WAVE PROPAGATION ANALYSIS IN BEAMS USING SHEAR DEFORMABLE BEAM THEORIES CONSIDERING SECOND SPECTRUM

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ABSTRACT

In this study, wave propagation in beams is studied using different beam theories like Euler-Bernoulli, Timoshenko and Reddy beam theories. Dispersion curves obtained for these beam theories are compared with the exact plane elasticity solutions. It is obtained that, there are two branches for Reddy beam theory similar to the Timoshenko beam theory. However, one branch is obtained for Euler-Bernoulli beam theory. The effects of in-plane load on Timoshenko and Reddy beam theories are examined and dispersion curves of the Timoshenko and Reddy beams are compared with exact plane elasticity solution. In Timoshenko beam theory, qualitative difference between the two spectrums has been lost with in-plane loads for some wave numbers.

Keywords: Wave propagation, Timoshenko beam theory, Reddy beam theory, Second spectrum.

1. INTRODUCTION

Different beam theories have been used in order to study dynamic analysis of beam type structures. Euler-Bernoulli theory is the first theory in order to explain deformation of beam type structures. In this theory, it is assumed that normal lines to the mid-plane of the beam are normal before and after deformation of the beam. With the development in structural engineering, deviations of the Euler-Bernoulli (EB) beam theory from experimental results have been observed. In order to solve this problem Timoshenko beam theory (TBT) has been proposed. Shear deformation and rotary inertia effects are included in TBT.

The TBT has been applied to many statics and dynamics problem of beam type structures. While studying flexural wave propagation in beams, two branches are observed for Timoshenko beam, whereas one branch is observed in EB beam theory. The first branch of TBT is a correction to the EB beam theory. Second branch agrees with the second curve of the exact plane elasticity solution for long wavelength.

The validity of the second branch (or second spectrum) of the Timoshenko theory is considered by some authors in the previous studies. The possibility of the second spectrum has been firstly considered by Traill-Nash and Collar [1]. Stephen [2] argued that the second spectrum of TBT is "an inevitable but meaningless consequence of the structure of an otherwise excellent approximate theory." This conclusion is obtained from a comparison of phase velocity results of TBT with an exact elasticity solution of Pochhammer-Chree theory. Different reasons have been asserted for recognition of the second spectrum "unphysical". One of these arguments, the second spectrum can be determined only in SS-SS boundary conditions that the frequency equation factorizes. Another argument, numerical comparison of the second spectrum predictions (TBT2) to the exact plane stress elastodynamics theory has demonstrated that TBT2 predictions do not agree well with any single mode of vibration [3].

Renton [4] discussed the accuracy of Timoshenko theory using wave propagation. Dispersion curves were given for EB, Timoshenko and plane stress elasticity solution. It is obtained that when the wavelengths are greater than beam depth, lower TBT solution gives acceptable results. The second branch agrees with an elasticity solution for long wavelength, but difference increases for lower wave length. Then it is concluded that the second spectrum should be disregarded. Bhaskar [5] studied elastic waves in Timoshenko beams and concluded that the second spectrum should not be ignored. Elishakoff and Lubliner [6] investigated the random vibrations of Timoshenko beams with classical and non-classical theories. Excitations were discussed for lower and upper cut-off frequencies considering the effects of shear deformation and rotary inertia. They concluded that lower cut-off frequency plays a significant role for lower modes of vibration. Some closed form solutions of the random vibration of Timoshenko

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beams have been investigated in [7-8] and highlighted the significance of higher natural frequencies.

Bhasyam and Prathap [9] and Abbas and Thomas [10] have examined finite element methods to obtain the natural frequencies of TBT for various end conditions. Recently, Manevich [11] explained the significance of the second spectrum frequency of Timoshenko beam using linear and nonlinear elastic foundation. He showed that qualitative difference of two branches theories change with elastic medium. On the other hand, Elishakoff *et al*. [12] investigated the effects of shear deformation and rotary inertia on Timoshenko beam model based on asymptotic justification. Elishakoff and co-workers have been used Bresse-Timoshenko beam theory in the analysis of some nonlocal beam problems [13-15].

Considering above references it can be seen that there are three groups of studies related to the dynamics of the TBT.

1. Group 1 (G1): Timoshenko second spectrum should be disregarded.
2. Group 2 (G2): Timoshenko second spectrum should be considered in order to correct the description of dynamics.
3. Group 3 (G3): Timoshenko second spectrum has not been considered in one group of studies given in literature.

This conflict has not been completely solved in the previous studies. There are some other shear deformation theories in open literature like Reddy theory [16]. The vibration problem of Reddy beams has been investigated in some of the previous studies [17-19]. It is natural to consider second spectrum in higher order shear deformation theories. According to the best knowledge of the authors this point has not been considered in the previous studies.

The aims of this study are:

1. To investigate the existence of the second spectrum for a higher order shear deformation theory: Reddy beam theory.
2. How spectrums of Timoshenko and Reddy beam theories are qualitatively affected by in-plane forces?
3. To analyze and compare wave characteristics of the exact elasticity solution and EB, TBT and Reddy beam theory.
4. Effect of the shear coefficient (κ) on the spectrums of Timoshenko beam theory.

To answer these questions, dispersion curves for flexural wave are obtained for different beam theories, namely, EB, Timoshenko and Reddy beam theories. Results obtained by using the different beam theories are compared with the plane elasticity solutions. We considered flexural wave propagation in Timoshenko and Reddy beams applying the in-plane load, studying properties of the first and second frequency branches. The presence of the in-plane load creates additional challenges and opportunities for comparative analysis of the both frequency branches. We showed that the differences between the two branches are identified by the phase relations between the total rotation of the cross section and rotation of the cross section due to shear. Moreover, we compared the different predictions of the Timoshenko second spectrum results for different values of the shear coefficient (κ) with the exact plane elasticity solutions. Based on this analysis the importance of the second spectrum is discussed, and features of dispersion relations for the frequency are explained.

2. ANALYSIS

2.1 General Beam Theory

Consider a beam of width b and thickness 2h made of isotropic material. x-axis is chosen in the length direction of the beam and z- axis is chosen in the thickness direction of the beam. Deformation of beam takes place in x-z plane and following general shear deformable displacement field is assumed [17]:

$$\begin{aligned} U(x,z;t) &= u(x;t) - zw_{,x} + f(z)u_1(x;t), \\ V(x,z;t) &= 0, \\ W(x,t) &= w(x;t). \end{aligned} \tag{1}$$

Where U, V and W are the displacement components of an arbitrary point of the beam in the x, y and z directions respectively. u and w are the displacement components of a point in the mid-plane of the beam. "$_{,x}$" denotes the partial derivative with respect to x. u_1 denotes shear deformation. $f(z)$ is called shape function denoting shear stress distribution along the thickness of the beam.

