Ergod. Th. & Dynam. Sys. (2018), **38**, 1564–1587 © *Cambridge University Press, 2016*
doi:10.1017/etds.2016.77

The dynamical zeta function for commuting automorphisms of zero-dimensional groups

RICHARD MILES† and THOMAS WARD‡

† *MASH, University of Sheffield, Sheffield S10 2TN, UK*
(e-mail: r.miles@sheffield.ac.uk)
‡ *Level 13 Ziff Building, University of Leeds, Leeds LS2 9JT, UK*
(e-mail: t.b.ward@leeds.ac.uk)

(*Received* 20 *April* 2015 *and accepted in revised form* 20 *July* 2016)

Abstract. For a $\mathbb{Z}^d$-action α by commuting homeomorphisms of a compact metric space, Lind introduced a dynamical zeta function that generalizes the dynamical zeta function of a single transformation. In this article, we investigate this function when α is generated by continuous automorphisms of a compact abelian zero-dimensional group. We address Lind's conjecture concerning the existence of a natural boundary for the zeta function and prove this for two significant classes of actions, including both zero entropy and positive entropy examples. The finer structure of the periodic point counting function is also examined and, in the zero entropy case, we show how this may be severely restricted for subgroups of prime index in $\mathbb{Z}^d$. We also consider a related open problem concerning the appearance of a natural boundary for the dynamical zeta function of a single automorphism, giving further weight to the Pólya–Carlson dichotomy proposed by Bell and the authors.

1. *Introduction*

For a $\mathbb{Z}^d$-action α generated by d commuting homeomorphisms of a compact metric space X, Lind [**16**] introduced a dynamical zeta function that generalizes the well-known dynamical zeta function of a single transformation introduced by Artin and Mazur [**1**]. Lind's dynamical zeta function is defined as follows. Denote the set of finite index subgroups of $\mathbb{Z}^d$ by $\mathcal{L}_d$ and, for any $\Lambda \in \mathcal{L}_d$, let $[\Lambda] = |\mathbb{Z}^d/\Lambda|$. For any $\mathbf{n} \in \mathbb{Z}^d$, let $\alpha^{\mathbf{n}}$ denote the element of the action corresponding to $\mathbf{n}$ and let $\mathsf{F}_\alpha(\Lambda)$ denote the cardinality of the set of points $x \in X$ with $\alpha^{\mathbf{n}}(x) = x$ for all $\mathbf{n} \in \Lambda$. The *dynamical zeta function* of α is defined as a formal power series by

$$\zeta_\alpha(z) = \exp\left(\sum_{\Lambda \in \mathcal{L}_d} \frac{\mathsf{F}_\alpha(\Lambda)}{[\Lambda]} z^{[\Lambda]}\right).$$

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

When α is generated by a single transformation (that is, a $\mathbb{Z}$-action), ζ_α agrees with the dynamical zeta function considered by Artin and Mazur.

The number of subgroups of index n in $\mathbb{Z}^d$ is polynomially bounded in n and so, as observed by Lind [**16**, Theorem 5.3], it follows from the usual Hadamard formula that ζ_α has radius of convergence $e^{-\mathsf{g}(\alpha)}$, where

$$\mathsf{g}(\alpha) = \limsup_{[\Lambda]\to\infty} \frac{1}{[\Lambda]} \log \mathsf{F}_\alpha(\Lambda)$$

is the *upper growth rate of periodic points*, which may or may not coincide with the topological entropy $\mathsf{h}(\alpha)$. Lind conjectures [**16**, §7] that for a $\mathbb{Z}^d$-action with $d \geqslant 2$, the circle $|z| = e^{-\mathsf{h}(\alpha)}$ is a natural boundary for ζ_α and that ζ_α is meromorphic inside this circle. Part of this complexity comes from the subgroup structure of the acting group: as Lind points out, for $d \geqslant 2$ the zeta function of the trivial $\mathbb{Z}^d$-action on a point has a natural boundary.

An important class of $\mathbb{Z}^d$-actions particularly amenable to study are the algebraic ones (see the monograph of Schmidt [**24**] for an overview), that is, those generated by commuting automorphisms of compact metric abelian groups. We assume throughout that the group X under consideration is metrizable. When α is mixing and X is connected with finite topological dimension, the first author has shown that Lind's conjecture holds [**22**]. Motivated in part by examples such as Ledrappier's $\mathbb{Z}^2$-action [**14**] that do not fall into this class, our purpose here is to investigate ζ_α for algebraic systems where X is zero-dimensional (equivalently, X is totally disconnected). Notably, Ledrappier's example plays an important role in Lind's initial investigation [**14**, Example 3.4] and in many subsequent studies of algebraic $\mathbb{Z}^d$-actions. For example, it is a prototypical example of a $\mathbb{Z}^d$-action of a compact abelian zero-dimensional group where every element of the action has finite entropy, that is, an *entropy rank one action* [**8**]. If α is an entropy rank one action, then clearly $\mathsf{h}(\alpha) = 0$ if $d \geqslant 2$.

As well as having the motivation of Lind's conjecture, we also consider a separate problem that concerns the dynamical zeta function for a $\mathbb{Z}$-action generated by a single automorphism. In this case, if ζ_α is not rational, there is strong evidence to suggest that ζ_α admits a natural boundary. For automorphisms of connected groups this problem is considered in [**2**], and the conjectured dichotomy discussed there is further supported by the following.

THEOREM 1.1. *Let X be a compact abelian zero-dimensional group. Suppose α is generated by an ergodic automorphism of X that commutes with an expansive automorphism of X and suppose that α has finite entropy. Then the zeta function ζ_α has radius of convergence $e^{-\mathsf{h}(\alpha)}$, and either*

$$\zeta_\alpha(z) = (1 - e^{\mathsf{h}(\alpha)}z)^{-1}$$

or the circle $|z| = e^{-\mathsf{h}(\alpha)}$ is a natural boundary for the function ζ_α.

In [**27**], the second author considered a natural uncountable family of ergodic automorphisms of zero-dimensional groups parameterized by a probability space, and showed that an irrational zeta function should be expected almost surely for an

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

automorphism in this family. Furthermore, the first author shows in [**20**] that any ergodic finite entropy automorphism of a compact abelian zero-dimensional group has a sequence of periodic point counts that is simply a finite product of the sequences dealt with in [**27**]. It is also shown in [**27**] that ζ_α has radius of convergence $e^{-\mathsf{h}(\alpha)}$ almost surely, so, from this perspective, Theorem 1.1 points to the occurrence of a natural boundary for ζ_α as typical behaviour, even for a single automorphism.

A crucial step in the proof of the main result in [**22**] is the beautiful theorem of Pólya and Carlson which states that a power series with integer coefficients and radius of convergence equal to one is either rational or has the unit circle as a natural boundary (see [**3**, **23**, **25**]). Using this result, it may be shown that if ζ_α has radius of convergence equal to one and $d \geqslant 2$, then ζ_α admits the unit circle as a natural boundary (see Proposition 4.1). In using the Pólya–Carlson theorem in these contexts, both the integrality of the coefficients and the radius of convergence create problems, and some argument is needed. Consequently, Lind's conjecture is shown to hold in the setting of [**22**] by demonstrating that the upper growth rate of periodic points is zero. This is achieved with the help of an upper estimate for $\mathsf{F}_\alpha(\Lambda)$ obtained using techniques from [**21**] and a theorem of Corvaja and Zannier [**6**] concerning bounds on quantities related to greatest common divisors for rings of S-integers. In the setting of this article, similar results are unavailable (see [**7**] and [**26**]). However, by further developing existing periodic point counting techniques for entropy rank one actions [**19**] and by appealing more directly to the geometry of $\mathcal{L}_d$, we adopt an alternative approach to show that $\mathsf{g}(\alpha) = 0$ for all mixing entropy rank one $\mathbb{Z}^d$-actions by automorphisms of compact abelian zero-dimensional groups, with $d \geqslant 2$. Thus, we are able to prove the following.

THEOREM 1.2. *Let α be a mixing entropy rank one $\mathbb{Z}^d$-action by automorphisms of a compact abelian zero-dimensional group X, with $d \geqslant 2$. Then ζ_α has radius of convergence equal to one and the unit circle is a natural boundary for the function.*

In addition to showing that the upper growth rate of periodic points is zero for entropy rank one actions, we also investigate the finer structure of the periodic point counting function $\mathsf{F}_\alpha : \mathcal{L}_d \to \mathbb{N}$, with a view to identifying a significant class of subgroups upon which this function is, in fact, bounded. This emerged as a possibility from examples such as Ledrappier's. The following result applies to algebraic $\mathbb{Z}^d$-actions that are Noetherian, which, in our zero-dimensional setting, equates to the dynamical property of expansiveness [**12**] (see §2 for a formal definition).

THEOREM 1.3. *Let α be a mixing Noetherian entropy rank one $\mathbb{Z}^d$-action by automorphisms of a compact abelian zero-dimensional group X, with $d \geqslant 2$. Then there is a set of indices Q of positive density in the rational primes such that $\{\mathsf{F}_\alpha(\Lambda) : [\Lambda] \in Q\}$ is a finite set. When $d \geqslant 3$, such a set Q may be found with density one.*

The proof of Theorem 1.3 relies on a strong upper estimate for $\mathsf{F}_\alpha(\Lambda)$ obtained in the proof of Theorem 1.2 (see Corollary 2.1 and Example 4.1) and contesting lower estimates provided by quantities analogous to greatest common divisors for subrings of global function fields. The proof also involves multiplicative orders of rational primes, and this accounts for the statement involving prime densities (see Remark A).

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

Finally, we delve deeper into the dual algebraic structure behind $\mathbb{Z}^d$-actions in order to obtain a product decomposition (Corollary 5.1) for counting periodic points for $\mathbb{Z}^2$-actions with positive entropy. This allows us to exploit our earlier estimates for entropy rank one actions in an essential way and to prove Lind's conjecture in the following setting.

THEOREM 1.4. *Let α be a mixing Noetherian $\mathbb{Z}^2$-action by automorphisms of a compact abelian zero-dimensional group. Then ζ_α has radius of convergence $e^{-\mathsf{h}(\alpha)}$, and the circle of convergence is a natural boundary for the function.*

Notably, in the proof of Theorem 1.4, the Pólya–Carlson theorem cannot be used in the same way as in the proof of Theorem 1.2, as ζ_α may not have the unit circle as the circle of convergence. Nonetheless, we are able to use a functional equation relating the dynamical zeta function to an ordinary generating function, such that, in combination with our product formula for counting periodic points, the Pólya–Carlson result can be applied.

The paper is organized as follows. In §2, we develop the periodic point counting results that lead to Theorems 1.2 and 1.3. In §3, using existing periodic point counting formulae, we show how the dichotomy of Theorem 1.1 arises by directly exhibiting singularities on the circle of convergence, instead of via the Pólya–Carlson result. In §4, we describe the final steps in the proof of Theorem 1.2 and consider some examples. In particular, Example 4.2 is a mixing $\mathbb{Z}^3$-action α on a zero-dimensional group with $\mathsf{h}(\alpha) = 0$ and $\mathsf{g}(\alpha) = \frac{2}{3}\log 2$. The dynamical zeta function for this example also has a natural boundary, and hence further supports Lind's conjecture. However, it arises in an essentially different way to all the other cases we consider, and cannot be dealt with by similar methods. It is also interesting to note that the value of $\mathsf{g}(\alpha)$ for this example is smaller than any possible non-zero entropy of a $\mathbb{Z}^d$-action by automorphisms of a compact abelian zero-dimensional group. Finally, §5 is devoted to the proof of Theorem 1.4.

2. *Periodic points*

In this section we assemble and develop periodic point counting results in order to describe the dynamical zeta function sufficiently explicitly to demonstrate a natural boundary in a variety of settings.

The commutative algebra used to study a $\mathbb{Z}^d$-action α by continuous automorphisms of a compact abelian group X is now familiar (see, for example, Schmidt's monograph [**24**] and Einsiedler and Lind's paper [**8**] which is particularly useful when each automorphism $\alpha^{\mathbf{n}}$ has finite entropy, in which case α is said to be of *entropy rank one*). The starting point is to notice that the Pontryagin dual of X, denoted by $M = \widehat{X}$, becomes a module over the Laurent polynomial ring $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ by identifying application of the dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$, and extending this in a natural way to polynomials. Conversely, using this multiplication rule, any R_d-module M defines an action α_M of $\mathbb{Z}^d$ by continuous automorphisms of $X = \widehat{M}$.

