Canad. J. Math. 2025, pp. 1–20
http://dx.doi.org/10.4153/S0008414X25101661
© The Author(s), 2025. Published by Cambridge University Press on behalf of Canadian Mathematical Society

# Inequalities for the sequences with unified form

Li-Mei Dou, Zhen-Yu Gao, and Larry X.W. Wang

*Abstract.* We consider a class of sequence $c(n)$ with the following asymptotic form:

$$c(n) \sim \frac{C}{n^{\kappa}} \exp\left(\sum_{\lambda\in\mathcal{S}} A_\lambda n^\lambda\right) \sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{n^\mu} \qquad (n\to\infty).$$

We give criteria for the Turán inequality of any order, the double Turán inequality, and the Laguerre inequality of any order of $c(n)$ for sufficiently large $n$. We also give the companion inequalities for the Turán inequality and the Laguerre inequality of any order for $c(n)$. As applications, we will show that the numbers of commuting $\ell$-tuples in $S_n$, the partition without sequence, the plane partition, the partition into $k$-gonal numbers, the finite-dimensional representations of groups $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$, and the coefficients of infinite product generating functions asymptotically satisfy these inequalities. Some of them settle open problems proposed by Bringmann, Franke, and Heim.

## 1 Introduction

The main purpose of this article is to consider sequences $c(n)$ of real numbers of the form

$$c(n) \sim \frac{C}{n^{\kappa}} \exp\left(\sum_{\lambda\in\mathcal{S}} A_\lambda n^\lambda\right) \sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{n^\mu} \qquad (n\to\infty). \tag{1.1}$$

Here, $\kappa\in\mathbb{R}$, $\mathcal{S}\subset\mathbb{Q}^+\cap(0,1)$ is finite, $\mathcal{T}\subset\mathbb{Q}_0^+$ is finite, $C, A_\lambda, \beta_\mu\in\mathbb{R}$ with $\beta_0 = 1$ if $0\in\mathcal{T}$. This general form was introduced by Bringmann, Franke, and Heim [5], including the asymptotical expression of the numbers of the unimodal sequences [3], the strongly unimodal sequences [23], the partition without sequence [18], the plane partition [28], the partition into $k$-gonal numbers [2], the finite-dimensional representations of groups $\mathfrak{su}(3)$ [4] and $\mathfrak{so}(5)$ [2] and the coefficients of the infinite product generating functions [5]. Bringmann, Franke, and Heim [5] proved that $c(n)$ satisfies the Turán inequality for sufficiently large $n$ and put forward an open problem concerning with the Turán inequalities of higher order.

Received by the editors November 19, 2024; revised May 13, 2025; accepted June 9, 2025.
Published online on Cambridge Core November 20, 2025.
This work was supported by the National Science Foundation of China (Grant No. 12171254).
AMS subject classification: **11P82, 11E45, 11M41, 05A20.**
Keywords: Turán inequality, Laguerre inequality, symmetric group, plane partition, finite representation of group.

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Recall that a real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{1.2}$$

is said to be in the Laguerre–Pólya class, denoted by $\psi(x) \in \mathcal{LP}$, if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} (1 + x/x_k)\, e^{-x/x_k},$$

where $c$, $\beta$, and $x_k$ are real numbers, $\alpha \geq 0$, $m$ is a nonnegative integer, and $\sum x_k^{-2} < \infty$. The $\mathcal{LP}$ class has attracted much attention in view of its connection with Riemann hypothesis.

The Riemann $\Xi$-function is defined as

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

Following [8], the Riemann $\Xi$-function can be written in Taylor series form as

$$F(z) := \frac{1}{8}\Xi\left(\frac{z}{2}\right) = \sum_{n=0}^{\infty} \frac{\gamma(n)}{n!} z^{2n}.$$

We say that a polynomial with real coefficients is hyperbolic if all of its zeros are real. The $d$-th associated Jensen polynomial with shift $n$ of an arbitrary sequence $\{\alpha(0), \alpha(1), \alpha(2), \ldots\}$ of real numbers is the polynomial

$$J_{\alpha}^{d,n}(x) = \sum_{k=0}^{d} \binom{d}{k} \alpha_{k+n} x^k. \tag{1.3}$$

The hyperbolicity of degree $d$ Jensen polynomial associated with the sequence $\{\alpha(n)\}$ is equivalent to the Turán inequality for $\{\alpha(n)\}$ of order $d$. Pólya and Schur proved that the Riemann hypothesis holds if and only if the function $F(z)$ belongs to $\mathcal{LP}$ class, i.e., having only real zeros, or equivalently, all Jensen polynomials having only real zeros.

Griffin, Ono, Rolen, and Zagier [12] proved that under some conditions, the degree $d$ Jensen polynomials associated with the sequence $\{\alpha(n)\}$ are hyperbolic for any $d$ and sufficiently large $n$. In fact, they established the following theorem.

***Theorem 1.1*** (Griffin, Ono, Rolen, and Zagier) *Let $\alpha(n)$, $A(n)$, and $\delta(n)$ be sequences of positive real numbers with $\delta(n)$ tending to zero and satisfying*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = A(n)j - \delta(n)^2 j^2 + \sum_{i=3}^{d} g_i(n) j^i + o\left(\delta(n)^d\right) \quad \text{as} \quad n \to \infty$$

*for some $d \geq 1$, all $0 \leq j \leq d$ and some $g_i(n) = o\left(\delta(n)^i\right)$. Then, we have*

$$\lim_{n\to\infty}\left(\frac{\delta(n)^{-d}}{\alpha(n)} J_{\alpha}^{d,n}\left(\frac{\delta(n)X - 1}{\exp(A(n))}\right)\right) = H_d(X)$$

*uniformly for $X$ in any compact subset of $\mathbb{R}$, where $H_d(X)$ are the Hermite polynomials.*

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Recall that a sequence $\{a_n\}_{n\geq 0}$ is called to satisfy the Turán inequality, if

$$T(a_n) := {a_n}^2 - a_{n-1}a_{n+1} \geq 0.$$

We say that $\{a_n\}_{n\geq 0}$ satisfies the double Turán inequality, if

$$T(T(a_n)) := ({a_n}^2 - a_{n-1}a_{n+1})^2 - ({a_{n-1}}^2 - a_{n-2}a_n)({a_{n+1}}^2 - a_n a_{n+2}) \geq 0.$$

A sequence $\{a_n\}_{n\geq 1}$ is called to satisfy the Laguerre inequality of order $m$ [26], if

$$L_m(a_n) := \frac{1}{2}\sum_{k=0}^{2m}(-1)^{k+m}\binom{2m}{k}a_{n+k}a_{n+2m-k} \geq 0. \tag{1.4}$$

The Laguerre inequalities have profound connections to the Riemann hypothesis and the Laguerre–Pólya class (see [9, 11, 15, 21, 22]). Recently, the Laguerre inequalities for the Maclaurin coefficients of the Riemann $\Xi$-function, the partition function, and some celebrated sequences have been extensively studied (see [1, 7, 10, 25–27]). Denote the sum of the terms with positive (negative, respectively) coefficients of $L_m(a_n)$ by $L_m^+(a_n)$ ($L_m^-(a_n)$, respectively). One can see that (1.4) equals to $\frac{L_m^+(a_n)}{L_m^-(a_n)} \geq 1$. It is also interested to consider the companion Laguerre inequalities, i.e., give an upper bound for $\frac{L_m^+(a_n)}{L_m^-(a_n)}$. For more recent work, see [1, 7, 10].

In this article, we will investigate these inequalities for $c(n)$. This article is organized as follows. In Section 2, we shall prove that as $n \to \infty$, $c(n)$ satisfies the Turán inequality of any order and the double Turán inequality. In Section 3, we will show that $c(n)$ satisfies the companion Turán inequality for sufficiently large $n$. In Section 4, we will show that $c(n)$ asymptotically satisfies the Laguerre inequality of any order and its companion version. In Section 5, as applications, we will show that the numbers of commuting $\ell$-tuples in $S_n$, the partition without sequence, the plane partition, the partition into $k$-gonal numbers, the finite-dimensional representations of groups $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$, and the coefficients of infinite product generating functions asymptotically satisfy these inequalities.

## 2 Turán-type inequalities

In this section, we will prove that as $n \to \infty$, $c(n)$ meets the requires of the Turán inequality of any order and the double Turán inequality.

To prove that $c(n)$ fulfills the Turán inequality of any order, we just need to verify $c(n)$ conform to the conditions of Theorem 1.1.

