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RESEARCH ARTICLE

Weighted decoupling estimates and the Bochner-Riesz means

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Abstract

We prove new weighted decoupling estimates. As an application, we give an improved sufficient condition for almost everywhere convergence of the Bochner-Riesz means of arbitrary L^p functions for $1 < p < 2$ in dimensions 2 and 3.

Contents

1 Introduction **1**
2 Weighted and refined decoupling inequalities **3**
2.1 A multilinear weighted decoupling estimate . . . 4
2.2 A weighted refined decoupling estimate . . . 5
3 Broad-narrow analysis: Proof of Theorem 1.3 **6**
3.1 Broad case . . . 7
3.2 Narrow case . . . 9
4 Convergence of Bochner-Riesz means: Proof of Theorem 1.1 **11**
4.1 Linearization . . . 11
4.2 Decompositions . . . 12
4.3 Proof of Proposition 4.2 . . . 15
4.4 An improved bound for $d = 3$ . . . 16
References **17**

1. Introduction

Let $d \geq 2$. For $\lambda, t > 0$, the Bochner-Riesz means $S_t^\lambda f$ are defined as

$$\widehat{S_t^\lambda f}(\xi) = \left(1 - \frac{|\xi|^2}{t^2}\right)_+^\lambda \widehat{f}(\xi), \ \xi \in \mathbb{R}^d.$$

We consider the pointwise convergence problem of the Bochner-Riesz means; for which λ, $S_t^\lambda f$ converges to f almost everywhere as $t \to \infty$ for arbitrary $f \in L^p(\mathbb{R}^d)$? For $p > 2$, this is the case if $\lambda > \max(d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, 0)$. This result is due to Carbery [4] for $d = 2$ and Carbery, Rubio de Francia and Vega [5] for every $d \geq 2$ (see also [18] for endpont results). Note that the given range of λ is precisely the one for which S_t^λ is known to be bounded on $L^p(\mathbb{R}^2)$ and is conjectured to be bounded on

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$L^p(\mathbb{R}^d)$ for $d > 2$ (see [6, 7, 12] for $d = 2$ and [14] and references therein for partial results for $d \geq 3$). Meanwhile, it is not known what happens when $\lambda = 0$ and $p = 2$, unlike the one-dimensional case where the Carleson-Hunt theorem is available.

For the case $1 \leq p < 2$, by Stein's maximal principle [20], almost everywhere convergence of an arbitrary L^p function is equivalent to the following weak-type estimate:

$$\| \sup_{t>0} |S_t^\lambda f| \|_{L^{p,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^p}, \tag{1.1}$$

for test functions f. Here, by $A \lesssim B$, we mean $A \leq CB$ for an absolute positive constant C. Examples of Fefferman [11] and Tao [22] show that (1.1) fails unless $\lambda > \max(0, d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2p})$. We prove a new partial result on this problem in dimension $d = 2$.

Theorem 1.1. *Let $d = 2$ and $p = \frac{86}{57}$. Then (1.1) holds for any $\lambda > \frac{9}{86}$.*

By an interpolation of classical L^1 and L^2 estimates, (1.1) holds if $\lambda > (d-1)(\frac{1}{p} - \frac{1}{2})$. For $d = 2$, Tao [23] improved this classical sufficient condition by giving a $L^{10/7}$-estimate. Building on [23], Li and Wu [19] gave a new $L^{18/13}$-estimate using the Bourgain-Demeter decoupling theorem [2]. More recently, Gan and Wu [13] obtained an improved $L^{10/7}$-estimate by using a weighted version of the ℓ^p-decoupling theorem. Moreover, they gave a new partial result in dimension $d = 3$. See Theorem 4.9 for an improved bound in $\mathbb{R}^3$.

Combining Theorem 1.1 with the classical L^1 and L^2 estimates and the $L^{18/13}$-estimate [19] yields the following corollary, which improves previously known results for $18/13 < p < 2$.

Corollary 1.2. *Let $d = 2$, $1 < p < 2$ and*

$$\lambda > \max\left(\frac{24}{23p} - \frac{27}{46}, \frac{9}{14p} - \frac{9}{28}, \frac{6}{5p} - \frac{7}{10} \right).$$

Then (1.1) holds. Consequently, $\lim_{t\to\infty} S_t^\lambda f(x) = f(x)$ for almost every $x \in \mathbb{R}^2$ for any $f \in L^p(\mathbb{R}^2)$.

Our new estimate is based on weighted ℓ^p-decoupling inequalities to be discussed below. We first recall the ℓ^p-decoupling theorem, which follows from the Bourgain-Demeter ℓ^2-decoupling theorem [2]. Let $S \subset \mathbb{R}^d$ be a strictly convex C^2 hypersurface with Gaussian curvature comparable to 1. For $R \gg 1$, cover the R^{-1}-neighborhood of S by finitely overlapping rectangles θ of dimensions $R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1}$. Suppose that the Fourier transform of f_θ is supported on θ and $f = \sum_\theta f_\theta$. The ℓ^p-decoupling inequality says that, for $2 \leq p \leq \frac{2(d+1)}{d-1}$ and any ball $B_R \subset \mathbb{R}^d$ of radius R,

$$\|f\|_{L^p(B_R)} \lessapprox R^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})} \Big(\sum_\theta \|f_\theta\|_{L^p}^p \Big)^{\frac{1}{p}},$$

where we denote by $A \lessapprox B$ the inequality $A \leq C_\epsilon R^\epsilon B$, which holds for any $\epsilon > 0$.

Gan and Wu [13] observed that the ℓ^p-decoupling estimate can be improved when the integration of f is taken over a subset $Y \subset B_R$ for the exponent $p = \frac{2d}{d-1}$ and showed that it yields an improved weak-type estimate (1.1). Here we give a refinement of their weighted ℓ^p-decoupling estimates for a wider range of p.

Theorem 1.3. *Let $p_d = \frac{2(d+1)}{d-1}$, $2 \leq p \leq p_d$, and*

$$\alpha(p) = \min\left(\frac{d+1}{2d}\Big(\frac{1}{p} - \frac{1}{p_d}\Big), \frac{1}{d-1}\Big(\frac{1}{2} - \frac{1}{p}\Big) \right). \tag{1.2}$$

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Then for any $Y \subset B_R$,

$$\|f\|_{L^p(Y)} \lessapprox (|Y|R^{-d})^{\alpha(p)} R^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})} \Big(\sum_{\theta} \|f_\theta\|_{L^p}^p \Big)^{\frac{1}{p}}. \tag{1.3}$$

Note that $\alpha(p) > 0$ for $2 < p < p_d$, so the factor $(|Y|R^{-d})^{\alpha(p)}$ represents a gain over the ℓ^p-decoupling inequality. Gan and Wu [13] proved (1.3) for the exponent $p = \frac{2d}{d-1}$ with $\alpha(p) = \frac{d+1}{2d}\left(\frac{1}{p} - \frac{1}{p_d}\right) = \frac{1}{16}$ for $d = 2$ and with $\alpha(p) = \frac{1}{3d-1}\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{1}{2d(3d-1)}$ for $d \geq 3$.

We next discuss the sharpness of Theorem 1.3. The exponent $\alpha(p)$ given in (1.2) is optimal when $\frac{2(d^2+2d-1)}{d^2+1} \leq p \leq p_d$, which is the range where $\alpha(p) = \frac{d+1}{2d}(\frac{1}{p} - \frac{1}{p_d})$. Indeed, if (1.3) holds for any $Y \subset B_R$, then $\alpha(p)$ must obey

$$\alpha(p) \leq \min\left(\frac{d+1}{2d}\Big(\frac{1}{p} - \frac{1}{p_d}\Big), \frac{1}{2} - \frac{1}{p} \right). \tag{1.4}$$

To see this, one takes $\widehat{f_\theta} = \frac{1}{|\theta|}\chi_\theta$ for a bump function χ_θ supported on θ and lets $Y = B_c(0)$ for sufficiently small $c \sim 1$ (cf. [13]). Then $|\sum_\theta f_\theta(x)| \sim \#\theta \sim R^{\frac{d-1}{2}}$ on Y. This example gives $\alpha(p) \leq \frac{d+1}{2d}\Big(\frac{1}{p} - \frac{1}{p_d}\Big)$. To see the other upper bound, it suffices to take $f = f_\theta$ for a fixed θ and take $Y = \theta^*$, the rectangle dual to θ.

