Why does (4) only 'suggest'? Can you be more precise? (Here, try multiplying the 'contribution' $\sqrt{1+x} - \left(1 + \frac{1}{2}x\right)$ by $\sqrt{1+x} + \left(1 + \frac{1}{2}x\right)$ and discussing the value of the multiplier.)

Mental arithmetic: what is $\sqrt{7.4}$ to 3 significant figures? (If you know your squares up to, say, $32^2 = 1024$, start with $740 = 27^2 + 11$.) What is the square root algorithm in binary?

Lastly, or as an appetiser, one can work out the formula for the distance of the horizon in Imperial measure, namely $\sqrt{\frac{3h}{2}}$ miles, where h is the height in feet. Why are the numbers so simple?

References

1. Peter Shiu, Fermat's method of factorisation, *Math. Gaz.* **99** (March 2015) pp. 97-103.
2. Charles Hutton, *Mathematical Tables*, London (1804) pp. 64-65.

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When is the sum of two triangular numbers triangular?

It was good to see Erick Gooding's 30 year involvement with *Mathematical Pie* celebrated in the June 2015 *MA News*. His *Pie Notes* are a constant source of inspirational ideas and the Summer 2015 issue was certainly no exception, providing the starting-point for this article. The item 'Sum of the squares become triangles' concerns triangular numbers, $T(n) = \frac{1}{2}n(n+1)$, satisfying the relationship

$$T(x) + T(y) = T(z) \qquad (*)$$

for whole numbers x, y, z.

Erick notes that an infinite set of solutions is provided by

$$T\big(T(n) - 1\big) + T(n) = T\big(T(n)\big)$$

and, for this set, $T(n) - 1$, n and $T(n)$ form the sides of an obtuse-angled triangle which becomes increasingly right-angled as $n \to \infty$. He also lists seven other solutions of $(*)$ which are not part of this set.

Our first result is that for *any* solution of $(*)$, the lengths x, y, z form the sides of an obtuse-angled triangle which is increasingly right-angled as the perimeter $x + y + z$ increases.

For if $T(x) + T(y) = T(z)$, then $\frac{1}{2}x(x+1) + \frac{1}{2}y(y+1) = \frac{1}{2}z(z+1)$ so that $x^2 + y^2 - z^2 = z - x - y$ or $(x+y)^2 - z^2 + x + y - z = 2xy$ whence $(x + y - z)(x + y + z + 1) = 2xy$.

It follows that $x + y > z$ giving the required triangle inequality; also

$$\cos Z = \frac{x^2 + y^2 - z^2}{2xy} = \frac{z - x - y}{2xy} = \frac{-1}{x + y + z + 1}.$$

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Thus Z is always obtuse and approaches 90° as $x + y + z$ increases.

We next look at general solutions to (∗). From

$$\tfrac{1}{2}x(x+1) + \tfrac{1}{2}y(y+1) = \tfrac{1}{2}z(z+1)$$

we obtain

$$x(x+1) = z(z+1) - y(y+1) = (z-y)(z+y+1).$$

Matching factors then shows that there are positive integers a, b, c, d with $x + 1 = ab$, $x = cd$, $z - y = ac$, $z + y = bd - 1$ so that $x = cd$, $y = \frac{1}{2}(bd - ac - 1)$, $z = \frac{1}{2}(ac + bd - 1)$ subject to $ab - cd = 1$; we also need bd greater than, and of opposite parity to, ac to ensure that y and z are positive integers.

Erick's infinite set of solutions corresponds to the choice $a = 1$, $b = n + 1$, $c = 1$, $d = n$. His list of seven other solutions is reproduced in Table 1 below with corresponding values of a, b, c, d.

x	y	z	a	b	c	d
5	6	8	2	3	1	5
6	9	11	1	7	2	3
8	10	13	3	3	1	8
9	13	16	1	10	3	3
9	21	23	2	5	1	9
11	20	23	3	4	1	11
14	18	23	5	3	1	14

TABLE 1: Notice that a, b, c, d are not unique here – for example $a = 2, b = 11, c = 7, d = 3$ also generates the triple $\{9, 21, 23\}$.

Some other attractive infinite sets of solutions are given below.

