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ON A CONTINUITY METHOD FOR DIRICHLET PROBLEM OF HESSIAN EQUATIONS

RIRONG YUAN

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Abstract

In this paper, we develop a continuity method for the Dirichlet problem of Hessian equations on Riemannian manifolds. Such equations, introduced by Caffarelli *et al.* ['The Dirichlet problem for nonlinear second-order elliptic equations III: functions of eigenvalues of the Hessians', *Acta Math.* **155** (1985), 261–301] are defined in terms of the eigenvalues of the Hessian and a given pair (f, Γ), where f is a symmetric function defined in a symmetric cone $\Gamma \subset \mathbb{R}^n$ and Γ specifies the set of admissible eigenvalues for the solution. Our method combines techniques from Morse theory with a characterization of the pair (f, Γ). More precisely, in the type-2 case, we first construct admissible functions using Morse theory and then solve the Dirichlet problem without any additional assumptions on the boundary or the subsolution. Building on this characterization of the pair, we can approximate the type-1 equation by a family of type-2 equations.

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1. Introduction

Let $(\overline{M}, g)$ be an n-dimensional compact connected Riemannian manifold with smooth boundary ∂M. Denote $\overline{M} := M \cup \partial M$, where $n \geq 2$ and M is the interior of $\overline{M}$. Let χ be a smooth symmetric $(0,2)$-tensor on $\overline{M}$, and let $\psi \in C^\infty(\overline{M})$ and $\varphi \in C^\infty(\partial M)$. Moreover, denote $\vec{\mathbf{1}} = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

In this paper, we propose a continuity method for the Dirichlet problem

$$F(\nabla^2 u + \chi) = \psi \quad \text{in } \overline{M}, \tag{1-1}$$

$$u = \varphi \quad \text{on } \partial M, \tag{1-2}$$

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where $\nabla^2 u$ is the real Hessian of u with respect to the Levi–Civita connection ∇ of g. The nonlinear operator $F(\nabla^2 u + \chi)$ is generated by a pair (f, Γ) and has the form

$$F(\nabla^2 u + \chi) = f(\lambda[\nabla^2 u + \chi]),$$

where $\lambda[\nabla^2 u + \chi]$ is an n-tuple of eigenvalues of $\nabla^2 u + \chi$ with respect to g, f is a smooth symmetric function of n real variables, defined in an open symmetric convex cone Γ with vertex at the origin, and

$$\Gamma_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \text{ each } \lambda_i > 0\} \subseteq \Gamma,$$

as well as $\partial\Gamma \neq \emptyset$. The function f is supposed to satisfy

$$f \text{ is a concave function in } \Gamma, \tag{1-3}$$

$$f_i(\lambda) = f_{\lambda_i}(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \quad \text{in } \Gamma \text{ for all } 1 \leq i \leq n. \tag{1-4}$$

DEFINITION 1.1. For (1-1), following [2], we say that $u \in C^2(\overline{M})$ is an admissible function if $\lambda[\nabla^2 u + \chi] \in \Gamma$.

Equation (1-1) includes the Monge–Ampère equation and the more general k-Hessian equation corresponding to

$$f = \sigma_k^{1/k}, \quad \Gamma = \Gamma_k,$$

where σ_k is the kth elementary symmetric function and Γ_k is the kth Gårding cone. Recently, an example introduced by [12] has attracted considerable interest:

$$f = \frac{\sigma_k}{\sigma_{k-1}} - \sum_{j=0}^{k-2} \frac{\beta_j \sigma_j}{\sigma_{k-1}} \quad \text{for all } \beta_j \geq 0, \sum_{j=0}^{k-2} \beta_j > 0, \Gamma = \Gamma_{k-1}. \tag{1-5}$$

Moreover, (1-1) also includes the following as a special case:

$$f(\lambda[\Delta u \cdot g - \nabla^2 u + \chi]) = \psi. \tag{1-6}$$

When $f(\lambda) = \sum_{i=1}^n \log \lambda_i$, it is a Monge–Ampère equation for $(n-1)$-PSH functions in the sense of [14]. Such a type of equation is of special interest in complex geometry, since its close connection to the form-type Calabi–Yau equation [5] and Gauduchon's conjecture [22, 24].

When $\chi = 0$ and $M = \Omega \subset \mathbb{R}^n$, the Dirichlet problem was initially treated by Caffarelli *et al.* [2] under some additional assumptions on f, including

$$\lim_{t \to +\infty} f(t\lambda) = +\infty \quad \text{for any } \lambda \in \Gamma, \tag{1-7}$$

and the unbounded condition

$$\lim_{t \to +\infty} f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + t) = +\infty \quad \text{for all } \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma. \tag{1-8}$$

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Similar results were also obtained in the literature [16, 17]. Caffarelli *et al.*'s result has been further extended by Trudinger [25] to the bounded case. In addition, their works require a certain assumption on the principal curvatures of $\partial\Omega$.

Without geometric restriction to the boundary, as developed by [7, 11] and others, in general, the existence result for the Dirichlet problem requires an appropriate subsolution assumption.

DEFINITION 1.2 (Subsolution). We say that $\underline{u} \in C^2(\overline{M})$ is an admissible subsolution to the Dirichlet problem (1-1)–(1-2) if it is an admissible function and satisfies

$$f(\lambda[\nabla^2 \underline{u} + \chi]) \geq \psi \quad \text{in } \overline{M}, \underline{u} = \varphi \quad \text{on } \partial M.$$

Condition (1-4) ensures that (1-1) be elliptic at any admissible solutions, while (1-3) implies that the operator $F(A) = f(\lambda(A))$ is concave with respect to A when $\lambda(A) \in \Gamma$. Therefore, under the assumption

$$\inf_M \psi > \sup_{\partial\Gamma} f \quad \text{where } \sup_{\partial\Gamma} f = \sup_{\lambda_0 \in \partial\Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda), \tag{1-9}$$

once the following *a priori* estimate of the Hessian is in place,

$$\sup_{\overline{M}} |\nabla^2 u| \leq C, \tag{1-10}$$

the higher regularities follow from the Evans–Krylov theorem and Schauder theory. Building on this, one can use the standard continuity method to derive the existence.

This paper is devoted to developing a continuity method for the Dirichlet problem. First, we review some notation. For a given pair (f, Γ), as in [34], denote

$$\Gamma_{\mathcal{G}}^f = \left\{\lambda \in \Gamma : \lim_{t \to +\infty} f(t\lambda) > -\infty\right\}.$$

In addition, $\mathring{\Gamma}_{\mathcal{G}}^f$ denotes the interior of $\Gamma_{\mathcal{G}}^f$. For $\Gamma_{\mathcal{G}}^f$, as in [33, 36], define

$$(1, \ldots, 1, 1 - \varrho_{\Gamma_{\mathcal{G}}^f}) \in \partial\Gamma_{\mathcal{G}}^f. \tag{1-11}$$

Following the convention in [2], as in [34, 36], we say that (f, Γ) is a type-2 pair if

$$(0, \ldots, 0, 1) \in \mathring{\Gamma}_{\mathcal{G}}^f. \tag{1-12}$$

Otherwise, it is a type-1 pair. Accordingly, we call (1-1) a type-2 equation if the corresponding pair (f, Γ) satisfies (1-12). Otherwise, it is a type-1 equation.