***Theorem 2.1*** *Assume that $c(n)$ is a sequence of the form (1.1), let $\mathcal{S} := \{\lambda_1, \ldots, \lambda_s\}$, where $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ and $\mathcal{T} := \{\mu_1, \ldots, \mu_t\}$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t \geq 0$, $\kappa$, $C$, $A_\lambda$, $\beta_\mu \in \mathbb{R}$ with $\beta_0 = 1$ if $0 \in \mathcal{T}$. If $A_{\lambda_1} > 0$, then for sufficiently large $n$, $c(n)$ fulfills the Turán inequality of any order.*

**Proof** By the Taylor expansion of $\log x$, we have that

$$\log(x) = x - 1 - \frac{(x-1)^2}{2} + \cdots + (-1)^{r-1}\frac{(x-1)^r}{r} + o(x^r),\ x \to 1. \tag{2.1}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

From (1.1), we get

$$\log\frac{c(n+j)}{c(n)} \sim \sum_{\lambda\in\mathcal{S}} A_\lambda (n+j)^\lambda - \sum_{\lambda\in\mathcal{S}} A_\lambda n^\lambda + \log\frac{n^\kappa}{(n+j)^\kappa} + \log\frac{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n+j)^\mu}}{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}}. \tag{2.2}$$

We proceed to estimate $\sum_{\lambda\in\mathcal{S}} A_\lambda\left((n+j)^\lambda - n^\lambda\right)$3, $\log\frac{n^\kappa}{(n+j)^\kappa}$ and $\log\frac{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n+j)^\mu}}{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}}$, respectively.

First, we estimate $\sum_{\lambda\in\mathcal{S}} A_\lambda\left((n+j)^\lambda - n^\lambda\right)$. Since

$$(n+j)^\lambda - n^\lambda = \sum_{i=1}^{N}\binom{\lambda}{i} n^{-i+\lambda} j^i + o(n^{-N+\lambda}) \qquad (N \geq r). \tag{2.3}$$

Thus,

$$\sum_{\lambda\in\mathcal{S}} A_\lambda\left((n+j)^\lambda - n^\lambda\right) = \sum_{\lambda\in\mathcal{S}} A_\lambda \sum_{i=1}^{N}\binom{\lambda}{i} n^{-i+\lambda} j^i + o(n^{-N+\lambda}) \qquad (N \geq r). \tag{2.4}$$

Next, we turn to estimating $\log\frac{n^\kappa}{(n+j)^\kappa}$. Since

$$\frac{n^\kappa}{(n+j)^\kappa} = \sum_{i=0}^{N}\binom{-\kappa}{i} n^{-i} j^i + O(n^{-N-1}), \tag{2.5}$$

by (2.1), we obtain

$$\log\frac{n^\kappa}{(n+j)^\kappa} = \sum_{i=1}^{N} f_i(\kappa) n^{-i} j^i + O(n^{-N-1}), \tag{2.6}$$

where $f_i(\kappa) = \frac{(-1)^i\kappa}{i}$.

Finally, we only need to estimate $\log\frac{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n+j)^\mu}}{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}}$. Since

$$\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu} = \sum_{i=1}^{t}\beta_{\mu_i} n^{-\mu_i}, \quad \sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n+j)^\mu} = \sum_{i=1}^{t}\beta_{\mu_i}(n+j)^{-\mu_i}, \tag{2.7}$$

from (2.1), we deduce that

$$\log\frac{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n+j)^\mu}}{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}} = \sum_{i=1}^{N} G_i(n) j^i + o(G_N(n)), \tag{2.8}$$

where $G_i(n)$ are certain polynomials of $n$ with degree no more than $-i$ and $|G_i(n)| \leq \frac{1}{i}\mu_1^i n^{-i}$.

Combining (2.4), (2.6), and (2.8), as $n\to\infty$, we get that

$$\begin{aligned}\log\frac{c(n+j)}{c(n)} &\sim \sum_{\lambda\in\mathcal{S}} A_\lambda \sum_{i=1}^{N}\binom{\lambda}{i} n^{-i+\lambda} j^i + \sum_{i=1}^{N} f_i(\kappa) n^{-i} j^i + \sum_{i=1}^{N} G_i(n) j^i + o(n^{-N+\lambda}) \\ &=: \sum_{i=1}^{N}\left(\sum_{\lambda\in\mathcal{S}} B_{\lambda,i}(n) + F_i(n)\right) j^i + o(n^{-N+\lambda}),\end{aligned} \tag{2.9}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

where $B_{\lambda,i}(n) = A_\lambda\binom{\lambda}{i}n^{-i+\lambda}$, $F_i(n) = f_i(\kappa)n^{-i} + G_i(n)$. Employing (2.6) and (2.8), we get $|F_i(n)| \le \frac{1}{i}(\mu_1^i + \kappa)n^{-i}$.

Let $A(n) = \sum_{\lambda\in\mathcal{S}} B_{\lambda,1}(n) + F_1(n)$, $\delta(n) = \sqrt{-\sum_{\lambda\in\mathcal{S}} B_{\lambda,2}(n) - F_2(n)}$, $g_i(n) = \sum_{\lambda\in\mathcal{S}} B_{\lambda,i}(n) + F_i(n)(i \ge 3)$. Then, $i \ge 3$, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{g_i(n)}{\delta(n)^i} &= \lim_{n\to\infty}\frac{\sum_{\lambda\in\mathcal{S}} B_{\lambda,i}(n) + F_i(n)}{\left(-\sum_{\lambda\in\mathcal{S}} B_{\lambda,2}(n) - F_2(n)\right)^{\frac{i}{2}}}\\ &= \lim_{n\to\infty}\frac{\sum_{\lambda\in\mathcal{S}} A_\lambda\binom{\lambda}{i}n^{-i+\lambda}}{\left(-\sum_{\lambda\in\mathcal{S}} A_\lambda\binom{\lambda}{2}n^{-2+\lambda}\right)^{\frac{i}{2}}}\\ &= \lim_{n\to\infty}\frac{\sum_{\lambda\in\mathcal{S}} A_\lambda\binom{\lambda}{i}}{\left(-\sum_{\lambda\in\mathcal{S}} A_\lambda\binom{\lambda}{2}\right)^{\frac{i}{2}}}n^{\lambda(1-\frac{i}{2})}\\ &= 0.\end{aligned} \tag{2.10}$$

In summary, we have that $c(n)$ satisfies the conditions of Theorem 1.1. Thus, for sufficiently large $n$, $c(n)$ satisfies the Turán inequality of any order. ■

Now, we turn to proving the double Turán inequality for $c(n)$.

***Theorem 2.2*** *For sufficiently large $n$, $c(n)$ satisfies the double Turán inequality, i.e.,*

$$\begin{aligned}T(T(c(n))) &= (c(n)^2 - c(n-1)c(n+1))^2\\ &\quad - (c(n-1)^2 - c(n-2)c(n))(c(n+1)^2 - c(n)c(n+2)) \ge 0.\end{aligned} \tag{2.11}$$

**Proof** Following the framework in the proof of Theorem 1.3 in [5], we claim that for all $\lambda \in \mathcal{S}$,

$$\exp\left(A_\lambda\left(2n^\lambda - (n+1)^\lambda - (n-1)^\lambda\right)\right) = 1 - \frac{\gamma_{\lambda,1}}{n^{2-\lambda}} + \frac{\gamma_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right),$$

where $\gamma_{\lambda,1} = 2A_\lambda\binom{\lambda}{2}$,

$$\exp\left(A_\lambda\left(n^\lambda + (n+2)^\lambda - 2(n+1)^\lambda\right)\right) = 1 + \frac{\mu_{\lambda,1}}{n^{2-\lambda}} + \frac{\mu_{\lambda,2}}{n^{3-\lambda}} + \frac{\mu_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right),$$

where $\mu_{\lambda,1} = 2A_\lambda\binom{\lambda}{2}$ and $\mu_{\lambda,2} = 6A_\lambda\binom{\lambda}{3}$,

$$\exp\left(A_\lambda\left(n^\lambda + (n-2)^\lambda - 2(n-1)^\lambda\right)\right) = 1 + \frac{\omega_{\lambda,1}}{n^{2-\lambda}} + \frac{\omega_{\lambda,2}}{n^{3-\lambda}} + \frac{\omega_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right),$$

where $\omega_{\lambda,1} = 2A_\lambda\binom{\lambda}{2}$ and $\omega_{\lambda,2} = -6A_\lambda\binom{\lambda}{3}$.

