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# RECURSIVE FORMULA FOR THE DOUBLE-BARRIER PARISIAN STOPPING TIME

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**Abstract**

In this paper we obtain a recursive formula for the density of the double-barrier Parisian stopping time. We present a probabilistic proof of the formula for the first few steps of the recursion, and then a formal proof using explicit Laplace inversions. These results provide an efficient computational method for pricing double-barrier Parisian options.

*Keywords:* Brownian excursion; Parisian stopping time; double-sided Parisian option

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## 1. Introduction

Parisian options are a type of path-dependent option, where the payoff depends not only on the final value of the underlying asset, but also on the path trajectory of the underlying asset above or below a predetermined barrier $L$. In particular, the owner of a Parisian down-and-out call loses the option when the underlying asset price $S$ reaches the level $L$ and remains constantly below this level for a time interval longer than $D$, while for a Parisian down-and-in call, the same event gives the owner the right to exercise the option. The concept of Parisian options was first introduced by Chesney *et al.* [3], where the Laplace transforms of the prices of single-sided Parisian options were obtained using Azéma martingales. The pricing of Parisian options was also studied in [7], [8], and [10]. There are several motivations for the interest in these options. Parisian options are cheaper than the vanilla option of the same maturity and strike, since the value of the option depends on a barrier event occurring. Parisian options also have the added advantage over a barrier option, as it is not as easily manipulated by an influential agent since the barrier event requires more than just a touch of the barrier. Furthermore, since Parisian options are triggered only when the underlying asset has spent an amount of time beyond the barrier, this smooths the delta and gamma values near the barrier and makes hedging easier. Parisian options also have other practical applications, for instance to real option problems [2] and to insurance [6].

Double-barrier Parisian options are a two-barrier version of the standard Parisian options described above. For example, a double-barrier Parisian min-in call gives the owner the right to exercise the option if the underlying asset price $S$ either makes an excursion above the upper barrier, or below the lower barrier for a continuous period longer than $D$, while the owner of a double-barrier Parisian min-out call will lose the right to exercise the option when the same event occurs. Pricing of double-barrier Parisian options has been studied in [1], [5], and [9].

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In these papers the authors have focused on obtaining explicit expressions for the Laplace transforms of the prices, but numerical inversions of these Laplace transforms are sometimes unstable.

In this paper we derive a recursive formula for the density of the double-barrier Parisian stopping time. In [4], an explicit solution for the density of the Parisian stopping time with a single barrier was obtained. But here, we consider excursions both above the upper barrier and below the lower barrier. We define the double-barrier Parisian stopping time as the first time the Brownian motion remains continuously below the lower barrier $b_1$ or above the upper barrier $b_2$ for a fixed amount of time. It turns out that the density is a finite sum of recursive terms, which are convolutions of the previous terms, and, hence, are fast and easy to compute. This provides us with an explicit expression for the price of a double-barrier Parisian option, which does not require any numerical inversion of Laplace transforms. Furthermore, our approach is intuitive and easy to understand from a probabilistic viewpoint. Since $t$ is the first time the length of an excursion reaches $D$, if $kD < t < (k+1)D$, the probability is the same as that of the current excursion starting at time $t - D$, which will be between $(k-1)D < t - D < kD$, and that there are no excursions outside the barriers of length greater than $D$ before this. Hence, we can decompose the Brownian path into intervals of length $D$, and if there has been no excursions of length greater than $D$, the density for the stopping time where $t$ is between $kD < t < (k+1)D$ can be computed from the density of the previous step. To illustrate this further, we provide a probabilistic proof for the first few steps of the recursion. This also suggests that the method can be generalised to obtain explicit formulas for densities of the Parisian stopping times of other Markov processes, of which the first and last passage time densities are known. Finally, we use the density to present an efficient computational method for pricing double-barrier Parisian options.

In Section 2 we define the excursions and the double-barrier Parisian stopping time and option. In Section 3 we present the result on the density of the double-barrier Parisian stopping time. We first provide a heuristic proof for the first few steps of the recursion, and then provide a formal proof of the formula for $t \geq 0$. In Section 4 we derive the pricing formulas for the Parisian double-barrier in call options and show how the prices of the double-barrier Parisian out call options can be obtained using the in-out parity relationships. In Section 5 we provide numerical examples to demonstrate the accuracy of our results.

## 2. Definitions

We will use the same definitions for the excursions as in [3]. Let $S$ be the price process for the underlying asset, and $\mathbb{Q}$ denote the risk neutral probability measure. We assume that $S$ follows a geometric Brownian motion and its dynamics under $\mathbb{Q}$ is

$$\mathrm{d}S_t = S_t(r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t), \qquad S_0 = x,$$

where $W$ is a standard Brownian motion under $\mathbb{Q}$, and $r$ and $\sigma$ are positive constants. We also introduce the notation

$$m \coloneqq \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right), \qquad b \coloneqq \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \qquad k \coloneqq \frac{1}{\sigma}\ln\left(\frac{K}{x}\right),$$

so that the asset price can be written as $S_t = x\mathrm{e}^{\sigma(mt+W_t)}$. We define

$$g^S_{L,t} \coloneqq \sup\{s \leq t \mid S_s = L\}, \qquad d^S_{L,t} \coloneqq \inf\{s \geq t \mid S_s = L\},$$

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with the usual convention that $\sup \varnothing = 0$ and $\inf \varnothing = \infty$. The trajectory of $S$ between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion which straddles time $t$. We are interested here in $t - g^S_{L,t}$, which is the age of the excursion at time $t$. For $D > 0$, we now define

$$\tau^+_{L,D}(S) \coloneqq \inf\{t \geq 0 \mid \mathbf{1}(S_t > L)(t - g^S_{L,t}) \geq D\},$$
$$\tau^-_{L,D}(S) \coloneqq \inf\{t \geq 0 \mid \mathbf{1}(S_t < L)(t - g^S_{L,t}) \geq D\}.$$

We have denoted as $\tau^+_{L,D}(S)$ the first time that the length of an excursion of process $S$ above the barrier $L$ reaches $D$, while $\tau^-_{L,D}(S)$ is the first time that the length of an excursion of process $S$ below the barrier $L$ reaches $D$. We also introduce the following notation for the stopping times where we refer to the standard Brownian motion $W$ instead of $S$. Furthermore, without loss of generality, since any time $t$ of interest can be expressed in units of the window length $D$, we let $D = 1$ from now on. Then

$$\tau^+_b \coloneqq \inf\{t \geq 0 \mid \mathbf{1}(W_t > b)(t - g^W_{b,t}) \geq 1\},$$
$$\tau^-_b \coloneqq \inf\{t \geq 0 \mid \mathbf{1}(W_t < b)(t - g^W_{b,t}) \geq 1\}. \tag{2.1}$$

We now look at the double-barrier Parisian option, which is defined as (for $b_1 < b_2$)

$$\tau^{b_2}_{b_1} \coloneqq \tau^-_{b_1} \wedge \tau^+_{b_2}. \tag{2.2}$$

This is the first time that, for the Brownian motion $W$, the length of an excursion above $b_2$, or an excursion below $b_1$, reaches length 1. We note that we have taken the window length of both sides to be the same (i.e. 1 in our case).

The owner of a double-barrier Parisian min-in option receives the payoff only if there is an excursion below the level $L_1$ or above the level $L_2$ which is of length greater than $D = 1$. This will be the case if $\tau^{L_2}_{L_1}(S) \leq T$. Denoting $C^{\text{double}}_i(x, T, L_1, L_2, K)$ as the price of a Parisian min-in call with initial underlying price $x$, maturity $T$, strike price $K$, lower barrier $L_1$, and upper barrier $L_2$, we have the risk-neutral price of the option

$$C^{\text{double}}_i(x, T, L_1, L_2, K) = \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-rT}\mathbf{1}(\tau^{L_2}_{L_1}(S) \leq T)(x\mathrm{e}^{\sigma(mT+W_T)} - K)^+].$$

We now introduce a new probability measure $\mathbb{P}$ with Radon–Nikodym derivative $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q} = \mathrm{e}^{-mW_t - m^2t/2}$. Applying Girsanov's theorem and a change of measure from $\mathbb{Q}$ to $\mathbb{P}$, we have

$$\begin{aligned} C^{\text{double}}_i(x, T, L_1, L_2, K) &= \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\mathrm{e}^{-rT}\mathbf{1}(\tau^{L_2}_{L_1}(S) \leq T)(x\mathrm{e}^{\sigma(mT+W_T)} - K)^+\right] \\ &= \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-(r+m^2/2)T}\mathbf{1}(\tau^{b_2}_{b_1} \leq T)\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+], \end{aligned}$$

where $Z_t = W_t + mt$ is a standard Brownian motion under $\mathbb{P}$. To simplify things, we also let

$${}^*C^{\text{double}}_i(x, T, L_1, L_2, K) = \mathrm{e}^{(r+m^2/2)T}C^{\text{double}}_i(x, T, L_1, L_2, K).$$

In the next section, we will first look at the density function of $\tau^{b_2}_{b_1}$, which we will denote by $f^{b_2}_{b_1}(t)$, and then show how it can be used to obtain the prices of a Parisian min-in call option.

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## 3. Density of the double-barrier Parisian stopping time

We are interested in deriving the density of the double-barrier Parisian stopping time $\tau_{b_1}^{b_2}$. We first look at the case when the excursion has not started ($b_1 \le 0 \le b_2$) and then discuss results for the case when we are already within an excursion ($b_1 < b_2 \le 0$ or $0 \le b_1 < b_2$). We look at two cases, one where the excursion of length 1 occurs above the upper barrier first ($\tau_{b_2}^+ < \tau_{b_1}^-$) and the other where the excursion of length 1 occurs below the lower barrier first ($\tau_{b_1}^- < \tau_{b_2}^+$).

