(s) &= C^*(s + \lambda(1 - Y_{k-1}^*(s))), \quad k = 2,3,\ldots. \end{aligned} \tag{3.4}$$

PROOF: Before permanent customer $P^k$ departs for the $i$th time, permanent customers $P^1, P^2, \ldots, P^{k-1}$ have departed $i$ times and permanent customers $P^{k+1}, P^{k+2}, \ldots, P^M$ have departed $i-1$ times. The time of the $n$th departure is then the sum of the length of the first $i$ subcycles for permanent customers $P^1, P^2, \ldots, P^k$ and the length of the first $i-1$ subcycles for permanent customers $P^{k+1}, P^{k+2}, \ldots, P^M$, so

$$S_n = \sum_{j=1}^{k} S_i^j + \sum_{j=k+1}^{M} S_{i-1}^j,$$

where $S_i^j$ is the sum of the first $i$ subcycle lengths for permanent customer $j$. Because the subcycles for the different permanent customers are independent, the LST for the distribution of the time of the $n$th departure of a permanent customer for a mixed-traffic queueing system with $M$ permanent customers, which is initially empty of ordinary customers, is

$$S_n^*(s) = \prod_{j=1}^{k} S_i^{j*}(s) \prod_{j=k+1}^{M} S_{i-1}^{j*}(s). \tag{3.5}$$

Because permanent customer $j$ begins with $c_j$ ordinary customers directly ahead of him in the queue, Theorem 3 implies

$$S_i^{j*}(s) = (Y_n^*(s))^{c_j} \prod_{l=1}^{i} X_l^*(s), \quad j = 1,\ldots,M. \tag{3.6}$$

Substituting in (3.5),

$$\begin{aligned} S_n^*(s) &= \prod_{j=1}^{k} \left( (Y_i^*(s))^{c_j} \prod_{l=1}^{i} X_l^*(s) \right) \prod_{j=k+1}^{M} \left( (Y_{i-1}^*(s))^{c_j} \prod_{l=1}^{i-1} X_l^*(s) \right) \\ &= (Y_i^*(s))^{\sum_{j=1}^{k} c_j} \left( \prod_{l=1}^{i} X_l^*(s) \right)^k (Y_{i-1}^*(s))^{\sum_{j=k-1}^{M} c_j} \left( \prod_{l=1}^{i-1} X_l^*(s) \right)^{M-k} \\ &= (Y_i^*(s))^{\sum_{j=1}^{k} c_j} (Y_{i-1}^*(s))^{\sum_{j=k-1}^{M} c_j} (X_i^*(s))^k \left( \prod_{l=1}^{i-1} X_l^*(s) \right)^M. \end{aligned}$$

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THEOREM 7: *For the mixed-traffic $M/G/1$ queue where the permanent and ordinary customers have the same service-time distribution containing $M$ permanent customers initially adjacent to each other at the end of the queue behind a geometric (starting from 0) number of ordinary customers, the LST of the time of the nth departure of a permanent customer is*

$$S_n^*(s) = \frac{\rho}{1-\rho}\left(\frac{1}{1-(1-\rho)X_i^*(s)}\right)(X_i^*(s))^k\left(\prod_{l=1}^{i-1}X_l^*(s)\right)^M,$$

*where $i = \lceil n/M \rceil$ and $k = n - \lfloor n/M \rfloor * M$, $P^*(s)$ is the LST of the service-time distribution*,

$$X_1^*(s) = P^*(s),$$

*and*

$$X_k^*(s) = P^*(s + \lambda(1 - X_{k-1}^*(s))), \quad k = 2,3,\ldots.$$

PROOF: If the $M$ permanent customers are initially adjacent to each other at the end of the queue behind a geometric (starting from 0) number of ordinary customers, (3.5) still holds and, by Theorem 3,

$$S_i^{j*}(s) = \prod_{l=1}^{i} X_l^*(s), \quad j = 2,\ldots,M. \tag{3.7}$$

Now,

$$S_i^{1*}(s) = \frac{\rho}{1-\rho}\,\frac{\prod_{l=1}^{i} X_l^*(s)}{1-(1-\rho)X_i^*(s)} \tag{3.8}$$

is the LST of the time of the $i$th departure of the permanent customer in a mixed-traffic queueing system with one permanent customer that is initially in the queue behind a geometric (starting from 0) number of ordinary customers. Substituting,

$$\begin{aligned} S_n^*(s) &= \frac{\rho}{1-\rho}\,\frac{\prod_{l=1}^{i} X_l^*(s)}{1-(1-\rho)X_i^*(s)}\left(\prod_{l=1}^{i}X_l^*(s)\right)^{k-1}\left(\prod_{l=1}^{i-1}X_l^*(s)\right)^{M-k} \\ &= \frac{\rho}{1-\rho}\left(\frac{1}{1-(1-\rho)X_i^*(s)}\right)(X_i^*(s))^k\left(\prod_{l=1}^{i-1}X_l^*(s)\right)^M. \end{aligned}$$

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## 4. CONCLUSIONS

A single-server queueing system with mixed traffic has been examined. For a system containing one permanent customer, the distribution of the length of the $i$th cycle was obtained. Also, the distributions for the time of the $n$th departure of the perma-

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nent customer and the number of departures up to time $t$ were found. For a system containing $M$ permanent customers, the LST for the distribution of the length of the $i$th cycle and the LST of the distribution for the time of the $n$th departure were also obtained.

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