Then, we get that

$$\begin{aligned}&\exp\left(\sum_{\lambda\in\mathcal{S}} A_\lambda\left(2n^\lambda - (n-1)^\lambda - (n+1)^\lambda\right)\right)\\ &= \prod_{\lambda\in\mathcal{S}}\left(1 - \frac{\gamma_{\lambda,1}}{n^{2-\lambda}} + \frac{\gamma_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right)\right)\end{aligned}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

$$
\begin{aligned}
&= 1 - \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} - \cdots - \frac{\gamma_{\lambda_2,1}}{n^{2-\lambda_2}} + \frac{\gamma_{\lambda_1,1}^2}{2n^{4-2\lambda_1}} + o\left(n^{-4+2\lambda_1}\right), \\
&\quad \exp\left(\sum_{\lambda\in\mathcal{S}} A_\lambda \left(n^\lambda + (n+2)^\lambda - 2(n+1)^\lambda\right)\right) \\
&= \prod_{\lambda\in\mathcal{S}} \left(1 + \frac{\mu_{\lambda,1}}{n^{2-\lambda}} + \frac{\mu_{\lambda,2}}{n^{3-\lambda}} + \frac{\mu_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right)\right) \\
&= 1 + \frac{\mu_{\lambda_1,1}}{n^{2-\lambda_1}} + \cdots + \frac{\mu_{\lambda_s,1}}{n^{2-\lambda_s}} + \frac{\mu_{\lambda_1,2}}{n^{3-\lambda_1}} + \cdots + \frac{\mu_{\lambda_{s'},2}}{n^{3-\lambda_{s'}}} + \frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}} + o\left(n^{-4+2\lambda_1}\right), \\
&\quad \exp\left(\sum_{\lambda\in\mathcal{S}} A_\lambda \left(n^\lambda + (n-2)^\lambda - 2(n-1)^\lambda\right)\right) \\
&= \prod_{\lambda\in\mathcal{S}} \left(1 + \frac{\omega_{\lambda,1}}{n^{2-\lambda}} + \frac{\omega_{\lambda,2}}{n^{3-\lambda}} + \frac{\omega_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right)\right) \\
&= 1 + \frac{\omega_{\lambda_1,1}}{n^{2-\lambda_1}} + \cdots + \frac{\omega_{\lambda_s,1}}{n^{2-\lambda_s}} + \frac{\omega_{\lambda_1,2}}{n^{3-\lambda_1}} + \cdots + \frac{\omega_{\lambda_{s'},2}}{n^{3-\lambda_{s'}}} + \frac{\omega_{\lambda_1,1}^2}{2n^{4-2\lambda_1}} + o\left(n^{-4+2\lambda_1}\right).
\end{aligned} \tag{2.12}
$$

Here, $\lambda_{s'} > 2\lambda_1 - 1$.

A direct calculation gives that for $n > 1$,

$$
\begin{aligned}
\frac{1}{n^{2\kappa}} &= \frac{1}{(n-1)^\kappa (n+1)^\kappa}\left(1 + \sum_{j\geq 2} \frac{a_j}{n^j}\right), \\
\frac{1}{n^\kappa (n+2)^\kappa} &= (n+1)^{-2\kappa}\left(1 + \sum_{j\geq 2} \frac{b_j}{n^j}\right), \\
\frac{1}{n^\kappa (n-2)^\kappa} &= (n-1)^{-2\kappa}\left(1 + \sum_{j\geq 2} \frac{c_j}{n^j}\right).
\end{aligned} \tag{2.13}
$$

According to (2.12) and (2.13), we have

$$
T(T(c(n))) \sim C^4 \frac{\exp\left(2\sum_{\lambda\in\mathcal{S}} A_\lambda\left((n-1)^\lambda + (n+1)^\lambda\right)\right)}{(n-1)^{2\kappa}(n+1)^{2\kappa}}\left(C'^2 - A'B'\right),
$$

where

$$
\begin{aligned}
A' &= \left(\sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{(n+1)^\mu}\right)^2 \\
&\quad - \left(1 + \frac{\mu_{\lambda_1,1}}{n^{2-\lambda_1}} + \cdots + \frac{\mu_{\lambda_s,1}}{n^{2-\lambda_s}} + \frac{\mu_{\lambda_1,2}}{n^{3-\lambda_1}} + \cdots + \frac{\mu_{\lambda_{s'},2}}{n^{3-\lambda_{s'}}} + \frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}} + o\left(n^{-4+2\lambda_1}\right)\right) \\
&\quad \times \left(1 + O\left(n^{-2}\right)\right) \sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{n^\mu} \sum_{\nu\in\mathcal{T}} \frac{\beta_\nu}{(n+2)^\nu},
\end{aligned}
$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

$$
\begin{aligned}
B' &= \left(\sum_{\mu\in\mathfrak{J}}\frac{\beta_\mu}{(n-1)^\mu}\right)^2 \\
&\quad -\left(1+\frac{\omega_{\lambda_1,1}}{n^{2-\lambda_1}}+\cdots+\frac{\omega_{\lambda_s,1}}{n^{2-\lambda_s}}+\frac{\omega_{\lambda_1,2}}{n^{3-\lambda_1}}+\cdots+\frac{\omega_{\lambda_{s'},2}}{n^{3-\lambda_{s'}}}+\frac{\omega_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}+o\left(n^{-4+2\lambda_1}\right)\right) \\
&\quad \times\left(1+O\left(n^{-2}\right)\right)\sum_{\mu\in\mathfrak{J}}\frac{\beta_\mu}{n^\mu}\sum_{\nu\in\mathfrak{J}}\frac{\beta_\nu}{(n-2)^\nu}, \\
C' &= \sum_{\mu\in\mathfrak{J}}\frac{\beta_\mu}{(n-1)^\mu}\sum_{\nu\in\mathfrak{J}}\frac{\beta_\nu}{(n+1)^\nu} \\
&\quad -\left(1-\frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}}-\cdots-\frac{\gamma_{\lambda_s,1}}{n^{2-\lambda_s}}-\cdots+\frac{\gamma_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}+o\left(n^{-4+2\lambda_1}\right)\right) \\
&\quad \times\left(1+O\left(n^{-2}\right)\right)\left(\sum_{\mu\in\mathfrak{J}}\frac{\beta_\mu}{n^\mu}\right)^2.
\end{aligned}
$$

As $\beta_0 = 1$, we take $A \bmod \left(o\left(n^{-4+2\lambda_1}\right)\right)$,

$$
\begin{aligned}
&\sum_{\substack{\mu,\nu\in\mathfrak{J}\\ 0\le\mu+\nu\le 4-2\lambda_1}}\frac{\beta_\mu\beta_\nu}{(n+1)^{\mu+\nu}}-\left(\sum_{\substack{\mu,\nu\in\mathfrak{J}\\ 0\le\mu+\nu\le 4-2\lambda_1}}\frac{\beta_\mu\beta_\nu}{n^\mu(n+2)^\nu}+\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}\sum_{\substack{\mu,\nu\in\mathfrak{J}\\ 0\le\mu+\nu\le 2-2\lambda_1+\lambda_i}}\frac{\beta_\mu\beta_\nu}{n^\mu(n+2)^\nu}\right. \\
&\quad \left.+\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}}+\frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}\right) \\
&= 2\sum_{\substack{\mu\in\mathfrak{J}\\ 0\le\mu\le 2-\lambda_1}}\beta_\mu^2\left(\frac{1}{(n+1)^{2\mu}}-\frac{1}{n^\mu(n+2)^\mu}\right) \\
&\quad +\sum_{\substack{\mu\in\mathfrak{J}\\ 0\le\mu<\nu\\ 1\le\mu+\nu\le 4-2\lambda_1}}\beta_\mu\beta_\nu\left(\frac{2}{(n+1)^{\mu+\nu}}-\frac{1}{n^\mu(n+2)^\nu}-\frac{1}{n^\nu(n+2)^\mu}\right) \\
&\quad -\left(\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}+\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}\sum_{\substack{\mu,\nu\in\mathfrak{J}\\ 1\le\mu+\nu\le 2-2\lambda_1+\lambda_i}}\frac{\beta_\mu\beta_\nu}{n^\mu(n+2)^\nu}+\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}}+\frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}\right) \\
&= \begin{cases}
-\dfrac{2\beta_1}{n^3}-\displaystyle\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}-2\beta_1\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,1}}{n^{3-\lambda_i}}-\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}}-\frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}, & 0<\lambda_1\le\dfrac{1}{2},\\
-\displaystyle\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}-2\beta_1\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,1}}{n^{3-\lambda_i}}-\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}}-\frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}, & \dfrac{1}{2}<\lambda_1<1.
\end{cases}
\end{aligned}
\tag{2.14}
$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Similarly,

$$B'\left(\mathrm{mod}\left(o\left(n^{-4+2\lambda_1}\right)\right)\right) = \begin{cases} -\dfrac{2\beta_1}{n^3} - \displaystyle\sum_{i=1}^{s} \frac{\omega_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\omega_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}, & 0 < \lambda_1 \le \dfrac{1}{2}, \\ -\displaystyle\sum_{i=1}^{s} \frac{\omega_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\omega_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}, & \dfrac{1}{2} < \lambda_1 < 1 \end{cases} \tag{2.15}$$

and

$$C'\left(\mathrm{mod}\left(o\left(n^{-4+2\lambda_1}\right)\right)\right) = \begin{cases} \dfrac{2\beta_1}{n^3} + \displaystyle\sum_{i=1}^{s} \frac{\gamma_{\lambda_i,1}}{n^{2-\lambda_i}} + 2\beta_1 \sum_{i=1}^{s'} \frac{\gamma_{\lambda_i,1}}{n^{3-\lambda_i}} - \frac{\gamma_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}, & 0 < \lambda_1 \le \dfrac{1}{2}, \\ \displaystyle\sum_{i=1}^{s} \frac{\gamma_{\lambda_i,1}}{n^{2-\lambda_i}} + 2\beta_1 \sum_{i=1}^{s'} \frac{\gamma_{\lambda_i,1}}{n^{3-\lambda_i}} - \frac{\gamma_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}, & \dfrac{1}{2} < \lambda_1 < 1. \end{cases} \tag{2.16}$$