Comparing (1.4) and (1.2), we see that the exponent $\alpha(p)$ given in Theorem 1.3 is optimal for $d = 2$ for all $2 \leq p \leq 6$. Note that $\alpha(p)$ takes the maximum value $\alpha(p) = \frac{1}{7}$ at $p = \frac{14}{5}$; we will use Theorem 1.3 with this particular p for our application to the Bochner-Riesz means. However, in dimensions $d \geq 3$, we do not currently have any reason to believe that the exponent $\alpha(p) = \frac{1}{d-1}\big(\frac{1}{2} - \frac{1}{p}\big)$ from (1.2) for p close to 2 is optimal. Moreover, if we assume further conditions on $Y \subset B_R$, then it is possible that (1.3) may hold with $\alpha(p)$, which does not obey (1.4); see [13] for a discussion of such a possibility when $|Y| \sim R^{1/2}$ and $d = 2$.

Our proof of Theorem 1.3 builds on [13], which is based on a broad-narrow analysis going back to the work of Bourgain and Guth [3] (see also [15]). In the broad case, we use the Bennett-Carbery-Tao multilinear restriction estimate [1] to exploit the transversality and combine it with the ℓ^p-decoupling theorem as in [13]. In the narrow case, we do induction on scales in a way similar to the one developed in the proof of the refined Strichartz estimate by Du, Guth and Li [8]. The novelty of our proof lies on the analysis of the narrow case. In order to perform induction on scales more efficiently, we utilize a weighted decoupling estimate (see Theorem 2.5), which originates from the refined and weighted decoupling estimates obtained by Guth, Iosevich, Ou and Wang [16] and Du, Ou, Ren and Zhang [9]. Given Theorem 1.3, Theorem 1.1 follows from the argument of [13].

Plan of the Paper

In Section 2, we prove a multilinear weighted decoupling estimate, which will be used to handle the broad part. We also state a weighted refined decoupling estimate to be used in the narrow part. In Section 3, we prove Theorem 1.3. In Section 4, we give an exposition of the paper [13] on the implication of weighted ℓ^p-decoupling estimates to the weak-type estimate (1.1) for the Bochner-Riesz means, completing the proof of Theorem 1.1. In addition, we give an improved bound in dimension $d = 3$. Throughout the paper, we denote $p_d = \frac{2(d+1)}{d-1}$ with the convention $p_1 = \infty$.

2. Weighted and refined decoupling inequalities

In this section, we prove a multilinear weighted refined decoupling inequality, Theorem 2.3, and then introduce a weighted refined decoupling estimate, Theorem 2.5, which will be used in the proof of Theorem 1.3.

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2.1. A multilinear weighted decoupling estimate

We first recall the multilinear restriction estimate [1]. Suppose that $U_j \subset \mathbb{S}^{d-1}$ for $j = 1, 2, \cdots, d$, such that for any $v_j \in U_j$,

$$|v_1 \wedge v_2 \wedge \cdots \wedge v_d| \geq \nu, \tag{2.1}$$

for some $0 < \nu \leq 1$.

Theorem 2.1 (Multilinear restriction). *For each $1 \leq j \leq d$, let $S_j \subset S$ be a cap such that the set $U_j \subset \mathbb{S}^{n-1}$ of vectors normal to S_j satisfies (2.1). Suppose that the Fourier transform of f_j is supported on the R^{-1}-neighborhood of S_j. Then*

$$\| \prod_{j=1}^{d} |f_j|^{1/d} \|_{L^{\frac{2d}{d-1}}(B_R)} \lesssim \nu^{-O(1)} R^{\epsilon} R^{-1/2} \prod_{j=1}^{d} \|f_j\|_{L^2}^{1/d}.$$

Next, we recall the refined decoupling estimate, which is stated in terms of wave packets. We recall the setup. Given $0 < \epsilon \ll 1$, let $\epsilon_1 = \epsilon^{100}$. For each θ, let $\mathbb{T}_\theta$ denote a collection of finitely overlapping tubes T covering $\mathbb{R}^d$ of length $\sim R^{1+\epsilon_1}$ and radius $\sim R^{(1+\epsilon_1)/2}$ with long axis orthogonal to θ. Let $\mathbb{T}(R) = \cup_\theta \mathbb{T}_\theta$. Given $T \in \mathbb{T}(R)$, let $\theta(T)$ denote the θ for which $T \in \mathbb{T}_\theta$.

We say that a function f_T is microlocalized to (T, θ) if f_T and $\widehat{f_T}$ are essentially supported on $2T$ and 2θ, respectively, in the sense that the L^p norms of the restrictions of f_T to $(2T)^c$ in the physical space and to $(2\theta)^c$ in the frequency space are both $O(R^{-100d}\|f_T\|_{L^p})$. Terms involving $O(R^{-100d})$ can be absorbed into the main term of the estimate, so we will ignore these terms in the following statements and proofs. We recall that a function f whose Fourier transform is supported on the R^{-1}-neighborhood of S admits a wave packet decomposition $f = \sum_\theta f_\theta = \sum_{T \in \mathbb{T}(R)} f_T$, where f_T is microlocalized to $(T, \theta(T))$.

We can now state the refined decoupling theorem, [16, Theorem 4.2].

Theorem 2.2 (Refined decoupling). *Let $\mathbb{W} \subset \mathbb{T}(R)$ and suppose that $f = \sum_{T \in \mathbb{W}} f_T$, where f_T is microlocalized to $(T, \theta(T))$. Let $Y \subset B_R$ be the union of a collection of $R^{1/2}$-cubes Q, each of which is contained in $3T$ for at most $M \geq 1$ tubes $T \in \mathbb{W}$. Then for $2 \leq p \leq p_d$,*

$$\|f\|_{L^p(Y)} \lessapprox M^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{T \in \mathbb{W}} \|f_T\|_{L^p}^p \right)^{1/p}.$$

The following is a multilinear refinement of Theorem 2.2.

Theorem 2.3. *For each $j = 1, 2, \cdots, d$, let $S_j \subset S$ be a cap such that the set $U_j \subset \mathbb{S}^{n-1}$ of vectors normal to S_j satisfies (2.1). Let $\mathbb{W}_j \subset \mathbb{T}(R)$ such that $2\theta(T)$ is contained in the $O(R^{-1})$-neighborhood of S_j for every $T \in \mathbb{W}_j$ and let $f_j = \sum_{T \in \mathbb{W}_j} f_T$, where f_T is microlocalized to $(T, \theta(T))$. Let $\{Q\}$ be a collection of $R^{1/2}$-cubes contained in B_R such that each cube in the collection is contained in $3T$ for at most M_j tubes $T \in \mathbb{W}_j$ for each $1 \leq j \leq d$. Let Y be a subset of $\cup_Q Q$. Then for $2 \leq p \leq p_d$,*

$$\left\| \prod_{j=1}^{d} |f_j|^{\frac{1}{d}} \right\|_{L^p(Y)} \lesssim \nu^{-O(1)} (|Y| R^{-d})^{\frac{d+1}{2d}(\frac{1}{p}-\frac{1}{p_d})} \prod_{j=1}^{d} \left[M_j^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_{T \in \mathbb{W}_j} \|f_T\|_{L^p}^p \Big)^{\frac{1}{p}} \right]^{\frac{1}{d}}.$$

For the proof, we interpolate Theorem 2.1 and Theorem 2.2. We remark that the use of *refined* decoupling is optional here as we shall use Theorem 2.3 with $M_j \lesssim R^{\frac{d-1}{2}}$. We note that Theorem 2.3 for the case $p = \frac{2d}{d-1}$ was essentially proved in [13]. Our proof is slightly more direct than the one given there in that it does not go through the multilinear Kakeya estimate.

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Proof. We prove the result for $2 \le p \le \frac{2d}{d-1}$; the result for the remaining p follows from interpolation with the statement for $p = p_d$, which is a consequence of Hölder and Theorem 2.2.

