Math. Proc. Camb. Phil. Soc. (1988), **103**, 389
Printed in Great Britain

Multiplicative functions at consecutive integers. II

ADOLF HILDEBRAND
Department of Mathematics, University of Illinois, Urbana, Illinois 61801, *U.S.A.*

(*Received* 24 *August* 1987)

1. *Introduction*

The global behaviour of multiplicative arithmetic functions has been extensively studied and is now well understood for a large class of multiplicative functions. In particular, Halász [5] completely determined the asymptotic behaviour of the means

$$\frac{1}{x}\sum_{n\leqslant x} g(n) \tag{1·1}$$

for multiplicative functions g satisfying $|g| \leqslant 1$, and gave necessary and sufficient conditions for the existence of the 'mean value'

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x} g(n). \tag{1·2}$$

In contrast to this, very little is known about the *local* behaviour of multiplicative functions, i.e. the behaviour on short sequences of consecutive integers $n, n+1, \ldots, n+k$, with fixed or slowly increasing k. One expects that the values $g(n), \ldots, g(n+k)$, are, in an appropriate sense, mutually independent for a 'typical' multiplicative function g. However, proving concrete results in this direction is a very difficult problem, even in the simplest case $k = 1$. A natural approach here would be to consider the averages

$$\frac{1}{x}\sum_{n\leqslant x} \overline{g(n+1)}\,g(n) \tag{1·3}$$

and try to obtain, for the case $|g| \leqslant 1$, say, analogues of Halász' results on the behaviour of the ordinary averages (1·1). Unfortunately, Halász' analytic method cannot be used to deal with (1·3), since the corresponding Dirichlet series do not have an Euler product representation, and it seems that the problem lies very deep.

A particularly interesting example is given by the Liouville function $\lambda(n)$, defined as 1 if n has an even number of prime factors (counted with multiplicity), and -1 if n has an odd number of prime factors. Here one expects that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}\lambda(n+1)\lambda(n) = 0 \tag{1·4}$$

holds, but at present it has not even been proved that

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}\lambda(n)\lambda(n+1) < 1. \tag{1·5}$$

In other words, it is conceivable that for 'most' n we have $\lambda(n) = \lambda(n+1)$, in which case the functions $\lambda(n)$ and $\lambda(n+1)$ would be far from being independent. It is

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

possible that (1·4) lies as deep as the twin prime conjecture, for it amounts to resolving, in a certain sense, the 'parity problem' in sieve theory, which constitutes the main obstacle to proving the twin prime conjecture by sieve methods (cf. [4, 9]).

In this paper we shall consider the problem of characterizing the multiplicative functions g that satisfy $|g| = 1$ and

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \overline{g(n+1)}\, g(n) = 1. \tag{1·6}$$

Although a complete solution is not known, this problem is presumably easier than that of determining the behaviour of (1·3) in general, since (1·6) is equivalent (in the case $|g| = 1$) to

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} |g(n+1)-g(n)| = 0, \tag{1·7}$$

which is a rather strong condition.

Trivially, (1·7) holds for functions of the type $g(n) = n^{i\alpha}$, where α is a real number, and it has been conjectured (cf. [11]) that these are the only multiplicative functions of modulus 1 that satisfy (1·7). This conjecture, if true, would imply (1·5).

Although the conjecture remains open, a few partial results are known. Mauclaire and Murata [14] showed that any multiplicative function g of modulus 1 satisfying (1·7) must be completely multiplicative. In [7] we showed that, if g assumes only the values ± 1, then

$$\limsup_{x\to\infty} \frac{(\log\log x)^4}{x} \sum_{n\leqslant x} |g(n+1)-g(n)| > 0$$

holds unless $g \equiv 1$. Also, it is known (cf. Lemma 1 below) that the conjecture is true for functions g that are, in a suitable sense, sufficiently close to one of the functions $n^{i\alpha}$.

Our object here is to prove the conjecture under a relatively mild restriction on the values $g(p)$.