Additionally, if X is assumed to be zero-dimensional, via duality, this means that every element of M has finite additive order, so every associated prime ideal $\mathfrak{p} \in \mathrm{Ass}(M)$ contains a rational prime. It will also be useful to refer to the coheight of a prime ideal $\mathfrak{p} \subset R_d$, denoted by $\mathrm{coht}(\mathfrak{p})$; this coincides with the Krull dimension of the domain $R_d/\mathfrak{p}$. Einsiedler and Lind show that if α has entropy rank one and if X is zero-dimensional, then

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

$\operatorname{coht}(\mathfrak{p}) \leqslant 1$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. Unless otherwise stated, we will always assume from now on that X is zero-dimensional. If the R_d-module M is Noetherian, then we also say that $\alpha = \alpha_M$ is Noetherian. In this case, there is a chain of submodules

$$M_0 \subset M_1 \subset \cdots \subset M_r = M, \tag{1}$$

for which $M_0 = \{0\}$ and $M_i/M_{i-1} \cong R_d/\mathfrak{p}_i$ for a list $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \subset R_d$ of prime ideals which are either associated primes of M, or contain such a prime. In addition, we may always arrange such a filtration so that the associated primes appear first and so that those associated primes with the greatest coheight appear foremost amongst the associated primes.

Recall that a *global field* $\mathbb{K}$ of characteristic $p > 0$ is a finite extension of the rational function field $\mathbb{F}_p(t)$, where t is an indeterminate. The *places* of $\mathbb{K}$ are the equivalence classes of absolute values on $\mathbb{K}$, which are all non-Archimedean. For example, the *infinite place* of $\mathbb{F}_p(t)$ is given by $|f/g|_\infty = p^{\deg(f)-\deg(g)}$ and all other places of $\mathbb{F}_p(t)$ correspond, in the usual way, to valuation rings obtained by localizing the domain $\mathbb{F}_p[t]$ at its non-trivial prime ideals (generated by irreducible polynomials). The places of $\mathbb{K}$ are extensions of those just described, and the set of all such places is denoted by $\mathcal{P}(\mathbb{K})$. Such sets of places provide a foundation for the periodic point counting formulae presented here.

For ease of notation, we write $\mathsf{F}_\alpha(\langle \mathbf{n} \rangle) = \mathsf{F}_\alpha(\mathbf{n})$ for any $\mathbf{n} \in \mathbb{Z}^d$. Suppose that α is an entropy rank one $\mathbb{Z}^d$-action and that the module M is Noetherian. If there exists $\mathbf{n} \in \mathbb{Z}^d$ for which $\alpha^{\mathbf{n}}$ is ergodic, then [**19**, §2] shows that $\operatorname{coht}(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. In this case, the field of fractions of $R_d/\mathfrak{p}$ is a global field of positive characteristic which we denote by $\mathbb{K}(\mathfrak{p})$, and the corresponding set of places is denoted by $\mathcal{P}(\mathbb{K}(\mathfrak{p}))$. We assume throughout that the associated absolute values are normalized so that the Artin product formula for global fields holds. For any $\mathbf{n} \in \mathbb{Z}^d$ for which $\alpha^{\mathbf{n}}$ is ergodic, [**19**, §3] shows that

$$\mathsf{F}_\alpha(\mathbf{n}) = \prod_{\mathfrak{p} \in \operatorname{Ass}(M)} \prod_{v \in S(\mathfrak{p})} |\overline{u}^{\mathbf{n}} - 1|_v^{m(\mathfrak{p})}, \tag{2}$$

where $\overline{u}^{\mathbf{n}}$ denotes the image of $u_1^{n_1} \cdots u_d^{n_d}$ in the domain $R_d/\mathfrak{p}$, $m(\mathfrak{p})$ is the dimension of the $\mathbb{K}(\mathfrak{p})$-vector space $M \otimes \mathbb{K}(\mathfrak{p})$ and

$$S(\mathfrak{p}) = \{v \in \mathcal{P}(\mathbb{K}(\mathfrak{p})) : |R_d/\mathfrak{p}|_v \text{ is an unbounded subset of } \mathbb{R}\}. \tag{3}$$

Note that the set $S(\mathfrak{p})$ is a finite because $R_d/\mathfrak{p}$ is finitely generated.

In a more general setting, it is difficult to obtain such an explicit formula, although coarser estimates are also sometimes useful, such as the following.

LEMMA 2.1. *Let α be an entropy rank one $\mathbb{Z}^d$-action by automorphisms of a compact abelian zero-dimensional group X and let M denote the dual R_d-module. Then there is a $\mathbb{Z}^d$-action β by automorphisms of a compact abelian zero-dimensional group corresponding via duality to a Noetherian submodule of M such that, for each $\mathbf{n} \in \mathbb{Z}^d$ for which $\alpha^{\mathbf{n}}$ is ergodic, $\beta^{\mathbf{n}}$ is ergodic and*

$$\mathsf{F}_\alpha(\mathbf{n}) \leqslant \mathsf{F}_\beta(\mathbf{n}) < \infty.$$

Proof. If there is no $\mathbf{n} \in \mathbb{Z}^d$ for which $\alpha_M^{\mathbf{n}}$ is ergodic, or if M is Noetherian, then there is nothing to prove. Hence, assume that $\alpha_M^{\mathbf{n}}$ is ergodic and M is not Noetherian. Since M

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

corresponds to an entropy rank one action, so does every submodule $L \subset M$ and quotient M/L. Also, since M can be expressed as a countable increasing union of Noetherian submodules, [**8**, Propositions 4.4 and 6.1] show that every prime ideal $\mathfrak{p} \in \mathrm{Ass}(M)$ has $\mathrm{coht}(\mathfrak{p}) \leqslant 1$. If $\mathrm{coht}(\mathfrak{p}) = 0$, then $\mathfrak{p}$ is maximal, $R_d/\mathfrak{p}$ is a finite field and, subsequently, $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is not ergodic. Since there is a submodule of M isomorphic to $R_d/\mathfrak{p}$, this contradicts ergodicity of $\alpha^{\mathbf{n}}_M$, by [**24**, Proposition 6.6]. Therefore, $\mathrm{coht}(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$.

If $L \subset M$ is a Noetherian submodule, $\mathfrak{p} \in \mathrm{Ass}(M/L)$ and $\mathrm{coht}(\mathfrak{p}) = 1$, then, for some $x \in M$, $\mathrm{Ann}(x + L) = \mathfrak{p}$. So $\mathfrak{p}$ contains $\mathrm{Ann}(x)$ and $\mathfrak{p} \in \mathrm{Supp}(Rx)$. Since all associated primes of Rx have coheight one, $\mathrm{Ass}(Rx)$ coincides with the elements of $\mathrm{Supp}(Rx)$ with coheight one, by [**18**, Theorem 6.5(iii)]. Hence,

$$\mathfrak{p} \in \mathrm{Ass}(M/L) \quad \text{and} \quad \mathrm{coht}(\mathfrak{p}) = 1 \Rightarrow \mathfrak{p} \in \mathrm{Ass}(M). \tag{4}$$

We claim that there is a Noetherian submodule $L \subset M$ such that $\mathrm{Ass}(M/L)$ comprises entirely maximal ideals. To see this, choose a submodule $L \subset M$ such that $L \cong R_d/\mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$. Since $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is an ergodic automorphism of a zero-dimensional group, $\mathsf{h}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}) \geqslant \log 2$. Now if there exists $\mathfrak{p}' \in \mathrm{Ass}(M/L)$ with $\mathrm{coht}(\mathfrak{p}') = 1$, then (4) shows that $\mathfrak{p}' \in \mathrm{Ass}(M)$, and there is a Noetherian submodule $L' \subset M$ containing L with $L'/L \cong R/\mathfrak{p}'$. Furthermore, the entropy addition formula shows that $\mathsf{h}(\alpha^{\mathbf{n}}_{L'}) \geqslant 2 \log 2$. If we can find no such prime $\mathfrak{p}'$, then $\mathrm{Ass}(M/L)$ comprises entirely maximal ideals. If we repeat the process inductively with L' in place of L, it must eventually terminate with a Noetherian submodule $L \subset M$ with the claimed property, otherwise we contradict the assumption that $\mathsf{h}(\alpha^{\mathbf{n}}_M) < \infty$. Note also that L is independent of the specific choice of ergodic automorphism $\alpha^{\mathbf{n}}_M$.

We may now find a chain of modules

$$L = M_0 \subset M_1 \subset M_2 \subset \cdots$$

with $M = \bigcup_{i \geqslant 0} M_i$ and M_i/M_{i-1} finite for all $i \geqslant 1$. Following the approach of [**19**, §3], $\mathsf{F}_\alpha(\mathbf{n}) = |M/(u^{\mathbf{n}} - 1)M|$,

$$M/(u^{\mathbf{n}} - 1)M \cong \varinjlim M_i/((u^{\mathbf{n}} - 1)M \cap M_i), \tag{5}$$

and

$$|M_i/((u^{\mathbf{n}} - 1)M \cap M_i)| \leqslant |M_i/(u^{\mathbf{n}} - 1)M_i)|. \tag{6}$$

Since each M_i is a Noetherian R_d-module corresponding to an entropy rank one action, and since each factor M_i/M_{i-1} is finite, the proof of [**19**, Theorem 3.2] shows that

$$|M_i/(u^{\mathbf{n}} - 1)M_i)| = |L/(u^{\mathbf{n}} - 1)L|$$

for all $i \geqslant 0$. It follows from (5) and (6) that $|M/(u^{\mathbf{n}} - 1)M|$ is bounded above by $|L/(u^{\mathbf{n}} - 1)L)|$. If we let $\beta = \alpha_L$ be the the action corresponding to the Noetherian R_d-module L, then $\mathsf{F}_\beta(\mathbf{n}) = |L/(u^{\mathbf{n}} - 1)L|$, and the required result follows. □

This leads to the following uniformity result.

THEOREM 2.1. *If α is an entropy rank one $\mathbb{Z}^d$-action by continuous automorphisms of a compact abelian zero-dimensional group X, then there is a constant $C > 0$ such that, for all $\mathbf{n} \in \mathbb{Z}^d$ for which $\alpha^{\mathbf{n}}$ is ergodic,*

$$\mathsf{F}_\alpha(\mathbf{n}) \leqslant C^{\|\mathbf{n}\|}.$$

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

Proof. Since we simply require an upper estimate for $\mathsf{F}_\alpha(\mathbf{n})$, using Lemma 2.1, we may as well assume that (X, α) corresponds to a Noetherian module M. In this case, we may apply (2) to calculate $\mathsf{F}_\alpha(\mathbf{n})$.

Considering any place v appearing in (2), the ultrametric inequality shows that

$$|\overline{u}^{\mathbf{n}} - 1|_v \leqslant \prod_{i=1}^{d} \max\{|\overline{u}_i|_v, |\overline{u}_i|_v^{-1}\}^{|n_i|}.$$

Let

$$\lambda = \max\{\max\{|\overline{u}_i|_v, |\overline{u}_i|_v^{-1}\} : i = 1, \ldots, d,\ v \in S(\mathfrak{p}),\ \mathfrak{p} \in \mathrm{Ass}(M)\}$$

and $E = \sum_{\mathfrak{p}\in\mathrm{Ass}(M)} m(\mathfrak{p})|S(\mathfrak{p})|$. Then the right-hand side of (2) is at most

$$\lambda^{E\cdot\|\mathbf{n}\|_1},$$

where $\|\mathbf{n}\|_1 = \sum_{i=1}^{d} |n_i|$. Since $\|\mathbf{n}\|_1 \leqslant \sqrt{d}\|\mathbf{n}\|$, the required result follows from (2) with $C = \lambda^{E\sqrt{d}}$. □

We now turn our attention to estimating the size of $\mathsf{F}_\alpha(\Lambda)$, $\Lambda \in \mathcal{L}_d$. For any $\Lambda \in \mathcal{L}_d$, let

$$\mathfrak{b}_\Lambda = (\overline{u}^{\mathbf{n}} - 1 : \mathbf{n} \in \Lambda) \subset R_d. \tag{7}$$

Note that this ideal is identical to $(\overline{u}^{\mathbf{n}} - 1 : \mathbf{n} \in \Lambda')$, where Λ' is any set of generators for the group Λ.