**Theorem 3.1.** *Let $f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-)$ and $f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+)$ denote the probability density function of $\tau_{b_1}^{b_2}$ on the set $\tau_{b_2}^+ < \tau_{b_1}^-$ and $\tau_{b_1}^- < \tau_{b_2}^+$, respectively. Then, for $b_1 \le 0 \le b_2$, we have, for $t > 1$, $n < t \le n+1$, $n = 1, 2, \ldots$,*

$$f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-) = \sum_{k=0}^{n-1} (-1)^k L_k(t-1), \qquad f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+) = \sum_{k=0}^{n-1} (-1)^k \tilde{L}_k(t-1), \tag{3.1}$$

*where $L_k(t)$ and $\tilde{L}_k(t)$ are defined recursively as follows for $t > k+1$:*

$$L_{k+1}(t) = \int_1^{t-k} (L_k(t-s)\varphi_1(s) + \tilde{L}_k(t-s)\varphi_2(s, b_2 - b_1))\,\mathrm{d}s, \tag{3.2}$$

$$\tilde{L}_{k+1}(t) = \int_1^{t-k} (\tilde{L}_k(t-s)\varphi_1(s) + L_k(t-s)\varphi_2(s, b_2 - b_1))\,\mathrm{d}s, \tag{3.3}$$

*with initial conditions*

$$L_0(t) = \frac{1}{2\pi\sqrt{t}} \exp\left(-\frac{b_2^2}{2t}\right), \qquad \tilde{L}_0(t) = \frac{1}{2\pi\sqrt{t}} \exp\left(-\frac{b_1^2}{2t}\right) \quad \text{for } t > 0, \tag{3.4}$$

*and the functions $\varphi_1(s)$ and $\varphi_2(s, b)$ are defined as*

$$\varphi_1(s) := \frac{\sqrt{s-1}}{2\pi s}, \tag{3.5}$$

$$\varphi_2(s, b) := \frac{\sqrt{s-1}}{2\pi s} \exp\left(-\frac{b^2}{2(s-1)}\right) - \frac{b}{\sqrt{2\pi} s^{3/2}} \exp\left(-\frac{b^2}{2s}\right) \mathcal{N}\left(-\frac{b}{\sqrt{s(s-1)}}\right). \tag{3.6}$$

Before we begin the formal proof of Theorem 3.1, we will first provide, in the following subsection, an intuitive proof for $1 < t < 3$.

### 3.1. A probabilistic explanation for the recursion

Here, we explain the above result using excursions. We will prove the result for small values of $t$ by using a path decomposition of the Brownian motion around time $t = 1$. The general result will then follow by induction. We look only at $f_{b_1}^{b_2}(t,\ \tau_{b_2}^+ < \tau_{b_1}^-)$, the case when the excursion above $b_2$ occurs before the excursion below $b_1$, but the same idea applies to $f_{b_1}^{b_2}(t,\ \tau_{b_1}^- < \tau_{b_2}^+)$, the case when the excursion below $b_1$ occurs before the excursion above $b_2$. We denote by $\mathbb{P}^x$ the law of a Brownian motion starting at $x$ at time 0, $p_{x,y}(t)$ the transition density of a Brownian motion from $x$ to $y$ in time $t$, and $T_x$ the first hitting time of level $x$ of the Brownian motion. Recall the notation that $g_{b_1,t}$ is the last time the Brownian motion hits level $b_1$ before time $t$. We want to find the density of $\tau_{b_1}^{b_2}$ when $\tau_{b_2}^+ < \tau_{b_1}^-$. First, we note that there is no density for $\tau_{b_1}^{b_2}$ when $t < 1$. For $1 < t < 2$, if $\{\tau_{b_1}^{b_2} \in \mathrm{d}t\}$, the excursion must start

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at $t-1$, where $0 < t-1 < 1$, and it must be the first excursion. Hence, we need to find $\nu_{b_2}(t) \coloneqq \mathbb{P}^0(\tau_{b_1}^{b_2} - 1 \in \mathrm{d}t,\ \tau_{b_2}^+ < \tau_{b_1}^-)$, the probability of $t$ being the start of the excursion above $b_2$ greater than length 1 for a Brownian motion starting at 0, by decomposing it into the part of the excursion between $g_{b_2,1}$ and 1, and between 1 and $g_{b_2,1}+1$. We have, for $0 < t < 1$,

$$\mathbb{P}(\tau_{b_1}^{b_2} - 1 \in \mathrm{d}t,\ \tau_{b_2}^+ < \tau_{b_1}^-) = \int_{b_2}^{\infty} \mathbb{P}^0(g_{b_2,1} \in \mathrm{d}t,\ W_1 \in \mathrm{d}x)\mathbb{P}^x(T_{b_2} \geq 1-(1-t)).$$

Using the time reversal property of Brownian motion, we have

$$\begin{aligned}\mathbb{P}^0(g_{b_2,1} \in \mathrm{d}t,\ W_1 \in \mathrm{d}x) &= \mathbb{P}^x(T_{b_2} \in 1-t)\,p_{b_2,0}(t)\\ &= \frac{x-b_2}{\sqrt{2\pi(1-t)^3}}\exp\left(-\frac{(x-b_2)^2}{2(1-t)}\right)\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{b_2^2}{2t}\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{P}(\tau_{b_1}^{b_2} - 1 \in \mathrm{d}t,\ \tau_{b_2}^+ < \tau_{b_1}^-)\\ &\quad= \int_{b_2}^{\infty} \frac{x-b_2}{\sqrt{2\pi(1-t)^3}}\exp\left(-\frac{(x-b_2)^2}{2(1-t)}\right)\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{b_2^2}{2t}\right)\\ &\qquad\quad\times\int_t^{\infty}\frac{x-b_2}{\sqrt{2\pi u^3}}\exp\left(-\frac{(x-b_2)^2}{2u}\right)\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_t^{\infty}\frac{1}{2\pi\sqrt{t(1-t)^3}}\frac{1}{\sqrt{2\pi u^3}}\exp\left(-\frac{b_2^2}{2t}\right)\\ &\qquad\quad\times\int_0^{\infty}x^2\exp\left(-\frac{x^2}{2(1-t)}\right)\exp\left(-\frac{x^2}{2u}\right)\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}t\\ &\quad= \int_t^{\infty}\frac{1}{2\pi\sqrt{t(1-t)^3}}\frac{1}{\sqrt{2\pi u^3}}\exp\left(-\frac{b_2^2}{2t}\right)\sqrt{\frac{\pi}{2}}\left(\frac{u(1-t)}{1-t+u}\right)^{3/2}\mathrm{d}u\,\mathrm{d}t\\ &\quad= \frac{1}{2\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right)\mathrm{d}t.\end{aligned}$$

This is the first term in the recursion, which we have denoted by $L_0(t)$. Note that, due to the symmetry of Brownian motion, this probability is the same for the excursion below $b_1$, and only depends on the difference between the barrier and the starting point. Hence, it follows that the probability of $t$ being the start of the excursion below $b_1$ greater than length 1, for $0 < t < 1$, is

$$\mathbb{P}^0(\tau_{b_1}^{b_2} - 1 \in \mathrm{d}t,\ \tau_{b_1}^- < \tau_{b_2}^+) = \nu_{b_1}(t) = \frac{1}{2\pi\sqrt{t}}\exp\left(-\frac{b_1^2}{2t}\right)\mathrm{d}t,$$

which we denote by $\tilde{L}_0(t)$, and corresponds to the first term of the second recursion (3.4). This proves (3.1) for $1 < t < 2$.

Now, for $2 < t < 3$, the same interpretation for $L_0(t-1)$ as the start of the excursion greater than length 1 for $1 < t-1 < 2$ still applies, but now there can be up to two excursions greater than length 1. Hence, we need to subtract the probability of $t$ being the start of the second excursion greater than length 1. We denote by $\varphi_1(t-s+1)$ the probability that we will start another excursion above $b_2$ greater than length 1 at time $t$, given that at time $s$ we are already

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length 1 into an excursion above $b_2$. We have, for $1 < s < t$,

$$\varphi_1(t-s+1) = \int_{b_2}^{\infty} \mathbb{P}^{b_2}(W_s \in \mathrm{d}x \mid g_{b_2,s} = s-1) \int_s^t \mathbb{P}^x(T_{b_2} \in \mathrm{d}u) v_0(t-u),$$

where we have decomposed the excursion, conditioning on the value of the Brownian motion at time $s$, and the time when it comes back to level $b_2$, and $v_0(t-u)$ is the probability that the Brownian motion will start another excursion above $b_2$ of length 1 at time $t$, given that it is at level $b_2$ at time $u$. Now, the Brownian motion conditioned to stay above $b_2$ up to time 1 is a Brownian meander, which has density

$$\mathbb{P}^{b_2}(W_s \in \mathrm{d}x \mid g_{b_2,s} = s-1) = (x-b_2)\exp\left(-\frac{(x-b_2)^2}{2}\right)\mathbf{1}(x > b_2)\,\mathrm{d}x.$$

Hence, we have

$$\begin{aligned}\varphi_1(t-s+1) &= \int_{b_2}^{\infty} (x-b_2)\exp\left(-\frac{(x-b_2)^2}{2}\right) \\ &\qquad \times \int_s^t \frac{x-b_2}{\sqrt{2\pi(u-s)^3}} \exp\left(-\frac{(x-b_2)^2}{2(u-s)}\right) \frac{1}{2\pi\sqrt{t-u}}\,\mathrm{d}u\,\mathrm{d}x \\ &= \frac{1}{2\pi}\frac{\sqrt{t-s}}{t-s+1}.\end{aligned}$$