REMARK 1.3. When f satisfies (1-7), we know $\Gamma_{\mathcal{G}}^f = \Gamma$. Notably, $\mathring{\Gamma}_{\mathcal{G}}^f \neq \Gamma_n \iff \varrho_{\Gamma_{\mathcal{G}}^f} > 1$. For (1-5), we have $\mathring{\Gamma}_{\mathcal{G}}^f = \Gamma_k$ and $\varrho_{\Gamma_{\mathcal{G}}^f} = (n/k)$.

The type-2 pair is much simpler, as detailed in Lemma 2.2. This is a special case of the partial uniform ellipticity developed by the author in a series of works [31, 32, 36]. In this type-2 case, together with a Morse theory technique that was first introduced in [32] as a tool to construct admissible functions, we obtain the existence result for the Dirichlet problem without extra assumptions on the boundary or the subsolution.

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THEOREM 1.4. *In addition to* (1-3), (1-4), *and* (1-9), *we assume that* (f,Γ) *is of type 2. Then, the Dirichlet problem* (1-1)–(1-2) *has a unique smooth admissible solution.*

This is a fully nonlinear analogue of the existence theorem for Poisson's equation. Significantly, in the type-2 case, we can solve the Dirichlet problem without any extra assumption regarding an admissible subsolution or admissible function. This stands in contrast with the vast existing literature.

However, using the constant $\varrho_{\Gamma^f_{\mathcal{G}}}$, we can give a characterization of (f,Γ) in Lemmas 2.4 and 2.2. Thus, we can recast (1-1) in the following form:

$$\tilde{f}(\lambda[\Delta u\cdot g-\varrho\nabla^2 u+\tilde{\chi}])=\psi, \tag{1-13}$$

corresponding to $(\tilde{f},\tilde{\Gamma})$, and *vice versa*, where

$$\varrho\le\varrho_{\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}},\quad \varrho\neq 0\quad\text{and}\quad \tilde{\chi}=(\mathrm{tr}_g\chi)\cdot g-\varrho\chi.$$

In the critical case $\varrho=1$, it reads as (1-6). Significantly, when $\varrho<\varrho_{\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}}$ and $\varrho\neq 0$, (1-13) can be reformulated as an equation (1-1) of type 2, and *vice versa*. See Proposition 2.6. Motivated by this, in Theorem 2.5, we propose an approximation of a pair (f,Γ) via type-2 pairs (f^t,Γ^t). An analogous approximation was also considered in [13, 18]. Our strategy is different and new. It is based on the applications of Morse theory to fully nonlinear equations, which was first observed by the author [32] (see also [27, 33, 35–37]).

By integrating Theorems 1.4 and 2.5, we propose a continuity method for the Dirichlet problem and apply it to establish a proof of the following result. This is a real analogue of [34, Theorem 1.6].

THEOREM 1.5. *Let* f *satisfy* (1-3), (1-4), (1-9), *and* (1-8). *Suppose that there exists an admissible subsolution* $\underline{u}\in C^2(\overline{M})$ *to the Dirichlet problem* (1-1)–(1-2). *Then, the Dirichlet problem admits a unique smooth admissible solution.*

In contrast to the usual approaches, our method leverages the *a priori* estimates directly, eliminating the necessity of smooth approximations to the C^2-regularity admissible subsolution. In the unbounded-condition case, a similar approach also works on closed manifolds.

Definitely not too surprisingly, things become more subtle in the degenerate case

$$\inf_M\psi=\sup_{\partial\Gamma} f. \tag{1-14}$$

The optimal regularity one can expect is $C^{1,1}$. For a type-2 degenerate equation, we derive a $C^{1,1}$-weak solution to the Dirichlet problem. To this end, we further assume

$$f\in C^\infty(\Gamma)\cap C(\overline{\Gamma})\quad\text{where }\overline{\Gamma}=\Gamma\cup\partial\Gamma. \tag{1-15}$$

THEOREM 1.6. *Assume* (1-3), (1-4), (1-12), (1-14), *and* (1-15) *hold. Then, the Dirichlet problem* (1-1)–(1-2) *possesses a weak solution* $u\in C^{1,1}(\overline{M})$ *with* $\Delta u\in L^\infty(\overline{M})$ *and* $\lambda(\nabla^2 u+\chi)\in\overline{\Gamma}$ *in* $\overline{M}$.

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REMARK 1.7. Theorems 1.4 and 1.6 allow (1-6) when $\mathring{\Gamma}^f_{\mathcal{G}} \neq \Gamma_n$, in which case $\varrho_{\Gamma^f_{\mathcal{G}}} > 1$.

One of the key ingredients in deriving (1-10) is to derive the gradient estimate

$$\sup_M |\nabla u| \leq C.$$

For (1-1) with $\chi = g$, the gradient estimate has been studied by Li [20] when the sectional curvature of g is nonnegative, and by Urbas [26] when replacing such a curvature condition by certain extra assumptions on f including

$$f_i(\lambda) \geq \delta \sum f_j(\lambda) \quad \text{if } \lambda_i \leq 0.$$

See also [9]. In 2020, the author [31] confirmed such a key inequality when f satisfies (1-7), thereby extending (1-10) to a large number of real Hessian equations. A straightforward proof of the gradient estimate for (1-1) was carried out by Guan [10] under a rather weak assumption. The above-mentioned gradient estimate is based on the Bernstein method. Another effective method is the blow-up argument. Such an argument was developed by Chen [3] for the Dirichlet problem of a complex Monge–Ampère equation, and by Dinew and Kołodziej [4], building on Hou *et al.*'s [15] second-order estimate of the form

$$\sup_M |\Delta u| \leq C\Big(1 + \sup_M |\nabla u|^2\Big), \tag{1-16}$$

in the case of complex k-Hessian equations on closed (compact without boundary) Kähler manifolds. Recently, under the extra assumption (1-7), the blow-up argument was extended extensively by Székelyhidi [23] to Hessian equations on closed Hermitian manifolds. The technical assumption (1-7) was subsequently removed by the author [34], thereby extending Székelyhidi's $C^{2,\alpha}$-estimate to the optimal setting. Moreover, in the unbounded case (1-8), the author [34] also established a prior estimate (1-16) and then solved the Dirichlet problem of Hessian equations on complex manifolds. In this paper, we employ the strategy of [34] to derive a gradient estimate in the type-1 case.

The rest of this paper is organized as follows. In Section 2, we discuss the structure of a pair and then propose an approximation of the pair (f, Γ) by type-2 pairs. In Section 3, we prove Theorems 1.4 and 1.6 based on Morse theory and the characterization of type-2 pairs. In Section 4, we establish (1-16) and (1-10). In Section 5, we propose a continuity method for the Dirichlet problem and employ it to give a proof of Theorem 1.5. In Appendix A, we summarize some useful lemmas.