THEOREM 1. *There exists a positive constant c with the following property. Let g be a completely multiplicative function satisfying $|g| = 1$, $g \not\equiv 1$ and*

$$|g(p)-1| \leqslant c \quad \textit{for all primes } p. \tag{1·8}$$

Then

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} |g(n+1)-g(n)| > 0. \tag{1·9}$$

Stated differently (and in a slightly weaker form), the theorem asserts that under the condition (1·8), (1·7) can only hold if $g \equiv 1$. The condition (1·8) can be relaxed somewhat; for example, as will be clear from the proof, it suffices to require (1·8) for sufficiently large primes p. The constant c in (1·8) could be given a numerical value. An admissible value is $c = 10^{-3}$, but probably one can attain $c = 10^{-1}$ using sufficient care in the estimates.

2. *Reformulation*

By writing a multiplicative function g of modulus 1 in the form $g = e^{2\pi i f}$, where f is an additive function, one can reformulate results involving multiplicative functions of modulus 1 in terms of real-valued additive functions reduced modulo 1. One easily sees in this way that Theorem 1 is equivalent to the following result.

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

THEOREM 2. *There exists a positive constant c with the following property. Let f be a real-valued, completely additive function satisfying $\|f\| \not\equiv 0$ and*

$$\|f(p)\| \leqslant c \quad \text{for all primes } p, \tag{2·1}$$

where $\|u\|$ denotes the distance of u to the nearest integer. Then

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \|f(n+1)-f(n)\| > 0. \tag{2·2}$$

This result is related to the problem of characterizing the logarithm as an additive function. Erdös[2] showed in 1946 that an additive function satisfying

$$f(n+1)-f(n)\to 0 \quad (n\to\infty) \tag{2·3}$$

must be of the form $f = \lambda \log$ for some constant λ. Kátai[10] and Wirsing[16] later relaxed the condition (2·3) to

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} |f(n+1)-f(n)| = 0. \tag{2·4}$$

In [8] it was shown that (2·4) can be further relaxed to

$$f(n+1)-f(n)\to 0 \quad (n\to\infty \text{ through a set of positive density}). \tag{2·5}$$

Several authors have considered the problem of finding similar conditions involving $\|f(n+1)-f(n)\|$ that imply $\|f-\lambda \log\| \equiv 0$ for some λ, i.e. $f = \lambda \log + f_1$, where f_1 is an integer-valued additive function. Improving on earlier results of Kátai[12], Wirsing[17] recently showed that the condition

$$\lim_{n\to\infty} \|f(n+1)-f(n)\| = 0, \tag{2·6}$$

i.e. the analogue of (2·3), implies $\|f-\lambda \log\| \equiv 0$ for some λ. Kátai[11] has conjectured that (2·6) can be replaced by an average condition, namely

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \|f(n+1)-f(n)\| = 0, \tag{2·7}$$

as had been the case with Erdös' condition (2·3). While this conjecture remains open in its full scope, it follows from Theorem 2 that the conjecture is true for additive functions satisfying (2·1).

We conclude this section by giving an application of Theorem 2. Let E be a non-empty set of primes, and let $\Omega_E(n)$ denote the number of prime divisors of n (counted with multiplicity) belonging to the set E.

COROLLARY. *We have*

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \|\alpha(\Omega_E(n+1)-\Omega_E(n))\| > 0 \tag{2·8}$$

for all except at most finitely many numbers $\alpha\in(0,1)$. More precisely, (2·8) holds for all irrational α, as well as for $\alpha = r/s$, whenever r and s are coprime integers with $s > \max(2, c^{-1})$, where c is the constant in Theorem 2.

In the case $\alpha = 1/m$, (2·8) is equivalent to the statement that the set of integers n which satisfy

$$\Omega_E(n+1) \not\equiv \Omega_E(n) \pmod m.$$

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

has positive lower density. The corollary shows that this is the case, provided that m is sufficiently large. If the same were true for $m = 2$, we would obtain the relation (1·5) for the Liouville function.