Next, we denote by $\varphi_2(t-s+1, b_2-b_1)$ the probability that we will start an excursion above $b_2$ of length at least 1 at time $t$, given that at time $s$ we are already length 1 into an excursion below $b_1$. We have, for $1 < s < t$,

$$\begin{aligned}&\varphi_2(t-s+1, b_2-b_1) \\ &\quad = \int_{-\infty}^{b_1} \mathbb{P}^{b_1}(W_s \in \mathrm{d}x \mid g_{b_1,s} = s-1) \int_s^t \mathbb{P}^x(T_{b_1} \in \mathrm{d}u) v_{b_2-b_1}(t-u),\end{aligned}$$

where we have decomposed the excursion, conditioning on the value of the Brownian motion at time $s$, and the time when it comes back to level $b_1$. Then $v_{b_2-b_1}(t-u)$ is the probability that the Brownian motion will start an excursion above $b_2$ of length 1 at time $t$, given that it is at level $b_1$ at time $u$. Computations lead to

$$\begin{aligned}&\varphi_2(t-s+1, b) \\ &= \int_{-\infty}^{b_1} (b_1-x)\exp\left(-\frac{(b_1-x)^2}{2}\right) \\ &\qquad \times \int_s^t \frac{b_1-x}{\sqrt{2\pi(u-s)^3}} \exp\left(-\frac{(b_1-x)^2}{2(u-s)}\right) \frac{1}{2\pi\sqrt{t-u}} \exp\left(-\frac{b^2}{2(t-u)}\right)\mathrm{d}u\,\mathrm{d}x \\ &= \frac{1}{4\pi}\int_s^t \frac{1}{\sqrt{t-u}(1+u-s)^{3/2}} \exp\left(-\frac{b^2}{2(t-u)}\right)\mathrm{d}u \\ &= \frac{1}{4\pi}\left[-\frac{2\sqrt{t-u}}{(1+t-s)\sqrt{1+u-s}} \exp\left(-\frac{b^2}{2(t-u)}\right)\right. \\ &\qquad \left. - \frac{\sqrt{2\pi}\,b}{(1+t-s)^{3/2}} \exp\left(-\frac{b^2}{2(1+t-s)}\right)\left(1-2\mathcal{N}\left(-\frac{b\sqrt{1+u-s}}{\sqrt{(t-u)(1+t-s)}}\right)\right)\right]_s^t\end{aligned}$$

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$$= \frac{1}{2\pi}\frac{\sqrt{t-s}}{1+t-s}\exp\left(-\frac{b^2}{2(t-s)}\right) + \frac{b}{\sqrt{2\pi}(1+t-s)^{3/2}}\exp\left(-\frac{b^2}{2(1+t-s)}\right)\mathcal{N}\left(-\frac{b}{\sqrt{(t-s)(1+t-s)}}\right).$$

Note that due to the symmetry of Brownian motion, $\varphi_1(t-s+1)$ is also the probability that $t$ is the start of another excursion below $b_1$ greater than length 1, given that at time $s$ we are already in an excursion of length 1 below $b_1$. Likewise, $\varphi_2(t-s+1, b_2-b_1)$ is also the probability that $t$ is the start of an excursion below $b_1$ greater than length 1, given that at time $s$ we are already in an excursion of length 1 above $b_2$.

Since the first excursion can either be above $b_2$ or below $b_1$, there are two scenarios. The probability that $t$ is the start of the second excursion above $b_2$ greater than length 1 is the sum of the two cases:

$$\int_1^t L_0(s-1)\varphi_1(t-s+1)\,\mathrm{d}s + \int_1^t \tilde{L}_0(s-1)\varphi_2(t-s+1, b_2-b_1)\,\mathrm{d}s = \int_1^t L_0(t-s)\varphi_1(s)\,\mathrm{d}s + \int_1^t \tilde{L}_0(t-s)\varphi_2(s, b_2-b_1)\,\mathrm{d}s,$$

which is $L_1(t-1)$ in recursion equation (3.2). Similarly, the probability that $t$ is the start of the second excursion below $b_1$ greater than length 1 is

$$\int_1^t \tilde{L}_0(t-s)\varphi_1(s)\,\mathrm{d}s + \int_1^t L_0(t-s)\varphi_2(s, b_2-b_1)\,\mathrm{d}s,$$

which is $\tilde{L}_1(t-1)$ in recursion equation (3.3). Hence, for $2 < t < 3$, the density of $\tau_{b_1}^{b_2}$ for the $\tau_{b_2}^+ < \tau_{b_1}^-$ and $\tau_{b_1}^- < \tau_{b_2}^+$ cases are $L_0(t-1) - L_1(t-1)$ and $\tilde{L}_0(t-1) - \tilde{L}_1(t-1)$, respectively, and we proved (3.1) for $2 < t < 3$. The same argument would follow by induction for $t > 3$ and, thus, we obtain the recursion.

### 3.2. Formal proof

In this section we give a formal proof of the recursive formula based on Laplace transforms. We define the Laplace transform $\hat{h}(\beta)$ of a function $h(t)$ on the positive real line as

$$\mathcal{L}(h(t)) = \hat{h}(\beta) := \int_0^\infty \mathrm{e}^{-\beta t}h(t)\,\mathrm{d}t,$$

and the inverse Laplace transform operator is denoted by $\mathcal{L}^{-1}(\cdot)$. Furthermore, for ease of notation, we define, as in previous papers, the following function:

$$\Psi(x) := 1 + x\sqrt{2\pi}\mathrm{e}^{x^2/2}\mathcal{N}(x),$$

where $\mathcal{N}(x)$ is the cumulative distribution function for the standard normal distribution.

*Proof of Theorem 3.1.* We show the calculations only for the case when $\{\tau_{b_2}^+ < \tau_{b_1}^-\}$, but the case when $\{\tau_{b_1}^- < \tau_{b_2}^+\}$ can be proved in the same way. The Laplace transform of the double-barrier Parisian stopping time was studied in [1] and we use here the result of Theorem 3.2 of that paper. The Laplace transform for $\tau_{b_1}^{b_2}$ on the set $\{\tau_{b_2}^+ < \tau_{b_1}^-\}$ is

$$\mathbb{E}\left[\mathrm{e}^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}(\tau_{b_2}^+ < \tau_{b_1}^-)\right] = \frac{\mathrm{e}^{-\sqrt{2\beta}b_1}\Psi(\sqrt{2\beta}) - \mathrm{e}^{\sqrt{2\beta}b_1}\Psi(-\sqrt{2\beta})}{\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)}\Psi(\sqrt{2\beta})^2 - \mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\Psi(-\sqrt{2\beta})^2}.$$

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Factorising, we obtain

$$\mathbb{E}\big[\mathrm{e}^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}(\tau_{b_2}^{+}<\tau_{b_1}^{-})\big]=\frac{(\mathrm{e}^{\sqrt{2\beta}(b_1+b_2)/2}+\mathrm{e}^{-\sqrt{2\beta}(b_1+b_2)/2})/2}{\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}\Psi(\sqrt{2\beta})+\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)/2}\Psi(-\sqrt{2\beta})} \tag{3.7}$$

$$-\frac{(\mathrm{e}^{\sqrt{2\beta}(b_1+b_2)/2}-\mathrm{e}^{-\sqrt{2\beta}(b_1+b_2)/2})/2}{\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}\Psi(\sqrt{2\beta})-\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)/2}\Psi(-\sqrt{2\beta})}. \tag{3.8}$$

Now we refer the reader to Dassios and Lim [4] for the derivation of the following equality:

$$\mathrm{e}^{-\beta}\frac{1}{\beta}\Psi(\sqrt{2\beta})=2\sqrt{\frac{\pi}{\beta}}\bigg(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\bigg).$$

Similarly, we derive

$$\begin{aligned}\mathrm{e}^{-\beta}\frac{1}{\beta}\Psi(-\sqrt{2\beta})&=\frac{\mathrm{e}^{-\beta}}{\beta}-2\sqrt{\frac{\pi}{\beta}}\mathcal{N}(-\sqrt{2\beta})\\&=\int_1^\infty\mathrm{e}^{-\beta s}\,\mathrm{d}s-2\sqrt{\frac{\pi}{\beta}}\int_{-\infty}^{-\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x\\&=\int_1^\infty\mathrm{e}^{-\beta s}\,\mathrm{d}s-\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{\sqrt{s}}\,\mathrm{d}s\\&=2\sqrt{\frac{\pi}{\beta}}\bigg(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\bigg).\end{aligned}$$

Adding the two, we obtain an expression for the denominator in the expression on the right-hand side of (3.7),

$$\begin{aligned}&\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}\Psi(\sqrt{2\beta})+\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)/2}\Psi(-\sqrt{2\beta})\\&\quad=\mathrm{e}^{\beta}\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}2\sqrt{\pi\beta}\\&\qquad\times\bigg(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s+\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\bigg),\end{aligned}$$

and (3.8),

$$\begin{aligned}&\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}\Psi(\sqrt{2\beta})-\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)/2}\Psi(-\sqrt{2\beta})\\&\quad=\mathrm{e}^{\beta}\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}2\sqrt{\pi\beta}\\&\qquad\times\bigg(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s-\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\bigg).\end{aligned}$$

Denoting by $\hat{g}_0(\beta)$, $\hat{\tilde{g}}_0(\beta)$, $\hat{g}_k(\beta)$, and $\hat{\tilde{g}}_k(\beta)$ the following expressions:

$$\hat{g}_0(\beta)=\frac{\mathrm{e}^{\sqrt{2\beta}b_1}+\mathrm{e}^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}},\qquad\hat{\tilde{g}}_0(\beta)=\frac{\mathrm{e}^{\sqrt{2\beta}b_1}-\mathrm{e}^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}},$$

$$\hat{g}_1(\beta)=\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s+\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s, \tag{3.9}$$

$$\hat{\tilde{g}}_1(\beta)=\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s-\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s, \tag{3.10}$$