We may view f_T as essentially constant on T; this can be made precise by using the frequency localization property of f_T; cf. Section 3.1. By dyadic pigeonholing, we may reduce the matter to the case where, for each $j = 1, \cdots, d$, $|f_T| \sim \gamma_j$ on T for some $\gamma_j > 0$ for all $T \in \mathbb{W}_j$. By homogeneity, we may assume that $\gamma_j = 1$. Then for $1 \le q < \infty$,

$$\sum_{T \in \mathbb{W}_j} \|f_T\|_{L^q}^q \sim |T||\mathbb{W}_j|.$$

By Hölder, we get

$$\left\| \prod_{j=1}^d |f_j|^{\frac{1}{d}} \right\|_{L^p(Y)} \lesssim |Y|^{\frac{1}{p} - \frac{d-1}{2d}} \left\| \prod_{j=1}^d |f_j|^{\frac{1}{d}} \right\|_{L^{\frac{2d}{d-1}}(Y)}. \tag{2.2}$$

The multilinear restriction estimate and the L^2-orthogonality imply

$$\left\| \prod_{j=1}^d |f_j|^{\frac{1}{d}} \right\|_{L^{\frac{2d}{d-1}}(B_R)} \lessapprox R^{-\frac{1}{2}} \prod_{j=1}^d \|f_j\|_{L^2}^{\frac{1}{d}} \lessapprox R^{-\frac{1}{2}} \prod_{j=1}^d (|T||\mathbb{W}_j|)^{\frac{1}{2d}}. \tag{2.3}$$

However, by Hölder and the refined decoupling estimate, Theorem 2.2, we have

$$\begin{aligned} \left\| \prod_{j=1}^d |f_j|^{\frac{1}{d}} \right\|_{L^{\frac{2d}{d-1}}(Y)} &\lesssim |Y|^{\frac{d-1}{2d} - \frac{1}{p_d}} \left\| \prod_{j=1}^d |f_j|^{\frac{1}{d}} \right\|_{L^{p_d}(Y)} \\ &\lessapprox |Y|^{\frac{d-1}{2d} - \frac{1}{p_d}} \prod_{j=1}^d \left[M_j^{\frac{1}{2} - \frac{1}{p_d}} \left(\sum_{T \in \mathbb{W}_j} \|f_T\|_{L^{p_d}}^{p_d} \right)^{\frac{1}{p_d}} \right]^{\frac{1}{d}} \\ &\lessapprox |Y|^{\frac{d-1}{2d(d+1)}} \prod_{j=1}^d \left[M_j^{\frac{1}{d+1}} (|T||\mathbb{W}_j|)^{\frac{1}{p_d}} \right]^{\frac{1}{d}}. \end{aligned} \tag{2.4}$$

We combine the $(d+1)(\frac{1}{p} - \frac{1}{p_d})$-th power of (2.3) and $(d+1)(\frac{1}{2} - \frac{1}{p})$-th power of (2.4). Then we get

$$\left\| \prod_{j=1}^d |f_j|^{\frac{1}{d}} \right\|_{L^{\frac{2d}{d-1}}(Y)} \lessapprox R^{-\frac{d+1}{2}(\frac{1}{p} - \frac{1}{p_d})} |Y|^{\frac{d-1}{2d}(\frac{1}{2} - \frac{1}{p})} \prod_{j=1}^d \left[M_j^{\frac{1}{2} - \frac{1}{p}} \left(\sum_{T \in \mathbb{W}_j} \|f_T\|_{L^p}^p \right)^{\frac{1}{p}} \right]^{\frac{1}{d}}.$$

Finally, we combine this estimate with (2.2). □

2.2. A weighted refined decoupling estimate

Let $v(T) \in \mathbb{S}^{d-1}$ denote the direction of $T \in \mathbb{T}$. Following [9], for $R^{-1/2} \le r \le 1$ and $1 \le m \le d$, we say that $f = \sum_{T \in \mathbb{W}} f_T$ has (r, m)-concentrated frequencies if there exists a m-dimensional subspace $V \subset \mathbb{R}^d$ such that $\angle(v(T), V) \le r$ for any $T \in \mathbb{W}$.

When f has $(R^{-1/2}, m)$-concentrated frequencies, one can decouple f by ℓ^2-decoupling in dimension m (cf. [15, Lemma 9.3]). In the paper [9], it was observed that this continues to hold for the refined decoupling estimate as well.

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Theorem 2.4 ([9, Theorem 1.2(a)][1]). *Let $\mathbb{W} \subset \mathbb{T}(R)$ and suppose that $f = \sum_{T\in\mathbb{W}} f_T$, where f_T is microlocalized to $(T, \theta(T))$. Let $Y \subset B_R$ be the union of a collection of $R^{1/2}$-cubes Q, each of which is contained in $3T$ for at most $M \geq 1$ tubes $T \in \mathbb{W}$. Suppose that f has $(R^{-1/2}, m)$ concentrated frequencies for some $1 \leq m \leq d$. Then for $2 \leq p \leq p_m$,*

$$\|f\|_{L^p(Y)} \lessapprox M^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{T\in\mathbb{W}} \|f_T\|_{L^p}^p\right)^{1/p}.$$

A consequence of Theorem 2.4 is a weighted refined decoupling estimate with weights $w : Y \to [0, 1]$ such that $\int_Q w \lesssim R^{\alpha/2}$ holds for any $R^{1/2}$-cube Q in Y; see [9, Theorem 1.2(b)]). Here we state a slightly more general version, which does not require such an assumption.

Theorem 2.5 (Weighted refined decoupling). *Let $1 \leq m \leq d$ and $2 \leq p \leq p_m$. Suppose that $f = \sum_{T\in\mathbb{W}} f_T$, where $\mathbb{W} \subset \mathbb{T}(R)$ and each f_T is microlocalized to $(T, \theta(T))$. Assume that f has $(R^{-1/2}, m)$-concentrated frequencies. Let $\{Q\}$ be a collection of disjoint $R^{1/2}$-cubes in B_R such that each Q is contained in $3T$ for at most $M \geq 1$ tubes $T \in \mathbb{W}$. Let Y be a subset of $\cup_Q Q$. Let $w : Y \to [0, 1]$ and $w(T) := \int_T w$. Then*

$$\|f\|_{L^p(w)} \lessapprox \left(\max_{T\in\mathbb{W}} \frac{w(3T)}{|T|}\right)^{\frac{1}{p}-\frac{1}{p_m}} M^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{T\in\mathbb{W}} \|f_T\|_{L^p}^p\right)^{1/p}.$$

The power of $\max_{T\in\mathbb{W}} \frac{w(3T)}{|T|}$ represents a gain over the refined decoupling estimate. Theorem 2.5 essentially follows from the proof of [9, Theorem 1.2(b)]). We include a sketch of the proof for the sake of completeness.

Proof. By dyadic pigeonholing, we may assume that each $R^{1/2}$-cube Q in the collection is contained in $3T$ for $\sim M$ tubes $T \in \mathbb{W}$. By Hölder,

$$\|f\|_{L^p(w)} \lesssim w(Y)^{\frac{1}{p}-\frac{1}{p_m}} \|f\|_{L^{p_m}(Y)}.$$

Next, we use Theorem 2.4 with exponent $p = p_m$ and then replace the $\ell^{p_m}(L^{p_m})$ norm back to $\ell^p(L^p)$ as in the proof of Theorem 2.3, which gives

$$\|f\|_{L^p(w)} \lessapprox \left(\frac{w(Y)M}{|\mathbb{W}||T|}\right)^{\frac{1}{p}-\frac{1}{p_m}} M^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{T\in\mathbb{W}} \|f_T\|_{L^p}^p\right)^{1/p}. \tag{2.5}$$

Then

$$w(Y)M \sim \sum_Q \sum_{T\in\mathbb{W}:Q\subset 3T} w(Q) = \sum_{T\in\mathbb{W}} \sum_{Q:Q\subset 3T} w(Q) \leq |\mathbb{W}| \max_{T\in\mathbb{W}} w(3T).$$

Combining this inequality and (2.5) finishes the proof. □

3. Broad-narrow analysis: Proof of Theorem 1.3

In this section, we prove the following theorem, which implies Theorem 1.3 by the wave packet decomposition.

[1]The case $m = 1$ is omitted in [9], but it trivially holds as $O(1)$ boxes θ are involved.

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Theorem 3.1. *Suppose that* $\mathbb{W} \subset \mathbb{T}(R)$ *and* $f = \sum_{T \in \mathbb{W}} f_T$, *where* f_T *is microlocalized to* $(T, \theta(T))$. *Let* $p_d = \frac{2(d+1)}{d-1}$ *and* $Y \subset B_R$. *Then for* $2 \le p \le p_d$ *and* $\alpha(p)$ *defined in* (1.2),

$$\|f\|_{L^p(Y)} \lessapprox (|Y|R^{-d})^{\alpha(p)} R^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})} \left(\sum_{T \in \mathbb{W}} \|f_T\|_{L^p}^p \right)^{\frac{1}{p}}. \tag{3.1}$$

We turn to the proof of Theorem 3.1. We decompose f at a scale dictated by the parameter $K = R^{\epsilon_1} = R^{\epsilon^{100}}$. Consider a cover of $\mathcal{S}$ by rectangles τ of dimensions $K^{-1} \times \cdots \times K^{-1} \times K^{-2}$ such that for each $T \in \mathbb{W}$, there exist at least one and at most $O(1)$ rectangles τ containing $\theta(T)$. We fix one such $\tau = \tau(T)$ for each $T \in \mathbb{W}$. We let $\mathbb{W}_\tau = \{T \in \mathbb{W} : \tau(T) = \tau\}$ and $f_\tau = \sum_{\mathbb{T} \in \mathbb{W}_\tau} f_T$, so that $f = \sum_\tau f_\tau$.

