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An embedding theorem for HCF-rings

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Let D be an integral domain with identity. In (3), Cohn has shown that if D is a Schreier ring, then D can be inertly embedded in a pre-Bezout domain $\mathscr{B}(D)$, which Cohn calls the pre-Bezout hull of D. Further, D is an HCF-ring if and only if $\mathscr{B}(D)$ is a Bezout domain, and D is a UFD if and only if $\mathscr{B}(D)$ is a PID. Using a modification of Cohn's techniques, Samuel in (8) has shown that a Krull domain can be inertly embedded in a Dedekind domain. In this paper, we show that these embedding theorems are also obtained by considering the embedding of a v-ring in its Kronecker function ring with respect to the v-operation. To wit, if D is a v-ring and if D^v is the Kronecker function ring of D with respect to the v-operation, then D^v is a Bezout domain, and we show that D is inertly embedded in D^v if an only if D is an HCF-ring. Moreover, D is a UFD if and only if D^v is a PID, and if D is a Krull domain, D is inertly embedded (in Samuel's sense) in D^v, a Dedekind domain.

1. *Notation and terminology.* Let D be an integral domain with identity having quotient field K and let $\mathscr{I}(D)$ be the set of non-zero fractional ideals of D. If $F \in \mathscr{I}(D)$, then F^{-1} denotes the set of elements x in K such that $xF \subseteq D$. The fractional ideal $(F^{-1})^{-1}$ is denoted by F_v and is called the v-ideal associated with F.† The mapping $F \to F_v$ of $\mathscr{I}(D)$ into $\mathscr{I}(D)$ is called the *v-operation on* D; this mapping satisfies Krull's conditions in (5), p. 556, for a '-operation. We say that a v-ideal of the form F_v, where F is finitely generated, is of *finite type*. The v-operation on D is *endlich arithmetisch brauchbar* (e.a.b.) if $(F_1F_2)_v \subseteq (F_1F_3)_v$, with F_1 finitely generated, implies that $(F_2)_v \subseteq (F_3)_v$; if the same conclusion holds for all F_1, F_2, F_3 in $\mathscr{I}(D)$ with no restrictions on F_1, we say that the v-operation on D is *arithmetisch brauchbar* (a.b.). The notions e.a.b. and a.b. are equivalent for the v-operation, and domains on which the v-operation is e.a.b. are called *v-rings*;‡ any v-ring is integrally closed (4), p. 268. If D is a v-ring with quotient field K and if X is an indeterminate over K, then for any non-zero element $f(X)$ in $K[X]$, we denote by A_f the content of f—that is, the fractional ideal of D generated by the coefficients of f. The *Kronecker function ring* D^v *of* D *with respect to the v-operation*§ is the subring of $K(X)$ consisting of all elements $f(X)/g(X)$, where $g(X) \neq 0$ and $(A_f)_v \subseteq (A_g)_v$. It is well known that D^v is a Bezout domain with quotient field $K(X)$ (5); (4), section 26. Moreover, $(A_{fg})_v = (A_fA_g)_v$ for any f, $g \in K[X]$; this

† The preceding terminology, as well as that which follows, is that of Krull in (5) and (6). The v-operation was first considered by Artin and van der Waerden; see (9), section 103.

‡ In the terminology of Bourbaki (2), p. 89, a v-ring is a *regularly integrally closed domain.*

§ The Kronecker function ring of a '-operation can be defined with respect to any set of indeterminates over D in the obvious manner. The hypothesis that the v-operation is e.a.b. is needed in order to prove that D^v is a subring of $K(X)$.

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extremely useful result is what Krull in (5), p. 557, calls Gauss's Theorem (see also (7), section 9). D is said to be an *HCF-ring*† if each v-ideal of D of finite type is principal.