The principle of virtual displacements can be expressed as [17]:

$$\begin{aligned} 0 = \iint [& \rho_0 \left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t} \right) + \rho_{01} \left(\frac{\partial u}{\partial t}\frac{\partial \delta u_1}{\partial t} + \frac{\partial u_1}{\partial t}\frac{\partial \delta u}{\partial t} \right) \\ & + \rho_2 \left(\frac{\partial^2 w}{\partial x \partial t}\frac{\partial^2 \delta w}{\partial x \partial t} \right) - \rho_{11} \left(\frac{\partial u_1}{\partial t}\frac{\partial^2 \delta w}{\partial x \partial t} + \frac{\partial \delta u_1}{\partial t}\frac{\partial^2 w}{\partial x \partial t} \right) \\ & + \rho_{02}\frac{\partial u_1}{\partial t}\frac{\partial \delta u_1}{\partial t} - N_X \delta \varepsilon_x^0 - M_x^c \delta k_x^c - M_x^{sd} \delta k_x^{sd} \\ & - Q_x \delta u_1 - f\delta u + q\delta w + N_x^e \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} \Big] dt dx \end{aligned} \tag{2}$$

Here ε_x is the strain component and k_x is the curvature along the x-direction. Where the resultants of the force (Q_x) and the moment (M_x) can be written as:

$$\begin{aligned} (N_x^c) &= \int_{-h/2}^{h/2} (\sigma_x) dz, \quad (M_x^c) = \int_{-h/2}^{h/2} \sigma_x z dz, \\ (M_x^{sd}) &= \int_{-h/2}^{h/2} (\sigma_x) f(z) dz, \\ (Q_x^{sd}) &= \int_{-h/2}^{h/2} (\tau_{xz}) f'(z) dz. \end{aligned} \tag{3}$$

$$\begin{aligned} \rho_i &= \int_{-h/2}^{h/2} \rho z^i dz, \qquad (i = 0,1,2), \\ \rho_{jm} &= \int_{-h/2}^{h/2} \rho z^j f_j^m dz, \quad (j = 0.1; m = 1,2). \end{aligned} \tag{4}$$

Where superscript 'c' denotes the conventional force and moment resultants of the classical beam theories,

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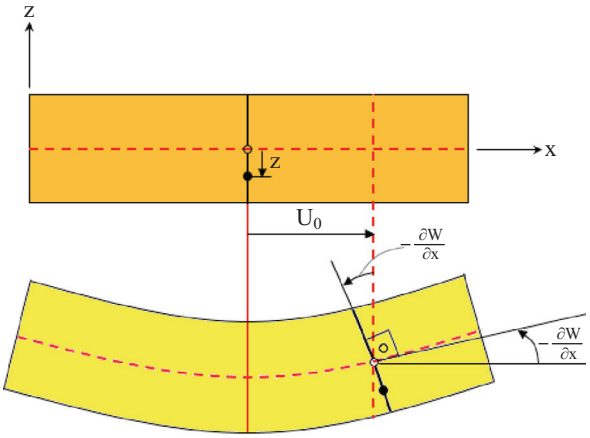


Fig. 1 Deformation in EB beam model.

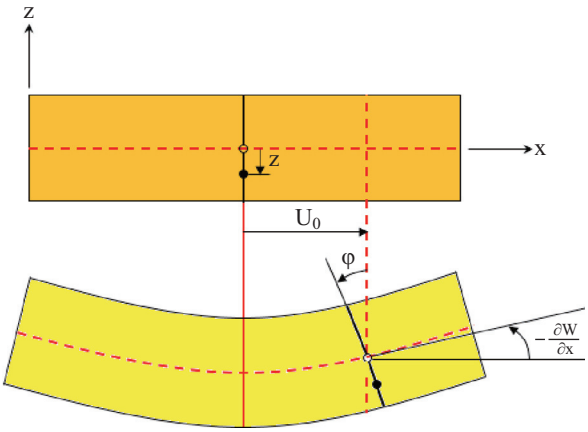


Fig. 2 Deformation in Timoshenko beam model.

whereas superscript '*sd*' is used for the shear deformation effects. ρ_i and ρ_{jm} denote the inertias, M_x^{sd} denotes the moment resultants of shear deformable theory and σ_x and τ_{xz} denote the stress components. By using the strain-stress relations with the force and moment resultants, the following equations can be found [17]:

$$\begin{bmatrix} N_x^c \\ M_x^c \\ M_x^{sd} \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & E_{11} \\ & D_{11} & F_{11} \\ Sym & & H_{11} \end{bmatrix} \begin{bmatrix} u_{,x} \\ -w_{,xx} \\ u_{1,x} \end{bmatrix} \begin{bmatrix} Q_x^{sd} \end{bmatrix} = [A_{55}][u_1]. \tag{5}$$

The extensional, coupling, bending and transverse shear rigidities can be defined as:

$$\begin{aligned} A_{11} &= \int_{-h/2}^{h/2} Q_{11}^{(k)} dz, \quad A_{55} = \int_{-h/2}^{h/2} Q_{55}^{(k)} (f')^2 dz, \\ B_{11} &= \int_{-h/2}^{h/2} Q_{ij}^{(k)} z dz, \quad E_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(k)} f(z) dz, \\ D_{11} &= \int_{-h/2}^{h/2} Q_{ij}^{(k)} z^2 dz, \quad F_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(k)} f(z) dz, \\ H_{11} &= \int_{-h/2}^{h/2} Q_{ij}^{(k)} (f)^2 dz. \end{aligned} \tag{6}$$

where $Q_{ij}^{(k)}$ denotes the reduced stiffnesses in each layer. Applying the principle of virtual displacement, the governing equations of the beam are obtained as

$$\begin{aligned} &N_{x,x}^c = (\rho_0 u + \rho_{01} u_1 - \rho_1 w_{,x})_{,tt}, \\ &M_{x,xx}^c = (\rho_1 u_{,x} + \rho_{11} u_{1,x} + \rho_0 w - \rho_2 w_{,xx})_{,tt} + N_x^e w_{,xx} + q, \\ &M_{x,x}^{sd} - Q_x^{sd} = (\rho_{01} u + \rho_{02} u_1 - \rho_{11} w_{,x})_{,tt}. \end{aligned} \tag{7}$$

where $_{,tt}$ denotes the time derivatives, N_x^c is the force resultants of classical beam theory, N_x^e is constant in-plane edge load and q is a transverse load.