For each lattice K^2-cube B intersecting Y, define the 'significant set'

$$S(B) = \{\tau : \|f_\tau\|_{L^p(B \cap Y)} \ge \frac{1}{100 \# \tau} \|f\|_{L^p(B \cap Y)}\}.$$

Then the definition of $S(B)$ ensures that for each K^2-cube B,

$$\|f\|_{L^p(B \cap Y)} \sim \| \sum_{\tau \in S(B)} f_\tau \|_{L^p(B \cap Y)}.$$

We say that a lattice K^2-cube B intersecting Y is narrow if there exists a $(d-1)$-dimensional subspace $V \subset \mathbb{R}^d$ such that for every $\tau \in S(B)$,

$$\angle(G(\tau), V) \le K^{-1},$$

where $G(\tau) \in \mathbb{S}^{d-1}$ denotes the direction normal to τ. Otherwise, we say that B is broad. If B is broad, then there exist $\tau_1, \tau_2, \cdots, \tau_d \in S(B)$ such that

$$|G(\tau_1) \wedge G(\tau_2) \wedge \cdots \wedge G(\tau_d)| \gtrsim K^{-(d-1)}. \tag{3.2}$$

Let $\mathcal{B}_{\text{broad}}$ denote the collection of broad K^2-cubes and $\mathcal{B}_{\text{narrow}}$ denote the collection of narrow K^2-cubes. We let

$$Y_{\text{broad}} = \cup_{B \in \mathcal{B}_{\text{broad}}} B \cap Y, \quad Y_{\text{narrow}} = \cup_{B \in \mathcal{B}_{\text{narrow}}} B \cap Y.$$

We say that we are in the broad case if $\|f\|_{L^p(Y_{\text{broad}})} \ge \|f\|_{L^p(Y_{\text{narrow}})}$. Otherwise, we say that we are in the narrow case. We handle each case in the following subsections.

3.1. Broad case

In this subsection, we denote by $A \lessapprox B$ expressions of the form $A \le K^{O(1)} B$; the loss of $K^{O(1)}$ is harmless as long as it is at most $R^{\epsilon/2}$.

In the broad case, we have $\|f\|_{L^p(Y)} \lesssim \|f\|_{L^p(Y_{\text{broad}})}$. Let $B \in \mathcal{B}_{\text{broad}}$. Then there exist $\tau_1, \tau_2, \cdots, \tau_d \in S(B)$ satisfying (3.2). We fix such a d-tuple and denote $\bar{\tau}(B) = (\tau_1, \cdots, \tau_d)$. Let

$$\Gamma = \{\bar{\tau}(B) : B \in \mathcal{B}_{\text{broad}}\}.$$

Since $\#\Gamma \lesssim K^{O(1)}$, by dyadic pigeonholing, there exist $\bar{\tau} = (\tau_1, \cdots, \tau_d) \in \Gamma$ and a sub-collection of broad K^2-cubes $\mathcal{B}^1_{\text{broad}}$ such that $\bar{\tau}(B) = \bar{\tau}$ for all $B \in \mathcal{B}^1_{\text{broad}}$ and

$$\|f\|_{L^p(Y_{\text{broad}})} \lessapprox \|f\|_{L^p(Y^1_{\text{broad}})}, \text{ where } Y^1_{\text{broad}} = \cup_{B \in \mathcal{B}^1_{\text{broad}}} B \cap Y.$$

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Let $B \in \mathcal{B}^1_{\text{broad}}$. Since $\tau_j \in S(B)$ for each j, we have

$$\|f\|^p_{L^p(B\cap Y)} \lessapprox \prod_{j=1}^d \|f_{\tau_j}\|^{\frac{p}{d}}_{L^p(B\cap Y)}.$$

Using the Fourier support property of f_j, we morally have the following reverse Hölder inequality (cf. [10]):

$$\prod_{j=1}^d \|f_{\tau_j}\|^{\frac{p}{d}}_{L^p(B\cap Y)} \lessapprox \|\prod_{j=1}^d |f_{\tau_j}|^{1/d}\|^p_{L^p(B\cap Y)}. \tag{3.3}$$

We shall first assume (3.3), and then make it precise later. Summing (3.3) over $B \in \mathcal{B}^1_{\text{broad}}$, we get

$$\|f\|_{L^p(Y_{\text{broad}})} \lessapprox \|\prod_{j=1}^d |f_{\tau_j}|^{1/d}\|_{L^p(Y^1_{\text{broad}})}.$$

By using Theorem 2.3 with the bound $M_j \lesssim R^{\frac{d-1}{2}}$ and $\mathbb{W}_{\tau_j} \subset \mathbb{W}$, we obtain

$$\begin{aligned}\|f\|_{L^p(Y_{\text{broad}})} &\lessapprox (|Y|R^{-d})^{\frac{d+1}{2d}(\frac{1}{p}-\frac{1}{p_d})} \prod_{j=1}^d \left[M_j^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{T\in\mathbb{W}_{\tau_j}} \|f_T\|^p_{L^p} \right)^{1/p} \right]^{1/d} \\ &\lessapprox (|Y|R^{-d})^{\frac{d+1}{2d}(\frac{1}{p}-\frac{1}{p_d})} R^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})} \left(\sum_{T\in\mathbb{W}} \|f_T\|^p_{L^p} \right)^{1/p}, \end{aligned} \tag{3.4}$$

which implies the claimed estimate (3.1) as $\alpha(p) \le \frac{d+1}{2d}(\frac{1}{p} - \frac{1}{p_d})$.

Now we make the step (3.3) rigorous following [17]. Since f_{τ_j} has compact Fourier support, we may write $f_{\tau_j} = f_{\tau_j} * \psi$ for some ψ with compact Fourier support. Therefore, we have $|f_{\tau_j}| \lesssim |f_{\tau_j}| * \Psi_1$, where $\Psi_\rho(x) = \rho(\rho + |x|)^{-(d+1)}$. Moreover,

$$|f_{\tau_j}|^r \lesssim_r |f_{\tau_j}|^r * \Psi_1 \tag{3.5}$$

for all $r > 0$. This follows from Hölder for $r > 1$. The case $0 < r < 1$ can be proved by using Bernstein's inequality; see [17, Lemma 5.9]. We also note that $\Psi_1(x) = K^{2d}\Psi_{K^2}(K^2x) \lessapprox \Psi_{K^2}(x)$. Using (3.5) with $r = p/d$, we get

$$|f_{\tau_j}|^{\frac{p}{d}} \lessapprox |f_{\tau_j}|^{\frac{p}{d}} * \Psi_{K^2}. \tag{3.6}$$

We are going to use the following two properties of Ψ_{K^2}; its L^1 norm is comparable to 1, and it is locally constant on balls of radius K^2. The latter implies that $|f_{\tau_j}|^{\frac{p}{d}} * \Psi_{K^2}(x) \sim |f_{\tau_j}|^{\frac{p}{d}} * \Psi_{K^2}(y)$ whenever $|x - y| \lesssim K^2$. Therefore, by (3.6),

$$\begin{aligned}\prod_{j=1}^d \left(\int_{B\cap Y} |f_{\tau_j}|^p(x)dx \right)^{1/d} &\lessapprox \prod_{j=1}^d \left(\int_{B\cap Y} \left(|f_{\tau_j}|^{\frac{p}{d}} * \Psi_{K^2}(x) \right)^d dx \right)^{1/d} \\ &\sim |B\cap Y| \prod_{j=1}^d |f_{\tau_j}|^{\frac{p}{d}} * \Psi_{K^2}(y), \text{ for every } y \in B \\ &\sim \int_{B\cap Y} \prod_{j=1}^d |f_{\tau_j}|^{\frac{p}{d}} * \Psi_{K^2}(x)dx. \end{aligned}$$

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Let $f_{\tau_j,y_j}(x) = f_{\tau_j}(x - y_j)$. Summing this estimate over $B \in \mathcal{B}^1_{\text{broad}}$, we get

$$\begin{aligned}\|f\|^p_{L^p(Y_{\text{broad}})} &\lessapprox \int_Y \prod_{j=1}^d |f_{\tau_j}|^{\frac{p}{d}} * \Psi_{K^2}(x)dx \\ &= \int_{(\mathbb{R}^d)^d} \| \prod_{j=1}^d |f_{\tau_j,y_j}|^{1/d} \|^p_{L^p(Y)} \prod_{j=1}^d \Psi_{K^2}(y_j) dy_1 \cdots dy_d.\end{aligned}$$

By Theorem 2.3, we obtain (3.4).

