Mathematika 63 (2017) 351–363 © 2017 University College London
doi:10.1112/S0025579316000280

HILBERT TRANSFORMS ALONG LIPSCHITZ DIRECTION FIELDS: A LACUNARY MODEL

SHAOMING GUO AND CHRISTOPH THIELE

Abstract. We prove bounds for the truncated directional Hilbert transform in $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$ under a combination of a Lipschitz assumption and a lacunarity assumption. It is known that a lacunarity assumption alone is not sufficient to yield boundedness for $p = 2$, and it is a major question in the field whether a Lipschitz assumption alone suffices, at least for some p.

§1. *Introduction.* The directional Hilbert transform and directional maximal operator in the plane have been recurrently studied in the literature, see [1–3, 6–11] and the references therein. A prominent question concerns suitable assumptions on the direction field, under which these operators are bounded in some $L^p(\mathbb{R}^2)$.

To be specific, define directional operators in the following form, parametrized by a measurable function $u : \mathbb{R}^2 \to (0, 1]$, and originally defined for functions f in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$:

$$H_u f(x, y) = p.v. \int_{-1}^{1} f(x - t, y - u(x, y)t) \frac{dt}{t}, \tag{1.1}$$

$$M_u f(x, y) = \sup_{\epsilon<1} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x - t, y - u(x, y)t)| \, dt. \tag{1.2}$$

One natural assumption is that u is Lipschitz with sufficiently small Lipschitz constant. One of the main open questions in the area is whether H_u and M_u are bounded in $L^2(\mathbb{R}^2)$ in this case. No $L^p(\mathbb{R}^2)$ bounds other than the trivial $L^\infty(\mathbb{R}^2)$ bound for M_u are known. An extensive discussion of this pair of conjectures appears in the work of Lacey and Li [9, 10].

Another natural assumption is that u takes values in a lacunary set. With such a mere assumption on the range of u, the truncation of the integral in H_u to $[-1, 1]$ and the constraint $\epsilon < 1$ in M_u are ineffective as they can be transformed by scaling. The following theorem addresses M_u and is a special case of a theorem proven by Nagel *et al* [11]. For simplicity we shall restrict attention to a particular lacunary set.

THEOREM 1.1 [11]. *For all $1 < p \leqslant \infty$ there is a constant C_p such that for all measurable functions u on $\mathbb{R}^2$ with values in the set $\{2^{-j}, j \in \mathbb{N}\}$ we have*

$$\left\| \sup_{\epsilon} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x - t, y - u(x, y)t)| \, dt \right\|_p \leqslant C_p \|f\|_p. \tag{1.3}$$

Received 22 August 2016, published online 9 February 2017.
MSC (2010): 42B20, 42B25 (primary).

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

It is known that a suitable generalization of lacunarity is the precise assumption on the range of u to make M_u bounded, see the work of Katz [8] and Bateman [1]. In contrast, Karagulyan [7] proved that H_u is not bounded for a lacunary set of directions from $L^2(\mathbb{R}^2)$ to $L^{2,\infty}(\mathbb{R}^2)$.

The purpose of this paper is to look at H_u for a combination of a Lipschitz assumption and a lacunarity assumption. Let $[x]$ denote the largest integer less than x.

THEOREM 1.2. *Let $u : \mathbb{R}^2 \to (0, 1]$ be Lipschitz with $\|u\|_{Lip} \leqslant 1$ and let $v : \mathbb{R}^2 \to (0, 1]$ be defined by*

$$\log_2 v = [\log_2 u]. \tag{1.4}$$

Then for every Schwartz function f and every $p \in (1, \infty)$ we have

$$\|H_v f\|_p \leqslant C_p \|f\|_p. \tag{1.5}$$

Here C_p is a constant depending only on p.

Note that the range of v is contained in the set of integer powers of two, a lacunary set. The function v itself cannot be Lipschitz, unless it is constant.

The above theorem has a dyadic model that we now present. For two points p_1 and p_2 in the unit line segment $(0, 1]$ (or the unit square $(0, 1]^2$) we define the dyadic distance to be the length of the smallest dyadic interval $(a, b]$ (or the side length of the smallest dyadic square) that contains both points. For a dyadic interval $I \subset (0, 1]$, denote by h_I its L^2 normalized Haar function. For a map v from $(0, 1]^2$ to $(0, 1]$ define the operator

$$H_{v,D} f(x, y) = \sum_{|J|/|I| \leqslant v(x,y)} \langle f, h_I \otimes h_J \rangle h_I(x) h_J(y), \tag{1.6}$$

where the sum runs over all dyadic rectangles $I \times J$ in the unit square with the stated bound on the eccentricity. We then have the following theorem.