Combining (2.14)–(2.16), we get that for $0 < \lambda_1 \le \frac{1}{2}$, $C'^2 - A'B'$ can be simplified to

$$\begin{aligned} &\left(\frac{2\beta_1}{n^3} + \sum_{i=1}^{s} \frac{\gamma_{\lambda_i,1}}{n^{2-\lambda_i}} + 2\beta_1 \sum_{i=1}^{s'} \frac{\gamma_{\lambda_i,1}}{n^{3-\lambda_i}} - \frac{\gamma_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}\right)^2 \\ &\quad - \left(-\frac{2\beta_1}{n^3} - \sum_{i=1}^{s} \frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\mu_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}\right) \\ &\quad \times \left(-\frac{2\beta_1}{n^3} - \sum_{i=1}^{s} \frac{\omega_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\omega_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}\right) \\ &= -\frac{16A_{\lambda_1}^3 \binom{\lambda_1}{2}^3}{n^{-6+3\lambda_1}} + o\left(n^{-6+3\lambda_1}\right). \end{aligned}$$

For $\frac{1}{2} < \lambda_1 < 1$, $C'^2 - A'B'$ can be simplified to

$$\begin{aligned} &\left(\sum_{i=1}^{s} \frac{\gamma_{\lambda_i,1}}{n^{2-\lambda_i}} + 2\beta_1 \sum_{i=1}^{s'} \frac{\gamma_{\lambda_i,1}}{n^{3-\lambda_i}} - \frac{\gamma_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}\right)^2 \\ &\quad - \left(-\sum_{i=1}^{s} \frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\mu_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}\right) \\ &\quad \times \left(-\sum_{i=1}^{s} \frac{\omega_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\omega_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}\right) \\ &= -\frac{16A_{\lambda_1}^3 \binom{\lambda_1}{2}^3}{n^{-6+3\lambda_1}} + o\left(n^{-6+3\lambda_1}\right). \end{aligned}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Since $A_{\lambda_1} > 0$ and $0 < \lambda_1 < 1$, we conclude that

$$\operatorname{sgn}\left(-16A_{\lambda_1}^3\left(\begin{matrix}\lambda_1\\2\end{matrix}\right)^3\right) = 1. \tag{2.17}$$

This completes the proof. ∎

## 3 The companion Turán inequality

In this section, we will show that $c(n)$ adheres the companion Turán inequality for sufficiently large $n$.

***Theorem 3.1*** *Assume that $c(n)$ satisfies the form (1.1), then $c(n)$ asymptotically satisfies the companion Turán inequality, i.e.,*

$$c(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right) c(n+1)c(n-1) \le 0, \tag{3.1}$$

*where $\gamma_{\lambda_1,1}$ is defined as in Theorem 2.2 and $\overline{\gamma}$ is any real number larger than $-\gamma_{\lambda_1,1}$.*

**Proof** We have

$$\begin{aligned} &c(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right) c(n+1)\, c(n-1) \\ &\sim C^2 \frac{\exp\left(2\sum_{\lambda\in S} A_\lambda n^\lambda\right)}{n^{2\kappa}} \Bigg( \left(\sum_{\mu\in T} \frac{\beta_\mu}{n^\mu}\right)^2 \\ &\quad - \left(1 + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right)\left(1 + \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} + o\left(n^{-2+\lambda_1}\right)\right)\left(1 + O\left(n^{-2}\right)\right) \sum_{\mu\in T} \frac{\beta_\mu}{(n+1)^\mu} \sum_{\nu\in T} \frac{\beta_\nu}{(n-1)^\nu} \Bigg). \end{aligned}$$

The sign of this is dictated by

$$\sum_{\mu,\nu\in T} \frac{\beta_\mu\beta_\nu}{n^{\mu+\nu}} - \left(1 + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right)\left(1 + \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} + o\left(n^{-2+\lambda_1}\right)\right)\left(1 + O\left(n^{-2}\right)\right) \sum_{\mu\in T} \frac{\beta_\mu}{(n+1)^\mu} \sum_{\nu\in T} \frac{\beta_\nu}{(n-1)^\nu}.$$

As $\beta_0 = 1$, the above formula mod $\left(o(n^{-2+\lambda_1})\right)$ equals,

$$\begin{aligned} &\sum_{\mu,\nu\in\mathcal{T}} \frac{\beta_\mu\beta_\nu}{n^{\mu+\nu}} - \left(\sum_{\mu,\nu\in\mathcal{T}} \frac{\beta_\mu\beta_\nu}{(n+1)^\mu(n-1)^\nu} + \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right) \\ &\quad = 2 \sum_{\substack{\mu\in\mathcal{T}\\ 0\le\mu\le 1-\frac{\lambda_1}{2}}} \beta_\mu^2 \left(\frac{1}{n^{2\mu}} - \frac{1}{(n+1)^\mu(n-1)^\mu}\right) \\ &\qquad + \sum_{\substack{\mu,\nu\in\mathcal{T}\\ 0\le\mu<\nu\\ 1\le\mu+\nu\le 2-\lambda_1}} \beta_\mu\beta_\nu \left(\frac{2}{n^{\mu+\nu}} - \frac{1}{(n+1)^\mu(n-1)^\nu} - \frac{1}{(n+1)^\nu(n+1)^\mu}\right) \\ &\qquad - \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} - \frac{\overline{\gamma}}{n^{2-\lambda_1}}. \end{aligned} \tag{3.2}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

For $n > 1$,

$$\frac{1}{(n+1)^{\nu}(n+1)^{\mu}} = n^{-2\mu} + O\left(n^{-2\mu-2}\right)$$

and

$$\frac{1}{(n+1)^{\mu}(n-1)^{\nu}} - \frac{1}{(n+1)^{\nu}(n+1)^{\mu}} = 2n^{-\mu-\nu}\left(1 + \sum_{r\geq 1}\frac{p_r}{n^{2r}}\right).$$

Thus, (3.2) becomes

$$O(n^{-2+\lambda_1}) - \frac{\gamma_{\lambda_1,1} + \overline{\gamma}}{n^{2-\lambda_1}}.$$

When $\overline{\gamma} > -\gamma_{\lambda_1,1}$, it is obvious that (3.1) holds for sufficiently large $n$. ∎

## 4 Laguerre inequalities

In this section, we demonstrate that $c(n)$ fulfills the Laguerre inequality and their companion inequality of any order. Recently, Wang and Yang [27] proved that for any $r$ and sufficiently large $n$, one has $L_r(\alpha(n)) > 0$.

***Theorem 4.1*** (Wang and Yang) *Let $\{\alpha(n)\}, \{\delta(n)\}, \{A_i(n)\}$ be sequences with $\alpha(n)$ positive, $\delta(n) \to 0^+$, $A_2(n) < 0$, $A_2(n) = \Theta(\delta(n)^t)$ and $A_{2i}(n) = o(\delta(n)^{it})$ $(i \geq 2)$ for some positive $t$, and for $-r < j < r$,*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = \sum_{i=1}^{2r} A_i(n) j^i + o\left(\delta(n)^{tr}\right), \tag{4.1}$$

*then for sufficiently large $n$,*

$$L_r(\alpha(n)) = \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\alpha(n+k)\alpha(n+2r-k) > 0. \tag{4.2}$$

Since the conditions of Theorem 1.1 are stronger than those of Theorem 4.1, we get that $c(n)$ meets the requirements of the above theorem from the proof of Theorem 2.1. Thus, $c(n)$ fulfills the Laguerre inequality of any order for sufficiently large $n$. Next, we prove that $c(n)$ satisfies the companion Laguerre inequalities. First, let us recall two combinatorial identities which will be used in our proofs (see [10, 27]).

***Lemma 4.2*** *For positive integers $m$ and $t \leq 2m$,*

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}(m-k)^t = \begin{cases} 0, & t < 2m, \\ (2m)!, & t = 2m. \end{cases} \tag{4.3}$$

***Lemma 4.3*** *For positive integers $r$,*

$$\sum_{k=0}^{2r}\binom{2r}{k}(r-k)^2 = 2^{2r-1}r. \tag{4.4}$$

Now, we are in a position to establish the companion Laguerre inequalities for $c(n)$.