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we can write the expression on the right-hand side of (3.7) as

$$\begin{aligned}
&\frac{(\mathrm{e}^{\sqrt{2\beta}(b_1+b_2)/2}+\mathrm{e}^{-\sqrt{2\beta}(b_1+b_2)/2})/2}{\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}\Psi(\sqrt{2\beta})+\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)/2}\Psi(-\sqrt{2\beta})}\\
&\quad=(\mathrm{e}^{\sqrt{2\beta}(b_1+b_2)/2}+\mathrm{e}^{-\sqrt{2\beta}(b_1+b_2)/2})\\
&\qquad\times\left[\mathrm{e}^{\beta}\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}2\sqrt{\pi\beta}\right.\\
&\qquad\quad\left.\times\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s+\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\right)\right]^{-1}\\
&\quad=\mathrm{e}^{-\beta}\frac{\hat{g}_0(\beta)}{1+\hat{g}_1(\beta)},
\end{aligned}\tag{3.11}$$

and (3.8) as

$$-\frac{(\mathrm{e}^{\sqrt{2\beta}(b_1+b_2)/2}-\mathrm{e}^{-\sqrt{2\beta}(b_1+b_2)/2})/2}{\mathrm{e}^{\sqrt{2\beta}(b_2-b_1)/2}\Psi(\sqrt{2\beta})-\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)/2}\Psi(-\sqrt{2\beta})}=-\mathrm{e}^{-\beta}\frac{\hat{\tilde{g}}_0(\beta)}{1+\hat{\tilde{g}}_1(\beta)}.\tag{3.12}$$

Since $\hat{g}_1(\beta)$ is a continuous and decreasing function of $\beta$, it goes to 0 when $\beta\to\infty$. Hence, there exists some $\beta>0$ such that $|\hat{g}_1(\beta)|<1$, and so (3.11) can be written as the sum of a convergent geometric series with first term $\hat{g}_0(\beta)$ and common ratio $-\hat{g}_1(\beta)$. Similarly, since

$$|\hat{\tilde{g}}_1(\beta)|\le\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s<1,$$

(3.12) can be written as the sum of a convergent geometric series with first term $\hat{\tilde{g}}_0(\beta)$ and common ratio $-\hat{\tilde{g}}_1(\beta)$. We obtain

$$\mathbb{E}\big[\mathrm{e}^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}(\tau_{b_2}^+<\tau_{b_1}^-)\big]=\mathrm{e}^{-\beta}\left(\hat{g}_0(\beta)\sum_{i=0}^\infty(-1)^k(\hat{g}_1(\beta))^k-\hat{\tilde{g}}_0(\beta)\sum_{i=0}^\infty(-1)^k(\hat{\tilde{g}}_1(\beta))^k\right).\tag{3.13}$$

Now, we invert the Laplace transform (3.13). If we denote the Laplace inversions of $\hat{g}_0(\beta)$, $\hat{\tilde{g}}_0(\beta)$, $\hat{g}_1(\beta)$, and $\hat{\tilde{g}}_1(\beta)$ by $g_0(t)$, $\tilde{g}_0(t)$, $g_1(t)$, and $\tilde{g}_1(t)$, we have, for $t>1$,

$$f_{b_1}^{b_2}(t,\tau_{b_2}^+<\tau_{b_1}^-)=\sum_{k=0}^\infty(-1)^k(g_k(t-1)-\tilde{g}_k(t-1)),$$

where $g_k(t)$ is the convolution of $g_0(t)$ with $k-1$ times of $g_1(t)$, and $\tilde{g}_k(t)$ is the convolution of $\tilde{g}_0(t)$ with $k-1$ times of $\tilde{g}_1(t)$. Next, we have the following explicit Laplace inversions:

$$\mathcal{L}^{-1}\left(\frac{\mathrm{e}^{\sqrt{2\beta}b_1}}{4\sqrt{\pi\beta}}\right)=\frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_1^2}{2t}\right),\qquad\mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}}\right)=\frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right),$$
$$\mathcal{L}^{-1}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\right)=\frac{\sqrt{t-1}}{2\pi t}\mathbf{1}(t>1),$$

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and

$$\mathcal{L}^{-1}\left(\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\right)$$
$$=\left(\frac{\sqrt{t-1}}{2\pi t}\exp\left(-\frac{(b_2-b_1)^2}{2(t-1)}\right)\right.$$
$$\left.-\frac{b_2-b_1}{\sqrt{2\pi}t^{3/2}}\exp\left(-\frac{(b_1-b_2)^2}{2t}\right)\mathcal{N}\left(-\frac{b_2-b_1}{\sqrt{t(t-1)}}\right)\right)\mathbf{1}(t>1).$$

The first three inversions were computed by Dassios and Lim [4], and the last one can be derived as the convolution of the following two functions:

$$\mathcal{L}^{-1}\left(\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\right)=\frac{1}{2\pi\sqrt{t}}\exp\left(-\frac{(b_1-b_2)^2}{2t}\right),$$
$$\mathcal{L}^{-1}\left(\int_1^\infty \frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\right)=\frac{1}{2t^{3/2}}\mathbf{1}(t>1),$$

so that

$$\mathcal{L}^{-1}\left(\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{\mathrm{e}^{-\beta s}}{2s^{3/2}}\,\mathrm{d}s\right)$$
$$=\mathbf{1}(t>1)\int_0^{t-1}\frac{1}{2\pi\sqrt{s}}\frac{1}{2(t-s)^{3/2}}\exp\left(-\frac{(b_1-b_2)^2}{2s}\right)\mathrm{d}s$$
$$=\mathbf{1}(t>1)\int_{-\infty}^{(b_1-b_2)/\sqrt{t-1}}\frac{-x(b_2-b_1)}{2\pi(tx^2-(b_1-b_2)^2)^{3/2}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x$$
$$=\mathbf{1}(t>1)\left(\frac{\sqrt{t-1}}{2\pi t}\exp\left(-\frac{(b_1-b_2)^2}{2(t-1)}\right)\right.$$
$$\left.-\int_{-\infty}^{(b_1-b_2)/\sqrt{t-1}}\frac{-x(b_2-b_1)}{2\pi t\sqrt{tx^2-(b_1-b_2)^2}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x\right)$$
$$=\mathbf{1}(t>1)\left(\frac{\sqrt{t-1}}{2\pi t}\exp\left(-\frac{(b_1-b_2)^2}{2(t-1)}\right)\right.$$
$$\left.-\frac{b_2-b_1}{\sqrt{2\pi}t^{3/2}}\exp\left(-\frac{(b_1-b_2)^2}{2t}\right)\mathcal{N}\left(-\frac{b_2-b_1}{\sqrt{t(t-1)}}\right)\right).$$

Thus, adding the appropriate terms, we have

$$g_0(t)=\frac{\mathrm{e}^{-b_1^2/2t}-\mathrm{e}^{-b_2^2/2t}}{4\pi\sqrt{t}},\qquad \tilde{g}_0(t)=\frac{\mathrm{e}^{-b_1^2/2t}+\mathrm{e}^{-b_2^2/2t}}{4\pi\sqrt{t}},$$
$$g_1(t)=\mathbf{1}(t>1)(\varphi_1(t)-\varphi_2(t,b_2-b_1)),\qquad \tilde{g}_1(t)=\mathbf{1}(t>1)(\varphi_1(t)+\varphi_2(t,b_2-b_1)),$$

where $\varphi_1(t)$ and $\varphi_2(t,b)$ are as defined in (3.5) and (3.6), and, for $k\geq 1$,

$$g_{k+1}(t)=\int_1^{t-k} g_k(t-s)(\varphi_1(s)-\varphi_2(s,b_2-b_1))\,\mathrm{d}s\quad\text{for } t>k+1,$$
$$\tilde{g}_{k+1}(t)=\int_1^{t-k} \tilde{g}_k(t-s)(\varphi_1(s)+\varphi_2(s,b_2-b_1))\,\mathrm{d}s\quad\text{for } t>k+1.$$

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We also note that, for $n < t \leq n+1$, $g_k(t)$ and $\tilde{g}_k(t)$ are 0 for $k > n$, and, thus, $g_k(t-1)$ and $\tilde{g}_k(t-1)$ are 0 for $k > n-1$, so we need a finite sum only up to $n-1$. Finally, we let $L_k(t) = g_k(t) - \tilde{g}_k(t)$ and $\tilde{L}_k(t) = g_k(t) + \tilde{g}_k(t)$ to obtain the result. For $n < t \leq n+1$, $n = 1, 2, \ldots$,

$$f_{b_1}^{b_2}(t,\ \tau_{b_2}^{+} < \tau_{b_1}^{-}) = \sum_{k=0}^{n-1}(-1)^k(g_k(t-1) - \tilde{g}_k(t-1)) = \sum_{k=0}^{n-1}(-1)^k L_k(t-1),$$

where We also note that, for $n < t \leq n+1$, $g_k(t)$ and $\tilde{g}_k(t)$ are 0 for $k > n$, and, thus, $g_k(t-1)$ and $\tilde{g}_k(t-1)$ are 0 for $k > n-1$, so we need a finite sum only up to $n-1$. Finally, we let $L_k(t) = g_k(t) - \tilde{g}_k(t)$ and $\tilde{L}_k(t) = g_k(t) + \tilde{g}_k(t)$ to obtain the result. For $n < t \leq n+1$, $n = 1, 2, \ldots$,

$$\begin{aligned} L_0(t) &= g_0(t) - \tilde{g}_0(t) = \frac{1}{2\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right), \\ \tilde{L}_0(t) &= g_0(t) + \tilde{g}_0(t) = \frac{1}{2\pi\sqrt{t}}\exp\left(-\frac{b_1^2}{2t}\right), \end{aligned} \quad \text{for } t > 0,$$

and

$$\begin{aligned} L_{k+1}(t) &= \int_1^{t-k}(L_k(t-s)\varphi_1(s) + \tilde{L}_k(t-s)\varphi_2(s, b_2-b_1))\,\mathrm{d}s, \\ \tilde{L}_{k+1}(t) &= \int_1^{t-k}(\tilde{L}_k(t-s)\varphi_1(s) + L_k(t-s)\varphi_2(s, b_2-b_1))\,\mathrm{d}s, \end{aligned} \quad \text{for } t > k+1,$$

which completes the proof. □

**Remark 3.1.** When $b_1 = b_2$, the above formula reduces to the formula for the two-sided Parisian stopping time for $b = b_1 = b_2$.