2. Approximate Hessian equations using type-2 equations

2.1. Structure of a type-2 pair. We review a property of $\Gamma^f_{\mathcal{G}}$ proposed in [34].

LEMMA 2.1 [34]. *Suppose that f satisfies (1-3) in Γ. Then,*

$$\sum_{i=1}^n f_i(\lambda)\mu_i \geq 0 \quad \textit{and} \quad f(\lambda+\mu) \geq f(\lambda) \quad \textit{for all } \lambda \in \Gamma, \quad \textit{for all } \mu \in \Gamma^f_{\mathcal{G}}. \tag{2-1}$$

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PROOF. We start with an observation

$$\sum_{i=1}^{n} f_i(\lambda)\mu_i \geq \limsup_{t\to+\infty} f(t\mu)/t \quad \text{for all } \lambda,\ \mu \in \Gamma. \tag{2-2}$$

This inequality follows from the concavity of f. Moreover, by (1-3), we get

$$\sum_{i=1}^{n} f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) \quad \text{for all } \lambda,\ \mu \in \Gamma. \tag{2-3}$$

By (2-2), $\sum f_i(\lambda)\mu_i \geq 0$ for all $\lambda \in \Gamma$, for all $\mu \in \Gamma_{\mathcal{G}}^{f}$. Combining (2-3), we get $f(\lambda+\mu) \geq f(\lambda)$. □

As in the introduction, $\vec{\mathbf{1}} = (1,\dots,1)$. According to [36, Theorem 5.6], we obtain the following lemma.

LEMMA 2.2 [36]. *In addition to* (1-3), *we assume* $(-\delta_0, 0, \dots, 0, 1) \in \mathring{\Gamma}_{\mathcal{G}}^{f}$ *for some* $\delta_0 > 0$. *Then,*

$$f_i(\lambda) \geq \frac{\delta_0}{n} \sum_{j=1}^{n} f_j(\lambda) \quad \text{in } \Gamma \quad \text{for all } 1 \leq i \leq n. \tag{2-4}$$

Moreover, $\lim_{t\to+\infty} f(\lambda_1,\dots,\lambda_{n-1},\lambda_n+t) = \sup_\Gamma f$ *for all* $\lambda = (\lambda_1,\dots,\lambda_n) \in \Gamma$.

REMARK 2.3. When f further satisfies (1-7), the result was obtained in [31].

PROOF. Suppose $\lambda_1 \leq \cdots \leq \lambda_n$. The concavity and symmetry of f implies $f_n \leq \cdots \leq f_1$ and $f_1 \geq (1/n)\sum f_j$. By Lemma 2.1, $f_n \geq \delta_0 f_1 \geq (\delta_0/n)\sum f_j$. Finally, for $R_0 > 0$ from $(0,\dots,0,1) \in \mathring{\Gamma}_{\mathcal{G}}^{f}$, one can find $t \gg 1$ so that $t(0,\dots,0,1) + \lambda - R_0\vec{\mathbf{1}} \in \mathring{\Gamma}_{\mathcal{G}}^{f}$. Thus, $f(\lambda_1,\dots,\lambda_{n-1},\lambda_n+t) \geq f(R_0\vec{\mathbf{1}})$ according to (2-1) from Lemma 2.1. □

2.2. Structure of a pair and an application to approximation. The results of this subsection are based on an observation. The observation is noted in [36] and closely aligns with the results presented in [35, Section 4]. Fix a pair (f,Γ). Let $\varrho_{\Gamma_{\mathcal{G}}^{f}}$ be as in (1-11), and pick a positive constant ϱ with $n - \varrho_{\Gamma_{\mathcal{G}}^{f}} < \varrho < n$. Define $\mu^\lambda := \sum_{j=1}^{n} \lambda_j\vec{\mathbf{1}} - \varrho\lambda$ for all $\lambda \in \mathbb{R}^n$. Let P be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $P(\lambda) = \mu^\lambda$. Since $0 < \varrho < n$, the linear map is well defined and invertible, and $P^{-1}(\mu) = (1/\varrho(n-\varrho))(\sum_{j=1}^{n}\mu_j\vec{\mathbf{1}} - (n-\varrho)\mu)$. Take $\tilde{\Gamma} := P(\Gamma)$ and define $\tilde{f} : \tilde{\Gamma} \to \mathbb{R}$ by

$$\tilde{f}(\mu^\lambda) = f(\lambda).$$

So, $\tilde{\Gamma}$ is an open, convex, symmetric cone with vertex at origin. We prove the following lemma, which is noted in [36].

LEMMA 2.4. *Given a pair* (f,Γ) *obeying* (1-3) *and* (1-4), *let* ϱ, $\tilde{\Gamma}$, *and* $\tilde{f}$ *be as above. Then,* $\tilde{\Gamma}$ *is an open symmetric convex cone with vertex at the origin and*

$$\Gamma_n \subseteq \mathring{\tilde{\Gamma}}_{\mathcal{G}}^{\tilde{f}} \subseteq \tilde{\Gamma} \subseteq \Gamma_1, \quad \varrho \leq \varrho_{\tilde{\Gamma}_{\mathcal{G}}^{\tilde{f}}},$$

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and $\tilde{f}$ satisfies (1-3) and (1-4) in $\tilde{\Gamma}$. Moreover, $(\tilde{f},\tilde{\Gamma})$ is of type 2 and we have the following.

(1) *If $(0,\dots,0,1)\in \mathring{\tilde{\Gamma}}^{\tilde{f}}_{\mathcal{G}}$, then $\varrho<\varrho_{\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}}$.*

(2) *If $(0,\dots,0,1)\in \partial\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}$, then $\varrho=\varrho_{\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}}$.*

PROOF. From $\sum_{j=1}^n \mu_j^\lambda=(n-\varrho)\sum_{j=1}^n\lambda_j$, we see $\tilde{\Gamma}\subseteq\Gamma_1$. By $n-\varrho_{\Gamma^f_{\mathcal{G}}}<\varrho$, we know $(1,\dots,1,1-n+\varrho)\in\mathring{\Gamma}^f_{\mathcal{G}}$, thereby $(0,\dots,0,1)\in\mathring{\tilde{\Gamma}}^{\tilde{f}}_{\mathcal{G}}$. By $\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}=P(\Gamma^f_{\mathcal{G}})$ and $(0,\dots,0,1)\in\overline{\Gamma}^f_{\mathcal{G}}$, we get $(1,\dots,1,1-\varrho)\in\overline{\tilde{\Gamma}}^{\tilde{f}}_{\mathcal{G}}$. Thus, $\varrho\le\varrho_{\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}}$. However, we can check the concavity of $\tilde{f}$ in $\tilde{\Gamma}$, thanks to

$$\begin{aligned}\tilde{f}(t\mu^\lambda+(1-t)\mu^{\lambda'})&=f(t\lambda+(1-t)\lambda')\ge tf(\lambda)+(1-t)f(\lambda')\\&=t\tilde{f}(\mu^\lambda)+(1-t)\tilde{f}(\mu^{\lambda'})\quad\text{for all } 0<t<1,\quad\text{for all }\mu^\lambda,\mu^{\lambda'}\in\tilde{\Gamma}.\end{aligned}$$