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

***Theorem 4.4*** *For any fixed $r \geq 2$ and sufficiently large $n$, $c(n)$ satisfies the companion Laguerre inequality of order $r$, i.e.,*

$$\left(1+\frac{(2r)!}{r!}2^{1-r}(-A_{\lambda_1})^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)}\right)L_r^-(c(n-r))>L_r^+(c(n-r)). \tag{4.5}$$

**Proof** The inequality (4.5) can be rewritten as

$$\begin{aligned}&\frac{(2r)!}{r!}2^{1-r}(-A_{\lambda_1})^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)}\frac{L_r^-(c(n-r))}{c(n)^2}\\ &\quad>\frac{L_r^+(c(n-r))}{c(n)^2}-\frac{L_r^-(c(n-r))}{c(n)^2}\\ &\quad=\frac{L_r(c(n-r))}{c(n)^2}.\end{aligned} \tag{4.6}$$

The right-hand side of the above inequality can be expressed as

$$\frac{L_r(c(n-r))}{c(n)^2}=\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\frac{c(n-r+j)c(n+r-j)}{c(n)^2}.$$

From (2.9), we get

$$\begin{aligned}&\frac{c(n+r-j)c(n-r+j)}{c(n)^2}\\ &\quad=\exp\left(\log\frac{c(n+r-j)}{c(n)}+\log\frac{c(n-r+j)}{c(n)}\right)\\ &\quad\sim\exp\left(\sum_{\lambda\in\mathcal{S}}A_\lambda\left((n+r-j)^\lambda-n^\lambda\right)+\log\frac{n^\kappa}{(n+r-j)^\kappa}+\log\frac{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n+r-j)^\mu}}{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}}\right.\\ &\qquad\left.+\sum_{\lambda\in\mathcal{S}}A_\lambda\left((n-r+j)^\lambda-n^\lambda\right)+\log\frac{n^\kappa}{(n-r+j)^\kappa}+\log\frac{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n-r+j)^\mu}}{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}}\right)\\ &\quad=\exp\left(\sum_{i=1}^{N}\left(\sum_{\lambda\in\mathcal{S}}B_{\lambda,i}(n)+F_i(n)\right)\left((r-j)^i+(r-j)^i(-1)^i\right)+o(n^{-N+\lambda})\right)\\ &\quad=\exp\left(2\sum_{i=1}^{N}\left(\sum_{\lambda\in\mathcal{S}}B_{\lambda,2i}(n)+F_{2i}(n)\right)(r-j)^{2i}+o(n^{-N+\lambda})\right)\\ &\quad=:\exp\left(2\sum_{i=1}^{N}B_{2i}(n)(r-j)^{2i}+o(B_{2N}(n))\right),\end{aligned}$$

where $B_{2i}(n)$ are certain polynomials of $n$ with degree no more than $-2i+\lambda_1$ and the coefficient of $n^{-2i+\lambda_1}$ is $A_{\lambda_1}\binom{\lambda_1}{2i}$.

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Let

$$\phi_r(x) := \sum_{i=1}^{2r} \frac{x^i}{i!} = \exp(x) + o\left(x^{2r}\right). \tag{4.7}$$

Then, as $n \to \infty$, we get

$$\begin{aligned}
&\frac{L_r(c(n-r))}{c(n)^2} \\
&= \frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\exp\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i} + o(B_{2N}(n)\right) \\
&= \frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\exp\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right)\exp\left(o(B_{2N}(n))\right) \\
&= \left(\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right) + o\left(B_{2N}(n)^{2Nr}\right)\right)\exp\left(o(B_{2N}(n))\right) \\
&= \frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right) + o\left(B_{2N}(n)\right).
\end{aligned}$$

Now, we focus on the expression

$$\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right). \tag{4.8}$$

The $\phi_r$-terms can be viewed as a polynomial in $(r-j)^2$, i.e., there exist coefficients $B'_{2i}(n)(0 \le i \le 2rN)$ such that

$$\phi_r\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right) = \sum_{i=0}^{2rN} B'_{2i}(n)(r-j)^{2i}, \tag{4.9}$$

where $B'_{2i}(n)$ are certain polynomials of $n$ with degree no more than $(-2+\lambda_1)i$ and the coefficient of $n^{(-2+\lambda_1)i}$ is $\frac{2^i A_{\lambda_1}^i \left(\frac{\lambda_1}{2}\right)^i}{i!}$.

By Lemma 4.2, one can see that

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right) \\
&\quad= \frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\sum_{i=0}^{2rN} B'_{2i}(n)(r-j)^{2i} \\
&\quad= \frac{1}{2}\sum_{i=0}^{2rN} B'_{2i}(n)\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}(r-j)^{2i} \\
&\quad= \frac{1}{2}\sum_{i=r}^{2rN} B'_{2i}(n)\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}(r-j)^{2i}.
\end{aligned} \tag{4.10}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

(The sum of terms with $i < r$ vanishes.) It implies that as $n \to \infty$, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N}B_{2i}(n)2(r-j)^{2i}\right)\\
&\quad=\frac{1}{2}B'_{2r}(n)\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}(r-j)^{2r}+o\left(B'_{2r}(n)\right)\\
&\quad=\frac{(2r)!}{r!}2^{r-1}(-1)^rA_{\lambda_1}^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)r}+o\left(n^{(-2+\lambda_1)r}\right).
\end{aligned}\tag{4.11}$$

On the other hand, we obtain

$$\begin{aligned}
\frac{c(n+2j+1)c(n-2j-1)}{c(n)^2}&\sim\exp\left(2\sum_{i=1}^{n}B_{2i}(n)(2j+1)^{2i}+o(B_{2N}(n))\right)\\
&=1+2A_{\lambda_1}\binom{\lambda_1}{2}(2j+1)^2n^{-2+\lambda_1}+o\left(n^{-2+\lambda_1}\right).
\end{aligned}\tag{4.12}$$

Thus, by Lemma 4.3, we get

$$\begin{aligned}
\frac{L_r^-(c(n-r))}{c(n)^2}&=\frac{1}{2}\sum_{j=-\lceil r-1/2\rceil}^{\lceil r-1/2\rceil}\binom{2r}{r+2j+1}\frac{c(n+2j+1)c(n-2j-1)}{c(n)^2}\\
&=2^{2r-2}+\frac{1}{2}\sum_{j=-\lceil r-1/2\rceil}^{\lceil r-1/2\rceil}\binom{2r}{r+2j+1}\left(2A_{\lambda_1}\binom{\lambda_1}{2}(2j+1)^2n^{-2+\lambda_1}+o\left(n^{-2+\lambda_1}\right)\right)\\
&=2^{2r-2}+\frac{1}{2}A_{\lambda_1}\binom{\lambda_1}{2}n^{-2+\lambda_1}\left(2\sum_{j=-\lceil r-1/2\rceil}^{\lceil r-1/2\rceil}\binom{2r}{r+2j+1}(2j+1)^2\right)+o\left(n^{-2+\lambda_1}\right)\\
&=2^{2r-2}+\frac{1}{2}A_{\lambda_1}\binom{\lambda_1}{2}n^{-2+\lambda_1}\left(\sum_{j=-\lceil r-1/2\rceil}^{\lceil r-1/2\rceil}\binom{2r}{r+2j+1}(2j+1)^2+\sum_{j=-\lceil r/2\rceil}^{\lceil r/2\rceil}\binom{2r}{r+2j}(2j)^2\right)\\
&\quad+o\left(n^{-2+\lambda_1}\right)\\
&=2^{2r-2}+\frac{1}{2}A_{\lambda_1}\binom{\lambda_1}{2}n^{-2+\lambda_1}\sum_{j=-r}^{r}\binom{2r}{r+j}j^2+o\left(n^{-2+\lambda_1}\right)\\
&=2^{2r-2}+2^{2r-2}rA_{\lambda_1}\binom{\lambda_1}{2}n^{-2+\lambda_1}+o\left(n^{-2+\lambda_1}\right).
\end{aligned}\tag{4.13}$$

Combining (4.11) and (4.13) gives

$$\begin{aligned}
&\left(1+\frac{(2r)!}{r!}2^{1-r}(-A_{\lambda_1})^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)}\right)\frac{L_r^-(c(n-r))}{c(n)^2}-\frac{L_r^+(c(n-r))}{c(n)^2}\\
&\quad=\frac{(2r)!}{r!}2^{r-1}(-A_{\lambda_1})^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)}+\frac{(2r)!}{(r-1)!}2^{r-1}(-1)^r(A_{\lambda_1})^{r+1}\binom{\lambda_1}{2}^{r+1}n^{(-2+\lambda_1)r}
\end{aligned}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

$$
\begin{aligned}
&-\frac{(2r)!}{r!}2^{r-1}(-1)^r A_{\lambda_1}^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)r}+o(n^{(-2+\lambda_1)r})\\
&=\frac{(2r)!}{r!}2^{r-1}A_{\lambda_1}^r(-1)^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)}+o(n^{(-2+\lambda_1)(r-1)}).
\end{aligned}
\tag{4.14}
$$

In view of $\lambda_1 \in (0,1)$, we get $(-1)^r\binom{\lambda_1}{2}^r > 0$. Hence, we conclude the positivity of (4.14) as $n \to \infty$. ∎

## 5 Applications

In this section, we will apply Theorems 2.1, 2.2, 3.1, and 4.4 to show that some sequences of the form $c(n)$ satisfy the companion Turán inequality, the double Turán inequality, the Turán inequality of any order, the Laguerre inequality of any order $r$, and its companion version.