To begin with, we have the following consequence of Theorem 2.1.

COROLLARY 2.1. *Let α be an entropy rank one $\mathbb{Z}^d$-action by continuous automorphisms of a compact abelian zero-dimensional group X. Then there is a constant $C > 0$ such that, for all $\Lambda \in \mathcal{L}_d$ with $\alpha^{\mathbf{n}}$ ergodic for all $\mathbf{n} \in \Lambda\backslash\{\mathbf{0}\}$,*

$$\mathsf{F}_\alpha(\Lambda) \leqslant C^{\sqrt{d}[\Lambda]^{1/d}}.$$

Proof. By a standard application of Minkowski's theorem, for any $\Lambda \in \mathcal{L}_d$, we can find a non-trivial $\mathbf{m} \in \Lambda$ such that

$$\|\mathbf{m}\| \leqslant \sqrt{d}[\Lambda]^{1/d}.$$

If $\Lambda \in \mathcal{L}_d$ is such that $\alpha^{\mathbf{n}}$ ergodic for all $\mathbf{n} \in \Lambda\backslash\{\mathbf{0}\}$, then, since

$$\mathsf{F}_\alpha(\Lambda) \leqslant \mathsf{F}_\alpha(\mathbf{m}),$$

the required result follows from Theorem 2.1. □

The following is a well-known consequence of Pontryagin duality.

LEMMA 2.2. *Let α be a $\mathbb{Z}^d$-action by continuous automorphisms of a compact abelian group X and let M be the corresponding dual module. Then, whenever either $\mathsf{F}_\alpha(\Lambda)$ or $|M/\mathfrak{b}_\Lambda M|$ is finite, both these quantities coincide.*

Proof. See [**17**, Lemma 7.2]. □

Using the isomorphism theorems (see [**17**, Lemma 7.5]), we also have the following basic factorization result.

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

LEMMA 2.3. *Let $L \subset M$ be R_d-modules and $\mathfrak{b} \subset R_d$ an ideal such that $M/\mathfrak{b}M$ is finite, then*

$$|M/\mathfrak{b}M| = \left|\frac{M/L}{\mathfrak{b}(M/L)}\right| \frac{|L/\mathfrak{b}L|}{|(\mathfrak{b}M \cap L)/\mathfrak{b}L|}.$$

When M is a module of the form $D = R_d/\mathfrak{p}$, with $\mathfrak{p}$ a prime ideal of coheight one, D is a finitely generated domain of Krull dimension one over $\mathbb{Z}$ or $\mathbb{F}_p$ for some rational prime p. Every proper quotient of D by an ideal is finite, the integral closure E of D is a finitely generated D-module, and the quotient E/D is finite. This leads to the following useful result.

LEMMA 2.4. *Let $\mathfrak{p} \subset R_d$ be a prime ideal with* $\operatorname{coht}(\mathfrak{p}) = 1$ *and let $D = R_d/\mathfrak{p}$. Then there exist constants $A, B > 0$ such that $A|E/\mathfrak{a}E| \leqslant |D/\mathfrak{a}| \leqslant B|E/\mathfrak{a}E|$ for any proper ideal $\mathfrak{a} \subset D$, where E is the integral closure of D in its field of fractions.*

Proof. Firstly, since D arises as a quotient ring of R_d, both D and E may be considered as R_d-modules in a natural way and Lemma 2.3 shows that

$$|E/\mathfrak{a}E| \leqslant |(E/D)/\mathfrak{a}(E/D)||D/\mathfrak{a}| \leqslant |E/D||D/\mathfrak{a}|.$$

Furthermore, since E is generated by a finite number of elements in the field of fractions of D, there exists $x \in D$ which will clear denominators so that xE is a finite index submodule of D. Therefore, in a similar way,

$$|D/\mathfrak{a}| \leqslant |D/xE||xE/\mathfrak{a}xE| = |D/xE||E/\mathfrak{a}E|,$$

where the final equality holds as $xE \cong E$. Hence,

$$A|E/\mathfrak{a}E| \leqslant |D/\mathfrak{a}| \leqslant B|E/\mathfrak{a}E|,$$

where $A = |E/D|^{-1}$ and $B = |D/xE|$ are independent of the ideal $\mathfrak{a}$. □

LEMMA 2.5. *Let $\mathfrak{p} \subset R_d$ be a prime ideal with* $\operatorname{coht}(\mathfrak{p}) = 1$ *and let $D = R_d/\mathfrak{p}$. Then there exist constants $A, B > 0$ such that, for any $\Lambda \in \mathcal{L}_d$ and any set of generators $\Lambda' \subset \Lambda$ with $\overline{u}^{\mathbf{n}} \neq 1$ for all $\mathbf{n} \in \Lambda'$,*

$$A \prod_{v \in T(\mathfrak{p})} \min_{\mathbf{n} \in \Lambda'} \{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\} \leqslant |D/\mathfrak{b}_\Lambda D| \leqslant B \prod_{v \in T(\mathfrak{p})} \min_{\mathbf{n} \in \Lambda'} \{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\},$$

where $T(\mathfrak{p}) = \mathcal{P}(\mathbb{K}(\mathfrak{p}))\backslash S(\mathfrak{p})$. Furthermore, if D is integrally closed, then

$$|D/\mathfrak{b}_\Lambda D| = \prod_{v \in T(\mathfrak{p})} \min_{\mathbf{n} \in \Lambda'} \{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\}.$$

Proof. If D is integrally closed in its field of fractions $\mathbb{K}(\mathfrak{p})$, the required result is obtained using a completely analogous method to [**21**, Theorem 4.1] which, with a straightforward adaptation to our setting, shows that

$$|D/\mathfrak{b}_\Lambda D| = \prod_{v \in T(\mathfrak{p})} \min_{\mathbf{n} \in \Lambda'} \{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\}. \tag{8}$$

The main idea in the proof of this result is to identify factors and estimate multiplicities in a composition series for $D/\mathfrak{b}_\Lambda D$ using places of the field $\mathbb{K}(\mathfrak{p})$. If D is not integrally

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

closed, then Lemma 2.4 shows that there exist constants $A, B > 0$, independent of $\mathfrak{b}_\Lambda$, such that

$$A|E/\mathfrak{b}_\Lambda E| \leqslant |D/\mathfrak{b}_\Lambda D| \leqslant B|E/\mathfrak{b}_\Lambda E|,$$

where E is the integral closure of D in $\mathbb{K}(\mathfrak{p})$. Since $|E/\mathfrak{b}_\Lambda E|$ is given by the product in (8), the required result follows. □

As in [**16**], using Hermite normal form, we recall that any $\Lambda \in \mathcal{L}_d$ has a unique representation as the image of $\mathbb{Z}^d$ under a matrix of the form

$$\begin{pmatrix} a_1 & b_{12} & b_{13} & \dots & b_{1d} \\ 0 & a_2 & b_{23} & \dots & b_{2d} \\ 0 & 0 & a_3 & \dots & b_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_d \end{pmatrix}, \tag{9}$$

where $a_m \geqslant 1$ for $1 \leqslant m \leqslant d$, $0 \leqslant b_{mn} \leqslant a_m - 1$ for $m + 1 \leqslant n \leqslant d$ and $a_1 a_2 \cdots a_d = [\Lambda]$. We now focus on identifying non-trivial collections of subgroups $\Lambda \subset \mathcal{L}_d$ for which $\mathsf{F}_\alpha(\Lambda)$ is severely restricted. For any rational primes p, q with $q \neq p$, let $m_p(q)$ denote the multiplicative order of p modulo q. Let $\mathbb{P}$ be the set of rational primes and for any finite set $P \subset \mathbb{P}$ and any $\varepsilon > 0$, define

$$\mathcal{L}_d(P, \varepsilon) = \{\Lambda \in \mathcal{L}_d : [\Lambda] \in \mathbb{P} \backslash P \text{ and } m_p([\Lambda]) > [\Lambda]^{1/d+\varepsilon} \text{ for all } p \in P\}$$

and let

$$\mathbb{P}(\alpha) = \{p \in \mathbb{P} : p \in \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Ass}(M)\}. \tag{10}$$

Remark A. It is a non-trivial problem to determine which subgroups of prime index comprise $\mathcal{L}_d(\mathbb{P}(\alpha), \varepsilon)$. Nonetheless, if we take, for example, $\varepsilon = 0.1$, the set of such prime indices will have positive density in $\mathbb{P}$. The authors would like to thank Pär Kurlberg for pointing out the following explanation. The background and relevant number theoretic results used are summarized in the introduction of [**13**].

To begin with, consider the case $d = 2$. For a given prime $p \in \mathbb{P}(\alpha)$, clearly, $m_p(q) \mid q - 1$. Furthermore, there is a sequence of primes q of positive density in $\mathbb{P}$ such that $q - 1$ has a prime divisor r larger than $q^{(1/2)+\varepsilon} = q^{0.6}$. If $r \nmid m_p(q)$, then $m_p(q) < q^{0.4}$, since $r m_p(q) \mid q - 1$. However, for a fixed prime p and given any $x > 0$, it is known that the number of primes $q < x$ such that $m_p(q) < q^{0.499}$, say, is $\mathrm{o}(x/\log(x))$. In particular, the set of such primes q has zero density in $\mathbb{P}$. Since $\mathbb{P}(\alpha)$ is finite, the set of such unwanted primes q is still of zero density if we consider all $p \in \mathbb{P}(\alpha)$ simultaneously, and we retain a set of primes q of positive density such that $m_p(q)$ has a prime divisor greater than $q^{0.6}$ and so satisfies $m_p(q) > q^{0.6}$ for all $p \in \mathbb{P}(\alpha)$. When $d \geqslant 3$ and ε is small, it follows more straightforwardly from such arguments that $\mathcal{L}_d(\mathbb{P}(\alpha), \varepsilon)$ is determined by a set of prime indices of density one in $\mathbb{P}$, as we are then interested in the less delicate case of primes q that satisfy $m_p(q) > q^{(1/3)+\varepsilon}$.

The observations above, together with the following, give Theorem 1.3.

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

THEOREM 2.2. *Let α be a mixing Noetherian entropy rank one $\mathbb{Z}^d$-action by continuous automorphisms of a compact abelian zero-dimensional group X, with $d \geqslant 2$. Then, for any $\varepsilon > 0$, $\{\mathsf{F}_\alpha(\Lambda) : \Lambda \in \mathcal{L}_d(\mathbb{P}(\alpha), \varepsilon)\}$ is a finite set.*

Proof. Firstly, Theorem 2.1 shows that there is a constant $C_0 > 0$ such that

$$\mathsf{F}_\alpha(\Lambda) \leqslant C_0^{\sqrt{d}[\Lambda]^{1/d}}. \tag{11}$$

Since the action is both mixing and of entropy rank one, $\mathrm{coht}(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$. Any remaining prime ideal $\mathfrak{q} \subset R_d$ appearing in a filtration of M is maximal, and $R_d/\mathfrak{q}$ is a finite field. By taking a prime filtration of M as in (1), and using Lemma 2.3 inductively, we also find a constant $C_1 > 0$ such that, for all $\Lambda \in \mathcal{L}_d$,

$$\mathsf{F}_\alpha(\Lambda) \leqslant C_1 \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} \left| \frac{R_d/\mathfrak{p}}{\mathfrak{b}_\Lambda(R_d/\mathfrak{p})} \right|^{m(\mathfrak{p})}, \tag{12}$$

where $m(\mathfrak{p})$ is the dimension of the $\mathbb{K}(\mathfrak{p})$-vector space $M \otimes \mathbb{K}(\mathfrak{p})$.