Finally, we see $(\tilde{f},\tilde{\Gamma})$ satisfies (1-4) thanks to

$$\frac{\partial\tilde{f}}{\partial\mu_i^\lambda}=\frac{1}{\varrho(n-\varrho)}\sum_{j=1}^n\frac{\partial f}{\partial\lambda_j}-\frac{1}{\varrho}\frac{\partial f}{\partial\lambda_i}>\frac{1}{\varrho\varrho_{\Gamma^f_{\mathcal{G}}}}\Big(\sum_{j=1}^n\frac{\partial f}{\partial\lambda_j}-\varrho_{\Gamma^f_{\mathcal{G}}}\frac{\partial f}{\partial\lambda_i}\Big)\ge 0.$$

The above two inequalities follow from $0<n-\varrho<\varrho_{\Gamma^f_{\mathcal{G}}}$ and Lemma 2.1. □

THEOREM 2.5. *Given a type-1 pair (f,Γ) with $(0,\dots,0,1)\in\partial\Gamma^f_{\mathcal{G}}$, let $(\tilde{f},\tilde{\Gamma})$ be the pair as asserted in Lemma 2.4. For $t\ge1$, denote*

$$f^t(\lambda)=\tilde{f}\Big(t\sum_{i=1}^n\lambda_i\vec{\mathbf{1}}-\varrho_{\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}}\lambda\Big),\quad \Gamma^t=\Big\{\lambda\in\mathbb{R}^n:t\sum_{i=1}^n\lambda_i\vec{\mathbf{1}}-\varrho_{\tilde{\Gamma}^{\tilde{f}}_{\mathcal{G}}}\lambda\in\tilde{\Gamma}\Big\}.\tag{2-5}$$

Then, we have the following.

(1) *$\Gamma\subseteq\Gamma^t$ and $\Gamma^f_{\mathcal{G}}\subseteq\Gamma^{tf^t}_{\mathcal{G}}$ for $t>1$. When $t=1$, (f^t,Γ^t) coincides with (f,Γ).*

(2) *For each $t>1$, (f^t,Γ^t) is a type-2 pair, that is, $(0,\dots,0,1)\in\mathring{\Gamma}^{tf^t}_{\mathcal{G}}$.*

(3) *$f^t(\lambda)\ge f(\lambda)$ for all $\lambda\in\Gamma$, for all $t\ge1$.*

From this, each equation of the form (1-1) can be reformulated as in the following form:

$$\tilde{f}(\lambda[\Delta u\cdot g-\varrho\nabla^2u+\tilde{\chi}])=\psi,\quad \varrho\ne0,$$

and *vice versa*.

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PROPOSITION 2.6. *Fix a pair* (f,Γ) *and pick a positive constant* ϱ *with* $n-\varrho_{\Gamma^f_{\mathcal{G}}}\le\varrho<n$. *Let* $(\tilde f,\tilde\Gamma)$ *be as above such that* $\varrho\le\varrho_{\tilde\Gamma^{\tilde f}_{\mathcal{G}}}$. *Given a symmetric* $(0,2)$*-tensor* χ, *denote* $\tilde\chi:=(\mathrm{tr}_g\chi)\cdot g-\varrho\chi$. *Then, for any* C^2*-function* u *with* $\lambda[\nabla^2u+\chi]\in\Gamma$,

$$f(\lambda[\nabla^2u+\chi])=\tilde f(\lambda[\Delta u\cdot g-\varrho\nabla^2u+\tilde\chi]).$$

Moreover, $(0,\dots,0,1)\in\mathring{\tilde\Gamma}^f_{\mathcal{G}}\iff\varrho<\varrho_{\tilde\Gamma^{\tilde f}_{\mathcal{G}}}$.

3. On the type 2 equations

3.1. C^0-estimate and construction of admissible functions. We use Morse theory to construct admissible functions. The strategy follows some ideas of [32] (see also [33, 36]). For our purpose, we need the following lemma.

LEMMA 3.1. *Let* $\overline{M}$ *be a compact connected manifold of real dimension* $n\ge2$ *with smooth boundary. Then, there is a smooth function* v *without any critical points.*

PROOF. We include a proof, following [32]. Let X be the double of $\overline{M}$. It is obtained by gluing two copies of $\overline{M}$ together along their common boundary. Let w be a smooth Morse function on X with the critical set $\{p_i\}_{i=1}^{m+k}$, among which $p_1,\dots,p_m$ are all the critical points being in $\overline{M}$. Pick $q_1,\dots,q_m\in X\setminus\overline{M}$, but not the critical point of w. By the homogeneity lemma (see [21]), one can find a diffeomorphism $h:X\to X$, which is smoothly isotopic to the identity, such that $h(p_i)=q_i$ for $1\le i\le m$ and $h(p_i)=p_i$ for $m+1\le i\le m+k$. Then, $v=w\circ h^{-1}|_{\overline{M}}$ is the desired function. □

By Lemma 3.1, we can take a smooth function v with $v\ge1$ and $dv\neq0$ on $\overline{M}$. Let

$$\underline{v}=e^{tv}. \tag{3-1}$$

The straightforward computation yields that

$$\chi+\nabla^2\underline{v}=\chi+te^{tv}(\nabla^2v+t\,dv\otimes dv).$$

Let R_0 be a positive constant such that $f(R_0\vec{\mathbf{1}})>\psi$ in $\overline{M}$. By $(0,\dots,0,1)\in\mathring{\Gamma}^f_{\mathcal{G}}$, we may pick $t\gg1$ such that

$$\lambda\Big[\chi+te^{tv}\Big(\nabla^2v+\frac{t}{2}\,dv\otimes dv\Big)\Big]\in\mathring{\Gamma}^f_{\mathcal{G}},\ \lambda\Big[\frac{t^2e^{tv}}{2}\,dv\otimes dv-R_0g\Big]\in\mathring{\Gamma}^f_{\mathcal{G}}.$$

In particular,

$$\lambda[\nabla^2\underline{v}+\chi]\in\mathring{\Gamma}^f_{\mathcal{G}}\quad\text{in }\overline{M}. \tag{3-2}$$

Using Lemma 2.1 twice, we conclude that

$$f(\lambda[\nabla^2\underline{v}+\chi])\ge f\Big(\lambda\Big[\frac{t^2e^{tv}}{2}\,dv\otimes dv\Big]\Big)\ge f(R_0\vec{\mathbf{1}})>\psi. \tag{3-3}$$

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The comparison principle then implies that

$$\inf_{\overline{M}}(u-\underline{v}) = \inf_{\partial M}(\varphi-\underline{v}). \tag{3-4}$$

The upper bound of u is standard. Let h be the solution to

$$\Delta h + \mathrm{tr}_g\chi = 0 \quad \text{in } \overline{M}, \quad h=\varphi \quad \text{on } \partial M.$$

The existence and regularity of h can be found in standard textbooks; see [6, Theorem 6.14] and [1, Theorem 4.8]. Then,

$$u \le h \quad \text{in } \overline{M}, \quad u=h=\varphi \quad \text{on } \partial M. \tag{3-5}$$