To deduce the corollary, we fix a real number α and suppose that it is not of the form $\alpha = r/s$ with $s \leqslant \max(2, c^{-1})$. Then there exists an integer k such that

$$0 < \|k\alpha\| \leqslant c. \tag{2·9}$$

The function

$$f(n) = k\alpha\Omega_E(n)$$

then satisfies the hypothesis (2·1) of Theorem 2. Moreover, $\|f\| \not\equiv 0$ since, by (2·9), $\|f(p)\| = \|k\alpha\| \neq 0$ for all $p \in E$ and E was assumed to be non-empty. Therefore Theorem 2 applies and yields

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \|k\alpha(\Omega_E(n+1) - \Omega_E(n))\| > 0,$$

and hence the asserted relation (2·8) follows, since

$$\|k\alpha(\Omega_E(n+1) - \Omega(n))\| \leqslant k\|\alpha(\Omega_E(n+1) - \Omega_E(n))\|.$$

3. *Outline of the proof*

We shall prove our result in the form of Theorem 2, which is slightly more convenient than proving Theorem 1. As already indicated, the two theorems are equivalent.

To prove (2·2), it suffices to obtain a non-trivial upper bound for the quantity

$$\sum_{\substack{n\leqslant x \\ \|f(n+1)-f(n)\|<\delta}} 1, \tag{3·1}$$

where δ is a fixed positive number that we can take arbitrarily small. To this end we shall use an idea that in similar form has been employed by Erdös, Pomerance and Sarközy in [3] to obtain an upper bound for the number of solutions $n \leqslant x$ of $d(n+1) = d(n)$. Namely, we shall represent n and $n+1$ in the form

$$n = mp, \quad n+1 = m'p', \tag{3·2}$$

where p and p' are the largest prime factors of n and $n+1$, respectively, and then estimate, for each pair (m, m'), the number of pairs (p, p') that may arise in this way. Using a standard sieve result, it is easy to obtain an upper for this number that has the expected order of magnitude, provided that $x/(mm')$ exceeds a fixed power of x. To ensure that this condition is always satisfied we shall therefore modify (3·1) by restricting the summation to integers n for which (in the representation (3·2)) both p and p' exceed $x^{\frac{3}{5}}$. Since there are $\geqslant x$ such integers $n \leqslant x$ (cf. Lemma 3 below), it will be sufficient to show that this modified form of (3·1) is small compared to x.

Taking into account the assumption (2·1), which implies that $\|f(n+1) - f(n)\|$ is approximately equal to $\|f(m) - f(m')\|$, we shall arrive in this way at an upper bound of the type

$$\frac{x}{\log^2 x} \sum_{\substack{m,m'\leqslant x \\ \|f(m)-f(m')\|\leqslant\delta+2c}} \frac{1}{\phi(m)\,\phi(m')}.$$

This quantity will be small compared to x, provided δ and c are small and the values

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

of $\|f\|$ are evenly distributed over the interval $(0, 1)$. Using a theorem of Halász and appealing again to the hypothesis (2·1), we shall see that the last condition is satisfied in the case the series

$$\sum_p \frac{\|f(p)\|^2}{p}$$

diverges. If, however, this series converges, then a different and much simpler argument will yield the desired conclusion.

4. *Lemmas*

LEMMA 1. *Let g be a multiplicative function satisfying $|g| = 1$ and suppose that for some real number α the series*

$$\sum_p \frac{|1-g(p)\,p^{i\alpha}|^2}{p} \tag{4·1}$$

converges. Then we have

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \overline{g(n+1)}\,g(n) = \prod_p \Phi_p, \tag{4·2}$$

where

$$\Phi_p = 1-\frac{2}{p}+2\left(1-\frac{1}{p}\right) \operatorname{Re} \sum_{m\geqslant 1} \frac{g(p^m)\,p^{im\alpha}}{p^m}.$$

This result was proved in [8]. The proof is comparatively simple; its basic idea is to approximate g by 'truncated' multiplicative functions g_r, for which the conclusion of the lemma is easy to verify, and to show, using the hypotheses of the lemma, that the error involved in this approximation can be made arbitrarily small.