2.2 Euler-Bernoulli (EB) Theory

By setting $f(z) = 0$ in Eq. (1) the displacement components for the EB beam theory (see Fig. 1), the well-recognized equation of motion for EB theory can be found from Hamilton Principle as follows:

$$EI \frac{\partial^4 W(x,t)}{\partial x^4} + \rho A \frac{\partial^2 W(x,t)}{\partial t^2} = 0 \tag{8}$$

where E is the Young modulus, I is the moment of inertia of the cross section of the beam, ρ is the density of beam material, A is the cross sectional area of the beam and t is the time.

Deflection given in Eq. (8) can be defined in the following form for transverse wave propagation:

$$W = W_m e^{i(\omega t - kx)} \tag{9}$$

where W_m is the amplitude of the wave, k is the wave number, ω is the natural frequency and ($i^2 = -1$). $E = 2G(1 + \nu)$. ν is the Poisson ratio, G is the shear modulus. Inserting Eq. (9) into Eq. (8) gives:

$$\omega = \sqrt{\frac{\frac{2}{3}(1+\nu)k^4 h^2 G}{\rho}} \tag{10}$$

Eq. (10) can be written in dimensionless form by defining $\gamma = \rho\omega^2/Gk^2$ [4]:

$$\gamma = \frac{2}{3}(1+\nu)k^2 h^2 \tag{11}$$

2.3 Timoshenko Beam Theory (TBT)

The TBT can be obtained by choosing $f(z) = z$ in Eq. (1) (see Fig. 2).

Using the Hamilton Principle, the equation of motion for the Timoshenko beam can be expressed as:

$$\begin{aligned} &-\kappa GA\left(\frac{\partial \varphi(x,t)}{\partial x} - \frac{\partial^2 W(x,t)}{\partial x^2}\right) = \rho A \frac{\partial^2 W(x,t)}{\partial t^2}, \\ &EI \frac{\partial^2 \varphi(x,t)}{\partial x^2} - \kappa GA\left(\varphi - \frac{\partial W(x,t)}{\partial x}\right) = \rho I \frac{\partial^2 \varphi(x,t)}{\partial t^2}. \end{aligned} \tag{12}$$

where κ is the shear factor of the Timoshenko beam, W is

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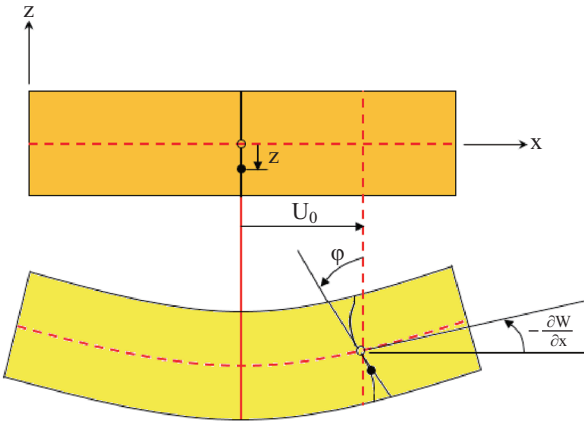


Fig. 3 Deformation in Reddy beam model.

the transverse displacement and φ is the angle of rotation due to the bending moment.

Deflection and rotation functions given in Eq. (12) can be expressed in the following form for transverse wave propagation:

$$W(x,t) = ae^{i(kx-\omega t)}, \quad \varphi(x,t) = be^{i(kx-\omega t)}. \tag{13}$$

where a and b are the amplitude of the waves. Inserting Eq. (13) into Eq. (12) gives:

$$\begin{bmatrix} -\kappa GAk^2 + \rho A\omega^2 & -\kappa GAik \\ \kappa GAik & -EIk^2 - \kappa GA + \rho I\omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{14}$$

The determinant of the matrix in Eq. (14) must be zero for non-trivial solution, and this condition leads to following dispersion equation:

$$(\rho A\rho I)\omega^4 - [\kappa GA\rho Ik^2 + \rho AEIk^2 + \kappa G\rho A^2]\omega^2 + \kappa GAEIk^4 = 0 \tag{15}$$

Solution of Eq. (15) can be written in dimensionless form by defining $\gamma = \rho\omega^2/Gk^2$:

$$\gamma_1, \gamma_2 = \frac{1}{2}\kappa + (1+\nu) + \frac{3\kappa}{2k^2h^2} \pm \sqrt{\frac{\kappa^2}{4} - \kappa(1+\nu) + \frac{3\kappa^2}{2k^2h^2} + (1+\nu)^2 + \frac{3\kappa(1+\nu)}{k^2h^2} + \frac{9\kappa^2}{4k^4h^4}} \tag{16}$$

2.4 Reddy Beam Theory

Reddy beam theory (see Fig. 3) can be obtained from Eq. (1) by writing $f(z) = z\left(1 - \frac{4z^2}{3h^2}\right)$.

Equations of motion for a parabolic shear deformation beam theory can be expressed as [18]:

$$-EI_1\frac{\partial^4 W}{\partial x^4} + E\left(I_1 - \frac{4}{3h^2}I_1'\right)\frac{\partial^3 U_1}{\partial x^3} = \rho A\frac{\partial^2 W}{\partial t^2} + \rho\left(I_1 - \frac{4}{3h^2}I_1'\right)\frac{\partial^3 U_1}{\partial^2 t\partial x} - \rho I_1\frac{\partial^4 W}{\partial x^2\partial t^2},$$

$$-E\left(I_1\frac{4}{3h^2}I_1'\right)\frac{\partial^3 W}{\partial x^3} + E\left(I_1 - \frac{8}{3h^2}I_1' + \frac{16}{9h^4}I_1''\right)\frac{\partial^2 U_1}{\partial x^2} - G\left(A - \frac{8}{h^2}I_1 + \frac{16}{h^4}I_1'\right)U_1 \tag{17}$$