3.2. Local and global estimates for type-2 equations.

PROPOSITION 3.2. *In addition to* (1-3), (1-4), *and* (1-9), *we assume* $(-\delta_0, 0, \dots, 0, 1) \in \mathring{\Gamma}^f_{\mathcal{G}}$ *for some* $\delta_0 > 0$. *Let* $B_r \subset M$ *be a geodesic ball with radius* $r > 0$. *Let* $u \in C^4(M) \cap C^2(\overline{M})$ *be an admissible solution to* (1-1). *Then,*

$$\sup_{B_{r/2}}(|\nabla^2 u| + |\nabla u|^2) \le C_1, \tag{3-6}$$

where C_1 *depends on* δ_0^{-1}, r^{-1}, $|u|_{C^0(B_r)}$, $|\psi|_{C^2(B_r)}$, *and geometric quantities on* B_r. *Moreover,*

$$\sup_{M}(|\nabla^2 u| + |\nabla u|^2) \le C_2\Big(1 + \sup_{\partial M}|\nabla^2 u| + \sup_{\partial M}|\nabla u|^2\Big), \tag{3-7}$$

where C_2 *depends on* δ_0^{-1}, $|u|_{C^0(M)}$, $|\psi|_{C^2(\overline{M})}$, *and other known data. Furthermore, both* C_1 *and* C_2 *are independent of* $\delta_{\psi,f}^{-1}$, *where and hereafter*

$$\delta_{\psi,f} := \inf_M \psi - \sup_{\partial\Gamma} f. \tag{3-8}$$

PROOF. By Lemma 2.2, f is of fully uniform ellipticity, that is, f satisfies (2-4). As in [8] (see also [31, 32]), we can derive local estimates (3-6) for second- and first-order derivatives. When taking the cutoff function $\eta \equiv 1$ in the proof, we get (3-7). Since the proof is somewhat standard, we omit the details here. □

3.3. Boundary estimate for first derivatives. Let σ be the distance function to ∂M, let $M_\delta := \{z \in M : \sigma(z) < \delta\}$. In the following, we use σ to construct a local $\mathring{\Gamma}^f_{\mathcal{G}}$-admissible function $\underline{w}$ subject to

$$f(\lambda[\nabla^2\underline{w} + \chi]) > \psi \quad \text{in } M_{\delta_0}, \tag{3-9}$$

$$u \ge \underline{w} \quad \text{in } M_{\delta_0}, \ \underline{w} = \varphi \quad \text{on } \partial M \tag{3-10}$$

for some $\delta_0 > 0$. As in [8], take $w(z) = 2\log(\delta^2/\delta^2 + \sigma(z))$. The computation gives

$$\chi + \nabla^2(w+\varphi) = \chi + \nabla^2\varphi + \frac{2\,d\sigma\otimes d\sigma}{(\delta^2+\sigma)^2} - \frac{2\nabla^2\sigma}{\delta^2+\sigma}.$$

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Notice $|\nabla\sigma| = 1$ on ∂M and $(0,\dots,0,1) \in \mathring{\Gamma}^f_{\mathcal{G}}$. Take $0 < \delta_1 \ll 1$ so that

$$\lambda\left[\chi + \nabla^2\varphi + \frac{d\sigma\otimes d\sigma}{(\delta^2+\sigma)^2} - \frac{2\nabla^2\sigma}{\delta^2+\sigma}\right] \in \mathring{\Gamma}^f_{\mathcal{G}} \quad \text{in } M_\delta$$

for $0 < \delta \le \delta_1$. By Lemma 2.1, there is $0 < \delta_2 \le \delta_1$ such that for $0 < \delta \le \delta_2$,

$$f(\lambda[\chi + \nabla^2(w+\varphi)]) \ge f\left(\lambda\left[\frac{d\sigma\otimes d\sigma}{(\delta^2+\sigma)^2}\right]\right) > \psi \quad \text{in } M_\delta.$$

From (3-4), there is a positive uniform constant δ_3 such that

$$\inf_M (u-\varphi) \ge 2\log\frac{\delta_3}{1+\delta_3}.$$

Denote $\delta_0 = \min\{\delta_2, \delta_3\}$ and take

$$\underline{w} = 2\log\frac{\delta_0^2}{\delta_0^2+\sigma} + \varphi. \tag{3-11}$$

Consequently, the comparison principle yields that

$$\inf_{M_{\delta_0}}(u-\underline{w}) = \inf_{\partial M_{\delta_0}}(u-\underline{w}) \ge 0.$$

Thus, $\underline{w}$ satisfies (3-10) and (3-9) in M_{δ_0}. From (3-5) and (3-10),

$$|\nabla u| \le C \quad \text{on } \partial M. \tag{3-12}$$

3.4. Boundary estimate for second derivatives. Now, we review some notation.

For $x_0 \in \partial M$, we choose a smooth local orthonormal frame $e_1,\dots,e_n$ around x_0 such that e_n is the unit inner normal vector field when restricted to ∂M. Denote $g_{ij} = g(e_i,e_j)$ and $\{g^{ij}\} = \{g_{ij}\}^{-1}$. Under the Levi–Civita connection of g, $\nabla_{e_i}e_j = \Gamma^k_{ij}e_k$ and Γ^k_{ij} denote the Christoffel symbols. For simplicity, we write

$$\nabla_i = \nabla_{e_i}, \nabla_{ij} = \nabla_i\nabla_j - \Gamma^k_{ij}\nabla_k, \quad \nabla_{ijk} = \nabla_i\nabla_{jk} - \Gamma^l_{ij}\nabla_{lk} - \Gamma^l_{ik}\nabla_{jl},\dots, \text{ and so forth.}$$

The linearized operator of (1-1) at the solution u, say $\mathcal{L}$, is locally given by

$$\mathcal{L}w = F^{ij}\nabla_{ij}w, \tag{3-13}$$

where $\mathfrak{g}_{ij} = \chi_{ij} + \nabla_{ij}u$ and $F^{ij} = (\partial F(\mathfrak{g})/\partial\mathfrak{g}_{ij})$.

Let $\underline{w}$ be as in (3-11), a local $\mathring{\Gamma}^f_{\mathcal{G}}$-admissible function obeying (3-10) and (3-9).

Applying such a local barrier function $\underline{w}$ to Lemma A.2, there is a positive constant ε (independent of $\delta_{\psi,f}$) such that either

$$F^{ij}(\nabla_{ij}\underline{w} - \nabla_{ij}u) \ge \varepsilon F^{ij}g_{ij} + \varepsilon \tag{3-14}$$

or

$$F^{ij} \ge \varepsilon(1 + F^{pq}g_{pq})g^{ij}. \tag{3-15}$$

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PROPOSITION 3.3. *Let $\delta_{\psi,f}$ be as in (3-8). In addition to (1-3), (1-4), and (1-9), we assume $(-\delta_0, 0, \dots, 0, 1) \in \mathring{\Gamma}^f_{\mathcal{G}}$ for some $\delta_0 > 0$. Let $u \in C^4(M) \cap C^2(\overline{M})$ be an admissible solution to the Dirichlet problem (1-1)–(1-2). Then,*

$$\sup_{\partial M} |\nabla^2 u| \le C \quad \text{independent of } \delta^{-1}_{\psi,f}, \tag{3-16}$$

where C is a positive constant depending only on δ_0^{-1}, $|u|_{C^0(M)}$, $|\nabla u|_{C^0(\partial M)}$, $|\underline{w}|_{C^2(M_{\delta_0})}$, $|\varphi|_{C^3(\partial M)}$, and other known data.