3.2. Narrow case

In the narrow case, we use induction on the scale R. In this subsection, we denote by $A \lessapprox B$ expressions of the form $A \le (\log R)^{O(1)} B$ and keep track of powers of K.

We start by considering the base case $R \sim 1$. Since $|\mathbb{W}| \sim 1$, we have

$$\|f\|_{L^p(Y)} \lesssim |Y|^{1/p} \max_{T\in\mathbb{W}} \|f_T\|_{L^\infty} \lesssim |Y|^{1/p} \max_{T\in\mathbb{W}} \|f_T\|_{L^p},$$

which is better than the claimed estimate since $\alpha(p) \le 1/p$ and $|Y| \lesssim 1$.

Let $B \in \mathcal{B}_{\text{narrow}}$. Recall that $\|f\|_{L^p(B\cap Y)} \sim \| \sum_{\tau\in S(B)} f_\tau \|_{L^p(B\cap Y)}$. For each τ, we use a wave packet decomposition

$$f_\tau = \sum_{T_1\in\mathbb{T}_\tau} f_{T_1},$$

where f_{T_1} is microlocalized to (T_1, τ). Note that $\mathbb{T}_\tau \subset \mathbb{T}(K^2)$ and each $T_1 \in \mathbb{T}_\tau$ has length $\sim K^{2(1+\epsilon_1)}$ and radius $\sim K^{(1+\epsilon_1)}$. For dyadic $R^{-1000d} \le \eta \le |T_1|$, let $\mathbb{T}_{\tau,\eta}$ denote the collection of $T_1 \in \mathbb{T}_\tau$ such that $|3T_1 \cap Y| \sim \eta$. The contribution of tubes $T_1 \in \mathbb{T}_\tau$ such that $|3T_1 \cap Y| \lesssim R^{-1000d}$ is negligible; it can be absorbed into the main term by crude estimates. By dyadic pigeonholing, there exists dyadic η such that

$$\sum_{B\in\mathcal{B}_{\text{narrow}}} \|f\|^p_{L^p(B\cap Y)} \lessapprox \sum_{B\in\mathcal{B}_{\text{narrow}}} \| \sum_{\tau\in S(B)} \sum_{T_1\in\mathbb{T}_{\tau,\eta}} f_{T_1} \|^p_{L^p(B\cap Y)}.$$

We fix this η and let $\mathbb{T}(K^2; B)$ denote the collection of $T_1 \in \cup_{\tau\in S(B)} \mathbb{T}_{\tau,\eta}$ such that $2T_1$ intersects B. Then

$$\|f\|^p_{L^p(Y)} \lessapprox \sum_{B\in\mathcal{B}_{\text{narrow}}} \| \sum_{T_1\in\mathbb{T}(K^2;B)} f_{T_1} \|^p_{L^p(B\cap Y)}.$$

For each $B \in \mathcal{B}_{\text{narrow}}$, $\sum_{T_1\in\mathbb{T}(K^2;B)} f_{T_1}$ has $(K^{-1}, d-1)$-concentrated frequencies and $|S(B)| \lesssim K^{d-2}$. Therefore, by the weighted refined decoupling Theorem 2.5,

$$\| \sum_{T_1\in\mathbb{T}(K^2;B)} f_{T_1} \|_{L^p(B\cap Y)} \lesssim K^{\epsilon_1} (\eta K^{-(d+1)})^{\frac{1}{p}-\frac{1}{p_{d-1}}} K^{(d-2)(\frac{1}{2}-\frac{1}{p})} \left(\sum_{T_1\in\mathbb{T}(K^2;B)} \|f_{T_1}\|^p_{L^p} \right)^{\frac{1}{p}}. \tag{3.7}$$

For each τ, we cover B_R by parallelepiped $\square$ of dimensions $K^{-1}R \times \cdots \times K^{-1}R \times R$ with the long axis perpendicular to τ. We denote the collection of $\square$ by $\mathbb{B}_\tau$ and $\mathbb{B} = \cup_\tau \mathbb{B}_\tau$. In addition, given $\square \in \mathbb{B}$, we denote by $\tau(\square)$ the τ for which $\square \in \mathbb{B}_\tau$.

Let $\mathbb{T}_\square$ denote the collection of all $T_1 \in \cup_{B\in\mathcal{B}_{\text{narrow}}} \mathbb{T}(K^2; B)$ such that $2T_1$ intersects $\square$ and $\tau(T_1) = \tau(\square)$. Let $Y_\square = \cup_{T_1\in\mathbb{T}_\square} 2T_1$. Let $\mathbb{W}_\square$ denote the collection of $T \in \mathbb{W}_{\tau(\square)}$ such that $2T$ intersects $Y_\square$ and define $f_\square = \sum_{T\in\mathbb{W}_\square} f_T$. We record here that

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$$\sum_{\square}\sum_{T\in\mathbb{W}_{\square}}\|f_T\|_{L^p}^p \lesssim \sum_{T\in\mathbb{W}}\|f_T\|_{L^p}^p.$$

Moreover, $f_{\square}$ is microlocalized to $(\square,\tau(\square))$, and we have

$$\sum_{B\in\mathcal{B}_{\mathrm{narrow}}}\sum_{T_1\in\mathbb{T}(K^2;B)}\|f_{T_1}\|_{L^p}^p \lesssim \sum_{\square}\sum_{T_1\in\mathbb{T}_{\square}}\|f_{T_1}\|_{L^p}^p \lesssim \sum_{\square}\|f_{\tau(\square)}\|_{L^p(Y_{\square})}^p \lesssim \sum_{\square}\|f_{\square}\|_{L^p(Y_{\square})}^p.$$

Summing the p-th power of (3.7) over narrow $B\in\mathcal{B}_{\mathrm{narrow}}$, we get

$$\|f\|_{L^p(Y)} \lessapprox K^{\epsilon_1}(\eta K^{-(d+1)})^{\frac{1}{p}-\frac{1}{p_{d-1}}}K^{(d-2)(\frac{1}{2}-\frac{1}{p})}\left(\sum_{\square}\|f_{\square}\|_{L^p(Y_{\square})}^p\right)^{1/p}. \tag{3.8}$$

Next, we apply the induction hypothesis to $f_{\square}$ after a parabolic rescaling. Without loss of generality, suppose that $\tau(\square)$ is contained in $B^{d-1}_{K^{-1}}(0)\times B^1_{K^{-2}}(0)$. Let $L(x_1,\cdots,x_d)=(Kx_1,\cdots,Kx_{d-1},K^2x_d)$. Then we may apply the induction hypothesis to $f_{\square}\circ L$ over $L^{-1}(Y_{\square})\subset L^{-1}(\square)$ at the scale $R_1=R/K^2$:

$$\|f_{\square}\|_{L^p(Y_{\square})} \lesssim R_1^{\epsilon/2}(|L^{-1}(Y_{\square})|R_1^{-d})^{\alpha(p)}(R_1)^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}\left(\sum_{T\in\mathbb{W}_{\square}}\|f_T\|_{L^p}^p\right)^{1/p}. \tag{3.9}$$

For every $\square$,

$$|\mathbb{T}_{\square}|\eta \lesssim \sum_{T_1\in\mathbb{T}_{\square}}|3T_1\cap Y| \lesssim |Y|.$$

Thus, we have

$$|L^{-1}(Y_{\square})| = K^{-(d+1)}|Y_{\square}| \sim K^{-(d+1)}|T_1||\mathbb{T}_{\square}| \lesssim K^{O(\epsilon_1)}\eta^{-1}|Y|.$$

By combining this estimate, (3.8) and (3.9), we obtain

$$\begin{aligned}\|f\|_{L^p(Y)} \lessapprox & R^{\epsilon/2}K^{O(\epsilon_1)}(\eta K^{-(d+1)})^{\frac{1}{p}-\frac{1}{p_{d-1}}}K^{(d-2)(\frac{1}{2}-\frac{1}{p})}(\eta^{-1}K^{2d})^{\alpha(p)}K^{-(d-1)(\frac{1}{2}-\frac{1}{p})}\\ & (|Y|R^{-d})^{\alpha(p)}R^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}\left(\sum_{T\in\mathbb{W}}\|f_T\|_{L^p}^p\right)^{1/p}\\ \lessapprox & R^{\epsilon/2}K^{O(\epsilon_1)}(\eta K^{-(d+1)})^{\frac{1}{p}-\frac{1}{p_{d-1}}-\alpha(p)}K^{(d-1)\alpha(p)-(\frac{1}{2}-\frac{1}{p})}\\ & (|Y|R^{-d})^{\alpha(p)}R^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}\left(\sum_{T\in\mathbb{W}}\|f_T\|_{L^p}^p\right)^{1/p}.\end{aligned}$$

Note that

$$(\eta K^{-(d+1)})^{\frac{1}{p}-\frac{1}{p_{d-1}}-\alpha(p)} = O(K^{O(\epsilon_1)})$$

since $\eta\lesssim|T_1|$ and $\frac{1}{p}-\frac{1}{p_{d-1}}-\alpha(p)\geq 0$. Moreover, $(d-1)\alpha(p)-(\frac{1}{2}-\frac{1}{p})\leq 0$ by the choice of $\alpha(p)$. Therefore, we have

$$\|f\|_{L^p(Y)} \lesssim R^{\epsilon}(|Y|R^{-d})^{\alpha(p)}R^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}\left(\sum_{T\in\mathbb{W}}\|f_T\|_{L^p}^p\right)^{1/p},$$

and the induction closes.