The mixing assumption means that in each domain $R_d/\mathfrak{p}$, $\overline{u}^{\mathbf{n}} \neq 1$ for all non-zero $\mathbf{n} \in \mathbb{Z}^d$ (see [**24**, §6]), so Lemma 2.5 applies and, for each $\mathfrak{p} \in \mathrm{Ass}(M)$, there exist constants $A(\mathfrak{p})$, $B(\mathfrak{p}) > 0$ such that

$$A(\mathfrak{p}) \prod_{v \in T(\mathfrak{p})} \min_{\mathbf{n} \in \Lambda'} \{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\} \leqslant \left| \frac{R_d/\mathfrak{p}}{\mathfrak{b}_\Lambda(R_d/\mathfrak{p})} \right| \leqslant B(\mathfrak{p}) \prod_{v \in T(\mathfrak{p})} \min_{\mathbf{n} \in \Lambda'} \{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\}, \tag{13}$$

where $T(\mathfrak{p}) = \mathcal{P}(\mathbb{K}(\mathfrak{p}))\backslash S(\mathfrak{p})$ and Λ' is any set of generators for Λ. Note that each set of places $S(\mathfrak{p})$ defined by (3) is finite.

Let $\mathfrak{p} \in \mathrm{Ass}(M)$ and let $p \in \mathbb{P}(\alpha)$ be the unique rational prime in $\mathfrak{p}$. Suppose the lattice $\Lambda \in \mathcal{L}_d(\alpha, \varepsilon)$ is given by the matrix (9) and $[\Lambda] = q$, for a prime q large enough to ensure that

$$q^{(1/d)+\varepsilon} \log p > q^{1/d} \sqrt{d} \log C_0 - \log A, \tag{14}$$

where $A = \min\{A(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Ass}(M)\}$.

Denote the image of u_i in $R_d/\mathfrak{p}$ by $\overline{u}_i$, $1 \leqslant i \leqslant d$. If $\overline{u}_1$ is algebraic over $\mathbb{F}_p$, then $\mathbb{F}_p(\overline{u}_1)$ is a finite field and $\overline{u}_1^m = 1$ for some $m \geqslant 1$, which contradicts the assumption that $\overline{u}^{\mathbf{n}} \neq 1$ for all non-zero $\mathbf{n} \in \mathbb{Z}^d$. Therefore, we may identify $\overline{u}_1$ with an indeterminate t so that $\mathbb{F}_p[\overline{u}_1] \cong \mathbb{F}_p[t]$. Let $T_1(\mathfrak{p})$ denote the set of places in $T(\mathfrak{p})$ that extend the place of $\mathbb{F}_p(t)$ corresponding to the polynomial $t - 1$, and let $T_2(\mathfrak{p}) = T(\mathfrak{p})\backslash T_1(\mathfrak{p})$.

By the definition of the matrix (9), $a_j = q$ for precisely one index $1 \leqslant j \leqslant d$ and $a_i = 1$ for all $i \neq j$. If $j \neq 1$, then $(1, 0, \ldots, 0) \in \Lambda$, and so, by (13), the right-hand side of (12) is at most

$$C_2 = C_1 \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} \left(B(\mathfrak{p}) \prod_{v \in T_1(\mathfrak{p})} |t - 1|_v^{-1} \right)^{m(\mathfrak{p})}$$

as $|t - 1|_v = 1$ for all $v \in T_2(\mathfrak{p})$, $\mathfrak{p} \in \mathrm{Ass}(M)$.

Suppose now that $j = 1$, so $(q, 0, \ldots, 0) \in \Lambda'$, where Λ' is the set of generators given by multiplying the standard basis vectors in $\mathbb{Z}^d$ by (9). In the product in (13), note that

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

each factor is a positive integer as $|\overline{u}^{\mathbf{n}}|_v = 1$ for all $v \in T(\mathfrak{p})$, by (3). Therefore, (13) also gives

$$\left|\frac{R_d/\mathfrak{q}}{\mathfrak{b}_\Lambda(R_d/\mathfrak{q})}\right| \geqslant A \prod_{v\in T_2(\mathfrak{q})} \min_{\mathbf{n}\in\Lambda'}\{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\}. \tag{15}$$

Suppose, for a contradiction, that the product on the right-hand side of (15) is not equal to one. Then there exists $v \in T_2(\mathfrak{q})$ such that $|\overline{u}^{\mathbf{n}} - 1|_v^{-1} > 1$ for all $\mathbf{n} \in \Lambda'$. In particular,

$$|\overline{u}^{(q,0,\ldots,0)} - 1|_v = |t^q - 1|_v > 1.$$

Let $\mathfrak{K}_v$ denote the residue field at v and consider the prime factorization of $t^q - 1$ in $\mathbb{F}_p[t]$, which comprises the factor $t - 1$ and the irreducible factors of the qth cyclotomic polynomial, each of which has degree $m_p(q)$, by [**15**, Theorem 2.47]. Since $v \in T_2(\mathfrak{p})$ and $|t^q - 1|_v > 1$, v must extend a place of $\mathbb{F}_p[t]$ corresponding to one of these these latter factors, f, say. Furthermore, since $\mathfrak{K}_v$ is an extension of the residue field $\mathfrak{K}_{(f)}$,

$$|\mathfrak{K}_v| \geqslant |\mathfrak{K}_{(f)}| = p^{m_p(q)}.$$

Therefore, for all $x \in \mathbb{K}$,

$$|x|_v > 1 \Rightarrow |x|_v \geqslant p^{m_p(q)}.$$

Hence, if $|\overline{u}^{\mathbf{n}} - 1|_v^{-1} > 1$ for all $\mathbf{n} \in \Lambda'$, then

$$\min_{\mathbf{n}\in\Lambda'}\{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\} \geqslant p^{m_p(q)}.$$

Furthermore, by the definition of $\mathcal{L}_d(\alpha, \varepsilon)$, $m_p(q) > q^{1/d+\varepsilon}$, so the inequalities (14) and (15) give

$$\left|\frac{R_d/\mathfrak{q}}{\mathfrak{b}_\Lambda(R_d/\mathfrak{q})}\right| \geqslant Ap^{q^{1/d+\varepsilon}} > C^{\sqrt{d}q^{1/d}},$$

which contradicts (11). Thus, for all $v \in T_2(\mathfrak{p})$, $\min_{\mathbf{n}\in\Lambda'}\{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\} = 1$ and (13) now gives

$$\begin{aligned}\left|\frac{R_d/\mathfrak{q}}{\mathfrak{b}_\Lambda(R_d/\mathfrak{q})}\right| &\leqslant B(\mathfrak{p}) \prod_{v\in T_1(\mathfrak{p})} \min_{\mathbf{n}\in\Lambda'}\{|\overline{u}^{\mathbf{n}} - 1|_v^{-1}\} \\ &\leqslant B(\mathfrak{p}) \prod_{v\in T_1(\mathfrak{p})} |t^q - 1|_v^{-1} = B(\mathfrak{p}) \prod_{v\in T_1(\mathfrak{p})} |t - 1|_v^{-1}.\end{aligned}$$

It follows that $\mathsf{F}_\alpha(\Lambda) \leqslant C_2$ by (12). Since there are only a finite number of primes q that do not satisfy (14), and only a finite number of subgroups of $\mathbb{Z}^d$ of any given index, the required result now follows. □

3. *The zeta function for a single automorphism*

In this section, we consider typical extensions of full shifts. Recall that the full shift σ on p symbols is an expansive automorphism of a zero-dimensional group with zeta function $\zeta_\sigma(z) = (1 - pz)^{-1}$. More generally, let X be a compact abelian zero-dimensional group and suppose that a $\mathbb{Z}$-action α is generated by an ergodic automorphism of X that commutes with an expansive automorphism. Since expansive automorphisms of compact abelian zero-dimensional groups have finite entropy, if we also assume that α

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

has finite entropy, then we can think of α as being generated by an ergodic element of an expansive entropy rank one $\mathbb{Z}^2$-action (note that we do not assume that α itself is expansive). Furthermore, the dual module corresponding to this $\mathbb{Z}^2$-action is Noetherian and the periodic point counting formula (2) applies. The fact that only a finite number of places appear in (2) allows us to adapt the methods of [**2**] to the zero-dimensional setting. Just as in that paper, the full result conjectured to hold would simply require that α have finite entropy and that the zeta function have radius of convergence $e^{-\mathsf{h}(\alpha)}$. As Theorem 1.1 falls short of the full conjectured result, and since the main emphasis of this paper is $\mathbb{Z}^d$-actions with $d \geqslant 2$, we simply indicate how the earlier methods of [**2**] may be used to prove the result, and discuss an example to make the argument clear.

This example is the invertible extension of [**5**, Example 8.5] and is naturally related to Ledrappier's example (Example 4.1). It may be considered to be one of the most fundamental examples given by a single automorphism of a compact abelian zero-dimensional group that is not a full shift.

Example 3.1. (An extension of a full p-shift) Consider the shift space

$$X = \{x = (x_{i,j}) \in \mathbb{F}_p^{\mathbb{Z}^2} : x_{i+1,j} - x_{i,j+1} - x_{i,j} = 0 \text{ for all } i,\ j \in \mathbb{Z}\}$$

together with the automorphism α given by the horizontal shift (that is, the map defined by $\alpha(x)_{i,j} = x_{i+1,j}$). We may also consider α as a non-expansive element of the expansive entropy rank one $\mathbb{Z}^2$-action generated by α and the vertical shift which is described in detail in [**8**, Example 6.8] and [**19**, Example 4.5] for the case $p = 2$. In particular, α commutes with all of the expansive elements of this action. Furthermore, the automorphism dual to α may be identified with multiplication by t on the integral domain $\mathbb{F}_p[t^{\pm 1}, (t-1)^{-1}] \cong R_2/(p, u_1 - u_2 - 1)$ and, by (2), the number of periodic points is

$$\mathsf{F}_\alpha(n) = |t^n - 1|_\infty |t^n - 1|_{(t)} |t^n - 1|_{(t-1)} = p^n |t^n - 1|_{(t-1)} = p^{n - \nu(n)},$$

where $\nu(n) = |n|_p^{-1}$. Therefore, ζ_α has radius of convergence p^{-1}. Let $a_n = p^{-\nu(n)}$ and $F(z) = \sum_{n \geqslant 1} a_n z^n$. Clearly, $\limsup_{n\to\infty} a_n^{1/n} = 1$ and, consequently, F has radius of convergence equal to one. Furthermore,

$$F(pz) = z\zeta_\alpha'(z)/\zeta_\alpha(z),$$

so if ζ_α has analytic continuation beyond the circle $|z| = p^{-1}$, then F has analytic continuation beyond the unit circle. Let $z = \lambda \exp(2a\pi i/p^s)$, where $s \in \mathbb{N}$ and $p \nmid a$. We wish to consider the behaviour of $F(z)$ as $\lambda \to 1^-$.

Write

$$\begin{aligned} F(z) &= \sum_{e \geqslant 0} \sum_{k \geqslant 1 : p \nmid k} a_{p^e k} z^{p^e k} \\ &= \sum_{e \geqslant 0} p^{-p^e} \sum_{k \geqslant 1 : p \nmid k} z^{p^e k} \\ &= \sum_{e \geqslant 0} p^{-p^e} \sum_{k \geqslant 1} z^{p^e k} - z^{p^{e+1} k} \end{aligned}$$

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

$$= \underbrace{\sum_{e \geqslant s} (p^{-p^e} - p^{-p^{e-1}}) \sum_{k \geqslant 1} (\lambda^{p^e})^k}_{G(z)}$$
$$+ \frac{z}{p(1-z)} + \sum_{1 \leqslant e < s} (p^{-p^e} - p^{-p^{e-1}}) \frac{z^{p^e}}{1 - z^{p^e}}.$$

There exists $M > 0$ such that, for all $1 \leqslant e < s$, $|1 - z^{p^e}| > M$. Hence, all the terms in the final line converge as $\lambda \to 1^-$. For the series $G(z)$, note that, for all $e \geqslant s$, $p^{-p^e} - p^{-p^{e-1}} < 0$ and $\sum_{k \geqslant 1} (\lambda^{p^e})^k \to +\infty$ as $\lambda \to 1^-$. Therefore, $G(z)$ diverges as $\lambda \to 1^-$ and hence so too does $F(z)$. Thus, the unit circle is a natural boundary for F.

Finally, since $\mathsf{F}_\alpha(n) = p^n a_n$, $\sum_{n \geqslant 1} \mathsf{F}_\alpha(n) z^n = F(pz)$, and it follows that the circle $|z| = 1/p$ is a natural boundary for this function.