PROOF. The proof of the boundary estimate for pure tangential derivatives

$$|\nabla_{\alpha\beta} u(x_0)| \le C_5 \quad \text{for all } \alpha, \beta \le n-1 \tag{3-17}$$

is standard by the boundary value condition. Here, C_5 depends only on $|\varphi|_{C^2(\partial M)}$, $|u|_{C^1(\partial M)}$, and geometric quantities of ∂M.

By a minor modification of [28, Proposition 3.1], we can employ (3-14) and (3-15) to derive boundary estimates for mixed derivatives

$$|\nabla_{\alpha n} u(x_0)| \le C_6\Big(1 + \sup_M |\nabla u|\Big) \quad \text{for all } 1 \le \alpha < n \quad \text{independent of } \delta^{-1}_{\psi,f},$$

where C_6 depends on $|u|_{C^0(\overline{M})}$, $|\varphi|_{C^3(\partial M)}$, $|\psi|_{C^1(\overline{M})}$, and other known data.

Let $\underline{v}$ be as in (3-1). From (2-3) and (3-3), we get

$$F^{nn}\nabla_{nn} u(x_0) \le -\sum_{i+j<2n} F^{ij}\nabla_{ij} u(x_0) + \sum_{i,j=1}^{n} F^{ij}\nabla_{ij}\underline{v}(x_0).$$

By Lemma 2.2 or (2-4), $F^{nn} \ge (\delta_0/n)\sum_{i=1}^n F^{ii}$ at x_0. Thus,

$$u_{nn}(x_0) \le \frac{n}{\delta_0}\sum_{i+j<2n} |\nabla_{ij} u(x_0)| + \frac{n}{\delta_0}\sum_{i,j=1}^{n} |\nabla_{ij}\underline{v}(x_0)|.$$

Finally, by (3-17) and $\Gamma \subseteq \Gamma_1$, we see $\nabla_{nn} u(x_0) \ge -C$. □

3.5. Completion of the proof of Theorem 1.4. Let $\underline{v}$ be a smooth function as constructed in (3-1), satisfying (3-2) and (3-3). For each $t \in [0,1]$, we consider

$$\begin{cases} f(\lambda[\nabla^2 u^t + \chi]) = t\psi + (1-t) f(\lambda[\nabla^2 \underline{v} + \chi]) & \text{in } \overline{M}, \\ u^t = t\varphi + (1-t)\underline{v} & \text{on } \partial M. \end{cases} \tag{3-18}$$

Denote

$$\mathrm{S} = \{t \in [0,1] : \text{ (3-18) has a unique smooth admissible solution for } t\}.$$

Obviously, for $t = 0$, the Dirichlet problem has a unique admissible solution $\underline{v}$. By the implicit function theorem and the theory of linear elliptic equations, S is an open subset of $[0,1]$. Together with the Evans–Krylov theorem and Schauder theory, (3-4), (3-5), (3-7), (3-12), and (3-16) imply that S is closed. Thus, $\mathrm{S} = [0,1]$.

We remark that the subsolution assumption is not needed in the present proof.

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3.6. Proof of Theorem 1.6. By Theorem 1.4, for each $\delta > 0$, there is a unique smooth admissible function u^δ satisfying

$$f(\lambda[\nabla^2 u^\delta + \chi]) = \psi + \delta \quad \text{in } \overline{M}, \quad u^\delta = \varphi \quad \text{on } \partial M.$$

From (3-4), (3-5), (3-7), (3-12), and (3-16), we deduce that

$$\sup_{\overline{M}} |\nabla^2 u^\delta| \le C \quad \text{independent of } \delta^{-1}.$$

Take $\delta \to 0^+$. By passing to a proper subsequence, we may obtain a weak $C^{1,1}$-solution to the Dirichlet problem for a degenerate equation.

4. Quantitative second-order estimate

We establish a real analogue of [34, Theorem 1.7] in the unbounded case.

PROPOSITION 4.1. *In addition to* (1-3), (1-4), (1-8), *and* (1-9), *we assume that the Dirichlet problem* (1-1)–(1-2) *admits an admissible subsolution* $\underline{u} \in C^2(\overline{M})$. *For any admissible solution* $u \in C^4(M) \cap C^2(\overline{M})$ *to the Dirichlet problem, we have a second estimate of the quantitative form*

$$\sup_{\overline{M}} |\nabla^2 u| \le C\Big(1 + \sup_M |\nabla u|^2\Big), \tag{4-1}$$

and the Hessian estimate (1-10).

We first summarize some notation. Fix a point $x_0 \in \partial M$. We choose a smooth local orthonormal frame $e_1, \dots, e_n$ around x_0 such that e_n is the unit inner normal vector field when restricted to ∂M. Denote

$$\Gamma_\infty = \{(\lambda_1, \dots, \lambda_{n-1}) : (\lambda_1, \dots, \lambda_{n-1}, R) \in \Gamma \text{ for some } R > 0\}.$$

Throughout this section, the Greek letters α, β run from 1 to $n-1$.

The main issue is to derive (4-1). Following an estimate presented in [10, Section 3], we can prove that

$$\sup_{\overline{M}} |\nabla^2 u| \le C\Big(1 + \sup_M |\nabla u|^2 + \sup_{\partial M} |\nabla^2 u|\Big).$$

To derive (4-1), it is only required to prove the quantitative boundary estimate

$$\sup_{\partial M} |\nabla^2 u| \le C\Big(1 + \sup_M |\nabla u|^2\Big).$$

From [28, Proposition 3.1], we have boundary estimate for mixed derivatives

$$|\nabla_{\alpha n}(x_0)| \le C\Big(1 + \sup_M |\nabla u|\Big) \quad \text{for all } 1 \le \alpha \le n-1.$$

By (3-17) and $\Gamma \subseteq \Gamma_1$, we see $\nabla_{nn} u(x_0) \ge -C$. So, it suffices to prove the following lemma. This is a real analogue of [34, Proposition 7.1].

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LEMMA 4.2. *In the setting of Proposition 4.1,*

$$\mathfrak{g}_{nn}(x_0) \le C\Big(1 + \sum_{\alpha=1}^{n-1} |\mathfrak{g}_{\alpha n}(x_0)|^2\Big).$$

Let σ be as in Subsection 3.3, the distance to the boundary. At $x_0 \in \partial M$, the boundary value condition yields

$$\mathfrak{g}_{\alpha\beta} = (1-t)\underline{\mathfrak{g}}_{\alpha\beta} + \{t\underline{\mathfrak{g}}_{\alpha\beta} + \nabla_n(u-\underline{u})\nabla_{\alpha\beta}\sigma\}. \tag{4-2}$$

Let t_0 be the first t as we decrease t from 1 such that

$$\lambda(t\underline{\mathfrak{g}}_{\alpha\beta} + \nabla_n(u-\underline{u})\nabla_{\alpha\beta}\sigma) \in \partial\Gamma_\infty$$

at x_0, where $\partial\Gamma_\infty$ is the boundary of Γ_∞. Obviously,

$$-T_0 < t_0 < 1$$

for some uniform positive constant T_0 under control. Following closely the outline of [2] (see also [19] in the complex case), one may obtain

$$(1-t_0)^{-1} \le C, \quad \text{possibly depending on } \delta_{\psi,f}^{-1}, \tag{4-3}$$

where C is a uniform positive constant depending on $|u|_{C^0(M)}$, $|\nabla u|_{C^0(\partial M)}$, $|\underline{u}|_{C^2(M)}$, $\sup_M \psi$, ∂M up to third derivatives, and other known data.