$$= \rho\left(\frac{\partial^2 U_1}{\partial t^2}\right)\left(I_1 - \frac{8}{3h^2}I_1' + \frac{16}{9h^4}I_1''\right) - \frac{\partial^3 W}{\partial x\partial t^2}\rho\left(I_1 - \frac{4}{3h^2}I_1'\right).$$

where W is the transverse displacement, U_1 is the shear deformation, x is the axial coordinate and h is the thickness of the beam. Inertia terms are defined as [18]:

$$(I_1, I_1', I_1'') = \int (z^2, z^4, z^6)dA. \tag{18}$$

Displacement functions given in Eq. (17) can be defined in the following form for transverse wave propagation:

$$U_1 = u_1e^{i(\omega t - kx)}, \quad W = w_me^{i(\omega t - kx)}. \tag{19}$$

Inserting the wave propagation (Eq. (19)) into the Eq. (17), wave number solution is:

$$\begin{bmatrix} -EI_1k^4 + \rho A\omega^2 + \rho I_1k^2\omega^2 & EI_2k^3i - \rho I_2\omega^2ki \\ -EI_2k^3i + \rho I_2\omega^2ki & -EI_3k^2 - GI_4 + \rho I_3\omega^2 \end{bmatrix} \begin{bmatrix} w_m \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

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The determinant of the matrix in Eq. (20) must be zero for non-trivial solution, so that the dispersion equation is obtained for Reddy beam theory.

$$\alpha\omega^4 - \beta\omega^2 + \varepsilon = 0 \tag{21}$$

α, β and ε are defined as:

$$\begin{aligned}
\alpha &= \rho^2 AI_3 + \rho^2 I_1 I_3 k^2 - \rho^2 I_2^2 k^2,\\
\beta &= 2EI_1 I_3 \rho k^4 + \rho AEI_3 k^2 + \rho AGI_4 + \rho GI_1 I_4 k^2\\
&\quad -2\rho EI_2^2 k^4,\\
\varepsilon &= E^2 I_1 I_3 k^6 + EGI_1 I_4 k^4 - E^2 I_2^2 k^6.
\end{aligned} \tag{22}$$

Using the wave dispersion Eq. (20), following equation can be written by defining $\gamma = \rho\omega^2/Gk^2$.

$$\gamma_1, \gamma_2 = \left(\frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\varepsilon}}{2\alpha}\right)\frac{\rho}{Gk^2} \tag{23}$$

Here, inertia terms appearing in Eq. (22) are defined as:

$$\begin{aligned}
&I_1 = \frac{b(h)^3}{12}, \quad I_1' = \frac{b(h)^5}{80}, \quad I_1'' = \frac{b(h)^7}{448},\\
&I_2 = \left(I_1 - \frac{4}{3h^2} I_1'\right), \quad I_3 = \left(I_1 - \frac{8}{3h^2} I_1' + \frac{16}{9h^4} I_1''\right),\\
&I_4 = \left(A - \frac{8}{h^2} I_1 + \frac{16}{h^4} I_1'\right), \quad A = bh.
\end{aligned} \tag{24}$$

2.5 Effects of In-Plane Load on Timoshenko's Second Spectrum Beam Theory

There are some controversial issues about two wave propagation curves of Timoshenko beam theory. Traill-Nash and Collar [1] investigated that characteristic equations of Timoshenko beam changed depending on the shear and rotary inertia values. The rotation of the cross section due to shear is in-phase with total rotation of the cross section for the first spectrum of the TBT and they are anti-phase for the second mode. Yet, Manevich [11] has shown that the rotation due to shear and cross sectional rotation due to bending can be anti-phase in case of an elastic foundation for the first branch, moreover they are always anti-phase for the second branch of the wave propagation. He showed that qualitative difference between two spectrums may be lost due to elastic foundation effect. It is shown that both spectrums are required in order to explain the dynamic behavior of the Timoshenko beams.

In this study, we will consider the effect of the in-plane loads on the two spectrums of the Timoshenko beams. Differential equations of motion of a Timoshenko beam model with the in-plane load (P) applied the ends can be expressed as follows:

$$\begin{aligned}
&\kappa GA\left(\frac{\partial\varphi(x,t)}{\partial x} - \frac{\partial^2 W(x,t)}{\partial x^2}\right) + P\frac{\partial^2 W(x,t)}{\partial x^2} + \rho A\frac{\partial^2 W(x,t)}{\partial t^2} = 0,\\
&-EI\frac{\partial^2\varphi(x,t)}{\partial x^2} + \kappa GA\left(\varphi - \frac{\partial W(x,t)}{\partial x}\right) + \rho I\frac{\partial^2\varphi(x,t)}{\partial t^2} = 0.
\end{aligned} \tag{25}$$

Differential equations of motion given in Eq. (25) can be solved by using the wave propagation Eq. (13) as follows:

$$\begin{aligned}
&-\rho A\omega^2 a - Pk^2 a + \kappa GAkbi + \kappa GAk^2 a = 0,\\
&-\rho I\omega^2 b + EIk^2 b + \kappa GAb - \kappa GAkai = 0.
\end{aligned} \tag{26}$$

By using Eq. (26) the ratio of shear angle ($W_{,x} - \varphi$) to the total cross sectional rotation angle (φ) of Timoshenko beam (RAR-T) can be obtained as:

$$\frac{W_{,x} - \varphi}{\varphi} = \frac{a}{b}ki - 1 = \frac{\kappa GAk^2}{(\kappa GAk^2 - \rho A\omega^2 - Pk^2)} - 1 \tag{27}$$

2.6 Effects of In-Plane Load on Reddy Beam Theory

After studying dispersion curves of TBT, similar problem is considered for Reddy beam theory. Differential equations of motion of a Reddy beam model where the in-plane load (P) applied at the ends of the beam can be expressed as follows:

$$\begin{aligned}
&-EI_1\frac{\partial^4 w}{\partial x^4} + EI_2\frac{\partial^3 u_1}{\partial x^3} - \rho A\frac{\partial^2 w}{\partial t^2} - \rho I_2\frac{\partial^3 u_1}{\partial x\partial t^2}\\
&\qquad + \rho I_1\frac{\partial^4 w}{\partial x^2\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = 0,\\
&-EI_2\frac{\partial^3 w}{\partial x^3} + EI_3\frac{\partial^2 u_1}{\partial x^2} - GI_4 u_1 - \rho I_3\frac{\partial^2 u_1}{\partial t^2}\\
&\qquad + \rho I_2\frac{\partial^3 w}{\partial x\partial t^2} = 0.
\end{aligned} \tag{28}$$