We shall use this lemma to dispose of an 'easy' case in the proof of Theorem 2. Our main task will be to deal with the situation when the condition of the lemma is not satisfied, i.e. when the series (4·1) diverges for all real numbers α. This case is also the 'hard' case in Halász' results.

The next lemma is a special case of Halász' mean value theorem (cf. theorem 6·3 of [1]).

LEMMA 2. *Let g be a multiplicative function satisfying $|g| \leqslant 1$ and suppose that the series (4·1) diverges for all real α. Then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} g(n) = 0.$$

In the remainder of this section we shall use the symbols '$\ll$' and 'O' with their usual meanings, the implied constants being absolute.

LEMMA 3. *For sufficiently large x we have*

$$|\{n \leqslant x : P(n) > x^{\frac{3}{5}}, P(n+1) > x^{\frac{3}{5}}\}| \gg x,$$

where $P(n)$ denotes the largest prime factor of n.

Proof. The quantity to be estimated is $\geqslant |A \cap (A-1)|$, where

$$A = \{n \leqslant x : P(n) > x^{\frac{3}{5}}\}, \quad A-1 = \{a-1 : a \in A\}.$$

The inclusion–exclusion formula gives

$$|A \cap (A-1)| = |A| + |A-1| - |A \cup (A-1)|.$$

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

Since

$$|A-1| = |A| = \sum_{x^{\frac{3}{5}} < p \leqslant x} \left[\frac{x}{p}\right] = x\left(\sum_{x^{\frac{3}{5}} < p \leqslant x} \frac{1}{p}\right) + O\left(\frac{x}{\log x}\right)$$
$$= x \log \tfrac{5}{3} + O\left(\frac{x}{\log x}\right)$$

and

$$|A \cup (A-1)| \leqslant x + O(1),$$

we obtain

$$|A \cap (A-1)| \geqslant x\left(2\log \tfrac{5}{3} - 1 + O\left(\frac{1}{\log x}\right)\right) \geqslant 0{\cdot}02x,$$

provided that x is sufficiently large. This proves the asserted result.

The next result can be deduced from any standard sieve upper bound, for example theorem 2·3 in [6].

LEMMA 4. *Let a_i, b_i $(i = 1, 2)$ be positive integers satisfying $a_1 b_2 - a_2 b_1 = 1$. Then we have, for any $y \geqslant 2$,*

$$|\{1 \leqslant n \leqslant y : a_i n + b_i \text{ prime for } i = 1, 2\}| \ll \frac{a_1 a_2}{\phi(a_1)\phi(a_2)} \frac{y}{\log^2 y},$$

where ϕ is the Euler function.

LEMMA 5. *Let a_n, $1 \leqslant n \leqslant N$, be arbitrary real numbers. Then, for any $\delta > 0$ and any real number γ,*

$$\frac{1}{N} \sum_{\substack{n \leqslant N \\ \|a_n - \gamma\| \leqslant \delta}} 1 \ll \delta + \sum_{1 \leqslant h \leqslant 1/\delta} \frac{1}{h} \left| \frac{1}{N} \sum_{n \leqslant N} e^{2\pi i h a_n} \right|.$$

Proof. The quantity to be estimated is $\leqslant 2\delta + D$, where D is the discrepancy of the sequence a_n, $1 \leqslant n \leqslant N$, in the usual sense (cf. p. 88 of [13]). By the Erdös–Turán inequality (cf. p. 112 of [12]) we have

$$D \ll \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h a_n} \right|$$

for every positive integer m. Taking $m = \max(1, [1/\delta])$, we obtain the asserted estimate.