### 3.3. Starting above ($b_1 < b_2 \leq 0$) or below ($0 \leq b_1 < b_2$) for both barriers

We denote by $T_b$ the first hitting time of level $b$ of a standard Brownian motion $W$. For the case when we start above both barriers ($b_1 < b_2 \leq 0$), we consider only the case when $T_{b_2} < D = 1$, because if this is not the case, then we would have $\tau_{b_1}^{b_2} = 1$ since we are already above the upper barrier.

**Theorem 3.2.** *When $b_1 < b_2 \leq 0$, we consider only the $T_{b_2} < 1$ case. Then considering the two cases $\tau_{b_2}^{+} < \tau_{b_1}^{-}$ and $\tau_{b_1}^{-} < \tau_{b_2}^{+}$ separately, we have the following formulas for the probability density function of $\tau_{b_1}^{b_2}$, for $t > 1$, $n < t \leq n+1$, and $n = 1, 2, \ldots$:*

$$f_{b_1}^{b_2}(t, \tau_{b_2}^{+} < \tau_{b_1}^{-}, T_{b_2} < 1) = \sum_{k=0}^{n-1}(-1)^k L_k(t-1),$$

$$f_{b_1}^{b_2}(t, \tau_{b_1}^{-} < \tau_{b_2}^{+}, T_{b_2} < 1) = \sum_{k=0}^{n-1}(-1)^k \tilde{L}_k(t-1),$$

*where $L_k(t)$ and $\tilde{L}_k(t)$ are defined recursively as follows for $t > k+1$:*

$$\begin{aligned} L_{k+1}(t) &= \int_1^{t-k}(L_k(t-s)\varphi_1(s) + \tilde{L}_k(t-s)\varphi_2(s, b_2-b_1))\,\mathrm{d}s, \\ \tilde{L}_{k+1}(t) &= \int_1^{t-k}(\tilde{L}_k(t-s)\varphi_1(s) + L_k(t-s)\varphi_2(s, b_2-b_1))\,\mathrm{d}s, \end{aligned}$$

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*with the initial conditions*

$$\begin{aligned} L_0(t) = {} & \mathbf{1}(0 \le t \le 1)\left(\frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_1^2}{2t}\right) + \frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right)\right) \\ & + \mathbf{1}(t > 1)\left(\frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_1^2}{2t}\right)\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}} + \frac{b_2 - b_1}{\sqrt{t}\sqrt{t-1}}\right)\right. \\ & \quad + \frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{(2b_2 - b_1)^2}{2t}\right)\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}} - \frac{b_2 - b_1}{\sqrt{t}\sqrt{t-1}}\right) \\ & \quad \left. + \frac{1}{2\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right)\mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}}\right)\right) \end{aligned}$$

*and*

$$\begin{aligned} \tilde{L}_0(t) = {} & \mathbf{1}(0 \le t \le 1)\left(\frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_1^2}{2t}\right) - \frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right)\right) \\ & + \mathbf{1}(t > 1)\left(\frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_1^2}{2t}\right)\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}} + \frac{b_2 - b_1}{\sqrt{t}\sqrt{t-1}}\right)\right. \\ & \quad + \frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{(2b_2 - b_1)^2}{2t}\right)\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}} - \frac{b_2 - b_1}{\sqrt{t}\sqrt{t-1}}\right) \\ & \quad \left. - \frac{1}{2\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right)\mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}}\right)\right), \end{aligned}$$

*where $\varphi_1(s)$ and $\varphi_2(s, b)$ are as defined in Theorem 3.1.*

*Proof.* When $b_1 < b_2 \le 0$, on the set $T_{b_2} < 1$, using the strong Markov property of the Brownian motion, we can restart the process the first time it hits $b_2$. Then $\tau_{b_1}^{b_2}$ can be decomposed into the sum of $T_{b_2}$ and $\tau_{b_1-b_2}^0$, which are independent of each other, and, furthermore, due to the symmetry of Brownian motion, $\tau_{b_1-b_2}^0 = \tau_0^{b_2-b_1}$. The Laplace transform of the stopping time on the sets we are interested in is, thus,

$$\begin{aligned} \mathbb{E}\big[\mathrm{e}^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}(\tau_{b_2}^+ < \tau_{b_1}^-)\mathbf{1}(T_{b_2} < 1)\big] &= \mathbb{E}\big[\mathrm{e}^{-\beta(T_{b_2}+\tau_{b_1-b_2}^0)}\mathbf{1}(\tau_0^+ < \tau_{b_1-b_2}^-)\mathbf{1}(T_{b_2} < 1)\big] \\ &= \mathbb{E}[\mathrm{e}^{-\beta T_{b_2}}\mathbf{1}(T_{b_2} < 1)]\mathbb{E}\big[\mathrm{e}^{-\beta\tau_{b_1-b_2}^0}\mathbf{1}(\tau_0^+ < \tau_{b_1-b_2}^-)\big] \\ &= \mathbb{E}[\mathrm{e}^{-\beta T_{b_2}}\mathbf{1}(T_{b_2} < 1)]\mathbb{E}\big[\mathrm{e}^{-\beta\tau_0^{b_2-b_1}}\mathbf{1}(\tau_{b_2-b_1}^+ < \tau_0^-)\big]. \end{aligned}$$

Using now (3.13), but with the two barriers now being 0 and $b_2 - b_1$, we have

$$\mathbb{E}\big[\mathrm{e}^{-\beta\tau_0^{b_2-b_1}}\mathbf{1}(\tau_{b_2-b_1}^+ < \tau_0^-)\big] = \mathrm{e}^{-\beta}\left(\hat{g}_0(\beta)\sum_{i=0}^{\infty}(-1)^k(\hat{g}_1(\beta))^k - \hat{\tilde{g}}_0(\beta)\sum_{i=0}^{\infty}(-1)^k(\hat{\tilde{g}}_1(\beta))^k\right),$$

with $\hat{g}_1(\beta)$ and $\hat{\tilde{g}}_1(\beta)$ the same as in (3.9) and (3.10), but $\hat{g}_0(\beta)$ and $\hat{\tilde{g}}_0(\beta)$ now become

$$\hat{g}_0(\beta) = \frac{\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)} + 1}{4\sqrt{\pi\beta}}, \qquad \hat{\tilde{g}}_0(\beta) = \frac{\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)} - 1}{4\sqrt{\pi\beta}}.$$

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Hence,

$$\mathbb{E}\big[\mathrm{e}^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}(\tau_{b_2}^{+}<\tau_{b_1}^{-})\mathbf{1}(T_{b_2}<1)\big] = \mathrm{e}^{-\beta}\bigg(\mathbb{E}[\mathrm{e}^{-\beta T_{b_2}}\mathbf{1}(T_{b_2}<1)]\hat{g}_0(\beta)\sum_{i=0}^{\infty}(-1)^k(\hat{g}_1(\beta))^k - \mathbb{E}[\mathrm{e}^{-\beta T_{b_2}}\mathbf{1}(T_{b_2}<1)]\hat{\tilde{g}}_0(\beta)\sum_{i=0}^{\infty}(-1)^k(\hat{\tilde{g}}_1(\beta))^k\bigg).$$

This can be inverted the same way as before, and each convolution term $L_1(t)$ and $\tilde{L}_1(t)$ is the same as before, but the initial conditions become

$$L_0(t) = \mathcal{L}^{-1}(\mathbb{E}[\mathrm{e}^{-\beta T_{b_2}}\mathbf{1}(T_{b_2}<1)]\hat{g}_0(t)), \qquad \tilde{L}_0(t) \quad = \mathcal{L}^{-1}(\mathbb{E}[\mathrm{e}^{-\beta T_{b_2}}\mathbf{1}(T_{b_2}<1)]\hat{\tilde{g}}_0(t)).$$

To find $L_0(t)$ and $\tilde{L}_0(t)$, we invert the following Laplace transforms:

$$\mathcal{L}^{-1}\bigg(\mathbb{E}[\mathrm{e}^{-\beta T_{b_2}}\mathbf{1}(T_{b_2}<1)]\frac{\mathrm{e}^{\sqrt{2\beta}(b_1-b_2)}}{4\sqrt{\pi\beta}}\bigg) \tag{3.14}$$

$$= \mathbf{1}(0\le t<1)\int_0^t \frac{-b_2}{\sqrt{2\pi s^3}}\exp\bigg(-\frac{b_2^2}{2s}\bigg)\frac{1}{4\pi\sqrt{t-s}}\exp\bigg(-\frac{(b_2-b_1)^2}{2(t-s)}\bigg)\mathrm{d}s \tag{3.15}$$

$$+ \mathbf{1}(t>1)\int_0^1 \frac{-b_2}{\sqrt{2\pi s^3}}\exp\bigg(-\frac{b_2^2}{2s}\bigg)\frac{1}{4\pi\sqrt{t-s}}\exp\bigg(-\frac{(b_2-b_1)^2}{2(t-s)}\bigg)\mathrm{d}s \tag{3.16}$$

$$= \mathbf{1}(0\le t<1)\frac{1}{4\pi\sqrt{t}}\exp\bigg(-\frac{b_1^2}{2t}\bigg) \tag{3.17}$$