In (3) and (8), three different notions of *inert* embeddings are considered; we use terminology here which distinguishes between these three concepts, and we add one new concept. If R is a ring with identity and if T is a unitary overring of R, we denote by $U(R)$ and $U(T)$, respectively, the groups of units of R and T. We say that R *is weakly inertly embedded in* T if for any non-zero element r of R and for any factorization $r = t_1t_2$ of r in T, there is a unit u of T such that t_1u and $u^{-1}t_2$ are in R. We say that R *is inertly embedded in* T if R is weakly inertly embedded in T and if $U(T) \cap R = U(R)$; R *is strongly inertly embedded in* T if R is weakly inertly embedded in T and if $U(R) = U(T)$.‡ If R and T are Krull domains, we say that R *is valuation-inertly embedded in* T if each essential valuation on R has a unique extension to an essential valuation on T, and if the ramification index of this extension is one.§

2. *The embedding theorems.* We proceed to prove the results stated in the introduction concerning embedding theorems for a v-ring in its Kronecker function ring.

THEOREM 2·1. *Suppose that D is a v-ring with quotient field K, and that D^v is the Kronecker function ring of D with respect to the v-operation. If D is an HCF-ring, then D is inertly embedded in D^v. Conversely, if D is weakly inertly embedded in D^v, then D is an HCF-ring.*

Proof. Suppose that D is an HCF-ring. We first observe that any non-zero element α of D^v is expressible in the form $\xi h(X)/k(X)$, where $\xi \in D$, $h(X)$, $k(X) \in D[X]$, and $(A_h)_v = (A_k)_v = D$. Thus, write α as $b(X)/c(X)$, where $b(X)$, $c(X) \in K[X] - \{0\}$, and $(A_b)_v \subseteq (A_c)_v$. Since D is an HCF-ring, $(A_b)_v = \beta D$ and $(A_c)_v = \gamma D$ are principal. Further, $\xi = \beta\gamma^{-1}$ is in D. Then $\alpha = \beta\gamma^{-1}[\beta^{-1}b(X)]/[\gamma^{-1}c(X)]$ where $\xi = \beta\gamma^{-1} \in D$, $(A_{\beta^{-1}b})_v = \beta^{-1}\beta D = D$, and $(A_{\gamma^{-1}c})_v = \gamma^{-1}(A_c)_v = \gamma^{-1}\gamma D = D$. Now consider a factorization $d = \alpha_1\alpha_2$ of a non-zero element d of D in D^v. We write α_i as $\xi_i f_i(X)/k_i(X)$, where $\xi_i \in D$ and $(A_{f_i})_v = (A_{k_i})_v = D$. Then $d = \alpha_1\alpha_2$ implies that $dk_1k_2 = \xi_1\xi_2 f_1 f_2$. Hence $(A_{dk_1k_2})_v = d((A_{k_1})_v(A_{k_2})_v)_v = d(DD)_v = (d) = (A_{\xi_1\xi_2 f_1 f_2})_v = \xi_1\xi_2(A_{f_1f_2})_v = (\xi_1\xi_2)$. It follows that there is a unit u of D such that $d = u\xi_1\xi_2$. We consider the unit $\mu = uk_1/f_1$ of D^v. We observe that $(uk_1/f_1)^{-1} = k_2/f_2$, for $d = u\xi_1\xi_2 = \xi_1\xi_2(f_1/k_1)(f_2/k_2)$ implies that $1 = (uk_1/f_1)(k_2/f_2) = \mu(k_2/f_2)$. Moreover, $\mu\alpha_1 = u\xi_1 \in D$, $\mu^{-1}\alpha_2 = \xi_2 \in D$, and $d = u\xi_1\xi_2 = \mu\alpha_1 . \mu^{-1}\alpha_2$. This proves that D is weakly inertly embedded in D^v. From the definition of D^v, it is then clear that D is inertly embedded in D^v.

Conversely, if D is a v-ring and if D is weakly inertly embedded in D^v, we prove that D is an HCF-ring. Thus, we consider any v-ideal $A = (a_0, a_1, \ldots, a_n)_v$ of finite type, where $a_0 \neq 0$. If $f(X) = a_0 + a_1X + \ldots + a_nX^n$, then $a_0 = [a_0/f(X)]f(X)$ is a factorization of a_0 in D^v. Since D is weakly inertly embedded in D^v, it follows that there is a unit

† This is the terminology of Cohn in (3); Bourbaki's terminology (2), p. 86, at this point is *pseudo-Bezoutian*. HCF-rings are considered by Prüfer in (7) under the designation *domains satisfying property $\mathfrak{B}A$*.