**The number of commuting $\ell$-tuples in $\mathbf{S_n}$.** Denote by $N_\ell(n)$ the number of $\ell$-tuples of elements in the symmetric group $S_n$ with commuting components, normalized by the order of $S_n$. Notice that $N_2(n) = p(n)$, where $p(n)$ denotes the number of partitions of $n$. Bringmann, Franke, and Heim [5] gave the asymptotical formulas for $N_\ell(n)$ for arbitrary $\ell$ as follows.

***Theorem 5.1*** (Bringmann, Franke, and Heim) *We have, as $n \to \infty$,*

$$
N_3(n) \sim \frac{e^{-\frac{\zeta'(-1)}{2}-\frac{\pi^2}{288\zeta(3)}}\zeta(3)^{\frac{11}{72}}}{2^{\frac{11}{24}}\cdot 3^{\frac{47}{72}}\cdot \pi^{\frac{11}{72}}\cdot n^{\frac{47}{72}}}\exp\left(\frac{(3\pi)^{\frac{2}{3}}\zeta(3)^{\frac{1}{3}}}{2}n^{\frac{2}{3}}-\frac{\pi^{\frac{4}{3}}}{4\cdot 3^{\frac{2}{3}}\cdot\zeta(3)^{\frac{1}{3}}}n^{\frac{1}{3}}\right)\left(1+\sum_{j=1}^{\infty}\frac{B_{3,j}}{n^{\frac{j}{3}}}\right)
$$

*for certain numbers $B_{3,j}$,*

$$
\begin{aligned}
N_4(n) &\sim \frac{e^{\frac{\zeta'(-2)}{24}}\pi^{\frac{1}{4}}\zeta(3)^{\frac{1}{8}}}{2^{\frac{13}{8}}\cdot 3^{\frac{1}{4}}\cdot 5^{\frac{1}{8}}\cdot n^{\frac{5}{8}}}\\
&\quad\times\exp\left(\frac{2^{\frac{7}{4}}\cdot\pi^{\frac{3}{2}}\cdot\zeta(3)^{\frac{1}{4}}}{3^{\frac{3}{2}}\cdot 5^{\frac{1}{4}}}n^{\frac{3}{4}}+A_{4,2}n^{\frac{1}{2}}+A_{4,3}n^{\frac{1}{4}}+A_{4,4}\right)\left(1+\sum_{j=1}^{\infty}\frac{B_{4,j}}{n^{\frac{j}{4}}}\right),\\
N_5(n) &\sim \frac{e^{\frac{\zeta'(-2)}{2880}}(\pi\zeta(3)\zeta(5))^{\frac{1}{10}}}{2^{\frac{2}{5}}\cdot 3^{\frac{1}{5}}\cdot 5^{\frac{3}{5}}\cdot n^{\frac{3}{5}}}\\
&\quad\times\exp\left(\frac{5^{\frac{4}{5}}\cdot\pi^{\frac{6}{5}}\cdot(\zeta(3)\zeta(5))^{\frac{1}{5}}}{2^{\frac{9}{5}}\cdot 3^{\frac{2}{5}}}n^{\frac{4}{5}}+A_{5,2}n^{\frac{3}{5}}+A_{5,3}n^{\frac{2}{5}}+A_{5,4}n^{\frac{1}{5}}+A_{5,5}\right)\left(1+\sum_{j=1}^{\infty}\frac{B_{5,j}}{n^{\frac{j}{5}}}\right),
\end{aligned}
$$

*with computable constants $A_{4,j}$ $(2 \le j \le 4)$ and $A_{5,j}$ $(2 \le j \le 5)$ and certain $B_{4,j}$ and $B_{5,j}$, and for $\ell \ge 6$,*

$$
N_\ell(n) \sim \frac{(\ell-1)!^{\frac{1}{2\ell}}\sqrt{Z_\ell}}{\sqrt{2\pi\ell}\,n^{\frac{\ell+1}{2\ell}}}\exp\left(\frac{\ell\Gamma(\ell)^{\frac{1}{\ell}}Z_\ell}{\ell-1}n^{\frac{\ell-1}{\ell}}+\sum_{k=2}^{\ell}A_{\ell,k}n^{\frac{\ell-k}{\ell}}\right)\left(1+\sum_{j=1}^{\infty}\frac{B_{\ell,j}}{n^{\frac{j}{\ell}}}\right),
$$

*where $Z_\ell := (\zeta(2)\cdot\zeta(3)\cdots\zeta(\ell))^{\frac{1}{\ell}}$ for certain $A_{\ell,k}$ and $B_{\ell,j}$.*

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Theorem 5.1 suggests that the asymptotical formulas for $N_\ell(n)$ satisfy the form of $c(n)$. Combining Theorems 2.1 and 2.2, we get that for sufficiently large $n$, $N_\ell(n)$ satisfies the Turán inequality of any order and the double Turán inequality.

***Theorem 5.2*** *For $\ell \geq 2$ and sufficiently large $n$, $N_\ell(n)$ satisfies the Turán inequality of any order and the double Turán inequality.*

Set $\lambda_1 = \frac{\ell-1}{\ell}$ and $A_{\lambda_1} = \frac{\ell\Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell-1}$. Theorem 3.1 gives that $N_\ell(n)$ satisfies the following companion Turán inequality.

***Theorem 5.3*** *For $\ell \geq 2$ and sufficiently large $n$, we have that $N_\ell(n)$ satisfies the companion Turán inequality, i.e.,*

$$N_\ell(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{1+\frac{1}{\ell}}}\right) N_\ell(n+1)N_\ell(n-1) \leq 0,$$

*where $\overline{\gamma}$ is any real number larger than $\frac{\Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell}$.*

Since $1 + \frac{1}{\ell} > 1$, from Theorem 5.3, we deduce that the log-convexity of $|C_{\ell,n}| = n!N_\ell(n)$, which give an affirmative answer to the open question (3) proposed by Bringmann, Franke, and Heim in [5].

***Corollary 5.4*** *For $\ell \geq 3$ and sufficiently large $n$, we have that $|C_{\ell,n}|$ is log-convex.*

For Laguerre inequalities, we get that $N_\ell(n)$ satisfies the Laguerre inequality of any order $r$ and its companion version from Theorem 4.4.

***Theorem 5.5*** *For $\ell \geq 2$, $r \geq 2$ and sufficiently large $n$, we have*

$$1 < \frac{L_r^+(N_\ell(n-r))}{L_r^-(N_\ell(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-\frac{\ell\Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell-1}\right)^r \left(\frac{\frac{\ell-1}{\ell}}{2}\right)^r n^{(-1-\frac{1}{\ell})(r-1)}.$$

The following sequences have also the asymptotical behavior of the form $c(n)$, then we can obtain results similar to $N_\ell(n)$.

**Partitions without sequences.** The partition without sequence is the partition of $n$ that do not contain any consecutive integers as parts. Let $p_2(n)$ denote the number of partitions without sequence of $n$. Bringmann and Mahlburg [6] first succeeded in proving an asymptotic formula for $p_2(n)$. Mauth [18] gave the following formula and proved the log-concavity of $p_2(n)$ for $n \geq 482$,

$$p_2(n) = \frac{e^{\frac{2\pi\sqrt{n}}{3}}}{n^{\frac{3}{4}}} \left(\sum_{i=0}^{9} \frac{\beta_i}{n^{\frac{i}{4}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right)\right),$$

where $\beta_i$ are constants. Note that $p_2(n)$ have the form (1.1) with $\lambda_1 = \frac{1}{2}$ and $A_{\lambda_1} = \frac{2\pi}{3}$. Hence, by the theorems above, we deduce the following result.

***Theorem 5.6*** *For any $r \geq 2$ and sufficiently large $n$, $p_2(n)$ satisfies the double Turán inequality, the Turán inequality of any order, and the following companion Turán inequality:*

$$p_2(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{3}{2}}}\right) p_2(n+1)p_2(n-1) \leq 0,$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

*where $\overline{\gamma}$ is any real number larger than $\frac{\pi}{6}$, the Laguerre inequality of any order $r$, and its companion version as follows:*

$$1 < \frac{L_r^+(p_2(n-r))}{L_r^-(p_2(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-\frac{2\pi}{3}\right)^r \left(\frac{\frac{1}{2}}{2}\right)^r n^{-\frac{3}{2}(r-1)}.$$

**Plane partition**. A plane partition of size $n$ is a two-dimensional array of non-negative integers $\pi_{j,k}$, for which $\sum_{j,k} \pi_{j,k} = n$, such that $\pi_{j,k} \geq \pi_{j,k+1}$ and $\pi_{j,k} \geq \pi_{j+1,k}$ for any $j, k \in N$, denote the number of plane partitions of size $n$ by $\mathrm{PL}(n)$. Wright [28] proved that

$$\mathrm{PL}(n) = \frac{C}{n^{\frac{25}{36}}} e^{A_1 n^{\frac{2}{3}}} \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{2(j-1)}{3}}} + O\left(n^{-\frac{2(N+1)}{3}}\right)\right),$$

where $A_1 > 0$, and $C$ and $B_j$ are constants.