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4. Convergence of Bochner-Riesz means: Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Given Theorem 3.1, it essentially follows from the argument of Gan and Wu [13]. The main goal of this section is to give an exposition of their argument. We fix $j \in \mathbb{N}$.

Definition 4.1. A multiplier m_j is type j if

$$|\partial^\alpha m_j(\xi)| \lesssim_{\alpha,N} 2^{j|\alpha|}(1+2^j|1-|\xi||)^{-N} \tag{4.1}$$

and

$$\widehat{m_j}(x) = 2^{-j\frac{d+1}{2}} e^{\pm 2\pi i|x|} a(2^{-j}x)$$

for some $a \in C_0^\infty$ supported on the annulus $|x| \sim 1$.

We use the notation $\widehat{m(D)f}(\xi) = m(\xi)\widehat{f}(\xi)$. The Bochner-Riesz means $S_t^\lambda f$ can be decomposed as

$$S_t^\lambda f = m_0(D/t)f + \sum_{j=1}^{\infty} 2^{-\lambda j} m_j(D/t)f,$$

where m_j is type j for $j \geq 1$ and $m_0(D/t)f$ is dominated by the Hardy-Littlewood maximal function uniformly in t (see [21, 22]). Tao [22] reduced the weak-type estimate (1.1) to a bound on the maximal function $f \mapsto \sup_{t\sim 1}|m_j(D/t)f|$. To be specific, if

$$\| \sup_{t\sim 1} |m_j(D/t)f| \|_{L^p(B(0,C2^j))} \lessapprox 2^{j\lambda_0}\|f\|_{L^p} \tag{4.2}$$

holds some λ_0 for type j multipliers m_j and test functions f for all $j \geq 1$, then the weak-type estimate (1.1) holds for $\lambda > \lambda_0$. Here and in the following, we denote by $A \lessapprox B$ the inequality $A \leq C_\epsilon 2^{\epsilon j} B$ or $A \leq C_\epsilon R^\epsilon B$, which holds for any $\epsilon > 0$.

4.1. Linearization

In this subsection, we reduce Theorem 1.1 to the following.

Proposition 4.2. *Fix a collection of disjoint intervals $\{I_1, \cdots, I_{2^j}\}$ of length $\sim 2^{-j}$ for which $\cup_l I_l = [1/2, 1]$. Let $c_l \in I_l$ and let $\{F_1, \cdots, F_{2^j}\}$ be a partition of $B(0, C2^j)$. For a type j multiplier m_j, let T denote the linear operator defined as*

$$Tf(x) = \sum_{l=1}^{2^j} 1_{F_l}(x) m_j(D/c_l)f(x).$$

Then

$$\|Tf\|_{L^{\frac{86}{57}}} \lessapprox 2^{\frac{9}{86}j}\|f\|_{L^{\frac{86}{57}}},$$

where the implicit constant is independent of the choice of $\{c_l, F_l\}$.

We postpone the proof of Proposition 4.2 to following subsections and first look at its implications. Clearly, T in Proposition 4.2 is dominated by the maximal function $f \mapsto \sup_{t\sim 1}|m_j(D/t)f|$. Conversely, we have the following:

Lemma 4.3. *Let T be as in Proposition 4.2. Assume that for any type j multiplier m_j, there exists a constant $A_j > 0$ independent of $\{F_l, c_l\}_{1\leq l\leq 2^j}$ such that*

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$$\|Tf\|_{L^p} \le A_j \|f\|_{L^p}.$$

Then for every type j multiplier m_j,

$$\|\sup_{t\sim 1} |m_j(D/t)f|\|_{L^p(B(0,C2^j))} \lesssim A_j \|f\|_{L^p}.$$

Proof. Let m_j be a type j multiplier. We linearize and discretize the maximal operator. Fix a measurable function $t : B(0,CR) \to [1/2,1]$. For each $l = 1, \cdots, 2^j$, define $F_l = \{x \in B(0,CR) : t(x) \in I_l\}$ so that

$$|m_j(D/t(x))f(x)|^p \le \sum_l 1_{F_l}(x) \sup_{t\in I_l} |m_j(D/t)f(x)|^p.$$

Let a_l denote the left endpoint of the interval I_l. By the fundamental theorem of calculus and Hölder, we have

$$\sup_{t\in I_l} |m_j(D/t)f(x)|^p \lesssim |m_j(D/a_l)f(x)|^p + |I_l|^{p-1} \int_{I_l} |\partial_t [m_j(D/t)f(x)]|^p dt.$$

For each $1 \le k \le d$, let $m_{j,k}(\xi) = |I_l|\xi_k \partial_k m_j(\xi)$. Then $m_{j,k}$ is a type j multiplier and

$$\sup_{t\in I_l} |m_j(D/t)f(x)|^p \lesssim |m_j(D/a_l)f(x)|^p + |I_l|^{-1} \int_{I_l} \sum_{1\le k\le d} |m_{j,k}(D/t)f(x)|^p dt.$$

Choose $b_l \in I_l$ for which

$$\int_{F_l} \sum_{1\le k\le d} |m_{j,k}(D/b_l)f|^p \gtrsim \sup_{t\in I_l} \int_{F_l} \sum_{1\le k\le d} |m_{j,k}(D/t)f|^p.$$

Then

$$\int_{F_l} \sup_{t\in I_l} |m_j(D/t)f|^p \lesssim \int_{F_l} |m_j(D/a_l)f|^p + \int_{F_l} \sum_{1\le k\le d} |m_{j,k}(D/b_l)f|^p.$$

By summing the inequality over l and using the assumption, we get

$$\int_{B(0,C2^j)} |m_j(D/t(x))f|^p \lesssim A_j^p \|f\|_{L^p}^p.$$

Since this holds for any measurable function t, the claim is proved. □

By Lemma 4.3 and Tao's reduction discussed earlier, we have reduced Theorem 1.1 to Proposition 4.2, which will be proved in the following subsections.

4.2. Decompositions

In this subsection, we work on $\mathbb{R}^d$ for $d \ge 2$. We fix $\{c_l, F_l\}_{l=1}^{2^j}$ as in Proposition 4.2 and let

$$Tf(x) = \sum_l 1_{F_l}(x) m_j(D/c_l)f(x).$$

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Given any exponent $1 \le p_0 \le \infty$, by dyadic pigeonholing, there exist $\mathcal{J} \subset \{1, 2, \cdots, 2^j\}$ and $2^{-100dj} \le \gamma \le 2^{jd}$ for which $|F_l| \sim \gamma$ for all $l \in \mathcal{J}$[2] and

$$\|Tf\|_{L^{p_0}} \lessapprox \|T_{\mathcal{J}} f\|_{L^{p_0}}, \tag{4.3}$$

where

$$T_{\mathcal{J}} f(x) = \sum_{l \in \mathcal{J}} 1_{F_l}(x) m_j(D/c_l) f(x).$$

By real interpolation, it suffices to work with $f = 1_E$, the characteristic function of $E \subset B(0, 2^j)$. Given $0 < \epsilon \ll 1$, let $0 < \epsilon_1 \ll \epsilon$. For a technical reason, we work with the parameter $R = 2^{(1-\epsilon_1)j}$, but one may identify R and 2^j as $R \approx 2^j$. The main result of this subsection is the following.