THEOREM 1.3. *Let $v : (0, 1]^2 \to (0, 1]$ have Lipschitz constant (with respect to dyadic metric both on domain and target space) at most $1/2$ and assume it is lacunary in the sense it takes values in $\{2^{-k}, k \in \mathbb{N}\}$. Then, for all $p \in (1, \infty)$, we have*

$$\|H_{v,D} f\|_p \leqslant C_p \|f\|_p. \tag{1.7}$$

Here C_p is a constant which depends only on p.

We round up the discussion by recalling a third type of assumptions on the direction field, where a Lipschitz assumption recently surfaced naturally, namely bi-parameter assumptions. Bateman [2] and Bateman along with the second author [3] proved L^p bounds for H_u under the assumption that u depends only on one variable, $u(x, y) = u(x, 0)$ for all y, and $3/2 < p < \infty$. The first author [6] generalized this to direction fields constant along families of Lipschitz curves, highlighting the role of Lipschitz assumptions in this context.

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

§2. *The dyadic model: Proof of Theorem 1.3.* We write $s = I \times J$ for the dyadic rectangles below and we write $h_s(x, y) = h_I(x)h_J(y)$. We write the pth power of the L^p norm of (1.6) as an iterated integral:

$$\int_0^1 \left[\int_0^1 \left| \sum_J \sum_{I : |J|/|I| \leqslant v(x,y)} \langle f, h_s \rangle h_s(x, y) \right|^p dy \right] dx. \tag{2.1}$$

LEMMA 2.1. *Assume v is as in Theorem 1.3. Let $I \times J$ be a dyadic rectangle in $(0, 1]^2$ and let $x \in I$. If for some $y \in J$ we have $|J|/|I| \leqslant v(x, y)$, then we have $|J|/|I| \leqslant v(x, y')$ for all $y' \in J$.*

Proof. Assume to get a contradiction that $v(x, y') < |J|/|I|$ for some $y' \in J$. Since $v(x, y) \geqslant |J|/|I|$, the smallest dyadic interval containing both $v(x, y')$ and $v(x, y)$ is $(0, v(x, y)]$ and we have $d(v(x, y), v(x, y')) = v(x, y)$. This gives a contradiction, since the Lipschitz assumption implies

$$d(v(x, y), v(x, y')) \leqslant d(y, y')/2 \leqslant |J|/2 \leqslant |J|/(2|I|). \tag{2.2}$$

This finishes the proof of Lemma 2.1. □

Denote by $\mathbf{I}(x, J)$ the set of all dyadic intervals $I \subset (0, 1]$ such that there exists $y \in J$ with $|J|/|I| \leqslant v(x, y)$. Only the intervals in $\mathbf{I}(x, J)$ have non-zero contribution to the inner sum of (2.1). By Lemma 2.1, the condition $|J|/|I| \leqslant v(x, y)$ becomes void for intervals in $\mathbf{I}(x, J)$ and we may write for (2.1):

$$\int_0^1 \left[\int_0^1 \left| \sum_J \Big(\sum_{I \in \mathbf{I}(x,J)} \langle f, h_s \rangle h_s \Big) \right|^p dy \right] dx. \tag{2.3}$$

We have gained independence of the inner summation constraint in y and may use Littlewood–Paley theory in this variable to estimate the last display by

$$\lesssim \int_0^1 \left[\int_0^1 \left(\sum_J \left| \sum_{I \in \mathbf{I}(x,J)} \langle f, h_s \rangle h_s \right|^2 \right)^{p/2} dy \right] dx. \tag{2.4}$$

The set $\mathbf{I}(x, J)$ is convex in the sense that if it contains two intervals I and I', then it also contains all I'' with $I \subset I'' \subset I$. Hence, we can telescope the Haar sum in the variable x into a difference of two martingale averages and estimate these averages by the maximal operator. Hence, the last display is bounded by

$$\lesssim \int_0^1 \left[\int_0^1 \left(\sum_J |M_1(\langle f, h_J \rangle_2 h_J)|^2 \right)^{p/2} dy \right] dx, \tag{2.5}$$

where M_1 denotes the one-dimensional maximal operator in the x-variable and $\langle f, h_J \rangle_2$ denotes the inner product of f in the second variable with h_J.

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

We interchange the order of integration in x and y and apply the Fefferman–Stein maximal inequality to estimate the last display by

$$\lesssim \int_0^1 \left[\int_0^1 \left(\sum_J |\langle f, h_J \rangle_2 h_J|^2 \right)^{p/2} dx \right] dy. \tag{2.6}$$

Changing the integration order back and applying the Littlewood–Paley theory in the second variable again estimate the last display by $\lesssim \|f\|_p^p$, as desired.