Heim, Neuhauser, and Tröger [14] conjectured that plane partition is log-concave and proved the conjecture for almost all $n$. Ono, Pujahari, and Rolen [19] derived the following explicit asymptotic formula and proved that $J_{\mathrm{PL}}^{d,n}(x)$ is hyperbolic for sufficiently large $n$.

***Theorem 5.7*** (Ono, Pujahari, and Rolen) *If $r \in \mathbb{Z}^+$, then for every integer $n \geq \max(n_r, \ell_r, 87)$, we have*

$$\mathrm{PL}(n) = \frac{e^{c+3AN_n^2}}{2\pi} \sum_{s=0}^{r+1} \sum_{m=0}^{r+1} \frac{(-1)^m \beta_s b : s, m\Gamma\left(m+\frac{1}{2}\right)}{A^{m+\frac{1}{2}} N_n^{2s+2m+\frac{25}{12}}} + E_r^{\mathrm{maj}}(n) + E^{\mathrm{min}}(n),$$

*where $|E_r^{\mathrm{maj}}(n)| \leq \widehat{E}_r^{\mathrm{maj}}(n)$, $N_n := \left(\frac{n}{2A}\right)^{\frac{1}{3}}$, and*

$$|E^{\mathrm{min}}(n)| \leq \exp\left(\left(3A - \frac{2}{5}\right) n^2/(2A)^{\frac{2}{3}}\right).$$

Pandey [20] found a lower bound such that $J_{\mathrm{PL}}^{d,n}(x)$ has all real roots for all $n \geq N_{\mathrm{PL}}(d)$, where

$$N_{\mathrm{PL}}(d) \leq 279928 \times d(d-1) \left(6d^3 (22.2)^{\frac{3(d-1)}{2}}\right)^{2d} e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d+2}}}.$$

Obviously, $\mathrm{PL}(n)$ have the form (1.1) with $\lambda_1 = \frac{2}{3}$ and $A_{\lambda_1} = A_1$. Thus, employing the theorems above, we get the following result.

***Theorem 5.8*** *For any $r \geq 2$ and sufficiently large $n$, $\mathrm{PL}(n)$ satisfies the double Turán inequality, the Turán inequality of any order, and the following companion Turán inequality:*

$$\mathrm{PL}(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{4}{3}}}\right) \mathrm{PL}(n+1)\mathrm{PL}(n-1) \leq 0,$$

*where $\overline{\gamma}$ is any real number larger than $\frac{2A_1}{9}$, the Laguerre inequality of any order $r$, and its companion version as follows:*

$$1 < \frac{L_r^+(\mathrm{PL}(n-r))}{L_r^-(\mathrm{PL}(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} (-A_1)^r \left(\frac{\frac{2}{3}}{2}\right)^r n^{-\frac{4}{3}(r-1)}.$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

**The coefficients of infinite product generating functions**. $\mathrm{p}_d(n)$ is defined by the generating function [13]

$$\sum_{n=0}^{\infty} \mathrm{p}_d(n)q^n := \prod_{n=1}^{\infty} (1-q^n)^{-n^{d-1}}.$$

Note that $\mathrm{p}_1(n) = p(n)$ and $\mathrm{p}_2(n) = \mathrm{PL}(n)$. Bringmann, Franke, and Heim [5] provided that

$$\mathrm{p}_d(n) \sim \frac{C}{n^b} e^{A_1 n^{\frac{d+1}{d+2}}} \left(1 + \sum_{j\geq 1} \frac{E_{d,j}}{n^{\frac{j}{d+2}}}\right),$$

where $C$, $A_1$, $b$, and $E_{d,j}$ are constants. One can see that $\mathrm{p}_d(n)$ have the form (1.1) with $\lambda_1 = \frac{d+1}{d+2}$ and $A_{\lambda_1} = A_1$. Thus, we can get the following result by the theorems above.

***Theorem 5.9*** *For any $r \geq 2$ and sufficiently large $n$, $\mathrm{p}_d(n)$ satisfies the double Turán inequality, the Turán inequality of any order, and the following companion Turán inequality:*

$$\mathrm{p}_d(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{1+\frac{1}{d+2}}}\right) \mathrm{p}_d(n+1)\mathrm{p}_d(n-1) \leq 0,$$

*where $\overline{\gamma}$ is any real number larger than $\frac{A_1(d+1)}{(d+2)^2}$, the Laguerre inequality of any order $r$, and its companion version as follows:*

$$1 < \frac{L_r^+(\mathrm{p}_d(n-r))}{L_r^-(\mathrm{p}_d(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} (-A_1)^r \binom{\frac{d+1}{d+2}}{2}^r n^{(-1-\frac{1}{d+2})(r-1)}.$$

**Partitions into k-gonal numbers**. We denote the number of partitions of $n$ into $k$-gonal numbers by $p_k(n)$. We have the generating function

$$\sum_{n\geq 0} p_k(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^{P_k(n)}},$$

where

$$P_k(n) := \frac{1}{2}\left((k-2)n^2 + (4-k)n\right)$$

is the $n$-th $k$-gonal number. Bridges, Brindle, Bringmann, and Franke [2] provided that

$$p_k(n) = \frac{C(k)e^{A(k)n^{\frac{1}{3}}}}{n^{\frac{5k-6}{6(k-2)}}} \left(1 + \sum_{j=1}^{N} \frac{B_{j,k}}{n^{\frac{j}{3}}} + O_N\left(n^{-\frac{N+1}{3}}\right)\right),$$

where $C(k)$, $A(k)$, and $B_{j,k}$ are constants.

Note that $p_k(n)$ have the form (1.1) with $\lambda_1 = \frac{1}{3}$ and $A_{\lambda_1} = A(k)$. Then, we can get the following result by the theorems above.

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

***Theorem 5.10*** *For any $r \geq 2$ and sufficiently large $n$, $p_k(n)$ satisfies the double Turán inequality, the Turán inequality of any order, and the companion Turán inequality*

$$p_k(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{5}{3}}}\right) p_k(n+1)p_k(n-1) \leq 0,$$

*where $\overline{\gamma}$ is any real number larger than $\frac{2A(k)}{9}$, the Laguerre inequality of any order $r$, and its companion version as follows:*

$$1 < \frac{L_r^+(p_k(n-r))}{L_r^-(p_k(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-A(k)\right)^r \binom{\frac{1}{3}}{2}^r n^{-\frac{5}{3}(r-1)}.$$

**The finite-dimensional representations of groups $\mathfrak{su}(3)$.** The unitary group $\mathfrak{su}(3)$, whose irreducible representations $W_{j,k}$ indexed by pairs of positive integers. The numbers $r_{\mathfrak{su}(3)}(n)$ of $n$-dimensional representations have the generating function

$$\sum_{n \geq 0} r_{\mathfrak{su}(3)}(n) q^n = \prod_{j,k \geq 1} \frac{1}{1 - q^{\frac{jk(j+k)}{2}}},$$

with $r_{\mathfrak{su}(3)}(0) := 1$. In [24], Romik proved that, as $n \to \infty$,

$$r_{\mathfrak{su}(3)}(n) \sim \frac{C_0}{n^{\frac{3}{5}}} \exp\left(A_1 n^{\frac{2}{5}} + A_2 n^{\frac{3}{10}} + A_3 n^{\frac{1}{5}} + A_4 n^{\frac{1}{10}}\right),$$

with explicit constants $C_0, A_1, A_2, A_3, A_4$ expressible in terms of zeta and gamma values. Romik asked for lower order terms in the asymptotical expansion of $r_{\mathfrak{su}(3)}(n)$. Bringmann and Franke [4] showed the following form of $r_{\mathfrak{su}(3)}(n)$.

***Theorem 5.11*** (Bringmann and Franke) *As $n \to \infty$, for any $N \in \mathbb{N}$,*

$$r_{\mathfrak{su}(3)}(n) = \frac{C_0}{n^{\frac{3}{5}}} e^{A_1 n^{\frac{2}{5}} + A_2 n^{\frac{3}{10}} + A_3 n^{\frac{1}{5}} + A_4 n^{\frac{1}{10}}} \left(1 + \sum_{j=1}^{N} \frac{C_j}{n^{\frac{j}{10}}} + O_N\left(n^{-\frac{N}{10} - \frac{3}{80}}\right)\right),$$

*where the constants $C_j$ do not depend on $N$ and $n$ and can be calculated explicitly.*

Note that $r_{\mathfrak{su}(3)}(n)$ have the form (1.1) with $\lambda_1 = \frac{2}{5}$ and $A_{\lambda_1} = A_1$. Hence, we can deduce the following result by the theorems above.