$$+ \mathbf{1}(t>1)\bigg(\exp\bigg(-\frac{b_1^2}{2t}\bigg)\frac{1}{4\pi\sqrt{t}}\mathcal{N}\bigg(\frac{b_2\sqrt{t-1}}{\sqrt{t}}+\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\bigg) + \exp\bigg(-\frac{(2b_2-b_1)^2}{2t}\bigg)\frac{1}{4\pi\sqrt{t}}\mathcal{N}\bigg(\frac{b_2\sqrt{t-1}}{\sqrt{t}}-\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\bigg)\bigg),$$

where the derivation from (3.15) to (3.17) is

$$\mathbf{1}(0\le t<1)\int_0^t \frac{-b_2}{\sqrt{2\pi s^3}}\exp\bigg(-\frac{b_2^2}{2s}\bigg)\frac{1}{4\pi\sqrt{t-s}}\exp\bigg(-\frac{(b_2-b_1)^2}{2(t-s)}\bigg)\mathrm{d}s$$

$$= \mathbf{1}(0\le t<1)\int_{-\infty}^{b_2/\sqrt{t}}\frac{1}{2\pi\sqrt{2\pi}}\frac{-x}{\sqrt{x^2t-b_2^2}}\exp\bigg(-\frac{x^2}{2}\bigg)\exp\bigg(-\frac{(b_2-b_1)^2x^2}{2(tx^2-b_2^2)}\bigg)\mathrm{d}x$$

$$= \mathbf{1}(0\le t<1)\exp\bigg(-\frac{b_2^2}{2t}\bigg)\exp\bigg(-\frac{(b_2-b_1)^2}{2t}\bigg) \times\int_0^\infty \frac{1}{2\pi\sqrt{2\pi}t}\exp\bigg(-\frac{1}{2t}\bigg(x^2+\frac{(b_2-b_1)^2b_2^2}{x^2}\bigg)\bigg)\mathrm{d}x$$

$$= \mathbf{1}(0\le t<1)\frac{1}{2}\exp\bigg(-\frac{b_2^2}{2t}\bigg)\exp\bigg(-\frac{(b_2-b_1)^2}{2t}\bigg) \times\bigg(\int_0^\infty \frac{1+(b_2-b_1)b_2/x^2}{2\pi\sqrt{2\pi}t}\exp\bigg(-\frac{1}{2t}\bigg(x-\frac{(b_2-b_1)b_2}{x}\bigg)^2\bigg) \times\exp\bigg(-\frac{1}{2t}(2(b_2-b_1)b_2)\bigg)\mathrm{d}x$$

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$$+\int_0^\infty \frac{1-(b_2-b_1)b_2/x^2}{2\pi\sqrt{2\pi}t}\exp\left(-\frac{1}{2t}\left(x+\frac{(b_2-b_1)b_2}{x}\right)^2\right)$$
$$\times\exp\left(\frac{1}{2t}(2(b_2-b_1)b_2)\right)\mathrm{d}x\Bigg)$$
$$=\mathbf{1}(0\le t<1)\frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_1^2}{2t}\right).$$

Likewise, we also have

$$\mathcal{L}^{-1}\left(\mathbb{E}[\mathrm{e}^{-\beta T_{b_2}}\mathbf{1}(T_{b_2}<1)]\frac{1}{4\sqrt{\pi\beta}}\right)$$
$$=\mathbf{1}(0\le t\le 1)\frac{1}{4\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right)+\mathbf{1}(t>1)\frac{1}{2\pi\sqrt{t}}\exp\left(-\frac{b_2^2}{2t}\right)\mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}}\right). \tag{3.18}$$

Finally, $L_0(t)$ is the sum of (3.14) and (3.18), and $\tilde{L}_0(t)$ is the difference of (3.14) and (3.18), completing the proof. □

**Corollary 3.1.** *For $0\le b_1<b_2$, on the set $T_{b_1}<1$, we have*

$$f_{b_1}^{b_2}(t,\ \tau_{b_2}^+<\tau_{b_1}^-,\ T_{b_1}<1)=f_{-b_2}^{-b_1}(t,\ \tau_{-b_2}^-<\tau_{-b_1}^+,\ T_{-b_1}<1),$$
$$f_{b_1}^{b_2}(t,\ \tau_{b_1}^-<\tau_{b_2}^+,\ T_{b_1}<1)=f_{-b_2}^{-b_1}(t,\ \tau_{-b_1}^+<\tau_{-b_2}^-,\ T_{-b_1}<1).$$

*Proof.* The results are due to the symmetry of Brownian motion. The positive barriers can be reflected to give the same result as in the case with negative barriers. □

## 4. Pricing a double-barrier Parisian call option

In the previous section we obtained a recursive formula for the density of the double-barrier Parisian stopping time for each of the three cases where we start in between the barrier ($b_1\le 0\le b_2$), above both barriers ($b_1<b_2\le 0$), and below both barriers ($0\le b_1<b_2$). In this section we will show how we can use the densities to compute the price of a double-barrier Parisian min-in call option.

### 4.1. Double-barrier Parisian min-in call

A double-barrier Parisian min-in call is a call option that gets knocked in if $\tau_{b_1}^{b_2}\le T$. We denote by

$$C_i^{\text{double}}(x,T,L_1,L_2,K)$$

the price of such an option with strike price $K$, barrier levels $L_1$ and $L_2$, where $L_1<L_2$, window length $D=1$, initial underlying price $x$, and maturity $T$. The payoff at maturity of such an option is

$$\mathbf{1}(\tau_{b_1}^{b_2}\le T)(S_T-K)^+.$$

When the underlying asset price follows a geometric Brownian motion, and when it is in between the two barriers $L_1$ and $L_2$, we have the following pricing formula.

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**Theorem 4.1.** *For $L_1 \le S_0 \le L_2$ ($b_1 \le 0 \le b_2$), the risk neutral price of a double-barrier Parisian min-in call with maturity $T > 1$ is given by*

$$
\begin{aligned}
&{}^*C_i^{\text{double}}(x, T, L_1, L_2, K) \\
&\quad = \sqrt{2\pi}\Bigg(\int_0^T f_{b_1}^{b_2}(t,\ \tau_{b_1}^- < \tau_{b_2}^+)(x\psi(\sigma + m, h_{b_1}, b_1, \rho, t) - K\psi(m, h'_{b_1}, b_1, \rho, t))\,\mathrm{d}t \\
&\qquad + \int_0^T f_{b_1}^{b_2}(t,\ \tau_{b_2}^+ < \tau_{b_1}^-)(x\psi(-(\sigma + m), h_{b_2}, -b_2, -\rho, t) \\
&\qquad\qquad - K\psi(-m, h'_{b_2}, -b_2, -\rho, t))\,\mathrm{d}t\Bigg),
\end{aligned} \tag{4.1}
$$

*where we have used the following functions in order to simplify the notation:*

$$
\begin{aligned}
&\psi(x, y, b, \rho, t) \\
&\quad := \exp\left(\frac{x^2(1 + T - t) + 2bx}{2}\right)\left(Z(-x)\mathcal{N}\left(\frac{-x\rho - y}{\sqrt{1 - \rho^2}}\right) - \rho Z(y)\mathcal{N}\left(\frac{-x - \rho y}{\sqrt{1 - \rho^2}}\right)\right. \\
&\qquad\qquad \left. - x(\mathcal{N}(-x) - \mathcal{N}_\rho(-x, y))\right),
\end{aligned} \tag{4.2}
$$

*with $\mathcal{N}_\rho(\cdot, \cdot)$ denoting the joint cumulative distribution of a pair of bivariate normal random variables with correlation coefficient $\rho$, and*

$$
h_b := \frac{1}{\sqrt{1 + T - t}}\left(k - b - (\sigma + m)(1 + T - t)\right), \tag{4.3}
$$

$$
h'_b := \frac{1}{\sqrt{1 + T - t}}\left(k - b - m(1 + T - t)\right), \qquad \rho := \frac{1}{\sqrt{1 + T - t}} \tag{4.4}
$$

*Proof.* As discussed in (2.1) and (2.2), we have

$$
{}^*C_i^{\text{double}}(x, T, L_1, L_2, K) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}(\tau_{b_1}^{b_2} \le T)\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+].
$$

We denote by $\mathcal{F}_t = \sigma(Z_s,\ s \le t)$ the natural filtration of the Brownian motion $(Z_t,\ t \ge 0)$. For ease of notation, since there is no ambiguity here, we refer to the double-barrier stopping time $\tau_{b_1}^{b_2}$ as just $\tau$. Then $\tau$ is an $\mathcal{F}_t$-stopping time, and by the strong Markov property of Brownian motion,

$$
\begin{aligned}
{}^*C_i^{\text{double}}(x, T, L_1, L_2, K) &= \mathbb{E}_{\mathbb{P}}[\mathbf{1}(\tau \le T)\mathbb{E}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]] \\
&= \mathbb{E}_{\mathbb{P}}[\mathbf{1}(\tau \le T)\mathbf{1}(\tau_{b_1}^- < \tau_{b_2}^+)\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]] \quad (4.5) \\
&\quad + \mathbb{E}_{\mathbb{P}}[\mathbf{1}(\tau \le T)\mathbf{1}(\tau_{b_2}^+ < \tau_{b_1}^-)\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]]. \quad (4.6)
\end{aligned}
$$

We have split this into two cases, $\tau_{b_1}^- < \tau_{b_2}^+$ and $\tau_{b_2}^+ < \tau_{b_1}^-$. In the first case, (4.5) is equal to

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[\mathbf{1}(\tau \le T)\mathbf{1}(\tau_{b_1}^- < \tau_{b_2}^+)\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]] \\
&\quad = \mathbb{E}_{\mathbb{P}}\Bigg[\mathbf{1}(\tau \le T)\mathbf{1}(\tau_{b_1}^- < \tau_{b_2}^+) \quad (4.7) \\
&\qquad \times \int_{-\infty}^{\infty} \mathrm{e}^{my}(x\mathrm{e}^{\sigma y} - K)^+ \frac{1}{\sqrt{2\pi(T - \tau)}} \exp\left(-\frac{(y - Z_\tau)^2}{2(T - \tau)}\right)\mathrm{d}y\Bigg]. \quad (4.8)
\end{aligned}
$$