§3. *Proof of Theorem 1.2: Initial reductions.* We write the Fourier transform of f as

$$\widehat{f}(\xi, \eta) = \iint f(x, y) \mathrm{e}^{-2\pi \mathrm{i}(x\xi + \eta y)} \, dx \, dy.$$

Our first step is to pass from H_v to a directional Fourier multiplier operator. The directional Fourier multiplier for a bounded measurable function $m : \mathbb{R} \to \mathbb{R}$ is defined as

$$T[m] f(x, y) := \iint \widehat{f}(\xi, \eta) m(\xi + \eta v(x, y)) \mathrm{e}^{2\pi \mathrm{i}(x\xi + y\eta)} \, d\xi \, d\eta. \tag{3.1}$$

Let ϕ be a real even Schwartz function with $\phi(0) = 1$, whose Fourier transform vanishes on $[-1, 1]$ and outside $[-2, 2]$:

$$\operatorname{supp}(\widehat{\phi}) \subset [-2, 2] \setminus [-1, 1]. \tag{3.2}$$

A calculation gives

$$T[\widehat{\phi} * (\mathrm{i}\pi \operatorname{sign})] f(x, y) = p.v. \int f(x - t, y - v(x, y)t) \phi(t) \frac{dt}{t}. \tag{3.3}$$

Comparing with H_v, since $t^{-1}(1_{[-1,1]} - \phi(t))$ is bounded and rapidly decaying, we obtain by a standard superposition argument

$$|(H_v - T[\widehat{\phi} * (\mathrm{i}\pi \operatorname{sign})]) f(x, y)| \leqslant C \sup_\epsilon \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} |f(x - t, y - v(x, y)t)| \, dt.$$

By Theorem 1.1, the right-hand side is bounded in L^p. Hence, it suffices to prove Theorem 1.2 with $T[\widehat{\phi} * \operatorname{sign}]$ in place of H_v. In the following, we will denote

$$m = \widehat{\phi} * \operatorname{sign}. \tag{3.4}$$

Note that $T[m]$ is well defined for functions whose Fourier transform is rapidly decaying. Given a Schwartz function f, we split it as a sum of two functions, one whose Fourier transform is supported in $\eta \geqslant 0$ (that is, the set $\{(\xi, \eta) : \eta \geqslant 0\}$), and one whose Fourier transform is supported in $\eta \leqslant 0$. It suffices to prove the bound of Theorem 1.2 separately for the two functions. By symmetry it suffices to consider the first function, we may thus without loss of generality assume $\widehat{f}$ is supported in $\eta \geqslant 0$.

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

Next we split the function as a sum of a function whose Fourier transform is supported in $|\xi| \leqslant 100$ and one whose Fourier transform is supported in $|\xi| \geqslant 100$. We consider the two functions separately. First assume $\widehat{f}$ is supported in $|\xi| \leqslant 100$.

Let ϕ be an even real Schwartz function as in (3.2). Moreover, we assume the normalization

$$\int_0^\infty \widehat{\phi}(t)\,\frac{dt}{t} = 1. \tag{3.5}$$

Set

$$\phi_t(x) := t\phi(tx) \tag{3.6}$$

and define the Littlewood–Paley operator in the second variable

$$P_t f(x, y) = \int f(x, y - z)\phi_t(z)\,dz. \tag{3.7}$$

We have by the classical Calderòn reproducing formula

$$f = \int_0^\infty P_t f\,\frac{dt}{t}.$$

Hence,

$$\begin{aligned} T[m]f(x, y) &= \int_0^\infty T_m[P_t f](x, y)\,\frac{dt}{t} \\ &= \int_0^{1000/v(x,y)} T_m[P_t f](x, y)\,\frac{dt}{t} \\ &\quad + \int_{1000/v(x,y)}^\infty P_t f(x, y)\,\frac{dt}{t}. \end{aligned} \tag{3.8}$$

Here we have dropped the operator $T[m]$ in the second integral since for $t > 1000/v(x, y)$ the support of $\widehat{P_t f}$ is contained in $|\xi| \leqslant 100$ and $\eta \geqslant 1000/v(x, y)$, where we have $m(\xi + \eta v(x, y)) = 1$.

Note that there is a Schwartz function $\varphi(\cdot)$ which coincides with

$$\int_0^1 \phi_t(\cdot)\,\frac{dt}{t} \tag{3.9}$$

outside the origin. Moreover, we denote $\varphi_t(\cdot) = t\varphi(t\cdot)$. We estimate the second term in (3.8) by

$$\begin{aligned} &\left|\int_{1000/v(x,y)}^\infty P_t f(x, y)\,\frac{dt}{t} - f(x, y)\right| = \left|\int_0^{1000/v(x,y)} P_t f(x, y)\,\frac{dt}{t}\right| \\ &\quad = \left|1000/v(x, y)\int \varphi_{1000/v(x,y)}(z) f(x, y - z)\,dz\right| \lesssim M_2 f(x, y), \end{aligned}$$

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

where M_2 is the maximal operator in the second variable,

$$M_2 f(x,y) = \sup_{\epsilon>0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x,y-z)|\, dz.$$

The classical bound on M_2 in L^p takes care of the second term in (3.8).