Denote

$$\underline{\lambda}' = (\underline{\lambda}'_1, \dots, \underline{\lambda}'_{n-1}) \equiv \lambda(\underline{\mathfrak{g}}_{\alpha\beta}), \tag{4-4}$$

$$(A_{t_0})_{\alpha\beta} = t_0\underline{\mathfrak{g}}_{\alpha\beta} + \nabla_n(u-\underline{u})\nabla_{\alpha\beta}\sigma, \ \tilde{\lambda}' = (\tilde{\lambda}'_1, \dots, \tilde{\lambda}'_{n-1}) \equiv \lambda[(A_{t_0})_{\alpha\beta}]. \tag{4-5}$$

For $R > 0$, define the matrix

$$A(R) = \begin{pmatrix} \mathfrak{g}_{\alpha\beta} & \mathfrak{g}_{\alpha n} \\ \mathfrak{g}_{n\beta} & R \end{pmatrix}.$$

By (4-2), we can decompose $A(R)$ as follows:

$$A(R) = A'(R) + A''(R),$$

where

$$A'(R) = \begin{pmatrix} (1-t_0)(\underline{\mathfrak{g}}_{\alpha\beta} - \dfrac{\varepsilon_0}{4}\delta_{\alpha\beta}) & \mathfrak{g}_{\alpha n} \\ \mathfrak{g}_{n\beta} & R/2 \end{pmatrix}, \quad A''(R) = \begin{pmatrix} (A_{t_0})_{\alpha\beta} + \dfrac{(1-t_0)\varepsilon_0}{4}\delta_{\alpha\beta} & 0 \\ 0 & R/2 \end{pmatrix}.$$

Moreover, we need the following lemma, which was proved by [34] in the complex case.

LEMMA 4.3. *There exist uniform constants R_3 and $C_{A''}$ depending on $(1-t_0)^{-1}$, ε_0^{-1}, f, and other known data under control, such that*

$$f(2\lambda[A''(R_3)]) \ge -C_{A''}. \tag{4-6}$$

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PROOF. We include the proof, following [34]. Let $\underline{\lambda}'$ and $\tilde{\lambda}'$ be as in (4-4) and (4-5), respectively. One can see that there is a uniform constant $C_0 > 0$ depending on $|t_0|$, $\sup_{\partial M} |\nabla u|$, and other known data, such that $|\tilde{\lambda}'| \le C_0$, that is,

$$\tilde{\lambda}' \in K \equiv \{\lambda' \in \overline{\Gamma}_\infty : |\lambda'| \le C_0\}.$$

There is a positive constant R_2 depending on $((1-t_0)\varepsilon_0)^{-1}$, K, and other known data, such that

$$\lambda\left[\begin{pmatrix} (A_{t_0})_{\alpha\beta} + \dfrac{(1-t_0)\varepsilon_0}{8}\delta_{\alpha\beta} & 0 \\ 0 & R_2/2 \end{pmatrix}\right] \in \Gamma.$$

Let $R_3 = R_2 + ((1-t_0)\varepsilon_0/4)$. So,

$$\lambda[A''(R_3)] \in \Gamma + \frac{(1-t_0)\varepsilon_0}{8}\vec{\mathbf{1}}. \tag{4-7}$$

Notice that $A''(R_3)$ is bounded. There exists a uniform constant $C_{A''}$ depending on $(1-t_0)^{-1}$, ε_0^{-1}, R_3 and f, such that (4-6) holds. □

PROOF OF LEMMA 4.2. We follow closely [29, 30, 34]. By (1-8) and (4-3), there is a uniform positive constant R_1 such that

$$f(2(1-t_0)(\underline{\lambda}'_1 - \varepsilon_0/2), \ldots, 2(1-t_0)(\underline{\lambda}'_{n-1} - \varepsilon_0/2), 2R_1) \ge 2f(\underline{\lambda}) + C_{A''} \tag{4-8}$$

and $(\underline{\lambda}'_1 - \varepsilon_0/2, \ldots, \underline{\lambda}'_{n-1} - \varepsilon_0/2, (1-t_0)^{-1}R_1) \in \Gamma$.

Let us pick $\epsilon = ((1-t_0)\varepsilon_0/4)$ in Lemma A.1. Then, we set

$$\begin{aligned} R_c \equiv{}& \frac{8(2n-3)}{(1-t_0)\varepsilon_0}\sum_{\alpha=1}^{n-1}|\mathfrak{g}_{\alpha n}|^2 + 2(n-1)(1-t_0)\sum_{\alpha=1}^{n-1}|\underline{\lambda}'_\alpha| \\ &+ \frac{n(n-1)(1-t_0)\varepsilon_0}{2} + 2R_1 + 2R_2, \end{aligned}$$

where ε_0, R_1, and R_2 are the constants we fixed in (4-8) and (4-7). According to Lemma A.1, the eigenvalues $\lambda[A'(R_c)]$ (possibly with an appropriate order) behave like

$$\lambda_\alpha[A'(R_c)] \ge (1-t_0)(\underline{\lambda}'_\alpha - \varepsilon_0/2),\ 1 \le \alpha \le n-1;\ \ \lambda_n[A'(R_c)] \ge R_c/2. \tag{4-9}$$

In particular, $\lambda[A'(R_c)] \in \Gamma$. So, $\lambda[A(R_c)] \in \Gamma$.

Using the concavity of f, Lemma 4.3, and (4-8), (4-9), we deduce that

$$f(\lambda[A(R_c)]) \ge \tfrac{1}{2}f(2\lambda[A'(R_c)]) + \tfrac{1}{2}f(2\lambda[A''(R_c)]) \ge f(\underline{\lambda}) \ge \psi.$$

Thus, $\mathfrak{g}_{nn} \le R_c$. By (4-3), $(1-t_0)^{-1}$ is bounded. The proof is now complete. □

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PROOF OF PROPOSITION 4.1. By (3-5) and the subsolution assumption,

$$\sup_M |u| + \sup_{\partial M} |\nabla u| \le C.$$

Based on (4-1), without using the assumption (1-7), we use the blow-up argument developed in [34], extending that of [4, 23] to the optimal setting, to derive the gradient estimate

$$\sup_M |\nabla u| \le C.$$

(The details can also be found in [28, Section 5].) In conclusion, we get (1-10). □

5. A continuity method and proof of Theorem 1.5

We now give a continuity method for the Dirichlet problem (1-1)–(1-2). Our method is based on Theorems 1.4 and 2.5. Let (f, Γ) be the pair corresponding to (1-1). If $(0, \dots, 0, 1) \in \mathring{\Gamma}^f_{\mathcal{G}}$, then we are done as in Theorem 1.4. From now on, we assume $(0, \dots, 0, 1) \in \partial\Gamma^f_{\mathcal{G}}$ and then take the pair (f^t, Γ^t) defined as in (2-5).