Proposition 4.4 (cf. [13]). *Let $E \subset B(0, 2^j)$ and $2 \le p \le p_d$. Let $R = 2^{(1-\epsilon_1)j}$. Suppose that (3.1) holds with the exponent $\alpha(p)$. Then*

$$|\{x : |T_{\mathcal{J}} 1_E(x)| > \alpha\}| \lessapprox \alpha^{-2} R^{\frac{1}{2p-3}(\frac{d+1}{4}p - \frac{d-1}{2})-1} (\gamma R^{-d})^{\frac{p}{2p-3}\alpha(p)} |E|^{\frac{3p-5}{2p-3}}. \tag{4.4}$$

Remark 4.5. When $d = 2$ and $p = 6$, Proposition 4.4 gives the $L^{18/13}$ bound obtained in [19]. However, when $d \ge 3$ and $p = p_d$, Proposition 4.4 gives sufficient conditions for the weak-type estimate (1.1), which are no better than the classical sufficient condition $\lambda > (d-1)(\frac{1}{p} - \frac{1}{2})$.

Proposition 4.4 follows from the argument from [13]. We sketch the proof for the sake of exposition. We decompose E according to the density. For a parameter $\beta_2 > 0$ to be chosen, let $\{Q\}$ denote the collection of maximal dyadic cubes such that

$$|E \cap Q| \ge \beta_2 l(Q),$$

where $l(Q)$ is the side-length of Q. Let $E_2 = \cup_Q (E \cap Q)$ and $E_1 = E \setminus E_2$. Then

$$|E_1 \cap B| \le \beta_2 l(B)$$

for any dyadic cube B; if otherwise, there is a dyadic cube Q in the collection containing B leading to the contradiction $|E_1 \cap B| = 0$. These sets E_1 and E_2 are the low and the high density parts of E, respectively. Note that $T_{\mathcal{J}} 1_E = T_{\mathcal{J}} 1_{E_1} + T_{\mathcal{J}} 1_{E_2}$.

For the high-density part E_2, we have the following estimate from a local L^2-estimate, which goes back to [23].

Lemma 4.6 [13, Lemma 3.4]. *For any $\alpha > 0$,*

$$|\{x : |T_{\mathcal{J}} 1_{E_2}(x)| \ge \alpha\} \lesssim \alpha^{-2} 2^{-j} \beta_2^{-1} |E|^2.$$

For the low-density part E_1, we use a wave-packet decomposition. Let $R = 2^{j(1-\epsilon_1)}$, so that m_j is essentially supported on the R^{-1}-neighborhood of $\mathbb{S}^{d-1}$. Cover $\mathbb{S}^{d-1}$ by finitely overlapping caps $\{\theta\}$ of diameter $R^{-1/2}$ and let $\{\chi_\theta\}$ be an associated smooth partition of unity. Define $m_{j,\theta}(\xi) = m_j(\xi)\chi_\theta(\xi/|\xi|)$ so that $m_{j,\theta}(\xi/c_l)$ is essentially a bump function supported on a box θ_l of dimensions $R^{-1/2} \times \cdots R^{-1/2} \times R^{-1}$.

Let f_{θ_l} denote $m_{j,\theta}(D/c_l)f$. Let $\mathbb{T}_\theta$ denote the collection of tubes as in Section 2. Then f_{θ_l} is morally constant on each $T \in \mathbb{T}_\theta$. Indeed, we may choose an L^1-normalized non-negative function Ψ_θ such that $\Psi_\theta(x) \sim \Psi_\theta(y)$ for all $x, y \in T$ and

$$|f_{\theta_l}(x)| \lesssim 2^{O(\epsilon_1 j)} |f_{\theta_l}| * \Psi_\theta(x)$$

[2] Those F_l with $|F_l| \le 2^{-100dj}$ are ignored here as these can be handled by crude estimates.

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up to a negligible error term (cf. Section 3.1). Therefore, $|f_{\theta_l}| * \Psi_\theta(x) \sim |f_{\theta_l}| * \Psi_\theta(y)$ for all $x, y \in T$. Let $f = \chi_{E_1}$. Given $\beta_1 > 0$, we let

$$\mathbb{T}_{\theta_l,small} = \{T \in \mathbb{T}_\theta : \||f_{\theta_l}| * \Psi_\theta\|_{L^\infty(T)} < \beta_1\}$$

and $\mathbb{T}_{\theta_l,large} = \mathbb{T}_\theta \setminus \mathbb{T}_{\theta_l,small}$. Let $\{\chi_T\}_{T\in\mathbb{T}_\theta}$ denote a smooth partition of unity associated with the covering $\mathbb{T}_\theta$. Note that $f_{\theta_l}\chi_T$ is microlocalized to (T, θ_l) for each $T \in \mathbb{T}_\theta$. We define

$$T_{small}(f) = \sum_{l\in\mathcal{J}} 1_{F_l} \sum_\theta \sum_{T\in\mathbb{T}_{\theta_l,small}} f_{\theta_l}\chi_T.$$

We define $T_{large}(f)$ similarly, which gives the decomposition $T_{\mathcal{J}} f = T_{small}(f) + T_{large}(f)$.

Lemma 4.7 [13, Lemmas 3.2 and 3.3]. *Suppose that the weighted ℓ^p-decoupling estimate (3.1) holds with the exponent $\alpha(p)$. Then*

$$\begin{aligned}\int |T_{large}(1_{E_1})| &\lessapprox \beta_1^{-1}\beta_2 R^{-1/2}|E_1|,\\ \int |T_{small}(1_{E_1})|^p dx &\lessapprox \left[(\gamma R^{-d})^{\alpha(p)} R^{\frac{d-1}{2}(\frac12-\frac1p)}\right]^p \beta_1^{p-2}|E_1|. \end{aligned} \tag{4.5}$$

The assumption on the weighted ℓ^p-decoupling inequality (3.1) is utilized only for the bound (4.5). For the sake of completeness, we give the proof.

Proof of (4.5). By the weighted ℓ^p-decoupling estimate (3.1),

$$\begin{aligned}\int |T_{small}(f)|^p dx &= \sum_{l\in\mathcal{J}} \int_{F_l} |\sum_\theta \sum_{T\in\mathbb{T}_{\theta_l,small}} f_{\theta_l}\chi_T|^p dx\\ &\lessapprox \left[(\gamma R^{-d})^{\alpha(p)} R^{\frac{d-1}{2}(\frac12-\frac1p)}\right]^p \sum_{l\in\mathcal{J}}\sum_\theta \sum_{T\in\mathbb{T}_{\theta_l,small}} \|f_{\theta_l}\chi_T\|_{L^p}^p.\end{aligned}$$

Let $f = 1_{E_1}$. Note that

$$\|f_{\theta_l}\chi_T\|_{L^\infty} \lessapprox \||f_{\theta_l}| * \Psi_\theta\|_{L^\infty(T)} \lessapprox \beta_1.$$

Therefore,

$$\sum_{l\in\mathcal{J}}\sum_\theta \sum_{T\in\mathbb{T}_{\theta_l,small}} \|f_{\theta_l}\chi_T\|_{L^p}^p \lessapprox \beta_1^{p-2} \sum_{l\in\mathcal{J}}\sum_\theta \sum_{T\in\mathbb{T}_\theta} \|f_{\theta_l}\chi_T\|_{L^2}^2 \lessapprox \beta_1^{p-2}\|f\|_{L^2}^2$$

by the L^2-orthogonality, giving the claimed bound. □

We may now prove Proposition 4.4.

Proof of Proposition 4.4. By Lemma 4.7,

$$\begin{aligned}|\{x : |T_{\mathcal{J}} 1_{E_1}(x)| > \alpha\}| \le& |\{x : |T_{large}(1_{E_1})(x)| > \alpha/2\}| + |\{x : |T_{small}(1_{E_1})(x)| > \alpha/2\}|\\ \lessapprox& \alpha^{-1}\beta_1^{-1}\beta_2 R^{-1/2}|E| + \alpha^{-p}(\gamma R^{-d})^{p\alpha(p)} R^{\frac{d-1}{2}(\frac{p}{2}-1)}\beta_1^{p-2}|E|.\end{aligned}$$

We take β_1 so that the two terms are equal, which gives

$$|\{x : |T_{\mathcal{J}} 1_{E_1}(x)| > \alpha\}| \lessapprox \alpha^{-2}\beta_2^{\frac{p-2}{p-1}} R^{\frac{d-1}{4}\frac{p-2}{p-1}-\frac12+\frac{1}{2(p-1)}} (\gamma R^{-d})^{\frac{p}{p-1}\alpha(p)}|E_1|. \tag{4.6}$$

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By combining Lemma 4.6 and (4.6),

$$\begin{aligned}|\{x : |T_{\mathcal{J}} 1_E(x)| > \alpha\}| \le & |\{x : |T_{\mathcal{J}} 1_{E_1}(x)| > \alpha/2\}| + |\{x : |T_{\mathcal{J}} 1_{E_2}(x)| > \alpha/2\}| \\ \lessapprox & \alpha^{-2}\beta_2^{\frac{p-2}{p-1}} R^{\frac{d-1}{4}\frac{p-2}{p-1}-\frac{1}{2}+\frac{1}{2(p-1)}} (\gamma R^{-d})^{\frac{p}{p-1}\alpha(p)} |E| + \alpha^{-2} R^{-1} \beta_2^{-1} |E|^2.\end{aligned}$$