‡ The concepts of weak inert embedding and strong inert embedding appear in Cohn's paper (2), p. 259; the referee suggested our definition of an inert embedding.

§ This is the definition of an inert embedding which Samuel uses in (8).

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u of D^v such that $[a_0/f(X)]u$ and $u^{-1}f(X)$ are in D. We write u as $g(X)/h(X)$, where $(A_g)_v = (A_h)_v$. If $u^{-1}f(X) = d \in D$, then $f(X)h(X) = dg(X)$, and

$$(A_{fh})_v = ((A_f)_v(A_h)_v)_v = ((A_f)_v(A_g)_v)_v = (A_fA_g)_v = (A_{dg})_v = (dA_g)_v.$$

Since D is a v-ring and since A_g is finitely generated, it follows that

$$(A_f)_v = A = (d)_v = (d).$$

We conclude that D is an HCF-ring.

THEOREM 2·2. *Suppose that D is a v-ring with quotient field K, and let D^v be the Kronecker function ring of D with respect to the v-operation. Then D is a Krull domain if and only if D^v is a PID.*

Proof. If D^v is a PID, then $D = D^v \cap K$ is a Krull domain; the converse is proved as Proposition 36·13 of (4).

THEOREM 2·3. *Suppose that D is an HCF-ring. Then D is a UFD if and only if D^v is a PID.*

Proof. Theorem 2·2 shows that D^v is a PID if D is a UFD. And if D^v is a PID, Theorem 2·2 implies that D is a Krull domain. Since D is also an HCF-ring, it follows that each v-ideal of D is principal (4), Cor. 36·3. In particular, minimal primes of D are v-ideals, and are therefore principal; it then follows that D is a UFD (4), Prop. 35·14.

We have, therefore, proved that the analogue of Cohn's Theorem 3·4 of (3) is true if the hypothesis that R is a Schreier ring is replaced by the condition that R is a v-ring and if the pre-Bezout hull $\mathscr{B}(R)$ of R is replaced by R^v, the Kronecker function ring of R with respect to the v-operation. One further remark is in order. Cohn proves that the embedding of R in $\mathscr{B}(R)$, for R Schreier, is strongly inert, while we have proved that the embedding of R in R^v, for R a v-ring, is inert if and only if R is an HCF-ring. It is clear that the embedding of R in R^v is never strongly inert; for example, X is a unit of R^v which is not in R.

We next observe that we also obtain Samuel's theorem in (8) concerning the valuation-inert embedding of a Krull domain D in a Dedekind domain; in fact, D is valuation-inertly embedded in D^v, a PID. To prove this, let $\{V_\alpha\}$ be the family of essential valuations for D. For each α, we denote by v_α^* the trivial extension of v_α to $K(X)$, where K is the quotient field of D. That is, $v_\alpha^*(\Sigma_0^n a_i X^i)$, for any non-zero element $\Sigma_0^n a_i X^i$ of $K[X]$, is defined to be $\min\{v_\alpha(a_i)\}_0^n$. It is known (see (4), section 36) that $\{v_\alpha^*\}$ is the family of essential valuations for D^v. Hence each v_α has a unique extension to an essential valuation v_α^* of D^v, a PID by Theorem 2·2. Moreover, by definition of v_α^*, the ramification index of v_α^* over v_α is one. We have, therefore, proved the following result:

THEOREM 2·4. *If D is a Krull domain, then D is valuation-inertly embedded in D^v. Further, each essential valuation on D^v is the extension of an essential valuation on D.*†

† One important difference between Theorem 2·4 and Samuel's result needs to be pointed out. While the embedding of D in D^v, for D Krull, reduces the divisor class group to the identity, Samuel's embedding preserves the divisor class group.