***Theorem 5.12*** *For any $r \geq 2$ and sufficiently large $n$, $r_{\mathfrak{su}(3)}(n)$ satisfies the double Turán inequality, the Turán inequality of any order, and the companion Turán inequality*

$$r_{\mathfrak{su}(3)}(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{8}{5}}}\right) r_{\mathfrak{su}(3)}(n+1) r_{\mathfrak{su}(3)}(n-1) \leq 0,$$

*where $\overline{\gamma}$ is any real number larger than $\frac{6A_1}{25}$, the Laguerre inequality of any order $r$, and its companion version*

$$1 < \frac{L_r^+(r_{\mathfrak{su}(3)}(n-r))}{L_r^-(r_{\mathfrak{su}(3)}(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-A_1\right)^r \binom{\frac{2}{5}}{2}^r n^{-\frac{8}{5}(r-1)}.$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

**The finite-dimensional representations of groups $\mathfrak{so}(5)$.** This framework generalizes to other groups. For example, one can investigate the *Witten zeta function* for $\mathfrak{so}(5)$, which is (for more background to this function, see [16, 17])

$$\zeta_{\mathfrak{so}(5)}(s) := \sum_{\varphi} \frac{1}{\dim(\varphi)^s} = 6^s \sum_{n,m\geq 1} \frac{1}{m^s n^s (m+n)^s (m+2n)^s},$$

where the $\varphi$ are running through the finite-dimensional irreducible representations of $\mathfrak{so}(5)$. Bridges, Brindle, Bringmann, and Franke [2] showed the following form of $r_{\mathfrak{so}(5)}(n)$.

***Theorem 5.13*** (Bridges, Brindle, Bringmann, and Franke) *As $n \to \infty$, for any $N \in \mathbb{N}$,*

$$r_{\mathfrak{so}(5)}(n) = \frac{C}{n^{\frac{7}{12}}} e^{A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4} \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),$$

*where $C$, $A_1$, $A_2$, $A_3$, $A_4$, and $B_j$ are constants.*

Bringmann, Franke, and Heim [5] have proved that $r_{\mathfrak{su}(3)}(n)$ and $r_{\mathfrak{so}(5)}(n)$ asymptotically satisfy the Turán inequality. Note that $r_{\mathfrak{so}(5)}(n)$ have the form (1.1) with $\lambda_1 = \frac{1}{3}$ and $A_{\lambda_1} = A_1$. Thus, we can get the following result by the theorems above.

***Theorem 5.14*** *For any $r \geq 2$ and sufficiently large $n$, $r_{\mathfrak{so}(5)}(n)$ satisfies the double Turán inequality, the Turán inequality of any order, and the following companion Turán inequality:*

$$r_{\mathfrak{so}(5)}(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{5}{3}}}\right) r_{\mathfrak{so}(5)}(n+1) r_{\mathfrak{so}(5)}(n-1) \leq 0,$$

*where $\overline{\gamma}$ is any real number larger than $\frac{2A_1}{9}$, the Laguerre inequality of any order $r$, and its companion version*

$$1 < \frac{L_r^+(r_{\mathfrak{so}(5)}(n-r))}{L_r^-(r_{\mathfrak{so}(5)}(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} (-A_1)^r \binom{\frac{1}{3}}{2}^r n^{-\frac{5}{3}(r-1)}.$$

## References


[1] K. Banerjee, *Invariants of the quartic binary form and proofs of Chen's conjectures on inequalities for the partition function and the Andrews' spt function*. Res. Math. Sci. **12**(2025), no. 1.

[2] W. Bridges, B. Brindle, K. Bringmann, and J. Franke, *Asymptotic expansions for partitions generated by infinite products*. Math. Ann. **390**(2024), 2593–2632.

[3] W. Bridges and K. Bringmann, *Log-concavity for unimodal sequences*. Res. Number Theory **10**(2023), no. 6.

[4] K. Bringmann and J. Franke, *An asymptotic formula for the number of n-dimensional representations of* $\mathfrak{su}(3)$. Rev. Mat. Ibero. **39**(2023), no. 5, 1599–1640.

[5] K. Bringmann, J. Franke, and B. Heim, *Asymptotics of commuting l-tuples in symmetric groups and log-concavity*. Res. Number Theory **10**(2024), no. 83.

[6] K. Bringmann and K. Mahlburg, *An extension of the Hardy-Ramanujan circle method and applications to partitions without sequences*. Am. J. Math. **133**(2011), no. 4, 1157–1178.

[7] W. Y. C. Chen, *The spt-function of Andrews*. In: A. Claesson, M. Dukes, S. Kitaev, D. Manlove, and K. Meeks (eds.), Surveys in combinatorics 2017, London Mathematical Society Lecture Note Series, 440, Cambridge University Press, Cambridge, 2017, pp. 141–203.

[8] G. Csordas, T. S. Norfolk, and R. S. Varga, *The Riemann hypothesis and the Turán inequalities*. Trans. Amer. Math. Soc. **296**(1986), no. 2, 521–541.


https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

[9] G. Csordas and A. Vishnyakova, *The generalized Laguerre inequalities and functions in the Laguerre-Pólya class*. Cent. Eur. J. Math. **11**(2013), no. 9, 1643–1650.
[10] L.-M. Dou, L. X. W. Wang, and N. N. Y. Yang, *Unified coefficients for the reversed inequalities associated with class*. submitted.
[11] G. Gasper, *Positivity and special functions*. In: A. Askey (ed.), Proceedings of an advanced seminar sponsored by the mathematics research center, The University of Wisconsin-Madison, New York-London, 1975, pp. 375–433.
[12] M. Griffin, K. Ono, L. Rolen, and D. Zagier, *Jensen polynomials for the Riemann zeta function and other sequences*. Proc. Nat. Acad. Sci. USA **116**(2019), no. 23, 11103–11110.
[13] B. Heim and M. Neuhauser, *Log-concavity of infinite product generating functions*. Res. Number Theory **8**(2022), no. 53, 1–14.
[14] B. Heim, M. Neuhauser, and R. Tröger, *Inequalities for plane partitions*. Ann. Comb. **27**(2023), no. 1, 87–108.
[15] E. Laguerre, *Oeuvres*. Vol. 1. Gaauthier-Villars, Paris, 1989.
[16] K. Matsumoto, *On Mordell–Tornheim and other multiple zeta-functions*. In: D. R. Heath-Brown and B. Z. Moroz (eds.), Proceedings of the session in analytic number theory and Diophantine equations, Bonner Mathematische Schriften, The University of Bonn, Bonn, 360, 2003, p. 17.
[17] K. Matsumoto and H. Tsumura, *On Witten multiple zeta-functions associated with semisimple lie algebras I*. Ann. Inst. Fourier **56**(2006), no. 5, 1457–1504.
[18] L. Mauth, *Log-concavity for partitions without sequences*. Preprint. arXiv:2306.07459.
[19] K. Ono, S. Pujahari, and L. Rolen, *Turán inequalities for the plane partition function*. Adv. Math. **409**(2022), 108692.
[20] B. V. Pandey, *Higher Turán inequalities for the plane partition function*. Res. Number Theory **10**(2024), no. 69.
[21] M. L. Patrick, *Some inequalities concerning Jacobi polynomials*. SIAM J. Math. Anal. **2**(1971), no. 2, 213–220.
[22] M. L. Patrick, *Extensions of inequalities of the Laguerre and Turán type*. Pacific J. Math. **44**(1973), no. 2, 675–682.
[23] R. Rhoades, *Asymptotics for the number of strongly unimodal sequences*. Int. Math. Res. Notices **3**(2014), 700–719.
[24] D. Romik, *On the number of n-dimensional representations of* $\mathfrak{su}(3)$*, the Bernoulli numbers, and the Witten zeta function*. Acta Arith. **180**(2017), no. 2, 111–159.
[25] I. Wagner, *On a new class of Laguerre-Pólya type functions with applications in number theory*. Pacific J. Math. **320**(2022), 177–192.
[26] L. X. W. Wang and E. Y. Y. Yang, *Laguerre inequalities for discrete sequences*. Adv. Appl. Math. **139**(2022), 120357.
[27] L. X. W. Wang and N. N. Y. Yang, *Laguerre inequalities and complete monotonicity for the Riemann xi-function and the partition function*. Trans. Amer. Math. Soc. **377**(2024), 4703–4725.
[28] E. M. Wright, *Asymptotic partition formulae: I. Plane partitions*. Q. J. Math., Oxf. Ser. **2**(1931), 177–189.

*Center for Combinatorics, LPMC, Nankai University, China*
*e-mail*: 1120220002@mail.nankai.edu.cn 2120230007@mail.nankai.edu.cn wsw82@nankai.edu.cn

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press