Proof of Theorem 1.1. Let β be the $\mathbb{Z}^2$-action generated by the two automorphisms in the statement of the theorem. Since one of these is expansive (and hence has finite entropy), so too is β, and β has entropy rank one since both generating automorphisms have finite entropy. Therefore, the periodic point counting formula (2) applies to β and the right-hand side of (2) is a product over a finite set of places. Without loss of generality, we may assume that α is generated by the automorphism whose dual corresponds to multiplication by u_1 on the R_2-module $M = \widehat{X}$, and so (2) reduces to the form

$$\mathsf{F}_\alpha(n) = e^{\mathsf{h}(\alpha)n} \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} \prod_{v \in S'(\mathfrak{p})} |t^n - 1|_v^{m(\mathfrak{p})}, \tag{16}$$

where $S'(\mathfrak{p}) = \{v \in S(\mathfrak{p}) : |u_1|_v = 1\}$ and where we have identified the image of u_1 in $R_2/\mathfrak{p}$ with an indeterminate t for each $\mathfrak{p} \in \mathrm{Ass}(M)$.

If the product on the right-hand side of (16) is empty, then we immediately obtain $\zeta_\alpha(z) = (1 - e^{\mathsf{h}(\alpha)} z)^{-1}$. Otherwise, we consider the generating function $F(z) = \sum_{n \geqslant 1} a_n z^n$, where $a_n = \mathsf{F}_\alpha(n)/e^{\mathsf{h}(\alpha)n}$. Just as in the previous example,

$$F(e^{\mathsf{h}(\alpha)} z) = z\zeta'_\alpha(z)/\zeta_\alpha(z),$$

and we show that ζ_α has a natural boundary at the circle of convergence $|z| = e^{-\mathsf{h}(\alpha)}$ by demonstrating a natural boundary for F at the unit circle. The method for this is similar to that used in the example but also needs an inclusion–exclusion argument to evaluate the product of expressions of the form $|t^n - 1|_v$ over the finite list of places occurring in the definition of a_n. This procedure is completely analogous to the one used in the connected case, which may be found in [**2**, Theorem 9], so we do not repeat the steps here. □

4. *The zeta function for $\mathbb{Z}^d$-actions with zero entropy*

In this section, we consider $\mathbb{Z}^d$-actions of zero entropy on compact abelian zero-dimensional groups, where $d \geqslant 2$. These are easily classified using dual modules and the entropy formula of Lind *et al* [**17**]. In particular, the action α_M belongs to this class if and only if, for each $\mathfrak{p} \in \mathrm{Ass}(M)$, the ideal $\mathfrak{p}$ is non-principal and $\mathfrak{p} \cap \mathbb{Z} \neq \{0\}$. The most amenable examples in this class are the entropy rank one actions, where each $\mathfrak{p} \in \mathrm{Ass}(M)$ also satisfies $\mathrm{coht}(\mathfrak{p}) \leqslant 1$. For these actions, we deduce Theorem 1.2 from the estimates

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

obtained in §2. We also include an example which is not of entropy rank one, to emphasize subtler properties of the zeta function not present in the entropy rank one case. We begin with the well-known example due to Ledrappier [**14**], who used it to demonstrate a $\mathbb{Z}^2$-action that is mixing but not mixing of higher orders. The dynamical zeta function for this example was first considered by Lind in [**16**].

Example 4.1. (Ledrappier's example) Let α denote the $\mathbb{Z}^2$-action given by the restriction of the full $\mathbb{Z}^2$-shift to the compact abelian group

$$\{x = (x_{i,j}) \in \mathbb{F}_2^{\mathbb{Z}^2} : x_{i+1,j} + x_{i,j+1} + x_{i,j} = 0 \text{ for all } (i, j) \in \mathbb{Z}^2\}.$$

This corresponds to the dual R_2-module $R_2/\mathfrak{p}$, where $\mathfrak{p} = (2, 1 + u_1 + u_2)$ and

$$R_2/\mathfrak{p} \cong \mathbb{F}_2[t^{\pm 1}, (t+1)^{\pm 1}]$$

for an indeterminate t, where the isomorphism is given by $u_1 \mapsto t$ and $u_2 \mapsto t + 1$. Using a geometric argument, Lind shows that there is a constant $C > 0$ such that $\mathsf{F}_\alpha(\Lambda) \leqslant 2^{C\sqrt{[\Lambda]}}$ for all $\Lambda \in \mathcal{L}_2$. This means that the upper growth rate of periodic points for this is example is zero and, using the Pólya–Carlson theorem and the integrality of coefficients in the Taylor expansion (see Proposition 4.1), one may deduce that the unit circle is a natural boundary for ζ_α.

Note that Lind's estimate agrees with that of Theorem 2.1. In particular, following the methods of §2, we may calculate a value for the constant C. We have $S(\mathfrak{p}) = \{\infty, (t), (t+1)\}$ and, using a standard application of Minkowski's theorem, it can be shown that, for any $\Lambda \in \mathcal{L}_2$, there exists an $\mathbf{n} \in \Lambda$ with

$$\|\mathbf{n}\| \leqslant 2\pi^{-1/2}\sqrt{[\Lambda]}.$$

Hence

$$\mathsf{F}_\alpha(\Lambda) \leqslant \mathsf{F}_\alpha(\mathbf{n}) = \prod_{v \in S(\mathfrak{p})} |t^{n_1}(t+1)^{n_2} - 1|_v \leqslant 2^{|n_1|+|n_2|} \leqslant 2^{\sqrt{2}\|\mathbf{n}\|} \leqslant 2^{2\sqrt{2/\pi}\sqrt{[\Lambda]}}.$$

Lind's calculations also show that this bound cannot be significantly improved upon. In particular, if $\Lambda = \langle (2^n - 1, 0), (0, 2^n - 1)\rangle$, $n \geqslant 1$, then $\mathsf{F}_\alpha(\Lambda) = \frac{1}{2}2^{\sqrt{[\Lambda]}}$.

A phenomenon that seems peculiar to this example and minor variations thereof is that $\mathsf{F}_\alpha(\Lambda)$ is trivial for subgroups of $\mathbb{Z}^d$ of the following form. Suppose that Λ is a subgroup with $[\Lambda] = 2^n$, $n \geqslant 1$. By (9), $\Lambda = \langle (2^j, 0), (b, 2^k)\rangle$, where $j + k = n$ and $0 \leqslant b \leqslant 2^j - 1$. Since

$$\mathsf{F}_\alpha((2^j, 0)) = |t^{2^j} - 1|_\infty |t^{2^j} - 1|_{(t)} |t^{2^j} - 1|_{(t+1)} = 2^{2^j} \cdot 1 \cdot 2^{-2^j} = 1,$$

it follows that $\mathsf{F}_\alpha(\Lambda) = 1$. Note that Theorem 1.3 identifies a more homogeneous collection of subgroups on which $\mathsf{F}_\alpha(\Lambda)$ is similarly restricted for this and similar examples of entropy rank one.

More generally, under the hypotheses of Theorem 1.2, Corollary 2.1 shows that

$$\mathsf{g}(\alpha) = \limsup_{[\Lambda]\to\infty} \frac{1}{[\Lambda]} \log \mathsf{F}_\alpha(\Lambda) \leqslant \lim_{[\Lambda]\to\infty} \frac{1}{[\Lambda]} \log C^{\sqrt{d}[\Lambda]^{1/d}} = 0.$$

So, ζ_α has radius of convergence equal to one. Subsequently, Theorem 1.2 may be deduced as a consequence of Corollary 2.1 and the following result.

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

PROPOSITION 4.1. *Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact abelian group. If ζ_α has radius of convergence equal to one and if $d \geqslant 2$, then ζ_α admits the unit circle as a natural boundary.*

Proof. This is based on Lind's calculations in [**16**, Example 3.4] and is proved using the Pólya–Carlson theorem. A crucial step in the proof is to demonstrate that the Taylor series of ζ_α has integer coefficients; we refer to [**22**, Lemma 3.3] and [**16**] for details. □

In the setting above, a natural boundary for ζ_α arises via dense singularities on the circle of convergence $|z| = 1$. In the following example, $\mathsf{g}(\alpha) > \mathsf{h}(\alpha) = 0$, so the radius of convergence is $e^{-\mathsf{g}(\alpha)} < 1$, yet ζ_α can be analytically continued beyond $|z| = e^{-\mathsf{g}(\alpha)}$, ultimately encountering a natural boundary at the unit circle. Lind gives an example of an algebraic $\mathbb{Z}^2$-action of this type [**16**, Example 3.3] which is non-mixing and of entropy rank one. Since Theorem 1.2 precludes the possibility of such examples for mixing entropy rank one $\mathbb{Z}^d$-actions with $d \geqslant 2$, to potentially witness this phenomenon for mixing actions, one must either consider $\mathbb{Z}^2$-actions with positive entropy or consider actions of $\mathbb{Z}^3$. In light of Theorem 1.4, this leads us to consider the latter case.

Example 4.2. (A shift extension of Ledrappier's example) By adding an extra dimension to Ledrappier's example, we obtain the compact abelian group

$$\{x = (x_{i,j,k}) \in \mathbb{F}_2^{\mathbb{Z}^3} : x_{i,j,k} + x_{i+1,j,k} + x_{i,j+1,k} = 0 \text{ for all } (i, j, k) \in \mathbb{Z}^3\},$$

and we consider the restriction of the full $\mathbb{Z}^3$-shift to this group. Denote this action by α and note that the corresponding dual R_3-module is $D = R_3/\mathfrak{p}$, where $\mathfrak{p} = (2, 1 + u_1 + u_2)$. Let β denote the $\mathbb{Z}^2$-action given by the R_2-module $\mathbb{F}_2[\overline{u}_1^{\pm 1}, \overline{u}_2^{\pm 1}] \subset D$, which corresponds to Ledrappier's example. If $\Lambda \in \mathcal{L}_3$ is given by a 3×3 matrix of the form (9), then a straightforward algebraic calculation shows that

$$\mathsf{F}_\alpha(\Lambda) = \mathsf{F}_\beta(\Lambda')^{a_3}, \tag{17}$$

where $\Lambda' \in \mathcal{L}_2$ is given by the top left 2×2 matrix in (9), with the parameters a_1, b_{12}, a_2 inherited from Λ. Note that $[\mathbb{Z}^3 : \Lambda] = [\mathbb{Z}^2 : \Lambda']a_3$. Hence

$$\begin{aligned}\mathsf{g}(\alpha) &= \limsup_{\Lambda \in \mathcal{L}_3 : [\Lambda] \to \infty} \frac{1}{[\Lambda]} \log \mathsf{F}_\alpha(\Lambda) \\ &= \limsup_{\Lambda \in \mathcal{L}_2, n \geqslant 1 : [\Lambda] n \to \infty} \frac{1}{[\Lambda] n} \log \mathsf{F}_\beta(\Lambda)^n \\ &= \sup_{\Lambda \in \mathcal{L}_2} \log \mathsf{F}_\beta(\Lambda)^{1/[\Lambda]}.\end{aligned}$$

For ease of notation, we write $\Lambda \in \mathcal{L}_2$ in the form $\Lambda = \langle (a, 0), (b, c) \rangle$, where $a, c \geqslant 1$ satisfy $ac = [\Lambda]$ and $0 \leqslant b \leqslant a - 1$. Let $T(m)$ denote the set of places corresponding to irreducible polynomials of $\mathbb{F}_2[t]$ of degree at most m, excluding t and $t + 1$. By Lemma 2.5,

$$\mathsf{F}_\beta(\Lambda) = \prod_{v \in T(\max\{a, b+c\})} \min\{|t^a - 1|_v^{-1}, |t^b(t+1)^c - 1|_v^{-1}\}. \tag{18}$$

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

Using (18), it can easily be verified that $\mathsf{F}_\alpha(\Lambda) \in \{1, 4\}$ for all Λ with $[\Lambda] \leqslant 6$. For example, when $\Lambda = \langle (3, 0), (1, 1)\rangle$, $\mathsf{F}_\beta(\Lambda) = 4$ since

$$|t^3 - 1|^{-1}_{t^2+t+1} = |t(t+1) - 1|^{-1}_{t^2+t+1} = 4,$$

and this is the only place that gives a non-trivial contribution in (18). Furthermore, this choice of Λ also gives the largest value of $\mathsf{F}_\alpha(\Lambda)^{1/[\Lambda]} = 2^{2/3}$ for $[\Lambda] \leqslant 6$. When $[\Lambda] \geqslant 7$,

$$\mathsf{F}_\beta(\Lambda)^{1/[\Lambda]} \leqslant 2^{2\sqrt{2/(7\pi)}} < 2^{2/3}$$

by the estimate given in Example 4.1. Thus, $\mathsf{g}(\alpha) = \frac{2}{3}\log 2$ and ζ_α has radius of convergence $2^{-2/3}$.