To estimate the first term in (3.8), we note

$$\int_0^{1000/v(x,y)} T[m]P_t f(x,y)\, \frac{dt}{t} = \iint \widehat{f}(\xi,\eta)\widehat{\varphi}(\eta v(x,y)/1000) m(\xi+\eta v(x,y)) e^{2\pi i(x\xi+y\eta)}\, d\xi\, d\eta.$$

Note that $\widehat{f}(\xi,\eta)\widehat{\varphi}(\eta v(x,y)/1000)$ is non-zero only if $|\xi| \leqslant 100$ and $\eta v(x,y) \leqslant 2000$. Moreover, we have for $n \geqslant 0$ the symbol estimates

$$\begin{aligned} |\partial_\xi^n (\widehat{\varphi}(\eta v(x,y)/1000) m(\xi+\eta v(x,y)))| &\leqslant C_n, \\ |\partial_\eta^n (\widehat{\varphi}(\eta v(x,y)/1000) m(\xi+\eta v(x,y)))| &\leqslant C_n v(x,y)^n. \end{aligned} \tag{3.10}$$

This means that if we dilate the multiplier $\widehat{\varphi}(\eta v(x,y)/100) m(\xi+\eta v(x,y))$ by a factor $v(x,y)$ in direction η we obtain a ((x,y)-dependent) multiplier with uniform symbol estimates, which therefore can be controlled pointwise by the Hardy–Littlewood maximal function. The undilated multiplier can then be estimated by the strong maximal function. Hence, we obtain for the second term in (3.8) the estimate

$$\int_0^{1000/v(x,y)} T[m]P_t f(x,y)\, \frac{dt}{t} \lesssim M_1 M_2 f(x,y), \tag{3.11}$$

where M_1 and M_2 are the maximal operators in the first and second variable.

This concludes the case that $\widehat{f}$ is supported in $|\xi| \leqslant 100$. Henceforth, we assume that $\widehat{f}$ is supported in $|\xi| \geqslant 100$. We note that for $\xi > 100$ and $\eta > 0$ we have $m(\xi+\eta v(x,y)) = 1$ and, hence, $T[m]f = f$ if f is supported in $\xi > 100$. We may thus assume f is supported in $\xi < -100$.

In what follows, the truncation of the directional Hilbert transform turns out unnecessary and we shall for simplicity remove it as follows. By the fundamental theorem of calculus we write

$$m(\xi) = -1 + \int_{-10}^{10} m'(\tau) 1_{(\tau,\infty)}(\xi)\, d\tau, \tag{3.12}$$

where we have used that m' vanishes outside $[-10, 10]$. Since $T[-1]$ is clearly bounded, it suffices to prove bounds on $T[1_{(\tau,\infty)}]$ for any $|\tau| \leqslant 10$.

Define

$$H_{\tau,j} f(x,y) := \iint \widehat{f}(\xi,\eta) 1_{(\tau,\infty)}(\xi+\eta 2^{-j}) e^{2\pi i(x\xi+y\eta)}\, d\xi\, d\eta. \tag{3.13}$$

We need the Cordoba and Fefferman [4] inequality in the form of [5, Theorem 6.1]. It states that for any $1 < p < \infty$ and any collection of functions f_j in $L^p(\mathbb{R}^2)$ we have

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

$$\left\| \left(\sum_j |H_{0,j} f_j|^2 \right)^{1/2} \right\|_p \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_p. \tag{3.14}$$

This is proved in [5] using the maximal function bound of Theorem 1.1. Applying the above estimate with the modulated functions

$$f_j(x, y) \mathrm{e}^{2\pi \mathrm{i} \tau x},$$

we obtain more generally

$$\left\| \left(\sum_j |H_{\tau,j} f_j|^2 \right)^{1/2} \right\|_p \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_p. \tag{3.15}$$

Let ϕ_s be as in (3.2), (3.5), (3.6) and define the Littlewood–Paley operator in the first variable

$$P_{s,1} g(x, y) := \int g(x - z, y) \phi_s(z)\, dz. \tag{3.16}$$

Since $\widehat{f}$ is supported in $\xi < -100$, we have the Calderòn reproducing formula in the form

$$f = \int_{10}^{\infty} P_{s,1} f \frac{ds}{s}. \tag{3.17}$$

Now let ψ be an even real Schwartz function supported in $[-1, 1]$ with $\widehat{\psi}(0) = 0$ and

$$\int_0^{\infty} (\widehat{\psi}(t))^2 \frac{dt}{t} = 1,$$

and define

$$P_{t,2} g(x, y) := \int g(x, y - z) \psi_t(z)\, dz. \tag{3.18}$$

Then, by the reproducing formula again,

$$f = \int_{10}^{\infty} \int_0^{\infty} P_{t,2}^2 P_{s,1} f \frac{dt}{t} \frac{ds}{s}. \tag{3.19}$$