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Since the stopping time $\tau\mathbf{1}(\tau_{b_1}^- < \tau_{b_2}^+)$ is independent of the Brownian meander $Z_{\tau_{b_1}^-}$, and the density of $Z_{\tau_{b_1}^-}$ is

$$v(\mathrm{d}z) = \mathbb{P}(Z_{\tau_{b_1}^-} \in \mathrm{d}z) = (b_1 - z)\mathrm{e}^{-(z-b_1)^2/2}\mathbf{1}(z < b_1)\,\mathrm{d}z,$$

(4.7) is equal to

$$\int_0^T \int_{-\infty}^{b_1} f_{b_1}^{b_2}(t;\ \tau_{b_1}^- < \tau_{b_2}^+)v(\mathrm{d}z) \times \int_k^\infty \mathrm{e}^{my}(x\mathrm{e}^{\sigma y} - K)\frac{1}{\sqrt{2\pi(T-t)}}\exp\left(-\frac{(y-z)^2}{2(T-t)}\right)\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t$$

for $k = (1/\sigma)\ln(K/x)$. Making use of the calculations of [4], we have

$$\begin{aligned}
&x\int_{-\infty}^{b_1}\int_k^\infty \frac{1}{2\pi\sqrt{T-t}}\mathrm{e}^{(\sigma+m)y}(b_1 - z)\exp\left(-\frac{(z-b_1)^2}{2}\right)\exp\left(-\frac{(y-z)^2}{2(T-t)}\right)\mathrm{d}z\,\mathrm{d}y \\
&\quad = x\exp\left(\frac{(\sigma+m)^2(1+T-t)+2b_1(\sigma+m)}{2}\right)\frac{1}{2\pi\sqrt{1-\rho^2}} \\
&\qquad \times \int_{-\infty}^{-(\sigma+m)}\int_{h_{b_1}}^\infty (-v-(\sigma+m))\exp\left(-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right)\mathrm{d}u\,\mathrm{d}v \\
&\quad = x\exp\left(\frac{x^2(1+T-t)+2b_1x}{2}\right)\left(Z(-(\sigma+m))\mathcal{N}\left(\frac{-(\sigma+m)\rho - h_{b_1}}{\sqrt{1-\rho^2}}\right)\right. \\
&\qquad\qquad \left. + \rho Z(h_{b_1})\mathcal{N}\left(\frac{-(\sigma+m)-\rho}{\sqrt{1-\rho^2}}\right)\right) \\
&\qquad - x\exp\left(\frac{x^2(1+T-t)+2b_1x}{2}\right)(\sigma+m)(\mathcal{N}(-(\sigma+m)) - \mathcal{N}_\rho(-(\sigma+m), h_{b_1})) \\
&\quad = x\psi(\sigma+m, h_{b_1}, b_1, \rho, t) - K\psi(m, h'_{b_1}, b_1, \rho, t),
\end{aligned}$$

where we have used the transformation $u = (y - (b_1 + (\sigma+m)(1+T-t)))/\sqrt{1+T-t}$ and $v = z - (b_1 + (\sigma+m))$, and denoted the function $\psi(x, y, b, \rho, t)$, $h_b$, $h'_b$, and $\rho$ as in (4.2)–(4.4). Similar calculations for the expectation (4.6) leads to the price of the option given in (4.1). □

**Theorem 4.2.** *For $S_0 \le L_1 < L_2$ ($0 \le b_1 < b_2$), the price of a double-barrier Parisian in call with maturity $T > 1$ can be expressed as*

$$\begin{aligned}
&{}^*C_i^{\text{double}}(x, T, L_1, L_2, K) \\
&\quad = x\phi(\sigma+m) - K\phi(m) \\
&\qquad + \sqrt{2\pi}\left(\int_0^T f_{b_1}^{b_2}(t,\ \tau_{b_1}^- < \tau_{b_2}^+,\ T_{b_1} < 1)(x\psi(\sigma+m, h_{b_1}, b_1, \rho, t)\right. \\
&\qquad\qquad \left. - K\psi(m, h'_{b_1}, b_1, \rho, t))\,\mathrm{d}t\right.
\end{aligned}$$

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$$+\int_0^T f_{b_1}^{b_2}(t,\ \tau_{b_2}^+ < \tau_{b_1}^-,\ T_{b_1} < 1)(x\psi(-(\sigma+m), h_{b_2}, -b_2, -\rho, t) - K\psi(-m, h'_{b_2}, -b_2, -\rho, t))\,\mathrm{d}t\Bigg), \tag{4.9}$$

*and, for $L_1 < L_2 \le S_0$ ($b_1 < b_2 \le 0$), the price of a double-barrier Parisian in call with maturity $T > 1$ can be expressed as*

$$\begin{aligned}
{}^*C_i^{\text{double}}(x, T, L_1, L_2, K) &= x\phi'(\sigma+m) - K\phi'(m) \\
&\quad + \sqrt{2\pi}\Bigg(\int_0^T f_{b_1}^{b_2}(t,\ \tau_{b_1}^- < \tau_{b_2}^+,\ T_{b_2} < 1)(x\psi(\sigma+m, h_{b_1}, b_1, \rho, t) \\
&\qquad - K\psi(m, h'_{b_1}, b_1, \rho, t))\,\mathrm{d}t \\
&\quad + \int_0^T f_{b_1}^{b_2}(t,\ \tau_{b_2}^+ < \tau_{b_1}^-,\ T_{b_2} < 1)(x\psi(-(\sigma+m), h_{b_2}, -b_2, -\rho, t) \\
&\qquad - K\psi(-m, h'_{b_2}, -b_2, -\rho, t))\,\mathrm{d}t\Bigg).
\end{aligned} \tag{4.10}$$

*In the above, we have defined the functions $\phi(x)$ and $\phi'(x)$ to be*

$$\begin{aligned}
\phi(x) &:= \exp\left(\frac{x^2T}{2}\right)\left(\mathcal{N}(b-x) - \mathcal{N}_{1/\sqrt{T}}\left(b-x, \frac{k-xT}{\sqrt{T}}\right)\right) \\
&\quad - \exp\left(\frac{x^2T+4bx}{2}\right)\left(\mathcal{N}(-b-x) - \mathcal{N}_{1/\sqrt{T}}\left(-b-x, \frac{k-2b-xT}{\sqrt{T}}\right)\right),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\phi'(x) &:= \exp\left(\frac{x^2T}{2}\right)\left(\bar{\mathcal{N}}_\rho\left(b-x, \frac{k-xT}{\sqrt{T}}\right)\right. \\
&\quad \left. - \exp\left(\frac{x^2T+4bx}{2}\right)\bar{\mathcal{N}}_\rho\left(-b-x, \frac{k-(2b+xT)}{\sqrt{T}}\right)\right).
\end{aligned} \tag{4.12}$$

*Proof.* For $S_0 \le L_1 < L_2$ ($0 \le b_1 < b_2$),

$$\begin{aligned}
{}^*C_i^{\text{double}}(x, T, L_1, L_2, K) &= \mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_1} \ge 1)\mathbb{E}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]] \\
&\quad + \mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_1} < 1)\mathbf{1}(\tau \le T)\mathbb{E}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]].
\end{aligned} \tag{4.13}$$

If $T_{b_1} \ge 1$, we have $\tau = 1$. Furthermore, the law of $Z_1$ on the set $\{T_{b_1} \ge 1\}$ is

$$\begin{aligned}
\mathbb{P}(Z_1 \in \mathrm{d}z, T_{b_1} \ge 1) &= \mathbb{P}(Z_1 \in \mathrm{d}z) - \mathbb{P}(Z_1 \in \mathrm{d}z, T_{b_1} < 1) \\
&= \frac{1}{\sqrt{2\pi}}\left(\exp\left(-\frac{z^2}{2}\right) - \exp\left(-\frac{(z-2b_1)^2}{2}\right)\right)\mathrm{d}z.
\end{aligned}$$

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Hence, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_1} \geq 1)\mathbb{E}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]] \\
&\quad = \frac{1}{\sqrt{2\pi(T-1)}} \int_{-\infty}^{b_1} \int_k^{\infty} \mathrm{e}^{my}(x\mathrm{e}^{\sigma y} - K) \exp\left(-\frac{(y-z)^2}{2(T-1)}\right) \\
&\qquad \times \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{z^2}{2}\right) - \exp\left(-\frac{(z-2b_1)^2}{2}\right)\right) \mathrm{d}y\,\mathrm{d}z \\
&\quad = x\phi(\sigma + m) - K\phi(m)
\end{aligned}$$

for $\phi(x)$ defined in (4.11) and (4.12). Equation (4.13) is the same as before with the density of the stopping time now being restricted to the set $\{T_{b_1} < 1\}$, and the pricing formula in (4.9) thus follows. For $L_1 < L_2 \leq S_0$ ($b_1 < b_2 \leq 0$), we have

$$\begin{aligned}
{}^*C_i^{\text{double}}(x, T, L_1, L_2, K) &= \mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_2} \geq 1)\mathbb{E}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]] \qquad (4.14) \\
&\quad + \mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_2} < 1)\mathbf{1}(\tau \leq T)\mathbb{E}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]]. \qquad (4.15)
\end{aligned}$$

The law of $Z_1$ on the set $\{T_{b_2} \geq 1\}$ is

$$\mathbb{P}(Z_1 \in \mathrm{d}z, T_{b_2} \geq 1) = \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{z^2}{2}\right) - \exp\left(-\frac{(z+2b_2)^2}{2}\right)\right) \mathrm{d}z.$$