Differential equations of motion given in Eq. (28) can be solved by using the wave propagation Eq. (19) as follows:

$$\begin{aligned}
&-EI_1 w_m k^4 + EI_2 u_1 k^3 i + \rho A\omega^2 w_m - \rho I_2 u_1\omega^2 ki\\
&+\rho I_1 k^2\omega^2 w_m + Pk^2 w_m = 0,
\end{aligned} \tag{29}$$

$$\begin{aligned}
&-EI_2 w_m k^3 i - EI_3 u_1 k^2 - GI_4 u_1 + \rho I_3 u_1\omega^2\\
&+\rho I_2 w_m k\omega^2 i = 0.
\end{aligned} \tag{30}$$

Thus, the ratio of shear angle (Υ) to the total cross sectional rotation angle (ψ) of Reddy beam is obtained as:

$$\begin{aligned}
\frac{\Upsilon}{\psi} &= \frac{u_1 f(z)'}{u_1 f(z)' + w_m ki}\\
&= \frac{1}{1 + \dfrac{EI_2 k^4 - \rho I_2\omega^2 k^2}{-EI_1 k^4 + \rho A\omega^2 + \rho I_1 k^2\omega^2 + Pk^2}\dfrac{1}{f(z)'}}
\end{aligned} \tag{31}$$

where

$$f(z)' = 1 - \frac{4z^2}{h^2} \tag{32}$$

for Reddy beam theory.

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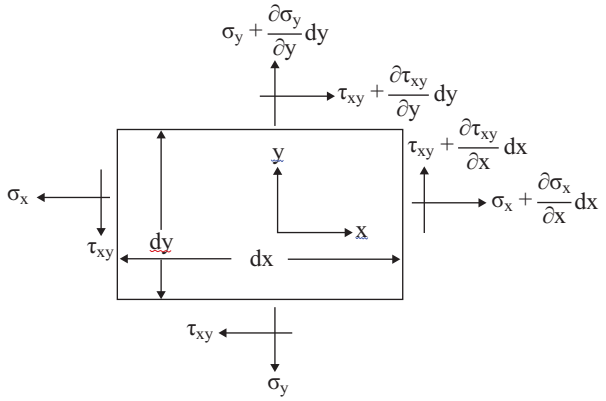


Fig. 4 The plane-stress condition of a differential element.

2.7 Effects of Shear Coefficient (κ) on Timoshenko's Second Spectrum Beam Theory

Several theoretical studies have been conducted to get the best value for the shear coefficient of TBT [20-24]. There are some popular approaches for the calculation of the shear coefficient in the TBT such as energy approach Corradi Dell'Acqua [25], and Cowper's approach Cowper [22]. Some commonly used values for shear coefficients are $\kappa = 5/6$, $\kappa = \pi^2/12$ etc. for the rectangular beam cross section.

To calculate the Timoshenko's shear coefficient, it is assumed that the cross sectional area is plane. This assumption is correct at low frequencies, but it seems to fail at frequencies higher than the critical frequency. It is expected that the value of the shear coefficient changes above the critical frequency. $\left(\omega_c = \sqrt{\frac{\kappa GA}{\rho I}}\right)$.

In this paper, we will use that the values of the shear coefficient above the critical frequency are different from the value below as suggested in [26]. Thus, it will give a new comparison for Timoshenko's second spectrum branch with the exact plane elasticity solution.

3. THE EXACT PLANE ELASTICITY SOLUTION

Consider a thin plate as given in Fig. 4 the equations of dynamic equilibrium can be written as [4]:

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho\omega^2 = 0 \quad \text{and} \\ &\frac{\partial \sigma_y}{\partial_y} + \frac{\partial \tau_{xy}}{\partial_x} + \rho\omega^2 = 0, \end{aligned} \tag{33}$$

where the stresses and the displacements are given by

$$\sigma_x = \frac{2G}{(1-\nu)}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right), \quad \sigma_y = \frac{2G}{(1-\nu)}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right), \quad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{34a}$$

$$u = \sum_i u_i \sinh m_i y \sin(\lambda x + \alpha)\sin(\omega t + \varepsilon), \quad v = \sum_i v_i \cosh m_i y \cos(\lambda x + \alpha)\sin(\omega t + \varepsilon). \tag{34b}$$

$$\begin{bmatrix} (2-\nu)\cosh\sqrt{1-\gamma}k\frac{h}{2} & 2\cosh\sqrt{1-\frac{\gamma}{2}(1-\nu)}k\frac{h}{2} \\ (2-2\gamma)\frac{1}{\sqrt{1-\gamma}k\frac{h}{2}}\sinh\sqrt{1-\gamma}k\frac{h}{2} & (2-\gamma)\frac{1}{\sqrt{1-\frac{\gamma}{2}(1-\nu)}k\frac{h}{2}}\sinh\sqrt{1-\frac{\gamma}{2}(1-\nu)}k\frac{h}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{35}$$

The determinant of the matrix in Eq. (35) must be zero for non-trivial solution. This gives a relationship between $k\frac{h}{2}$ and γ.

4. NUMERICAL RESULTS

Variation of dimensionless parameter (γ) with dimensionless wavelength (π/kh) is presented in Fig. 5 for different theories. The EB dispersion curve is denoted by EB, Reddy dispersion curves are denoted by R1 and R2, exact plane elasticity solutions are denoted by (1),(2),(3), TBT1 and TBT2 denote the first spectrum and second spectrum of Timoshenko model, respectively. $\kappa = 0.83$ is used for TBT unless otherwise stated.

It is clearly seen that EB, TBT1 and R1 branches converge to the first exact plane elasticity solution (1) for high dimensionless wavelength. R1 agrees well with the first plane elasticity solution (1) when $\pi/kh > 1$. Moreover, with decreasing the dimensionless wavelength (π/kh) difference between R1 and (1) increases very rapidly. TBT2 and R2 branches agree well with the second plane elasticity solution (2) for high dimensionless wavelength, especially for $\pi/kh > 3$. According to the author's best knowledge, the second spectrum of the Reddy beam theory has not been considered in the previous studies.