For $j \in \mathbb{Z}$ let E_j be the set of (x, y) such that $v(x, y) = 2^{-j}$. Then

$$\begin{aligned} T[1_{(\tau,\infty)}] f &= \sum_{j \geqslant 0} 1_{E_j} H_{\tau,j} f \\ &= \sum_{d \in \mathbb{Z}} \sum_{j \geqslant 0} 1_{E_j} \int_{10}^{\infty} \int_{2^{j+d}s}^{2^{j+d+1}s} H_{\tau,j} P_{t,2}^2 P_{s,1} f \frac{dt}{t} \frac{ds}{s}. \end{aligned} \tag{3.20}$$

We consider a summand in (3.20) for fixed d. Setting

$$f_j = \int_{10}^{\infty} \int_{2^{d+j}s}^{2^{d+j+1}s} P_{t,2}^2 P_{s,1} f \frac{dt}{t} \frac{ds}{s}, \tag{3.21}$$

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

we recognize with the Cordoba–Fefferman inequality (3.15)

$$\Big\| \sum_j 1_{E_j} \int_{10}^{\infty} \int_{2^{j+d}s}^{2^{j+d+1}s} H_{\tau,j} P_{t,2}^2 P_{s,1} f \, \frac{dt}{t} \frac{ds}{s} \Big\|_p$$
$$= \Big\| \sum_j 1_{E_j} H_{\tau,j} f_j \Big\|_p \leqslant \Big\| \Big(\sum_j |H_{\tau,j} f_j|^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_p .$$

To estimate the last term, by Khintchine's inequality, we need to estimate

$$\sum_j \epsilon_j f_j = \sum_j \epsilon_j \int_{10}^{\infty} \int_{2^{d+j}s}^{2^{d+j+1}s} P_{t,2}^2 P_{s,1} f \, \frac{dt}{t} \frac{ds}{s} \tag{3.22}$$

in L^p, uniformly in all choices of $|\epsilon_j| \leqslant 1$. However, the last expression is identified as a Marcinkiewicz multiplier applied to f, see [**12**, Ch. IV].

This estimates each term in (3.20) uniformly in d. To obtain summability in d, we proceed to refine this estimate, both for $d \leqslant -5$ and for $d \geqslant 5$.

Consider first $d \leqslant -5$. Let φ be a Schwartz function with $\widehat{\varphi}$ supported in $[-2, 2]$ and constantly equal to one on $[-1, 1]$. Define

$$A_{t,2} g(x, y) = \int g(x, y - z) \varphi_t(z) \, dz. \tag{3.23}$$

We claim that for $d \leqslant -5$ the summand in (3.20) is equal to

$$\sum_j 1_{E_j} \int_{10}^{\infty} \int_{2^{j+d}s}^{2^{j+d+1}s} H_{\tau,j} (1 - A_{2^{-d/2}t,2}) P_{t,2}^2 P_{s,1} f \, \frac{dt}{t} \frac{ds}{s}. \tag{3.24}$$

Namely, for fixed j and $(x, y) \in E_j$ and fixed $s > 10$ and $2^{j+d}s \leqslant t \leqslant 2^{j+d+1}s$ we have

$$H_{\tau,j} A_{2^{-d/2}t,2} P_{t,2}^2 P_{s,1} f$$
$$= \iint \widehat{P_{t,2}^2 f}(\xi, \eta) 1_{(\tau,\infty)}(\xi + \eta 2^{-j}) \widehat{\phi}(s^{-1}\xi) \widehat{\varphi}(2^{d/2} t^{-1} \eta) \mathrm{e}^{2\pi \mathrm{i}(x\xi + y\eta)} \, d\xi \, d\eta. \tag{3.25}$$

We claim the last expression vanishes, since the integrand vanishes. For $\widehat{P_{t,2}^2 f}(\xi, \eta)$ not to vanish we need $\xi < -100$. For $\widehat{\phi}(s^{-1}\xi)$ not to vanish we need $s < |\xi| < 2s$, and for $\widehat{\varphi}(2^{d/2} t^{-1} \eta)$ not to vanish we need $\eta \leqslant 2^{1-d/2} t$. Hence,

$$\xi + \eta 2^{-j} \leqslant -s + 2^{1-d/2-j} t \leqslant -s + 2^{1+d/2} s \leqslant -50. \tag{3.26}$$

Hence, $1_{(\tau,\infty)}(\xi + \eta 2^- j) = 0$ since $|\tau| \leqslant 10$ and the integrand vanishes.