1. Erick's infinite set of solutions is the only one with $z = y + 1$. The choice $a = k$ (fixed), $b > 1$ (odd), $c = 1$, $d = kb - 1$ gives the integers $x = kb - 1$, $y = \frac{1}{2}(b+1)(kb-k-1)$, $z = \frac{1}{2}\left(kb^2 - b + k - 1\right)$ for which $z = y + k$.
2. Problem 68.A from [1] (and the follow-up article [2]) concern the sequence of positive integers $(b_n)_{n=1}^{\infty}$ = 1,1,2,3,7,11,26,41,97, … generated by the recurrence relation $b_n b_{n+3} - b_{n+1}b_{n+2} = 1$ with initial conditions $b_1 = 1$, $b_2 = 1$, $b_3 = 2$. The choice $a = b_n$, $b = b_{n+3}$, $c = b_{n+2}$, $d = b_{n+1}$ is a valid one giving rise to $x = b_{n+1}b_{n+2}$, $y = \frac{1}{2}\left(b_{n+1}b_{n+3} - b_n b_{n+2} - 1\right)$, $z = \frac{1}{2}\left(b_n b_{n+2} + b_{n+1}b_{n+3} - 1\right)$.
3. The Fibonacci sequence $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ has the property that $F_{6n+1}^2 - F_{6n}F_{6n+2} = 1$ with F_{6n} even and F_{6n+1}, F_{6n+2} both odd. The choice $a = b = F_{6n+1}$, $c = F_{6n}$, $d = F_{6n+2}$ is then a valid one leading to

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$$x = F_{6n}F_{6n+2}, y = \tfrac{1}{2}\left(F_{6n+1}F_{6n+2} - F_{6n}F_{6n+1} - 1\right) = \tfrac{1}{2}\left(F_{6n+1}^2 - 1\right)$$

$$z = \tfrac{1}{2}\left(F_{6n}F_{6n+1} + F_{6n+1}F_{6n+2} - 1\right) = \tfrac{1}{2}\left(F_{6n+2}^2 - F_{6n}^2 - 1\right).$$

4. The case where x, y, z form an isosceles triangle with $x = y$ corresponds to $2T(x) = T(z)$ or $2(2x + 1)^2 - (2z + 1)^2 = 1$. Standard theory of the Pell equation shows that all solutions of $a^2 - 2b^2 = -1$ are given by the recurrence relation $a_1 = b_1 = 1$, $a_{n+1} = 3a_n + 4b_n$, $b_{n+1} = 2a_n + 3b_n$ with a_n, b_n always odd numbers. So all solutions to $2T(x) = T(z)$ are generated by $x_1 = z_1 = 0$, $x_{n+1} = 3x_n + 2z_n + 2$, $z_{n+1} = 4x_n + 3z_n + 3$. The first few are shown in Table 2: note how the successive convergents of the continued fraction expansion of $\sqrt{2}$ occur in the entries for a, b, c, d.

n	x	z	a	b	c	d
2	2	3	1	3	1	2
3	14	20	3	5	2	7
4	84	119	5	17	7	12
5	492	696	17	29	12	41
6	2870	4059	29	99	41	70

TABLE 2

References

1. K. May, Problem 68.A, *Math. Gaz.* **68** (March 1984) p. 57; Solution, *Math. Gaz.* **68** (October 1984) pp. 219-221.
2. T. Crilly, Double sequences of positive integers, *Math. Gaz.* **69** (December 1985) pp. 263-271.

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Constructing the 15th pentagon that tiles the plane

In July 2015, Casey Mann, Jennifer McLoud-Mann and David Von Derau from the University of Washington Bothell announced the discovery of the 15th type of pentagon that tiles the plane. This arose from a systematic computer search described in their paper [1]. Their discovery made the newspapers over the summer and Alex Bellos provided a very readable account on his mathematics blog [2]. It is known that tessellations with congruent copies of a single convex n-gon only exist for $n = 3, 4, 5, 6$ and that, for $n = 3, 4, 6$, these are fully classified [3]. For $n = 5, 15$ types of tiling pentagon have been found, but it is not known whether or not there are any more, [4]. Up to similarity, the new pentagon has the lengths and angles shown in Figure 1. This shape, together with its reflection, tiles the plane in the manner shown in Figure 2 (and on the cover).

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