Expression (4.14) is thus

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_2} \geq 1)\mathbb{E}[\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)^+ \mid \mathcal{F}_\tau]] \\
&\quad = \frac{1}{\sqrt{2\pi(T-1)}} \int_{b_2}^{\infty} \int_k^{\infty} \mathrm{e}^{my}(x\mathrm{e}^{\sigma y} - K) \exp\left(-\frac{(y-z)^2}{2(T-1)}\right) \\
&\qquad \times \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{z^2}{2}\right) - \exp\left(-\frac{(z+2b_2)^2}{2}\right)\right) \mathrm{d}y\,\mathrm{d}z \\
&\quad = x\phi'(\sigma + m) - K\phi'(m),
\end{aligned}$$

where $\phi'(x)$ is as defined in (4.12). Since (4.15) is the same as before with the density of the stopping time being restricted to the set $\{T_{b_2} < 1\}$, we have pricing formula (4.10). □

### 4.2. Double-barrier Parisian out call

The double-barrier Parisian out call is a call option which gets knocked out when the price of the underlying asset goes beyond the barriers. Hence, it has payoff $(S_T - K)^+\mathbf{1}(\tau \leq T)$ at time $T$. We denote by $C_o^{\text{double}}(x, T, L_1, L_2, K)$ the price of such an option with initial price $x$ and time to maturity $T$. Then since

$$\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-rT}(S_T - K)^+] = \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-rT}(S_T - K)^+\mathbf{1}(\tau \leq T)] + \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-rT}(S_T - K)^+\mathbf{1}(\tau > T)],$$

we have, for $L_1 \leq S_0 \leq L_2$ ($b_1 \leq 0 \leq b_2$),

$$C_o^{\text{double}}(x, T, L_1, L_2, K) = C_{BS}(x, T) - C_i^{\text{double}}(x, T),$$

where $C_{BS}(x, T)$ denotes the price of the vanilla call option with payoff $(S_T - K)^+$ at maturity $T$. For $S_0 \leq L_1 < L_2$ ($0 \leq b_1 < b_2$), the Parisian out call becomes useless if $T_{b_1} > 1$, hence,

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we have

$$\begin{aligned}
&C_o^{\text{double}}(x, T, L_1, L_2, K)\\
&\quad = \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-rT}\mathbf{1}(T_{b_1} < 1)\mathbf{1}(\tau > T)(S_T - K)^+]\\
&\quad = \mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_1} < 1)\mathrm{e}^{-rT}(S_T - K)^+] - \mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_1} < 1)\mathrm{e}^{-rT}\mathbf{1}(\tau \le T)(S_T - K)^+]\\
&\quad = \int_0^1 \frac{b_1}{\sqrt{2\pi t^3}} \exp\left(-\frac{b_1^2}{2t}\right) C_{BS}(L_1, T - t)\,\mathrm{d}t\\
&\qquad - (C_i^{\text{double}}(x, T, L_1, L_2, K) - (x\phi(\sigma + m) - K\phi(m))).
\end{aligned}$$

For $L_1 < L_2 \le S_0$ ($b_1 < b_2 \le 0$), the Parisian out call becomes useless if $T_{b_2} > 1$, hence, we have

$$\begin{aligned}
&C_o^{\text{double}}(x, T, L_1, L_2, K)\\
&\quad = \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{-rT}\mathbf{1}(T_{b_2} < 1)\mathbf{1}(\tau > T)(S_T - K)^+]\\
&\quad = \mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_2} < 1)\mathrm{e}^{-rT}(S_T - K)^+] - \mathbb{E}_{\mathbb{P}}[\mathbf{1}(T_{b_2} < 1)\mathrm{e}^{-rT}\mathbf{1}(\tau \le T)(S_T - K)^+]\\
&\quad = \int_0^1 \frac{-b_2}{\sqrt{2\pi t^3}} \exp\left(-\frac{b_2^2}{2t}\right) C_{BS}(L_2, T - t)\,\mathrm{d}t\\
&\qquad - (C_i^{\text{double}}(x, T, L_1, L_2, K) - (x\phi'(\sigma + m) - K\phi'(m))).
\end{aligned}$$

## 5. Numerical results

In this section we compute the prices of the two-sided Parisian in call options and compare the prices across different initial asset price $S_0$, and different window length $D$. We demonstrate the correctness of these results by comparing them with the numerical prices obtained by Anderluh and van der Weide [1] using Fourier transform inversion. We note that since we have chosen the window length $D$ as the unit of time, all parameters $(r, \sigma)$ are correspondingly normalised depending on the window length.

In Table 1 we present the results for a comparison of the prices of double-barrier Parisian in calls across different $S_0$ between 80 and 100, and $D$ between two weeks and three months, fixing the parameters $\sigma = 0.2$, $r = 0.05$, $T = 1$ year, $K = 90$, and $L_1 = 80$ and $L_2 = 100$.

TABLE 1: Price of the Parisian min-in call for $L_1 = 80$, $L_2 = 100$.

| $S_0$ | $D = \frac{1}{25}$ | $D = \frac{1}{12}$ | $D = \frac{1}{6}$ | $D = \frac{1}{4}$ |
|---|---|---|---|---|
| 80 | 3.894 968 | 3.540 956 | 2.844 183 | 2.210 231 |
| 82 | 4.650 542 | 4.245 481 | 3.478 998 | 2.782 763 |
| 84 | 5.526 743 | 5.082 531 | 4.256 382 | 3.498 445 |
| 86 | 6.525 658 | 6.055 637 | 5.182 448 | 4.366 187 |
| 88 | 7.644 386 | 7.161 490 | 6.255 264 | 5.387 473 |
| 90 | 8.877 623 | 8.394 327 | 7.470 518 | 6.561 364 |
| 92 | 10.218 160 | 9.746 352 | 8.821 621 | 7.884 263 |
| 94 | 11.657 360 | 11.208 180 | 10.299 920 | 9.349 889 |
| 96 | 13.185 670 | 12.769 300 | 11.895 020 | 10.949 430 |
| 98 | 14.793 030 | 14.418 470 | 13.595 140 | 12.671 890 |
| 100 | 16.396 830 | 16.037 190 | 15.231 930 | 14.310 760 |

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TABLE 2: Price of the Parisian min-in call $L_1 = 70$, $L_2 = 110$.

| $S_0$ | $D = \frac{1}{25}$ | $D = \frac{1}{12}$ | $D = \frac{1}{6}$ | $D = \frac{1}{4}$ |
|---|---|---|---|---|
| 80 | 2.192 441 | 1.780 915 | 1.240 433 | 0.857 615 |
| 82 | 2.814 304 | 2.332 982 | 1.679 245 | 1.199 394 |
| 84 | 3.563 176 | 3.007 037 | 2.229 091 | 1.639 840 |
| 86 | 4.444 798 | 3.811 887 | 2.902 650 | 2.194 032 |
| 88 | 5.462 882 | 4.754 442 | 3.711 202 | 2.876 336 |
| 90 | 6.618 767 | 5.839 241 | 4.663 993 | 3.699 677 |
| 92 | 7.911 251 | 7.068 144 | 5.767 730 | 4.674 874 |
| 94 | 9.336 592 | 8.440 194 | 7.026 212 | 5.810 075 |
| 96 | 10.888 640 | 9.951 636 | 8.440 122 | 7.110 332 |
| 98 | 12.559 120 | 11.596 080 | 10.006 980 | 8.577 325 |
| 100 | 14.337 960 | 13.364 780 | 11.721 250 | 10.209 270 |

TABLE 3: Computation time in seconds.

| | $D = \frac{1}{25}$ | $D = \frac{1}{12}$ | $D = \frac{1}{6}$ | $D = \frac{1}{4}$ |
|---|---|---|---|---|
| Elapsed time | 38.23 | 6.24 | 2.42 | 1.75 |

The prices decrease with longer window lengths, as it becomes more difficult to knock in the option. For comparison, we have computed the same call prices for the case when the barriers are widened to $L_1 = 70$ and $L_2 = 110$. As can be expected, the options become cheaper as it is now more difficult for the option to be knocked in. This is shown in Table 2.

The convolutions are evaluated using the `convolve` function in R. Due to the recursions, computation time decreases with the window length and are presented in Table 3.

## References


[1] ANDERLUH, J. H. M. AND VAN DER WEIDE, J. A. M. (2009). Double-sided Parisian option pricing. *Finance Stoch.* **13,** 205–238.

[2] BROEDERS, D. AND CHEN, A. (2010). Pension regulation and the market value of pension liabilities: a contingent claims analysis using Parisian options. *J. Banking Finance* **34,** 1201–1214.

[3] CHESNEY, M., JEANBLANC-PIQUE, M. AND YOR, M. (1997). Brownian excursions and Parisian barrier options. *Adv. Appl. Prob.* **29,** 165–184.

[4] DASSIOS, A. AND LIM, J. W. (2013). Parisian option pricing: a recursive solution for the density of the Parisian stopping time. *SIAM J. Financial Math.* **4,** 599–615.

[5] DASSIOS, A. AND WU, S. (2011). Double-barrier Parisian options. *J. Appl. Prob.* **48,** 1–20.

[6] DASSIOS, A. AND WU, S. (2009). On barrier strategy dividends with Parisian implementation delay for classical surplus processes. *Insurance Math. Econom.* **45,** 195–202.

[7] DASSIOS, A. AND WU, S. (2010). Perturbed Brownian motion and its application to Parisian option pricing. *Finance Stoch.* **14,** 473–494.

[8] HABER, R. J., SCHÖNBUCHER, P. J. AND WILMOTT, P. (1999). Pricing Parisian options. *J. Derivatives* **6,** 71–79.

[9] LABART, C. AND LELONG, J. (2009). Pricing double barrier Parisian options using Laplace transforms. *Internat. J. Theoret. Appl. Finance* **12,** 19–44.

[10] SCHRÖDER, M. (2003). Brownian excursions and Parisian barrier options: a note. *J. Appl. Prob.* **40,** 855–864.


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