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

Applying the Cordoba–Fefferman inequality as above and observing that, by the Marcinkiewicz multiplier theorem, for any choices of $|\epsilon_j| \leqslant 1$ we obtain

$$\Big\| \sum_j \epsilon_j \int_{10}^{\infty} \int_{2^{d+j}s}^{2^{d+j+1}s} P_{t,2} P_{s,1} (1 - A_{2^{d/2}t,2}) P_{t,2} f \, \frac{dt}{t} \frac{ds}{s} \Big\| \lesssim 2^{-d}, \quad (3.27)$$

that is, we obtain the desired decay in d. Here we have used that for $d \leqslant -5$ the portion $(1 - A_{2^{d/2}t,2}) P_{t,2}$ produces very small symbol estimates for the Marcinkiewicz multiplier.

To obtain good bounds for $d \geqslant 5$ we compare with the operator

$$Sf(x, y) := \sum_{d \geqslant 5} \sum_j 1_{E_j} \int_1^{\infty} \int_{2^{d+j}s}^{2^{d+j+1}s} P_{t,2}^2 P_{s,1} f \, \frac{dt}{t} \frac{ds}{s}, \quad (3.28)$$

which we will estimate in the next section.

We claim that

$$\sum_j 1_{E_j} \Big[\int_{10}^{\infty} \int_{2^{j+d}s}^{2^{j+d+1}s} H_{\tau,j} P_{t,2}^2 P_{s,1} f \, \frac{dt}{t} \frac{ds}{s} - \int_{10}^{\infty} \int_{2^{d+j}s}^{2^{d+j+1}s} P_{t,2}^2 P_{s,1} f \, \frac{dt}{t} \frac{ds}{s} \Big]$$
$$= \sum_j 1_{E_j} \int_1^{\infty} \int_{2^{j+d}s}^{2^{j+d+1}s} H_{\tau,j} A_{2^{-d/2}t,2} P_{t,2}^2 P_{s,1} f \, \frac{dt}{t} \frac{ds}{s}.$$

The argument is similar as before. The integrand on the right-hand side takes again the form (3.25). We need that for those $\eta > 0$ with $\widehat{\varphi}(2^{d/2} t^{-1} \eta) \neq 1$, that is $\eta > t 2^{-d/2}$, we have that either the integrand vanishes or $1_{(\tau,\infty)}(\xi + \eta 2^{-j}) = 1$. But if the integrand does not vanish, we have

$$\xi + \eta 2^{-j} \geqslant -2s + 2^{-d/2-j} t \geqslant -2s + 2^{d/2} s \geqslant 50. \quad (3.29)$$

This proves the claim.

We then obtain similarly the decay for Marcinkiewicz multiplier estimates, using that for $d > 5$ the portion $A_{2^{-d/2}t,2} P_{t,2}^2$ produces small symbol estimates since $\widehat{\psi}(0) = 0$. This finishes all steps of initial reductions.

§4. *Proof of Theorem 1.2: Main argument.* We rewrite the operator S as

$$\int_{10}^{\infty} \int_{32s/v(x,y)}^{\infty} P_{t,2}^2 P_{s,1} f(x, y) \, \frac{dt}{t} \frac{ds}{s}. \quad (4.1)$$

It suffices to prove for every pair of functions f, g with $\|f\|_p = 1$ and $\|g\|_{p'} = 1$ that

$$\iiint_{10}^{\infty} \int_{32s/v(x,y)}^{\infty} P_{t,2}^2 P_{s,1} f(x, y) \overline{g(x, y)} \, \frac{dt}{t} \frac{ds}{s} \, dx \, dy \lesssim 1.$$

We fix s and x and consider the integrand in these variables:

$$\iint_{32s/v(x,y)}^{\infty} P_{t,2}^2 P_{s,1} f(x, y) \overline{g(x, y)} \, \frac{dt}{t} \, dy. \quad (4.2)$$

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

We write out one of the $P_{t,2}$ convolution operators to obtain for the last display:

$$\iint_{32s/v(x,y)}^{\infty} \int P_{t,2}P_{s,1}f(x,y-z)\psi_t(z)\overline{g(x,y)}\,dz\,\frac{dt}{t}\,dy. \tag{4.3}$$

We pass the z integration outside the t integration. Then we compare the value $v(x,y)$ occurring in the specification of the integration domain with the value $v(x,y-z)$. The difference we consider as an error term

$$E(x,s) := \iint_{32s/v(x,y)}^{32s/v(x,y-z)} P_{t,2}P_{s,1}f(x,y-z)\psi_t(z)\overline{g(x,y)}\,\frac{dt}{t}\,dz\,dy, \tag{4.4}$$

which we estimate later. In the remaining integral, with t-integration in the domain $s/v(x,y-z) < t < \infty$, we use the variable $\tilde{y} = y - z$ in place of y and obtain

$$\iint_{32s/v(x,\tilde{y})}^{\infty} P_{t,2}P_{s,1}f(x,\tilde{y})\psi_t(z)\overline{g(x,\tilde{y}+z)}\,\frac{dt}{t}\,d\tilde{y}\,dz.$$