LEMMA 6. *Let f be a real-valued completely additive function. Then, for any real numbers $x \geqslant 2$, $\delta > 0$ and γ,*

$$\frac{1}{\log x} \sum_{\substack{n \leqslant x \\ \|f(n) - \gamma\| \leqslant \delta}} \frac{1}{\phi(n)} \ll \delta + \sup_{x^{\vartheta} \leqslant y \leqslant x} \sum_{1 \leqslant h \leqslant 1/\delta} \frac{1}{h} |s_h(y)|, \tag{4·3}$$

where

$$s_h(y) = \frac{1}{y} \sum_{n \leqslant y} e^{2\pi i h f(n)}$$

Proof. We first prove the asserted estimate with $\phi(n)$ replaced by n. Denoting by Σ' summation over integers n satisfying $\|f(n) - \gamma\| \leqslant \delta$, we have by partial summation

$$\frac{1}{\log x} \sum_{n \leqslant x}' \frac{1}{n} \leqslant \delta + \frac{1}{\log x} \int_{x^{\vartheta}}^{x} \sum_{n \leqslant u}' 1 \frac{du}{u^2} + \frac{1}{x \log x} \sum_{n \leqslant x}' 1.$$

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

Applying Lemma 5, we see that the right-hand side here is bounded by the right-hand side of (4·3).

To deduce from this the asserted estimate (4·3), we represent $1/\phi(n)$ in the form

$$\frac{1}{\phi(n)} = \frac{1}{n}\frac{n}{\phi(n)} = \frac{1}{n}(1*\psi)(n).$$

where ψ is a multiplicative function defined by

$$\psi(p^m) = \begin{cases} \dfrac{1}{p-1} & \text{if } m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\sum_{\substack{n\leqslant x \\ \|f(n)-\gamma\|\leqslant\delta}} \frac{1}{\phi(n)} = \sum_{d\leqslant x} \frac{\psi(d)}{d} \sum_{\substack{m\leqslant x/d \\ \|f(md)-\gamma\|\leqslant\delta}} \frac{1}{m} \leqslant \left(\sum_{d\geqslant 1}\frac{\psi(d)}{d}\right) \sup_{\gamma'\in\mathbf{R}} \sum_{\substack{m\leqslant x \\ \|f(m)-\gamma'\|\leqslant\delta}} \frac{1}{m},$$

and since the series

$$\sum_{d\geqslant 1}\frac{\psi(d)}{d} = \prod_p\left(1+\frac{1}{p(p-1)}\right)$$

is convergent, we obtain (4·3) from the above-proved estimate.

5. *Proof of Theorem* 2

We fix a completely additive function f satisfying (2·1) (with a constant c to be determined later), and set $g = e^{2\pi i f}$.

We first consider the case when the series

$$\sum_p \frac{|1-g(p)|^2}{p} \tag{5·1}$$

converges. In this case, we can apply Lemma 1 and get, using the complete multiplicativity of g,

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}\overline{g(n+1)}\,g(n) = \prod_p\left(1-\frac{2}{p}+2\left(1-\frac{1}{p}\right)\mathrm{Re}\frac{g(p)}{p-g(p)}\right).$$

The product here is strictly less than 1 except when $g\equiv 1$, or equivalently $\|f\|\equiv 0$. Since

$$\begin{aligned}\mathrm{Re}\frac{1}{x}\sum_{n\leqslant x}(1-\overline{g(n+1)}\,g(n)) &= \frac{1}{2x}\sum_{n\leqslant x}|1-\overline{g(n+1)}\,g(n)|^2 \\ &\leqslant \frac{1}{2x}\sum_{n\leqslant x}(2\pi\|f(n+1)-f(n)\|)^2,\end{aligned}$$

it follows that (2·2) holds whenever $\|f\|\not\equiv 0$, as asserted in the theorem.

Suppose now that the series (5·1) diverges. We shall first show that, for any positive integer $h\leqslant 1/(3c)$,

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}e^{2\pi i h f(n)} = \lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}g(n)^h = 0. \tag{5·2}$$

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

By Lemma 2, it suffices to show that the series

$$\sum_p \frac{|1-g(p)^h p^{i\alpha}|^2}{p} = \sum_p \frac{|1-e^{2\pi i h f(p)} p^{i\alpha}|^2}{p} \tag{5·3}$$

diverges for $1 \leqslant h \leqslant 1/(3c)$ and every real number α.