Given any $n \geqslant 1$, under the map $\Lambda \mapsto \Lambda'$, there are precisely n^2 subgroups $\Lambda \in \mathcal{L}_3$ such that $[\mathbb{Z}^3 : \Lambda] = [\mathbb{Z}^2 : \Lambda']n$. Hence, using (17),

$$\zeta_\alpha(z) = \exp\Bigg(\sum_{\Lambda\in\mathcal{L}_2, n\geqslant 1} \frac{n\mathsf{F}_\beta(\Lambda)^n}{[\Lambda]} z^{[\Lambda]n}\Bigg).$$

Writing $\mathcal{J}(N) = \{\Lambda \in \mathcal{L}_2 : [\Lambda] \leqslant N - 1\}$ and $\mathcal{K}(N) = \{\Lambda \in \mathcal{L}_2 : [\Lambda] \geqslant N\}$, we may split the series inside the brackets into the sum of two series $F_{\mathcal{J}(N)}(z)$ and $F_{\mathcal{K}(N)}(z)$ corresponding to this partition of $\mathcal{L}_2$. Since $\mathcal{J}(N)$ is finite, and since

$$\begin{aligned} F_{\mathcal{J}(N)}(z) &= \sum_{\Lambda\in\mathcal{J}(N), n\geqslant 1} \frac{n\mathsf{F}_\beta(\Lambda)^n}{[\Lambda]} z^{[\Lambda]n} \\ &= \sum_{\Lambda\in\mathcal{J}(N)} \frac{1}{[\Lambda]} \sum_{n\geqslant 1} n(\mathsf{F}_\beta(\Lambda) z^{[\Lambda]})^n \\ &= \sum_{\Lambda\in\mathcal{J}(N)} \frac{\mathsf{F}_\beta(\Lambda) z^{[\Lambda]}}{[\Lambda](1 - \mathsf{F}_\beta(\Lambda) z^{[\Lambda]})^2}, \end{aligned}$$

we see that $F_{\mathcal{J}(N)}(z)$ has an analytic continuation to a rational function with $[\Lambda]$ evenly distributed poles on the circle $|z| = \mathsf{F}_\beta(\Lambda)^{-1/[\Lambda]}$, $\Lambda \in \mathcal{J}(N)$. On the other hand,

$$\limsup_{[\Lambda]\geqslant N, n\geqslant 1:[\Lambda]n\to\infty} \left(\frac{n\mathsf{F}_\beta(\Lambda)^n}{[\Lambda]}\right)^{1/[\Lambda]n} \leqslant 2^{2\sqrt{2/(\pi N)}},$$

using the estimate obtained for $\mathsf{F}_\beta(\Lambda)$. So, the series $F_{\mathcal{K}(N)}(z)$ has radius of convergence at least $2^{-2\sqrt{2/(\pi N)}}$. Since N may be chosen to be arbitrarily large, ζ_α has an analytic continuation to the interior of the unit disk. Upon defining

$$\Lambda(n) = \langle (2^n - 1, 0), (0, 2^n - 1)\rangle,$$

by Example 4.1, $\mathsf{F}_\beta(\Lambda(n)) = \frac{1}{2}2^{2^n-1}$, as $[\Lambda(n)] = (2^n - 1)^2$. This means that if $N > [\Lambda(n)]$, then $F_{\mathcal{J}(N)}(z)$ has at least $[\Lambda(n)]$ poles evenly distributed on the circle $|z| = \mathsf{F}_\beta(\Lambda(n))^{-1/[\Lambda(n)]}$. Thus, the poles of $F_{\mathcal{J}(N)}(z)$ cluster towards a dense set on the unit circle as $N \to \infty$.

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

5. *The zeta function for $\mathbb{Z}^2$-actions with positive entropy*

In this section, we consider a positive entropy $\mathbb{Z}^2$-action α by automorphisms of a compact abelian zero-dimensional group X. Let $\mathbb{P}(\alpha)$ be given by (10) and let M denote the dual module. For each $p \in \mathbb{P}(\alpha)$, the local ring $M_{(p)}$ may be considered naturally as vector space over the field of fractions of $R_2/(p)$, which we denote by $\mathbb{K}(p)$, and $m(p) = \dim_{\mathbb{K}(p)} M_{(p)}$ is equal to the multiplicity of (p) in any prime filtration (1) of M. Furthermore, the entropy formula for $\mathbb{Z}^d$-actions shows that

$$\mathsf{h}(\alpha) = \log \prod_{p \in \mathbb{P}(\alpha)} p^{m(p)}. \tag{19}$$

THEOREM 5.1. *Let α be a mixing Noetherian $\mathbb{Z}^2$-action by automorphisms of a compact abelian zero-dimensional group X. Then there is a mixing Noetherian entropy rank one $\mathbb{Z}^2$-action β by automorphisms of a compact abelian zero-dimensional group and constants $A, B > 0$ such that, for any $\Lambda \in \mathcal{L}_2$,*

$$Ae^{\mathsf{h}[L]} \leqslant \mathsf{F}_\alpha(\Lambda) \leqslant Be^{\mathsf{h}[L]}\mathsf{F}_\beta(\Lambda),$$

where $\mathsf{h} = \mathsf{h}(\alpha)$ is given by (19).

The proof proceeds by an induction argument and a series of lemmas that develop the product estimate given by Lemma 2.3. A related problem is dealt with in [**17**, §7], but we require more refined estimates than the ones obtained there.

Recall that if $\Lambda \in \mathcal{L}_2$, then (9) shows that the ideal $\mathfrak{b}_\Lambda$ defined by (7) is of the form

$$\mathfrak{b}_\Lambda = (u_1^a - 1, u_1^b u_2^c - 1), \tag{20}$$

where $a, c \geqslant 1$, $0 \leqslant b \leqslant a - 1$ and $[\Lambda] = ac$. It is also helpful to note that if M is a Noetherian R_2-module such that each $\mathfrak{p} \in \mathrm{Ass}(M)$ contains a rational prime, then [**24**, Theorem 6.5(3)] shows that $M/\mathfrak{b}_\Lambda M$ is finite.

LEMMA 5.1. *Let $L \subset M$ be Noetherian R_2-modules such that $M/L \cong R_2/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset R_2$, and assume that $M/\mathfrak{b}_\Lambda M$ is finite. Then there is an ideal $\mathfrak{a} \subset R_2$ satisfying $\mathfrak{b}_\Lambda \mathfrak{p} \subset \mathfrak{a} \subset \mathfrak{b}_\Lambda \cap \mathfrak{p}$ such that*

$$|M/\mathfrak{b}_\Lambda M| = \left|\frac{R_2/\mathfrak{p}}{\mathfrak{b}_\Lambda(R_2/\mathfrak{p})}\right| \frac{|L/\mathfrak{b}_\Lambda L|}{|(\mathfrak{b}_\Lambda \cap \mathfrak{p})/\mathfrak{a}|}.$$

Proof. Firstly, using (20), we may assume $\mathfrak{b} = \mathfrak{b}_\Lambda$ is of the form $\mathfrak{b} = (b_1, b_2)$. As in Lemma 2.3,

$$|M/\mathfrak{b}M| = \left|\frac{M/L}{\mathfrak{b}(M/L)}\right| \left|\frac{L/\mathfrak{b}L}{(\mathfrak{b}M \cap L)/\mathfrak{b}L}\right|.$$

Hence, we need to show that $(\mathfrak{b}M \cap L)/\mathfrak{b}L \cong (\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}$, where the ideal $\mathfrak{a} \subset R_2$ has the desired property. The isomorphism $M/L \cong R_2/\mathfrak{p}$ implies that there exists $x \in M$ such that $M = L + R_2 x$, and $fx \in L$ if and only if $f \in \mathfrak{p}$. Notice that $(\mathfrak{b} \cap \mathfrak{p})x + \mathfrak{b}L \subset \mathfrak{b}M \cap L$. On the other hand, if $y \in \mathfrak{b}M \cap L$, then there exist $f_1, f_2 \in R_2$ and $l_1, l_2 \in L$ such that

$$y = b_1(f_1 x + l_1) + b_2(f_2 x + l_2) \Rightarrow (b_1 f_1 + b_2 f_2)x \in L.$$

So, $b_1 f_1 + b_2 f_2 \in \mathfrak{p}$, and hence $y \in (\mathfrak{b} \cap \mathfrak{p})x + \mathfrak{b}L$. Therefore,

$$\mathfrak{b}M \cap L = (\mathfrak{b} \cap \mathfrak{p})x + \mathfrak{b}L$$

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

and

$$\frac{\mathfrak{b}M \cap L}{\mathfrak{b}L} = F, \quad \frac{(\mathfrak{b} \cap \mathfrak{p})x + \mathfrak{b}L}{\mathfrak{b}L} \cong \frac{(\mathfrak{b} \cap \mathfrak{p})x}{\mathfrak{b}L \cap (\mathfrak{b} \cap \mathfrak{p})x}.$$

Finally, since $\mathfrak{b}\mathfrak{p}x \subset \mathfrak{b}L$, it follows that the natural surjection

$$\mathfrak{b} \cap \mathfrak{p} \to \frac{(\mathfrak{b} \cap \mathfrak{p})x}{\mathfrak{b}L \cap (\mathfrak{b} \cap \mathfrak{p})x}$$

has a kernel $\mathfrak{a}$ containing $\mathfrak{b}\mathfrak{p}$. □

LEMMA 5.2. *Let $\mathfrak{p} \subset R_2$ be a non-zero prime ideal and set $D = R_2/\mathfrak{p}$. Suppose that $D/\mathfrak{b}_\Lambda D$ is finite and that $\mathfrak{a} \subset R_2$ is an ideal satisfying $\mathfrak{b}_\Lambda \mathfrak{p} \subset \mathfrak{a} \subset \mathfrak{b}_\Lambda \cap \mathfrak{p}$. Then* $\mathrm{Ass}((\mathfrak{b}_\Lambda \cap \mathfrak{p})/\mathfrak{a}) \subset \mathrm{Ass}(D/\mathfrak{b}_\Lambda D)$ *and:*

(i) *if* $\mathrm{coht}(\mathfrak{p}) = 0$, *then* $|(\mathfrak{b}_\Lambda \cap \mathfrak{p})/\mathfrak{a}| = |R_2/\mathfrak{p}|^k$, *where* $k \in \{0, 1, 2\}$, *with* $k = 0$ *if* $\mathfrak{b}_\Lambda \not\subset \mathfrak{p}$;

(ii) *if* $\mathrm{coht}(\mathfrak{p}) = 1$, *then there exists a constant C depending only on $\mathfrak{p}$ such that*

$$|(\mathfrak{b}_\Lambda \cap \mathfrak{p})/\mathfrak{a}| \leqslant C|D/\mathfrak{b}_\Lambda D|;$$

(iii) *if $\mathfrak{p} = (p)$ for a rational prime p, then* $|(\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}| = 1$.