Finally, it suffices to choose β_2 so that the two terms are equal:

$$\beta_2^{-1} = |E|^{-\frac{p-1}{2p-3}} R^{\frac{1}{2p-3}(\frac{d+1}{4}p - \frac{d-1}{2})} (\gamma R^{-d})^{\frac{p}{2p-3}\alpha(p)}. \qquad \square$$

4.3. Proof of Proposition 4.2

By applying Proposition 4.4 and Theorem 3.1 with $p = \frac{14}{5}$ and $\alpha(p) = \frac{1}{7}$, we obtain the restricted weak-type estimate

$$|\{x : |T_{\mathcal{J}} 1_E(x)| > \alpha\}| \lessapprox \alpha^{-2} R^{-\frac{5}{13}} (\gamma R^{-2})^{\frac{2}{13}} |E|^{\frac{17}{13}},$$

or equivalently,

$$\|T_{\mathcal{J}} f\|_{L^{2,\infty}} \lessapprox R^{-\frac{5}{26}} (\gamma R^{-2})^{\frac{1}{13}} \|f\|_{L^{\frac{26}{17},1}}.$$

By Hölder's inequality (for Lorentz spaces), we get

$$\|T_{\mathcal{J}} f\|_{L^{\frac{26}{17}}} \lessapprox 2^{2(\frac{17}{26}-\frac{1}{2})j} 2^{-\frac{5}{26}j} (\gamma R^{-2})^{\frac{1}{13}} \|f\|_{L^{\frac{26}{17},1}} = 2^{\frac{3}{26}j} (\gamma R^{-2})^{\frac{1}{13}} \|f\|_{L^{\frac{26}{17},1}}. \tag{4.7}$$

In order to use the gain of $(\gamma R^{-2})^{\frac{1}{13}}$ in (4.7), we interpolate it with a $L^{4/3}$-estimate. We recall the $L^{4/3}$-estimate due to Carleson and Sjörin [6] and Córdoba [7]:

$$\|m_j(D) f\|_{L^{4/3}} \lessapprox \|f\|_{L^{4/3}}.$$

Since the $L^{4/3}$-operator norm of $m_j(D/t)$ is independent of $t > 0$, we have

$$\|T_{\mathcal{J}} f\|_{L^{4/3}} = \left(\sum_{l \in \mathcal{J}} \|m_j(D/c_l) f\|_{L^{4/3}(F_l)}^{4/3} \right)^{3/4} \lessapprox |\mathcal{J}|^{3/4} \|f\|_{L^{4/3}}. \tag{4.8}$$

We interpolate (4.7) and (4.8) so that we may apply $\gamma R^{-2}|\mathcal{J}| \lessapprox 1$. To be specific, take $\theta = \frac{39}{43}$ so that $\theta \cdot \frac{1}{13} = (1-\theta)\frac{3}{4}$. Then $\theta\frac{17}{26} + (1-\theta)\frac{3}{4} = \frac{57}{86}$. Real interpolation of (4.7) and (4.8) gives

$$\|T_{\mathcal{J}} f\|_{L^{\frac{86}{57}}} \lessapprox 2^{\frac{9}{86}j} \|f\|_{L^{\frac{86}{57}}},$$

which completes the proof of Proposition 4.2 by (4.3).

Remark 4.8. Consider the example $f(x) = e^{2\pi i x_2} \psi(x_1, 2^{-j/2} x_2)$ from [22]. In this example, if $x_1 \sim x_2 \sim 2^j$, then

$$\sup_{t \sim 1} |m_j(D/t) f(x)| \sim |m_j(D/t(x)) f(x)| \sim 2^{-j},$$

where $t(x) = |x|/x_2$. Thus, we may choose $\{c_l, F_l\}$ so that $\gamma \sim 2^j$, $|\mathcal{J}| \sim 2^j$, $\|T_{\mathcal{J}} f\|_{L^p} \gtrsim 2^{2j(\frac{1}{p}-\frac{1}{2})}$ and $\|f\|_{L^q} \sim 2^{j\frac{1}{2q}}$. In particular,

$$\|T_{\mathcal{J}} f\|_{L^{4/3}} \gtrsim |\mathcal{J}|^{1/8} \|f\|_{L^{4/3}}.$$

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This suggests that there may be significant room for improvement in the estimate (4.8). Any improvement to (4.8) would advance our knowledge of the almost everywhere convergence of the Bochner-Riesz means.

4.4. An improved bound for $d = 3$

Theorem 4.9. *Let* $d = 3$, $1 < p < 2$ *and*

$$\lambda > \max\left(\frac{174}{85p} - \frac{89}{85}, \frac{146}{77}\left(\frac{1}{p} - \frac{1}{2}\right)\right).$$

Then the weak-type estimate (1.1) holds. Consequently, $\lim_{t\to\infty} S_t^\lambda f(x) = f(x)$ *for almost every* $x \in \mathbb{R}^3$ *for any* $f \in L^p(\mathbb{R}^3)$.

Theorem 4.9 gives a small improvement to the result obtained in [13]; when $p = \frac{3}{2}$, (1.1) holds for $\lambda > \frac{27}{85} = 0.317\cdots$, improving the sufficient condition $\lambda > \frac{107}{325} = 0.329\cdots$ from [13].

Proof. Let $E \subset B(0, 2^j)$. By taking $p = \frac{14}{5}$ and $\alpha(p) = \frac{1}{14}$ in Proposition 4.4, we get

$$|\{x : |T_{\mathcal{J}} 1_E(x)| > \alpha\}| \lessapprox \alpha^{-2} R^{-\frac{4}{13}} (\gamma R^{-3})^{\frac{1}{13}} |E|^{\frac{17}{13}},$$

or equivalently,

$$\|T_{\mathcal{J}} f\|_{L^{2,\infty}} \lessapprox 2^{-\frac{2}{13}j} (\gamma R^{-3})^{\frac{1}{26}} \|f\|_{L^{\frac{26}{17},1}}.$$

By Hölder's inequality, we get

$$\|T_{\mathcal{J}} f\|_{L^{\frac{26}{17}}} \lessapprox 2^{\frac{4}{13}j} (\gamma R^{-3})^{\frac{1}{26}} \|f\|_{L^{\frac{26}{17},1}}. \tag{4.9}$$

However, by the sharp L^p estimate for the Bochner-Riesz means at the exponent $p = \frac{13}{9}$ due to Wu [24] (see also [14]), there holds

$$\|m_j(D) f\|_{L^{\frac{13}{9}}} \lessapprox 2^{(3(\frac{9}{13} - \frac{1}{2}) - \frac{1}{2})j} \|f\|_{L^{\frac{13}{9}}}.$$

Therefore,

$$\|T_{\mathcal{J}} f\|_{L^{\frac{13}{9}}} = \left(\sum_{l \in \mathcal{J}} \|m_j(D/c_l) f\|_{L^{\frac{13}{9}}(F_l)}^{\frac{13}{9}}\right)^{\frac{9}{13}} \lessapprox 2^{\frac{j}{13}} |\mathcal{J}|^{\frac{9}{13}} \|f\|_{L^{\frac{13}{9}}}. \tag{4.10}$$

We interpolate (4.9) and (4.10) so that we may apply $\gamma|\mathcal{J}|R^{-3} \lessapprox 1$. This gives

$$\|T_{\mathcal{J}} f\|_{L^{\frac{13\cdot 19}{9\cdot 18}}} \lessapprox 2^{\frac{73}{247}j} \|f\|_{L^{\frac{13\cdot 19}{9\cdot 18}}}.$$

This implies, by (4.3) and Lemma 4.3, that

$$\|\sup_{t\sim 1} |m_j(D/t) f|\|_{L^{\frac{247}{162}}(B(0,C2^j))} \lessapprox 2^{\frac{73}{247}j} \|f\|_{L^{\frac{247}{162}}}. \tag{4.11}$$

By interpolation with the standard L^1 and L^2 estimates, we get for $1 \le p \le 2$,

$$\|\sup_{t\sim 1} |m_j(D/t) f|\|_{L^p(B(0,C2^j))} \lessapprox \max(2^{j(\frac{174}{85p} - \frac{89}{85})}, 2^{j\frac{146}{77}(\frac{1}{p} - \frac{1}{2})}) \|f\|_{L^p}.$$

By the reduction of Tao, this implies Theorem 4.9. □

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