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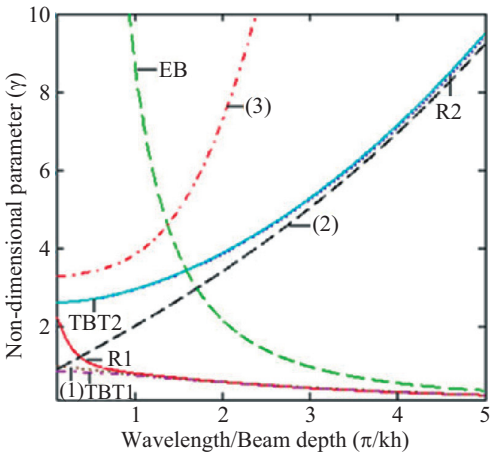


Fig. 5 Dispersion curves of Euler-Bernoulli (EB), Timoshenko (TBT1,TBT2) and Reddy (R1,R2) beam models comparing with exact plane elasticity solution (1,2,3) for $v = 0.3$.

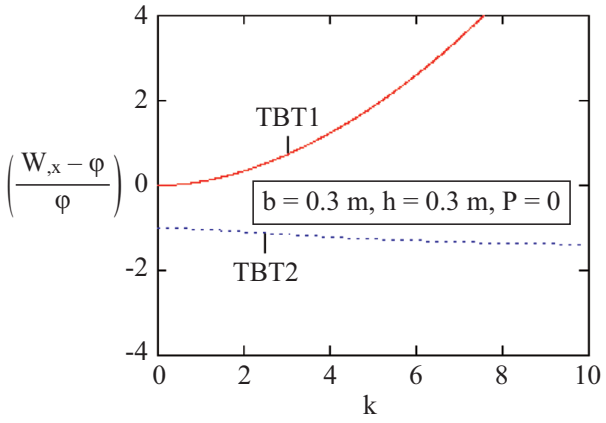


Fig. 6 Variation of $(W_{,x} - \varphi)/(\varphi)$ with wavenumber (k) for TBT. (TBT1: the first branch in TBT, TBT2: the second branch in TBT).

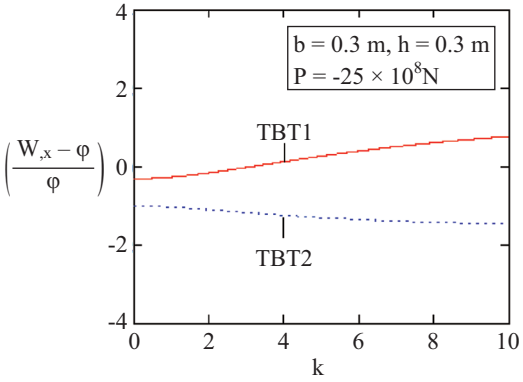


Fig. 7 Variation of $(W_{,x} - \varphi)/(\varphi)$ with wavenumber (k) for TBT. (TBT1: the first branch in TBT, TBT2: the second branch in TBT).

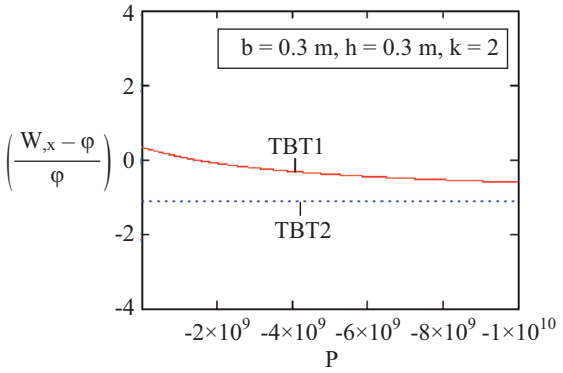


Fig. 8 Variation of $(W_{,x} - \varphi)/(\varphi)$ with in-plane load (P) for TBT. (TBT1: the first branch in TBT, TBT2: the second branch in TBT).

In order to see the effect of the in-plane loads initially RAR-T is plotted for $P = 0$ and depicted in Fig. 6. It is clearly seen that RAR-T is in-phase for the first frequency branch; on the contrary, it is anti-phase for the second branch. When we consider the effect of in-plane loads, the second frequency branch of the RAR-T takes always negative values, but the first frequency branch goes from negative values to positive values for different in-plane loads and wave numbers. This means that the signs of angles $(W_{,x} - \varphi)$ and (φ) are the same for the second branch and opposite for the first branch for some load values. This is illustrated in Fig. 7. In the second branch $(W_{,x} - \varphi)/(\varphi)$ ratio remains close to -1 for short waves, whereas this ratio approaches 1 in the first branch. Moreover, the variation of angle ratio with the in-plane load is shown in Figs. 8-9 for different wave numbers. The first spectrum is in-phase and anti-phase depending on in-plane load and the second spectrum is anti-phase for TBT.

In contrast to the TBT, in which the cross section remains flat and therefore is characterized with a single shear angle (constant shear angle), in Reddy theory cross sections are non-planar, the shear angle changes along

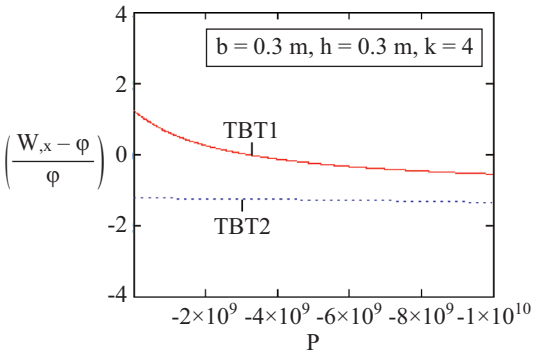


Fig. 9 Variation of $(W_{,x} - \varphi)/(\varphi)$ with in-plane load (P) for TBT. (TBT1: the first branch in TBT, TBT2: the second branch in TBT).

the beam height. Therefore, it is not easy to deduce a general conclusion for the ratio (rotation due to shear to total rotation of the cross section) for the Reddy beam theory (Figs. 10-17). Ratio is obtained for different locations along the thickness ($z = 2h/5$, $z = h/3$, $z = h/8$). When z is chosen close the mid-plane of the beam, R1

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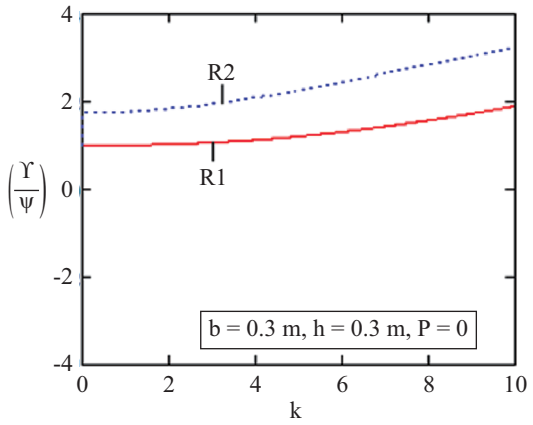


Fig. 10 Variation of (ϒ/ψ) with wavenumber (k) for Reddy beam theory. (R1: the first branch in Reddy beam theory, R2: the second branch in Reddy beam theory, $z = h/8$).