The z integral is recognized as a Littlewood–Paley operator acting on g. Writing again the x, s integrations and calling $\tilde{y}$ again y we need to estimate

$$\iiint_{10}^{\infty}\int_{32s/v(x,y)}^{\infty} P_{t,2}P_{s,1}f(x,y)\overline{P_{t,2}g(x,y)}\,\frac{dt}{t}\,\frac{ds}{s}\,dx\,dy. \tag{4.5}$$

We now interchange the s and t integrations and apply Cauchy–Schwarz and Hölder to estimate the last display by

$$\begin{aligned}
&\iint_{320/v(x,y)}^{\infty}\int_{10}^{tv(x,y)/32} P_{t,2}P_{s,1}f(x,y)\overline{P_{t,2}g(x,y)}\,\frac{ds}{s}\,\frac{dt}{t}\,dx\,dy\\
&\quad\leqslant \iint\Big(\int_0^{\infty}\Big|\int_{10}^{tv(x,y)/32} P_{t,2}P_{s,1}f(x,y)\,\frac{ds}{s}\Big|^2\,\frac{dt}{t}\Big)^{1/2}\\
&\qquad\times\Big(\int_0^{\infty}|\overline{P_{t,2}g(x,y)}|^2\,\frac{dt}{t}\Big)^{1/2}dx\,dy\\
&\quad\leqslant \Big\|\Big\|\Big(\int_0^{\infty}\Big|\int_{10}^{tv(x,y)/32} P_{t,2}P_{s,1}f(x,y)\,\frac{ds}{s}\Big|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(x)}\Big\|_{L^p(y)}.
\end{aligned}$$

Here we have used the Littlewood–Paley square function estimate for the function g in the second variable. Telescoping the s integral similarly to (3.9) and using the maximal function M_1 in the first direction, we estimate the last display by

$$\lesssim \Big\|\Big\|\Big(\Big(\int_0^{\infty} M_1(P_{t,2}f)(x,y)\Big)^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(x)}\Big\|_{L^p(y)}. \tag{4.6}$$

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

Applying the Fefferman–Stein vector-valued inequality for the maximal function in the first variable gives the bound

$$\lesssim \left\| \left\| \left(\int_0^\infty P_{t,2} f(x, y) \frac{ds}{s} \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(x)} \Big\|_{L^p(y)}. \tag{4.7}$$

Commuting the L^p norms and using the Littlewood–Paley square function estimate in the second variable controls the last display by $\lesssim \|f\|_p = 1$.

We turn to the estimate for the error term E. We distinguish the case $v(x, y-z) < v(x, y)$ and $v(x, y-z) > v(x, y)$, more precisely we write $E = E_< + E_>$ with $E_<(x, s)$ equal to

$$\iint 1_{s/v(x,y-z) > s/v(x,y)} \int_{32s/v(x,y)}^{32s/v(x,y-z)} \times P_{t,2} P_{s,1} f(x, y - z) \psi_t(z) \overline{g(x, y)} \frac{dt}{t} \, dz \, dy.$$

Define $w : \mathbb{R}^2 \to (0, 1]$ by

$$\log_2 w(x, y) = [1/2 + \log_2 u], \tag{4.8}$$

that is, w at any point is either equal to v or twice as large. We claim that $E_<(x, s)$ is equal to

$$\iiint_{32s/w(x,y-z)}^{64s/w(x,y-z)} P_{t,2} P_{s,1} f(x, y - z) \times \psi_t(z) \overline{g(x, y)} 1_{2v(x,y-z)=w(x,y-z)} 1_{v(x,y)=w(x,y)} \frac{dt}{t} \, dz \, dy. \tag{4.9}$$

To see the claim, note that the only change between the expressions concerns the coding of the domain of integration. Thus, we need to show that the domains of integration are equal. Consider a point in the domain of integration of the defining expression for $E_>$. We thus have $s/v(x, y - z) > s/v(x, y)$. We may also assume that $\psi_t(z) \neq 0$, or else the integrand vanishes. Then $|z| \leqslant t^{-1}$ from the support of ψ_t. By the Lipschitz assumption on u,

$$|u(x, y) - u(x, y - z)| \leqslant |z| \leqslant t^{-1} \leqslant u(x, y - z)/32, \tag{4.10}$$

where in the last inequality we have used $t \in [32s/v(x, y), 32s/v(x, y - z)]$ and $s > 1$. Thus, $\log_2 u(x, y)$ and $\log_2 u(x, y - z)$ differ by less than $1/4$. Since $v(x, y) > v(x, y - z)$ with a strict inequality, then $u(x, y)$ has to be slightly above an integer power of 2 and $u(x, y - z)$ has to be slightly below the integer power. Hence, we conclude

$$2v(x, y - z) = v(x, y) = w(x, y - z) = w(x, y). \tag{4.11}$$

This shows that the essential domain of integration in the defining expression of $E_<$ is contained in the essential domain of integration of (4.9). The converse implication is rather straight forward.