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Samuel in (8) actually proves that a Krull domain D can be valuation-inertly embedded in a Dedekind domain J such that J is a flat D-module. We obtain the same result for our embedding of D in D^v as a corollary to our next result, Theorem 2·5. The statement of Theorem 2·5 uses this terminology: An integrally closed domain J is said to be a *Prüfer v-multiplication ring* if the set of v-ideals of J of finite type is a group under v-multiplication. A Krull domain is completely integrally closed, and hence its set of v-ideals is a group under v-multiplication (2), p. 5; moreover, each v-ideal is of finite type (4), p. 534, so that a Krull domain is a Prüfer v-multiplication ring.

THEOREM 2·5. *Suppose that J is a v-ring and that J^v is the Kronecker function ring of J with respect to the v-operation. Then J^v is a quotient ring of $J[X]$ if and only if J is a Prüfer v-multiplication ring. In particular, if J is a Prüfer v-multiplication ring, then J^v is a flat J-module.*

Proof. We assume first that J is a Prüfer v-multiplication ring, and we prove that $J^v = J[X]_S$, where $S = \{f(X) \in J[X] | (A_f)_v = J\}$. First, S is a multiplicative system since for $f, g \in S$, $(A_{fg})_v = ((A_f)_v (A_g)_v)_v = (J)_v = J$ so that $fg \in S$. The containment $J[X]_S \subseteq J^v$ is clear; to prove the reverse containment, we take an element $\alpha = f/g$ of J^v, where $f, g \in J[X] - \{0\}$ and $(A_f)_v \subseteq (A_g)_v$. Since J is a Prüfer v-multiplication ring, there is a finite set $\{h_i\}_{i=0}^n$ of elements of K, the quotient field of J, such that $((h_0, h_1, \ldots, h_n) A_g)_v = J$. We set $h(X) = \sum_0^n h_i X^i \in K[X]$. Then $\alpha = fh/gh$, where $(A_{gh})_v = J$ by definition of $h(X)$, and $(A_{fh})_v = ((A_f)_v (A_h)_v)_v \subseteq ((A_g)_v (A_h)_v)_v = J$. Thus $fh \in J[X]$, $gh \in S$, and $\alpha \in J[X]_S$.

Conversely, if J^v is a quotient ring of $J[X]$, then of necessity, $J^v = J[X]_S$, where $S = \{f \in J[X] | (A_f)_v = J\}$ is the set of elements of $J[X]$ which are units of J^v. To prove that J is a Prüfer v-multiplication ring, we take any finite set $\{k_i\}_0^s$ of non-zero elements of K and set $F = (\{k_i\})_v$. If $k(X) = \sum_{i=0}^s k_i X^i$, then $k_0/k(X)$ is in $J^v = J[X]_S$ so that $k_0/k(X) = f/g$ for some $f, g \in J[X] - \{0\}$, with $(A_g)_v = J$. Then $k_0 g = kf$ and

$$k_0 J = (A_{k_0 g})_v = (A_{kf})_v = ((A_k)_v (A_f)_v)_v;$$

it follows that $(k_0^{-1} A_f)_v$ is the v-inverse of $F = (A_k)_v$, and by definition, $(k_0^{-1} A_f)_v$ is of finite type. We have therefore proved that J is a Prüfer v-multiplication ring.

Our final result shows exactly when a Krull domain D is inertly embedded in D^v.

THEOREM 2·6. *A Krull domain D is weakly inertly embedded in D^v if and only if D is a UFD.*

Proof. If D is weakly inertly embedded in D^v, then D is an HCF-ring by Theorem 2·1. But in the proof of Theorem 2·2, it was shown that a Krull domain which is an HCF-ring is a UFD.

Since a UFD is an HCF-ring, the converse is immediate from Theorem 2·1.

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Added in proof. In relation to Theorem 2·5, Arnold and Brewer have recently proved this result:

If D is an integrally closed domain, then (1) *D^v is the only Kronecker function ring of D which can be a flat $D[X]$-module, and* (2) *D^v is a flat $D[X]$-module if and only if D is a Prüfer v-multiplication ring.*

This result appears in a paper by Arnold and Brewer which is to appear in *Proceedings of the American Mathematical Society.*

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