For $h = 1$ and $\alpha = 0$ the divergence of (5·3) follows from our assumption. Moreover, for $1 \leqslant h \leqslant 1/(3c)$ we have, in view of (2·1),

$$|1-e^{2\pi i h f(p)}| \gg |1-e^{2\pi i f(p)}|$$

and hence deduce the divergence of (5·3) in the case $\alpha = 0$ for each of these values h. Finally, if $\alpha \neq 0$, then, since $\|hf(p)\| \leqslant \frac{1}{3}$ for $h \leqslant 1/(3c)$, we have

$$|1-e^{2\pi i h f(p)} p^{i\alpha}| \gg 1$$

whenever

$$\left\| \frac{1}{2\pi} \alpha \log p \right\| \geqslant \frac{2}{5}.$$

Therefore the series (5·3) is bounded from below by

$$\gg \sum_{\|(\alpha/2\pi)\log p\| \geqslant \frac{2}{5}} \frac{1}{p} = \sum_{k=0}^{\infty} \sum_{(|\alpha|/2\pi)\log p \in [k+\frac{2}{5}, k+\frac{3}{5}]} \frac{1}{p}. \tag{5·4}$$

A typical inner sum in the last expression is equal to

$$\log \frac{k+\frac{3}{5}}{k+\frac{2}{5}} + O(e^{-C\sqrt{k}})$$

by a sharp form of the well known asymptotic formula for the sum of reciprocals of primes $\leqslant x$ (see for example [**15**], p. 80), and hence is $\gg 1/k$ for sufficiently large k. It follows that the series (5·4) and (5·3) are divergent, as claimed.

Now, consider the sum

$$T = \sum_{\substack{n \leqslant x \\ \|f(n+1)-f(n)\| \leqslant c}}{}' 1$$

where in Σ' the summation is restricted to integers n satisfying $P(n) > x^{\frac{3}{5}}$ and $P(n+1) > x^{\frac{3}{5}}$. We shall show that if x is sufficiently large, then

$$T \ll cx \tag{5·5}$$

holds with an absolute implied constant. Since by Lemma 3

$$\sum_{n \leqslant x}{}' 1 \gg x,$$

we obtain from (5·5), after choosing the constant c small enough,

$$x \ll \sum_{\substack{n \leqslant x \\ \|f(n+1)-f(n)\| > c}}{}' 1 \leqslant \frac{1}{c} \sum_{n \leqslant x} \|f(n+1)-f(n)\|$$

and hence (2·2).

It remains to prove (5·5). We first observe that, for each integer n counted in the sum T, n and $n+1$ can be written uniquely in the form

$$n = mp, \quad n+1 = m'p'$$

with

$$x^{\frac{3}{5}} < p \leqslant x/m, \quad p' > x^{\frac{3}{5}}. \tag{5·6}$$

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

Using (2·1) and the complete additivity of f, we see that for each of these integers n,

$$\begin{aligned}\|f(m)-f(m')\| &= \|f(n)-f(p)-f(n+1)+f(p')\| \\ &\leqslant \|f(n+1)-f(n)\|+\|f(p)\|+\|f(p')\| \leqslant 3c.\end{aligned}$$

Therefore we obtain

$$T \leqslant \sum_{\substack{n\leqslant x \\ \|f(m)-f(m')\|\leqslant 3c}}{}' 1 = \sum_{\substack{m,\,m'\leqslant x \\ \|f(m)-f(m')\|\leqslant 3c}} T(m,m'), \tag{5·7}$$

where $T(m,m')$ is the number of pairs of primes (p,p') that satisfy (5·6) and

$$mp+1 = m'p'. \tag{5·8}$$

Next, we estimate $T(m,m')$ from above using Lemma 4. From (5·6) and (5·8) we see that $T(m,m') = 0$ unless

$$m \leqslant x^{\frac{2}{5}}, \quad m' \leqslant (x+1)\,x^{-\frac{3}{5}}, \quad (m,m') = 1. \tag{5·9}$$