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

Setting $\tilde{g}(x, y) = g(x, y)1_{v(x,y)=2w(x,y)}$ we thus obtain with the change of variables $\tilde{y} = y - z$ for $E_<(x, s)$

$$\iiint_{32s/w(x,\tilde{y})}^{64s/w(x,\tilde{y})} P_{t,2}P_{s,1}f(x, \tilde{y})\psi_t(z)\overline{P_{t,2}\tilde{g}(x, \tilde{y})}1_{v(x,\tilde{y})=w(x,\tilde{y})}\,\frac{dt}{t}\,dz\,d\tilde{y}. \tag{4.12}$$

Integrating in x and s with the s integration inside the t integration and applying Cauchy–Schwarz in the t integration and calling $\tilde{y}$ again y we estimate the last display similarly to before by

$$\begin{aligned}
&\leqslant \iint \left(\int_0^\infty \left|\int_{32tw(x,y)}^{64tw(x,y)} P_{t,2}P_{s,1}f(x, y)\,\frac{ds}{s}\right|^2 \frac{dt}{t}\right)^{1/2} \\
&\qquad \times \left(\int_0^\infty |\overline{P_{t,2}\tilde{g}(x, y)}|^2\,\frac{dt}{t}\right)^{1/2} dx\,dy \\
&\lesssim \left\| \left\| \left(\int_0^\infty \left|M_1(P_{t,2}f)(x, y)\,\frac{ds}{s}\right|^2 \frac{dt}{t}\right)^{1/2} \right\|_{L^p(x)} \right\|_{L^p(y)} \lesssim 1.
\end{aligned}$$

The finishes the estimate for $E_<$. The estimate for $E_>$ is very similar, hence we leave it out.

Acknowledgements. The second author acknowledges support by the Hausdorff Center for Mathematics. We thank Pavel Zorin-Kranich and Olli Saari for catching a typo.

References

1. M. Bateman, Kakeya sets and directional maximal operators in the plane. *Duke Math. J.* **147**(1) (2009), 55–77.
2. M. Bateman, Single annulus L^p estimates for Hilbert transforms along vector fields. *Rev. Mat. Iberoam.* **29**(3) (2013), 1021–1069.
3. M. Bateman and C. Thiele, L^p estimates for the Hilbert transforms along a one-variable vector field. *Anal. PDE* **6**(7) (2013), 1577–1600.
4. A. Cordoba and R. Fefferman, On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis. *Proc. Natl Acad. Sci. USA* **74**(2) (1977), 423–425.
5. C. Demeter and P. Silva, Some new light on a few classical results. *Colloq. Math.* **140**(1) (2015), 129–147.
6. S. Guo, Hilbert transform along measurable vector fields constant on Lipschitz curves: L^2 boundedness. *Anal. PDE* **8**(5) (2015), 1263–1288.
7. G. A. Karagulyan, On unboundedness of maximal operators for directional Hilbert transforms. *Proc. Amer. Math. Soc.* **135**(10) (2007), 3133–3141 (electronic).
8. N. Katz, Maximal operators over arbitrary sets of directions. *Duke Math. J.* **97**(1) (1999), 67–79.
9. M. Lacey and X. Li, Maximal theorems for the directional Hilbert transform on the plane. *Trans. Amer. Math. Soc.* **358**(9) (2006), 4099–4117.
10. M. Lacey and X. Li, On a conjecture of E. M. Stein on the Hilbert transform on vector fields. *Mem. Amer. Math. Soc.* **205**(965) (2010), viii+72 pp., ISBN: 978-0-8218-4540-0.
11. A. Nagel, E. Stein and S. Wainger, Differentiation in lacunary directions. *Proc. Natl Acad. Sci. USA* **75**(3) (1978), 1060–1062.
12. E. Stein, *Singular Integrals and Differentiability Properties of Functions* (*Princeton Mathematical Series* **30**), Princeton University Press (Princeton, NJ, 1970).

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press

Shaoming Guo,
Institute of Mathematics,
University of Bonn,
Endenicher Allee 60,
53115, Bonn,
Germany
E-mail: shaoguo@iu.edu

Current address:
Department of Mathematics,
Indiana University,
831 E Third St,
Bloomington, IN 47405,
U.S.A.

Christoph Thiele,
Institute of Mathematics,
University of Bonn,
Endenicher Allee 60,
53115, Bonn,
Germany
E-mail: thiele@math.uni-bonn.de

https://doi.org/10.1112/S0025579316000280 Published online by Cambridge University Press