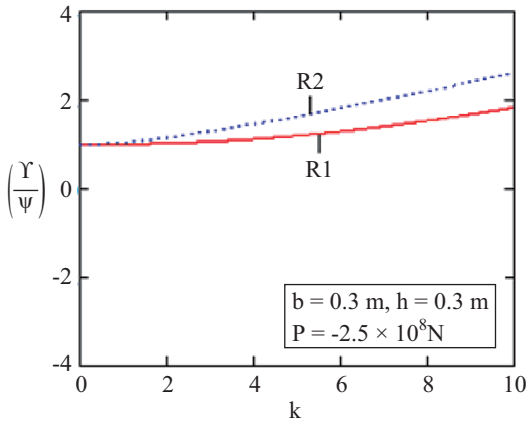


Fig. 11 Variation of (ϒ/ψ) with wavenumber (k) for Reddy beam theory. (R1: the first branch in Reddy beam theory, R2: the second branch in Reddy beam theory, $z = h/8$).

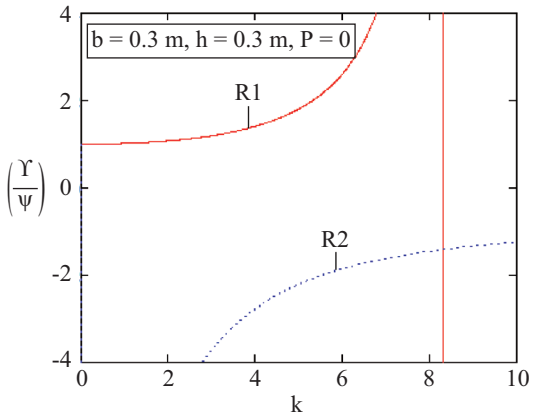


Fig. 12 Variation of (ϒ/ψ) with wavenumber (k) for Reddy beam theory. (R1: the first branch in Reddy beam theory, R2: the second branch in Reddy beam theory, $z = 2h/5$).

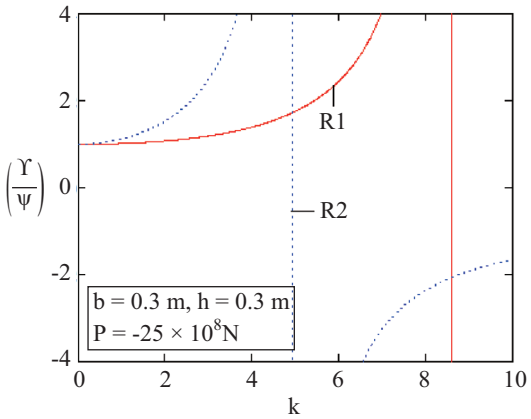


Fig. 13 Variation of (ϒ/ψ) with wavenumber (k) for Reddy beam theory. (R1: the first branch in Reddy beam theory, R2: the second branch in Reddy beam theory, $z = 2h/5$).

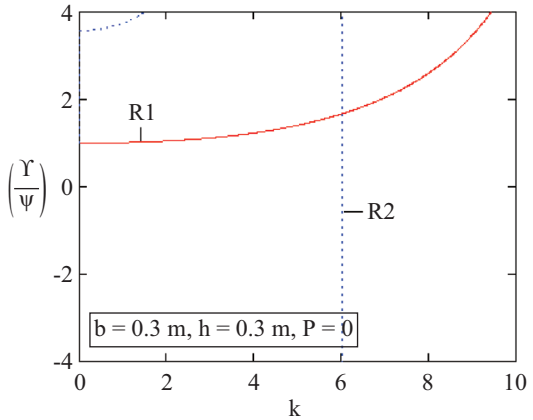


Fig. 14 Variation of (ϒ/ψ) with wavenumber (k) for Reddy beam theory. (R1: the first branch in Reddy beam theory, R2: the second branch in Reddy beam theory, $z = h/3$).

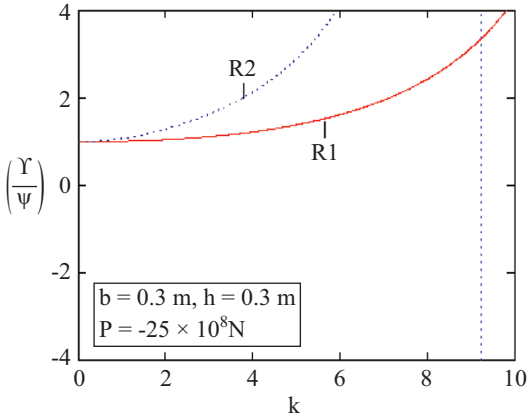


Fig. 15 Variation of (ϒ/ψ) with wavenumber (k) for Reddy beam theory. (R1: the first branch in Reddy beam theory, R2: the second branch in Reddy beam theory, $z = h/3$).

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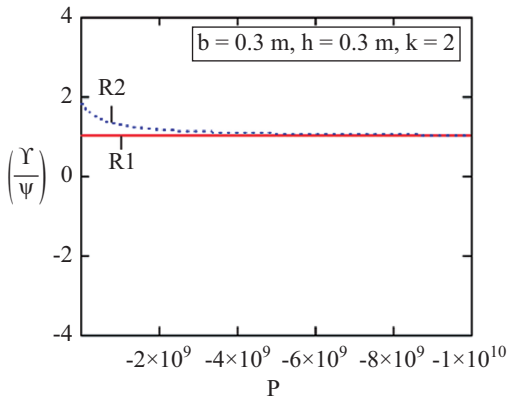


Fig. 16 Variation of (ϒ/ψ) with in-plane load (P) for Reddy beam theory. (R1: the first branch in Reddy beam theory, R2: the second branch in Reddy beam theory, $z = h/8$).