For each $t \in [1, 2]$, consider

$$f^t(\lambda[\nabla^2 u + \chi]) = \psi \quad \text{in } \overline{M}, \quad u = \varphi \quad \text{on } \partial M. \tag{5-1}$$

When $t > 1$, (f^t, Γ^t) is of type 2 according to Theorem 2.5. Then, by Theorem 1.4, for each $t > 1$, (5-1) has a unique smooth solution u^t with $\lambda[\nabla^2 u^t + \chi] \in \Gamma^t$.

Let $\underline{u}$ be an admissible subsolution to the Dirichlet problem (1-1)–(1-2). By items (3) and (1) of Theorem 2.5, $\underline{u}$ is also an admissible subsolution to (5-1) for all $1 \le t \le 2$,

$$f^t(\lambda[\nabla^2 \underline{u} + \chi]) \ge \psi, \quad \lambda[\nabla^2 \underline{u} + \chi] \in \Gamma^t \quad \text{in } \overline{M}.$$

By (1-8), Lemma 2.2, and Theorem 2.5 ($t > 1$), (f^t, Γ^t) uniformly obeys (1-8) for $t \in [1, 2]$. By Proposition 4.1, for $t \in (1, 2]$, there is some $\alpha \in (0, 1)$ such that

$$|u^t|_{C^{4,\alpha}(\overline{M})} \le C \quad \text{independently of } t.$$

Passing to a proper subsequence as $t \to 1^+$, we obtain a $C^{4,\alpha}$-admissible solution u to the Dirichlet problem (1-1)–(1-2). The Schauder theory implies that u is indeed smooth. The uniqueness follows from the maximum principle.

Appendix A. Useful lemmas

The following lemma is a key ingredient for the proof of Proposition 4.1.

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LEMMA A.1 [29, 30, 34]. *Let A be an $n \times n$ symmetric matrix*

$$\begin{pmatrix} d_1 & & & & a_1 \\ & d_2 & & & a_2 \\ & & \ddots & & \vdots \\ & & & d_{n-1} & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & \mathbf{a} \end{pmatrix}$$

with $d_1, \ldots, d_{n-1}, a_1, \ldots, a_{n-1}$ fixed, and with $\mathbf{a}$ variable. Denote the eigenvalues of A by $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let $\epsilon > 0$ be a fixed constant. Suppose that the parameter $\mathbf{a}$ in A satisfies the quadratic growth condition

$$\mathbf{a} \geq \frac{2n-3}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n-1) \sum_{i=1}^{n-1} |d_i| + \frac{(n-2)\epsilon}{2n-3}.$$

Then, the eigenvalues (possibly with a proper permutation) behave like

$$|d_\alpha - \lambda_\alpha| < \epsilon \quad \text{for all } 1 \leq \alpha \leq n-1; \quad 0 \leq \lambda_n - \mathbf{a} < (n-1)\epsilon.$$

Lemma A.1 was proposed in [29, 30]. We also refer the interested reader to [34] for a detailed proof.

In the following, we also present a lemma regarding a subsolution. Denote $\partial\Gamma^\sigma := \{\lambda \in \Gamma : f(\lambda) = \sigma\}$.

LEMMA A.2 [34]. *Assume (1-3) and (1-4) hold. Fix $\sigma \in (\sup_{\partial\Gamma} f, \sup_\Gamma f)$. Suppose $\mu \in \mathbb{R}^n$ obeys for some $\delta, R > 0$ that*

$$(\mu - 2\delta\vec{\mathbf{1}} + \Gamma_n) \cap \partial\Gamma^\sigma \subset B_R(0). \tag{A.1}$$

Then, there is a positive constant ε depending on σ, μ, δ, R such that if $\lambda \in \partial\Gamma^\sigma$, then either

$$\sum_{i=1}^n f_i(\lambda)(\mu_i - \lambda_i) \geq \varepsilon\Big(1 + \sum_{i=1}^n f_i(\lambda)\Big) \tag{A.2}$$

or

$$f_j(\lambda) \geq \varepsilon\Big(1 + \sum_{i=1}^n f_i(\lambda)\Big) \quad \text{for all } 1 \leq j \leq n. \tag{A.3}$$

PROOF. For self-sufficiency, we include the details, following the proof of [34]. Denote $\tilde{\mu} = \mu - \delta\vec{\mathbf{1}}$. By (A.1), there is a positive constant R_0 depending only on $\tilde{\mu}$ and R such that

$$f(\tilde{\mu} + R_0\vec{\mathbf{e}}_i) > \sigma, \quad \tilde{\mu} + R_0\vec{\mathbf{e}}_i \in \Gamma \quad \text{for all } 1 \leq i \leq n,$$

where $\vec{\mathbf{e}}_i$ is the ith standard basis vector in $\mathbb{R}^n$. Denote $\delta_0 = \min_{1\leq i\leq n}\{f(\tilde{\mu} + R_0\vec{\mathbf{e}}_i) - \sigma\}$. In the following, we prove the conclusion for $\varepsilon = \min\{(\delta/2), (\delta/2R_0), (\delta_0/2), (\delta_0/2R_0)\}$. Without loss of generality, we assume $\lambda_1 \leq \cdots \leq \lambda_n$. Then, $f_1(\lambda) \geq \cdots \geq f_n(\lambda)$. We get

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(A.3) if $f_n(\lambda) > (\delta_0/2R_0) + (\delta/2R_0)\sum_{i=1}^n f_i(\lambda)$. From now on, we assume $(\delta_0/2) + (\delta/2)\sum_{i=1}^n f_i(\lambda) \geq R_0 f_n(\lambda)$. Combining this assumption with (2-3),

$$\begin{aligned}\sum_{i=1}^n f_i(\lambda)(\mu_i - \lambda_i) &= \sum_{i=1}^n f_i(\lambda)(\tilde{\mu}_i - \lambda_i) + \delta \sum_{i=1}^n f_i(\lambda) \\ &\geq f(\tilde{\mu} + R_0\vec{\mathbf{e}}_n) - f(\lambda) - R_0 f_n(\lambda) + \delta \sum_{i=1}^n f_i(\lambda) \\ &\geq \frac{\delta_0}{2} + \frac{\delta}{2}\sum_{i=1}^n f_i(\lambda). \end{aligned}$$

□

REMARK A.3. According to [34, Lemma 3.1], the right-hand sides of both inequalities (A.2) and (A.3) can be strengthened to $\epsilon(1 + \sum_{i=1}^n f_i(\lambda) + |\sum_{i=1}^n f_i(\lambda)\lambda_i|)$.

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RIRONG YUAN, School of Mathematics,
South China University of Technology, Guangzhou 510641, PR China
e-mail: yuanrr@scut.edu.cn

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