We may therefore assume (5·9). In this case there exist unique integers l and l' satisfying

$$m'l-ml' = 1, \quad 1 \leqslant l' \leqslant m', \quad 1 \leqslant l \leqslant m,$$

and every solution (p,p') to (5·8) must be of the form

$$p = tm'+l', \quad p' = tm+l.$$

where $0 \leqslant t \leqslant x/(mm')$. Thus we have

$$T(m,m') \leqslant \left|\left\{0 \leqslant t \leqslant \frac{x}{mm'}\colon tm'+l',\, tm+l \text{ prime}\right\}\right|,$$

and applying Lemma 4 we obtain

$$T(m,m') \ll \frac{mm'}{\phi(m)\,\phi(m')}\,\frac{x/(mm')}{(\log(x/mm'))^2} \ll \frac{1}{\phi(m)\,\phi(m')}\,\frac{x}{\log^2 x}. \tag{5·10}$$

where in the last estimate we have used (5·9).

From (5·7) and (5·10) we get

$$T \ll \frac{x}{\log^2 x} \sum_{m\leqslant x} \frac{1}{\phi(m)} \sum_{\substack{m'\leqslant x \\ \|f(m)-f(m')\|\leqslant 3c}} \frac{1}{\phi(m')}.$$

Applying Lemma 6 with $\delta = 3c$ and $\gamma = f(m)$ and noting (5·2), we see that for sufficiently large x each of the inner sums is bounded by $\ll c\log x$. We therefore obtain the bound

$$T \ll \frac{cx}{\log x} \sum_{n\leqslant x} \frac{1}{\phi(m)},$$

which is $\ll cx$, as can be seen, for example, by applying Lemma 6 with $\delta = 1$. This proves (5·5), and hence the theorem.

The author's research was supported by N.S.F. grant DMS 8640693.

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press

REFERENCES

[1] P. D. T. A. ELLIOTT. *Probabilistic Number Theory* I (Springer-Verlag, 1979).
[2] P. ERDÖS. On the distribution function of additive functions. *Ann. of Math.* (2) **47** (1946), 1–20.
[3] P. ERDÖS, C. POMERANCE and A. SARKÖZY. On locally repeated values of arithmetic functions. III. *Proc. Amer. Math. Soc.* **101** (1987), 1–7.
[4] J. FRIEDLANDER and H. IWANIEC. On Bombieri's asymptotic sieve. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **5** (1978), 719–756.
[5] G. HALÁSZ. Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen. *Acta Math. Acad. Sci. Hungar.* **19** (1968), 365–403.
[6] H. HALBERSTAM and H.-E. RICHERT. *Sieve Methods* (Academic Press, 1974).
[7] A. HILDEBRAND. Multiplicative functions at consecutive integers. *Math. Proc. Cambridge Philos. Soc.* **100** (1986), 229–236.
[8] A. HILDEBRAND. An Erdös–Wintner theorem for differences of additive functions. *Trans. Amer. Math. Soc.*, (to appear).
[9] H. IWANIEC. Sieving limits. In *Séminaire de Théorie des Nombres, Paris* 1979–80 (Birkhäuser, 1981), pp. 151–169.
[10] I. KÁTAI. On a problem of P. Erdös. *J. Number Theory* **21** (1970), 1–6.
[11] I. KÁTAI. Some problems in number theory. *Studia Sci. Math. Hungar.* **16** (1981), 289–295.
[12] I. KÁTAI. Multiplicative functions with regularity properties. II. *Acta Math. Acad. Sci. Hungar.* **43** (1984), 105–130.
[13] L. KUIPERS and H. NIEDERREITER. *Uniform distribution of sequences* (Wiley, 1974).
[14] J.-L. MAUCLAIRE and L. MURATA. On the regularity of arithmetic multiplicative functions. I. *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), 438–440.
[15] K. PRACHAR. *Primzahlverteilung* (Springer-Verlag, 1957).
[16] E. WIRSING. A characterization of $\log n$ as an additive function. *Sympos. Math.* IV (1970), 45–57.
[17] E. WIRSING. Private communication (October 1986).

https://doi.org/10.1017/S0305004100065002 Published online by Cambridge University Press