Proof. Using (20), we may assume $\mathfrak{b} = \mathfrak{b}_\Lambda$ is of the form $\mathfrak{b} = (b_1, b_2)$. Furthermore, since $\mathfrak{b}\mathfrak{p} \subset \mathfrak{a}$, all elements of $(\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}$ are annihilated by $\mathfrak{b} + \mathfrak{p}$. Therefore,

$$\mathrm{Ass}((\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}) \subset \mathrm{Ass}(R_2/(\mathfrak{b} + \mathfrak{p})) = \mathrm{Ass}((R_2/\mathfrak{p})/\mathfrak{b}(R_2/\mathfrak{p})).$$

The surjective module homomorphism $R_2 \oplus R_2 \to \mathfrak{b}/\mathfrak{a}$ given by

$$f_1 \oplus f_2 \mapsto f_1 b_1 + f_2 b_2 + \mathfrak{a}$$

has $\mathfrak{p} \oplus \mathfrak{p}$ in the kernel as $\mathfrak{b}\mathfrak{p} \subset \mathfrak{a}$. Hence, this map also naturally defines a surjective module homomorphism $\phi : D \oplus D \to \mathfrak{b}/\mathfrak{a}$. Denote the images of b_1 and b_2 in D by $\overline{b}_1$ and $\overline{b}_2$, respectively. If

$$K = \{g_1 \oplus g_2 \in D \oplus D : g_1\overline{b}_1 + g_2\overline{b}_2 = 0\},$$

then $\phi(K)$ is also the image of $\mathfrak{b} \cap \mathfrak{p}$ in $\mathfrak{b}/\mathfrak{a}$. That is, $\phi(K) \cong (\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}$. Furthermore, $\mathfrak{b}K \subset \ker(\phi)$, as $\mathfrak{b}\mathfrak{p} \subset \mathfrak{a}$. Therefore,

$$|(\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}| \text{ divides } |K/\mathfrak{b}K|. \tag{21}$$

If $\mathrm{coht}(\mathfrak{p}) = 0$, then $\mathfrak{p}$ is maximal and D is hence a finite field. Therefore, $K/\mathfrak{b}K$ is a vector space of dimension at most two over D, and hence the same is true for the homomorphic image $(\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}$. Furthermore, since all elements of $(\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}$ are annihilated by $\mathfrak{b} + \mathfrak{p}$, if $\mathfrak{b} \not\subset \mathfrak{p}$, $\mathfrak{b} + \mathfrak{p} = R_2$ and so $(\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}$ must be trivial as it is annihilated by R_2. This gives (i).

Now assume that $\mathrm{coht}(\mathfrak{p}) = 1$. If $\mathfrak{c} \subset D$ is a non-trivial ideal, then Lemma 2.4 shows that there exist constants $A, B > 0$ independent of $\mathfrak{c}$ such that

$$A|E/\mathfrak{c}E| \leqslant |D/\mathfrak{c}| \leqslant B|E/\mathfrak{c}E|,$$

where E is the integral closure of D in its field of fractions. Furthermore, we may exploit the favourable ideal arithmetic in E, as E is a Dedekind domain. By assumption, the image

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

$\overline{\mathfrak{b}}$ of $\mathfrak{b}$ in D is non-trivial. This means that, at most, one of $\overline{b}_1$ and $\overline{b}_2$ is zero. Furthermore, if precisely one of $\overline{b}_1$ or $\overline{b}_2$ is zero, then $K \cong D$ and (ii) follows immediately from (21). Hence, assume that $\overline{b}_1 \neq 0$ and $\overline{b}_2 \neq 0$. In this case, $K \cong \mathfrak{c} = \{g \in D : g\overline{b}_1 \in (\overline{b}_2)\}$, which is a non-trivial ideal of D. Hence,

$$\left|\frac{K}{\mathfrak{b}K}\right| = \left|\frac{\mathfrak{c}}{\overline{\mathfrak{b}}\mathfrak{c}}\right| = \frac{|D/\overline{\mathfrak{b}}\mathfrak{c}|}{|D/\mathfrak{c}|} \leqslant \frac{B|E/\overline{\mathfrak{b}}\mathfrak{c}E|}{|D/\mathfrak{c}|} = \frac{B|E/\overline{\mathfrak{b}}E\|E/\mathfrak{c}E|}{|D/\mathfrak{c}|}, \tag{22}$$

where the final equality holds since E is a Dedekind domain. Since

$$|E/\overline{\mathfrak{b}}E\|E/\mathfrak{c}E| \leqslant A^{-2}|D/\overline{\mathfrak{b}}\|D/\mathfrak{c}|,$$

(21) and (22) give

$$|(\mathfrak{b} \cap \mathfrak{p})/\mathfrak{a}| \leqslant |K/\mathfrak{b}K| \leqslant BA^{-2}|D/\overline{\mathfrak{b}}|,$$

which yields (ii) with $C = BA^{-2}$.

Finally, if $\mathfrak{p} = (p)$ for a rational prime p, then it is straightforward to show that $\mathfrak{b} \cap \mathfrak{p} = \mathfrak{b}\mathfrak{p}$, so (iii) follows immediately (note that this is also shown more generally for principal ideals with a trivial complex unitary variety in [**17**, Lemma 7.6]). □

Remark B. From Lemma 5.2, it follows that if $L \subset M$ are Noetherian R_2-modules such that each $\mathfrak{p} \in \mathrm{Ass}(M)$ contains a rational prime, then there is a constant $C \geqslant 0$ such that, for all $\Lambda \in \mathcal{L}_2$,

$$|M/\mathfrak{b}_\Lambda M| > C|L/\mathfrak{b}_\Lambda L|.$$

This may be proved by induction as follows. Firstly, note that by lifting a prime filtration of M/L to M, we obtain a sequence of modules of the form (1) with $M_0 = L$. Then, $M_i/M_{i-1} \cong R_2/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal satisfying one of the three cases in Lemma 5.2. Furthermore, by Lemma 5.1,

$$|M_i/\mathfrak{b}_\Lambda M_i| = \left|\frac{R_2/\mathfrak{p}_i}{\mathfrak{b}_\Lambda(R_2/\mathfrak{p}_i)}\right| \frac{|M_{i-1}/\mathfrak{b}_\Lambda M_{i-1}|}{|(\mathfrak{b}_\Lambda \cap \mathfrak{p}_i)/\mathfrak{a}_i|},$$

where $\mathfrak{a}_i \subset R_2$ is an ideal satisfying $\mathfrak{b}_\Lambda\mathfrak{p}_i \subset \mathfrak{a}_i \subset \mathfrak{b}_\Lambda \cap \mathfrak{p}_i$. From Lemma 5.2, it follows that there is a constant $C_i > 0$, independent of Λ, such that

$$|M_i/\mathfrak{b}_\Lambda M_i| \geqslant C_i|M_{i-1}/\mathfrak{b}_\Lambda M_{i-1}|.$$

Therefore, $|M/\mathfrak{b}_\Lambda M| \geqslant (\prod_{i=1}^{r} C_i)|L/\mathfrak{b}_\Lambda L|$.

LEMMA 5.3. *Suppose that $L \subset M$ are Noetherian R_2-modules such that each $\mathfrak{p} \in \mathrm{Ass}(M)$ contains a rational prime, and suppose that $M/L \cong R_2/\mathfrak{q}$ for a prime ideal $\mathfrak{q} \subset R_2$ which contains a polynomial of the form $u^{\mathbf{n}} - 1$, $\mathbf{n} \in \mathbb{Z}^d\backslash\{0\}$. If α_M is mixing, then there exist constants A, $B > 0$ such that, for all $\Lambda \in \mathcal{L}_2$,*

$$A|L/\mathfrak{b}_\Lambda L| \leqslant |M/\mathfrak{b}_\Lambda M| \leqslant B|L/\mathfrak{b}_\Lambda L|.$$

Proof. First note that the lower estimate is provided immediately by Remark B. For the upper estimate, let $f = u^{\mathbf{n}} - 1$ and consider the R_2-module fM. Since α_M is mixing, multiplication by f on M is injective, so M and fM are isomorphic as R_2-modules. On the other hand, $f \in \mathfrak{p}$, so $fM \subset L$ and Remark B again provides a constant $C > 0$ such that

$$|L/\mathfrak{b}L| \geqslant C|fM/\mathfrak{b}_\Lambda fM| = C|M/\mathfrak{b}_\Lambda M|.$$

Dividing by C gives the required upper estimate. □

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

Proof of Theorem 5.1. We begin by taking a prime filtration of the dual module M of the form (1). Each prime $\mathfrak{p}_i$ appearing in such a filtration contains a rational prime, since X is zero-dimensional. If $u^{\mathbf{n}} - 1 \notin \mathfrak{p}_i$ for all $\mathbf{n} \in \mathbb{Z}^d \backslash \{0\}$, we say that $\mathfrak{p}_i$ is *mixing*. Note that whenever $\mathfrak{p}_i \in \mathrm{Ass}(M)$, $\mathfrak{p}_i$ is mixing, as α is mixing. When $\mathrm{coht}(\mathfrak{p}_i) = 0$, $\mathfrak{p}_i$ is maximal and $R_2/\mathfrak{p}_i$ is a finite field, so $\mathfrak{p}_i$ is not mixing and $\mathfrak{p}_i \notin \mathrm{Ass}(M)$. Hence, each prime $\mathfrak{p}_i$ is of one of the following three types.

(i) $\mathfrak{p}_i = (p_i)$ for a rational prime p_i and $\mathfrak{p}_i \in \mathrm{Ass}(M)$.

(ii) $\mathrm{coht}(\mathfrak{p}_i) = 1$ and $\mathfrak{p}_i$ is mixing.

(iii) $\mathrm{coht}(\mathfrak{p}_i) \leqslant 1$ and $\mathfrak{p}_i$ is not mixing.

If $\mathfrak{p}_i$ is of type (i), a straightforward calculation shows that, for any $\Lambda \in \mathcal{L}_2$,

$$\left| \frac{R_2/\mathfrak{p}_i}{\mathfrak{b}_\Lambda(R_2/\mathfrak{p}_i)} \right| = p_i^{[\Lambda]}$$

and hence Lemmas 5.1 and 5.2(iii) give

$$|M_i/\mathfrak{b}_\Lambda M_i| = p_i^{[\Lambda]} |M_{i-1}/\mathfrak{b}_\Lambda M_{i-1}|. \tag{23}$$

If $\mathfrak{p}_i$ is of type (ii), then Lemmas 5.1 and 5.2(ii) show that there are constants A_i, $B_i > 0$ such that, for all $\Lambda \in \mathcal{L}_2$,

$$A_i \leqslant |M_i/\mathfrak{b}_\Lambda M_i| \leqslant B_i \left| \frac{R_2/\mathfrak{p}_i}{\mathfrak{b}_\Lambda(R_2/\mathfrak{p}_i)} \right| |M_{i-1}/\mathfrak{b}_\Lambda M_{i-1}|. \tag{24}$$

Finally, if $\mathfrak{p}_i$ is of type (iii), then Lemmas 5.1 and 5.3 show that there are positive constants A_i, $B_i > 0$ such that, for all $\Lambda \in \mathcal{L}_2$,

$$A_i |M_{i-1}/\mathfrak{b}_\Lambda M_{i-1}| \leqslant |M_i/\mathfrak{b}_\Lambda M_i| \leqslant B_i |M_{i-1}/\mathfrak{b}_\Lambda M_{i-1}|. \tag{25}$$

Recall that any prime of type (i) appears in the filtration (1) with a fixed multiplicity, as given in the entropy formula (19). Hence, applying (23)–(25) inductively, the required result follows, where $\beta = \beta_L$ is simply defined by an R_2-module L which is a direct sum of the modules $R_2/\mathfrak{p}_i$, where $\mathfrak{p}_i$ runs through the primes of type (ii). □

When combined with Corollary 2.1, Theorem 5.1 has the following consequence, which we shall apply to demonstrate the existence of a natural boundary for the dynamical zeta function. For the rest of this section, denote the multiplicative group $\langle p : p \in \mathbb{P}(\alpha) \rangle \subset \mathbb{Q}^\times$ by $G(\alpha)$ and note that all values $\mathsf{F}_\alpha(\Lambda)$, $\Lambda \in \mathcal{L}_2$ lie in $G(\alpha) \cap \mathbb{N}$.