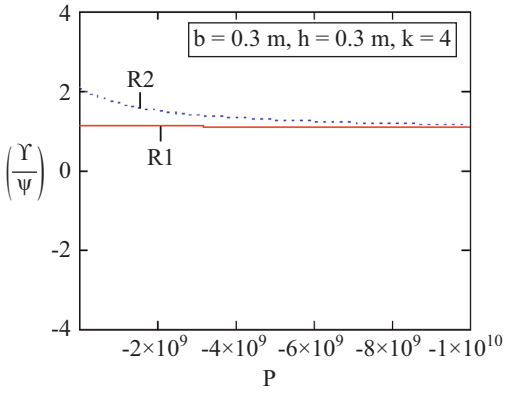


Fig. 17 Variation of (ϒ/ψ) with in-plane load (P) for Reddy beam theory. (R1: the first branch in Reddy beam theory, R2: the second branch in Reddy beam theory, $z = h/8$).

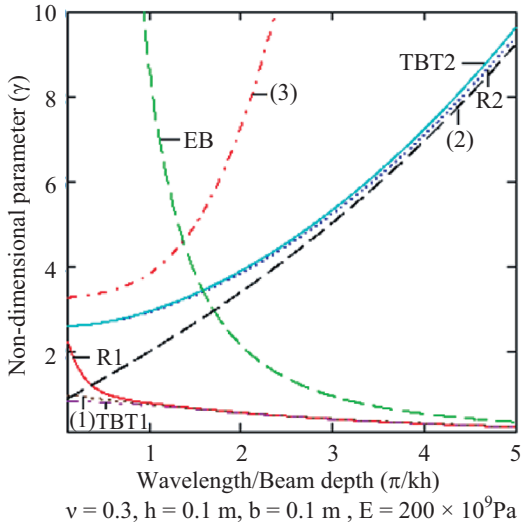


Fig. 18 Dispersion curves of Euler-Bernoulli (EB), Timoshenko (TBT1,TBT2) and Reddy (R1,R2) beam models comparing with exact plane elasticity solution (1,2,3) for $\kappa_1 = 0.833$ and $\kappa_2 = 0.84651$.

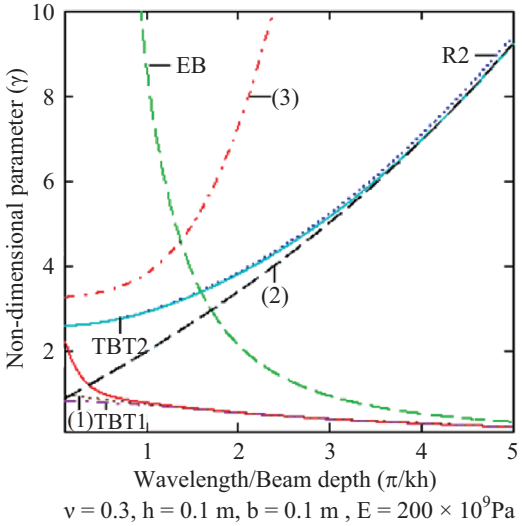


Fig. 19 Dispersion curves of Euler-Bernoulli (EB), Timoshenko (TBT1,TBT2) and Reddy (R1,R2) beam models comparing with exact plane elasticity solution (1,2,3) for $\kappa_1 = 0.833$ and $\kappa_2 = 0.80811$.

and R2 results are not effected qualitatively from in-plane loads (for present chosen load ($P = -25 \times 10^8$ N) and dimensions this point is at $z = h/8$). These effects can be seen in Figs. (10-11) and Figs. (16-17). Phase behavior of angles change near to top and bottom surface of the beam can be seen in Figs. (12-15). Shear deformation effects decrease near the surface of the beams and two frequency branches can be in-phase and/or anti-phase. However, two branches are in-phase when $z > h/3.38$ for $P = 0$. Furthermore, with considering the in-plane load ($P = -25 \times 10^8$ N) two frequency branches are always in-phase when $z > h/3.077$ for the given wave number, material and geometrical properties.

It is seen from the Figs. 18-20 that using different shear coefficient for different branches may improve the result when compared with the exact plane elasticity solution. It is seen that slightly better results are obtained for lower κ_2 coefficients.

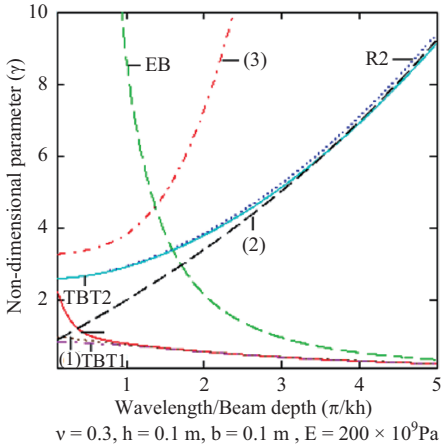


Fig. 20 Dispersion curves of Euler-Bernoulli (EB), Timoshenko (TBT1,TBT2) and Reddy (R1,R2) beam models comparing with exact plane elasticity solution (1,2,3) for $\kappa_1 = 0.833$ and $\kappa_2 = 0.79$.

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5. CONCLUSIONS

In this study, wave propagation analysis of EB, Timoshenko and Reddy beam theories have been studied. Following conclusions can be drawn:

- Similar to TBT there are two spectrums in Reddy beam theory in wave propagation.
- In TBT qualitative difference between the two spectrums has been lost with in-plane loads for some wave numbers. The phase relations of TBT1, TBT2, R1 and R2 branches are examined by considering the in-plane load for TBT and Reddy beam theory. TBT2 branch is in anti-phase (always take negative sign) but TBT1 branch is in anti-phase or in-phase (take positive or negative sign) for different in-plane loads. Therefore, the qualitative difference of the spectrums in TBT is not enough to omit the second spectrum. On the other hand, R1 and R2 branches results are position dependent along the thickness direction.
- TBT2 and R2 branches are in consistent agreement with the second plane elasticity solution (2) at long wavelengths.
- Dispersion curves of TBT are compared with the exact plane elasticity solution for using the different shear coefficients. The second plane elasticity solution (2) agrees well with the TBT2 branch, for lower $\kappa_2 = 0.80811$ and $\kappa_2 = 0.79$.
- Using different shear correction factor may improve results.

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