COROLLARY 5.1. *Let α be a mixing Noetherian $\mathbb{Z}^2$-action by automorphisms of a compact abelian zero-dimensional group X. Then there are constants $A \in G(\alpha)$ and $C > 0$ and a function $\psi : \mathcal{L}_2 \to G(\alpha) \cap \mathbb{N}$ such that, for all subgroups $\Lambda \in \mathcal{L}_2$, $\psi(\Lambda) \leqslant C^{\sqrt{[\Lambda]}}$ and*

$$\mathsf{F}_\alpha(\Lambda) = A e^{\mathsf{h}[\Lambda]} \psi(\Lambda). \tag{26}$$

Proof. Standard properties of torsion abelian groups show that we may write $X = \bigoplus_{p \in \mathbb{P}(\alpha)} X_p$, where each X_p is a p-group and is also α-invariant. Write α_p for the restriction of α to X_p so that, for all $\Lambda \in \mathcal{L}$,

$$\mathsf{F}_\alpha(\Lambda) = \prod_{p \in \mathbb{P}(\alpha)} \mathsf{F}_{\alpha_p}(\Lambda). \tag{27}$$

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

Theorem 5.1 shows that, given any $p \in \mathbb{P}(\alpha)$, there exists a constant $A_p > 0$ such that, for all $\Lambda \in \mathcal{L}_2$, $\mathsf{F}_{\alpha_p}(\Lambda) \geqslant A_p\ p^{m(p)[L]}$, where $m(p)$ is as in (19). Since $\mathsf{F}_{\alpha_p}(\Lambda)$ is a power of p, we can therefore assume that A_p is a power of p and that there is a function $\psi_p : \mathcal{L}_2 \to \{p^n : n \geqslant 0\}$ such that

$$\mathsf{F}_{\alpha_p}(\Lambda) = A_p\ p^{m(p)[L]}\psi_p(\Lambda).$$

So (26) follows from (27) and (19) with $A = \prod_{p\in\mathbb{P}(\alpha)} A_p$ and $\psi = \prod_{p\in\mathbb{P}(\alpha)} \psi_p$. Furthermore, for each $p \in \mathbb{P}(\alpha)$, the upper estimates provided by Theorem 5.1 and Corollary 2.1 show that there is a constant $C_p > 0$ such that, for all $\Lambda \in \mathcal{L}_2$,

$$\psi_p(\Lambda) \leqslant C_p^{\sqrt{[\Lambda]}}.$$

The required upper estimate for ψ now follows with $C = \prod_{p\in\mathbb{P}(\alpha)} C_p$. □

We now use Corollary 5.1 and Theorem 1.3 to prove Theorem 1.4. In order to convert an implausibility into an impossibility, we will also need some elementary bounds for the sum of divisors function σ along the lines of Gronwall's theorem [**11**]. In addition to showing that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma},$$

Gronwall showed that the sequence (n_k) defined by

$$n_k = (p_1 p_2 \cdots p_k)^{\lfloor \log p_n \rfloor},$$

where $p_1, p_2, \ldots$ are the rational primes in their natural order, has

$$\lim_{k\to\infty} \frac{\sigma(n_k)}{n_k \log\log n_k} = e^{\gamma}.$$

We need lower bounds an infinite number of times along arithmetic progressions for the same expression, but require very little information about exact values. We thank Stephan Baier for suggesting how to do this and Pieter Moree for further suggestions on how the methods of Choie *et al* [**4**, §4] giving explicit constants for the upper limit with n restricted to being odd, square-free, square-full and so on should give a complete picture.

LEMMA 5.4. *Given integers t, $q \geqslant 1$, there is a constant $C > 0$ with*

$$\frac{\sigma(n)}{n \log\log n} \geqslant C$$

for an infinite number of values of n congruent to t modulo q.

Proof. Write p for a prime running through the set of primes in their natural order. For $k \geqslant 1$, let

$$n_k = \prod_{\substack{p\leqslant k\\ p\nmid q}} p$$

be the product of all primes not dividing q that are less than or equal to k. By the Chinese remainder theorem, there is some $N_k \leqslant q n_k$ with $N_k \equiv t$ modulo q and $N_k \equiv 0$ modulo n_k. By construction,

$$\sigma(n_k) \geqslant N_k \prod_{\substack{p\leqslant k\\ p\nmid q}} \left(1 + \frac{1}{p}\right) \sim C(q) \log k$$

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

for some constant $C(q) > 0$, by Mertens' theorem. On the other hand,

$$k \asymp \log n_k \asymp \log N_k,$$

so

$$\sigma(N_k) \gg N_k \log\log N_k$$

for an infinite number of k, as required. □

Proof of Theorem 1.4. By Corollary 5.1,

$$\zeta_\alpha(z) = \exp\left(A \sum_{n \geqslant 1} a_n (e^{\mathsf{h}} z)^n\right),$$

where $a_n = \sum_{[L]=n} \psi(L)$. There are exactly $\sigma(n)$ subgroups $\Lambda \in \mathcal{L}_2$ such that $[\Lambda] = n$. Furthermore, Corollary 5.1 shows that $\psi(\Lambda) \leqslant C^{\sqrt{[\Lambda]}}$, so a_n satisfies

$$\sigma(n) \leqslant a_n \leqslant \sigma(n) \cdot C^{\sqrt{[\Lambda]}}, \tag{28}$$

whereby $\limsup_{n\to\infty} a_n^{1/n} = 1$. Thus the series $F(z) = \sum_{n\geqslant 1} a_n z^n$ has radius of convergence equal to one and ζ_α has radius of convergence $e^{-\mathsf{h}}$. Furthermore,

$$F(e^{\mathsf{h}} z) = \frac{z\zeta_\alpha'(z)}{A\zeta_\alpha(z)},$$

so, if ζ_α has analytic continuation beyond the circle $|z| = e^{-\mathsf{h}}$, then F has analytic continuation beyond the unit circle. We assume this to be the case for a contradiction.

Since F has integer coefficients and radius of convergence one, the Pólya–Carlson theorem implies that F is a rational function. Therefore, (a_n) is an integer linear recurrence sequence. Using [**9**, Theorem 1.2], the sequence (a_n) may be decomposed into a finite number of non-degenerate subsequences corresponding to a finite set of congruence classes, as no subsequence can be identically zero by (28). Each such subsequence is given by a linear recurrence with characteristic roots that are algebraic integers, as it comprises positive integers. None of these linear recurrence sequences can have a characteristic root of absolute value greater than one, by [**9**, Theorem 2.3] and the upper bound from (28). Now, by Theorem 1.3, there is some integer constant A with $a_p = A\sigma(p) = A(p+1)$ for an infinite number of primes p, so, by the pigeon-hole principle, one of the non-degenerate linear recurrence sequences, say, $(b_n = a_{qn+t})$ has integer constants A' and t' with an infinite number of terms satisfying

$$b_n = a_{qn+t} = A'n + t'. \tag{29}$$

The characteristic roots of the sequence (b_n) and their conjugates are algebraic integers, since the sequence is integral, and have modulus no greater than one. By Kronecker's lemma [**10**, Theorem 1.31], they must therefore be roots of unity, so there can only be one since the sequence is non-degenerate. Thus, by [**9**, equation (1.2)], we may write $b_n = P(n)\omega^n$ for some polynomial P and root of unity ω. On the other hand, (29) shows that $|P(n)\omega^n| \ll n$ for an infinite number of prime values of n, so P is linear. Therefore, $b_n = A'n + t'$ for all n, and so $a_n = A''n + t''$ for some constants A'' and t'' for all n lying in an arithmetic progression. However, according to the lower bound in (28) and Lemma 5.4, this is impossible, which thus gives the required contradiction. □

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

Acknowledgements. The authors gratefully acknowledge the support of the London Mathematical Society Scheme IV grant number 41352. The first author would also like to thank Durham University's Institute for Advanced Study and gratefully acknowledges the COFUND Senior Fellowship arising from the European Union's Seventh Framework Programme for research, technological development and demonstration (grant agreement number 609412).

REFERENCES

[**1**] M. Artin and B. Mazur. On periodic points. *Ann. of Math. (2)* **81** (1965), 82–99.
[**2**] J. Bell, R. Miles and T. Ward. Towards a Pólya–Carlson dichotomy for algebraic dynamics. *Indag. Math. (N.S.)* **25**(4) (2014), 652–668.
[**3**] F. Carlson. Über ganzwertige Funktionen. *Math. Z.* **11**(1–2) (1921), 1–23.
[**4**] Y. Choie, N. Lichiardopol, P. Moree and P. Solé. On Robin's criterion for the Riemann hypothesis. *J. Théor. Nombres Bordeaux* **19**(2) (2007), 357–372.
[**5**] V. Chothi, G. Everest and T. Ward. S-integer dynamical systems: periodic points. *J. Reine Angew. Math.* **489** (1997), 99–132.
[**6**] P. Corvaja and U. Zannier. A lower bound for the height of a rational function at S-unit points. *Monatsh. Math.* **144**(3) (2005), 203–224.
[**7**] P. Corvaja and U. Zannier. Greatest common divisors of $u - 1$, $v - 1$ in positive characteristic and rational points on curves over finite fields. *J. Eur. Math. Soc. (JEMS)* **15**(5) (2013), 1927–1942.
[**8**] M. Einsiedler and D. Lind. Algebraic $\mathbb{Z}^d$-actions of entropy rank one. *Trans. Amer. Math. Soc.* **356**(5) (2004), 1799–1831.
[**9**] G. Everest, A. van der Poorten, I. Shparlinski and T. Ward. *Recurrence Sequences (Mathematical Surveys and Monographs, 104)*. American Mathematical Society, Providence, RI, 2003.
[**10**] G. Everest and T. Ward. Heights of polynomials and entropy in algebraic dynamics. *Universitext*. Springer, London, 1999.
[**11**] T. H. Gronwall. Some asymptotic expressions in the theory of numbers. *Trans. Amer. Math. Soc.* **14**(1) (1913), 113–122.
[**12**] B. Kitchens and K. Schmidt. Automorphisms of compact groups. *Ergod. Th. & Dynam. Sys.* **9**(4) (1989), 691–735.
[**13**] P. Kurlberg and C. Pomerance. On the periods of the linear congruential and power generators. *Acta Arith.* **119**(2) (2005), 149–169.
[**14**] F. Ledrappier. Un champ markovien peut être d'entropie nulle et mélangeant. *C. R. Acad. Sci. Paris A–B* **287**(7) (1978), A561–A563.
[**15**] R. Lidl and H. Niederreiter. *Introduction to Finite Fields and their Applications,* 1st edn. Cambridge University Press, Cambridge, 1994.
[**16**] D. A. Lind. A zeta function for $\mathbb{Z}^d$-actions. *Ergodic Theory of $\mathbb{Z}^d$-actions (Warwick, 1993–1994) (London Mathematical Society Lecture Note Series, 228)*. Cambridge University Press, Cambridge, 1996, pp. 433–450.
[**17**] D. Lind, K. Schmidt and T. Ward. Mahler measure and entropy for commuting automorphisms of compact groups. *Invent. Math.* **101**(3) (1990), 593–629.
[**18**] H. Matsumura. *Commutative Ring Theory,* 2nd edn. *(Cambridge Studies in Advanced Mathematics, 8)*. Cambridge University Press, Cambridge, 1989.
[**19**] R. Miles. Zeta functions for elements of entropy rank-one actions. *Ergod. Th. & Dyn. Sys.* **27**(2) (2007), 567–582.
[**20**] R. Miles. Periodic points of endomorphisms on solenoids and related groups. *Bull. Lond. Math. Soc.* **40**(4) (2008), 696–704.
[**21**] R. Miles. Synchronization points and associated dynamical invariants. *Trans. Amer. Math. Soc.* **365**(10) (2013), 5503–5524.
[**22**] R. Miles. A natural boundary for the dynamical zeta function for commuting group automorphisms. *Proc. Amer. Math. Soc.* **143**(7) (2015), 2927–2933.
[**23**] G. Pólya. Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe. *Math. Ann.* **99**(1) (1928), 687–706.

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press

[**24**] K. Schmidt. *Dynamical Systems of Algebraic Origin (Progress in Mathematics, 128)*. Birkhäuser, Basel, 1995.
[**25**] S. L. Segal. *Nine Introductions in Complex Analysis (North-Holland Mathematics Studies, 208)*, revised edn. Elsevier Science B.V., Amsterdam, 2008.
[**26**] J. H. Silverman. Common divisors of $a^n - 1$ and $b^n - 1$ over function fields. *New York J. Math.* **10** (2004), 37–43.
[**27**] T. Ward. Dynamical zeta functions for typical extensions of full shifts. *Finite Fields Appl.* **5**(3) (1999), 232–239.

https://doi.org/10.1017/etds.2016.77 Published online by Cambridge University Press