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Periodic behavior on trees

Ll. ALSEDÀ†, D. JUHER‡ and P. MUMBRÚ§

† *Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08913 Cerdanyola del Vallès, Barcelona, Spain*
(e-mail: alseda@mat.uab.es)

‡ *Departament d'Informàtica i Matemàtica Aplicada, Universitat de Girona, Lluís Santaló s/n, 17071 Girona, Spain*
(e-mail: juher@ima.udg.es)

§ *Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain*
(e-mail: mumbru@mat.ub.es)

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Abstract. We characterize the set of periods for tree maps. More precisely, we prove that the set of periods of any tree map $f : T \to T$ is the union of finitely many initial segments of Baldwin's orderings ${}_p\geq$ and a finite set $\mathcal{F}$. The possible values of p and explicit upper bounds for the size of $\mathcal{F}$ are given in terms of the combinatorial properties of the tree T. Conversely, given any set $\mathcal{A}$ which is a union of finitely many initial segments of Baldwin's orderings ${}_p\geq$ with p of the above type and a finite set, we prove that there exists a tree map whose set of periods is $\mathcal{A}$.

1. *Introduction and statement of the main results*

In this paper we deal with the problem of determining the set of periods of all periodic orbits of a continuous map from a tree into itself. The widely known Sharkovskii's theorem (see [**24**]) which studies the set of periods of any continuous map defined on a closed interval was the first remarkable result in this setting. In order to state it, we introduce the Sharkovskii ordering $\trianglerighteq$ in the set $\mathbb{N} \cup \{2^\infty\}$:

$$3 \trianglerighteq 5 \trianglerighteq 7 \trianglerighteq \cdots \trianglerighteq 2\cdot 3 \trianglerighteq 2\cdot 5 \trianglerighteq 2\cdot 7 \trianglerighteq \cdots \trianglerighteq 4\cdot 3 \trianglerighteq 4\cdot 5 \trianglerighteq 4\cdot 7 \trianglerighteq \cdots \trianglerighteq \cdots \trianglerighteq 2^n\cdot 3 \trianglerighteq 2^n\cdot 5 \trianglerighteq 2^n\cdot 7 \trianglerighteq \cdots \trianglerighteq 2^\infty \trianglerighteq \cdots \trianglerighteq 2^n \trianglerighteq \cdots \trianglerighteq 16 \trianglerighteq 8 \trianglerighteq 4 \trianglerighteq 2 \trianglerighteq 1.$$

The Sharkovskii's theorem states that for each continuous map f from a closed interval into itself there exists some $n \in \mathbb{N}\cup\{2^\infty\}$ satisfying that the set of periods of f is the set of integers k such that $n \trianglerighteq k$. Conversely, given any $n \in \mathbb{N} \cup \{2^\infty\}$ there exists a continuous map defined on a closed interval whose set of periods is the set of all integers k such that $n \trianglerighteq k$.

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During the last three decades there have been several attempts to generalize Sharkovskii's theorem to one-dimensional spaces other than the interval (see, for instance, [**5**] about maps on Y, [**11**] for maps on the n-star, or [**15**] and [**14**] for maps on the circle; for further references on related works, see for example [**6**]).

An important step towards the characterization of the set of periods for maps on trees is Baldwin's theorem (see [**11**]), which describes the set of periods of any continuous map defined on a r-star (a tree with r edges and r endpoints) in terms of finitely many partial orderings. To state it, let us define the Baldwin partial orderings ${}_p\trianglerighteq$ for all $p \in \mathbb{N}$. If $p = 1$, then ${}_p\trianglerighteq$ is the Sharkovskii ordering. For $p > 1$ and $k, m \in \mathbb{N} \cup \{p2^\infty\}$, we write $m \ {}_p\trianglerighteq k$ if one of the following cases holds:

(i) $k = 1$ or m;
(ii) $k, m \in p\mathbb{N} \cup \{p2^\infty\}$ and $m/p \rhd k/p$;
(iii) $k \in p\mathbb{N} \cup \{p2^\infty\}$ and $m \notin \{1\} \cup p\mathbb{N} \cup \{p2^\infty\}$;
(iv) $k, m \notin \{1\} \cup p\mathbb{N} \cup \{p2^\infty\}$ and $k = im + jp$ with $i, j \in \mathbb{N}$;

where the arithmetic rule $p2^\infty/p = 2^\infty$ is assumed and $p\mathbb{N}$ stands for $\{pn : n \in \mathbb{N}\}$. It is not difficult to see that ${}_2\trianglerighteq$ is equivalent to the Sharkovskii ordering.

It may be useful to visualize ${}_p\trianglerighteq$ as a diagram where 1 is the least element, above it there are all the multiples of p arranged in the order induced by Sharkovskii ordering, and above them there are all the non-multiples of p separated into $p - 1$ branches. Each of these branches contains the integers (except 1) congruent with $j \in \{1, 2, \ldots, p\}$ modulo p, and is arranged in reverse ordering with respect to the usual ordering on the integers (see Figure 1 for an example where $p = 4$).

An *initial segment* of a given ordering $\trianglerighteq$ is a set $\mathcal{I}$ such that if $m \in \mathcal{I}$ and $m \trianglerighteq k$ then $k \in \mathcal{I}$.

Baldwin's theorem asserts that the set of periods of any continuous map from a r-star into itself is a union of finitely many initial segments of the orderings ${}_p\trianglerighteq$ for $1 \le p \le r$. Conversely, given such a union $\mathcal{A}$ there is a continuous map defined on a r-star whose set of periods is $\mathcal{A}$.

In the literature one can also find some partial attempts to characterize the set of periods for maps on trees (see, for instance, [**3, 10, 12, 17, 21**]). Additionally, there exists an unpublished preliminary report by Baldwin (Auburn University, 1992) stating (without proof) such a characterization.

In this paper we obtain a characterization of the set of periods of maps on trees in terms of the combinatorial structure of the tree under consideration (number and arrangement of vertices, edges and endpoints of T). More precisely, we prove that there exists a finite set of sequences $\underline{\mathtt{s}} = (p_1, p_2, \ldots, p_m)$ of positive integers such that the set of periods of f is (up to an explicitly bounded finite set) a finite union of sets of the form

$$\{p_1, p_1p_2, \ldots, p_1p_2 \cdots p_{m-1}\} \cup (\mathcal{I}_{\underline{\mathtt{s}}} \setminus p_1p_2 \cdots p_m\{2, 3, \ldots, \lambda_{\underline{\mathtt{s}}}\}),$$

where $\lambda_{\underline{\mathtt{s}}}$ is a non-negative integer and $\mathcal{I}_{\underline{\mathtt{s}}}$ is an initial segment of the Baldwin ordering ${}_{p_1p_2\cdots p_m}\trianglerighteq$. Conversely, given any finite union $\mathcal{A}$ of sets of the above form there exists a continuous map from a tree into itself whose set of periods is $\mathcal{A}$. To state more precisely the main results of our paper (Theorems 1.1 and 1.2), we introduce some notation and basic definitions.

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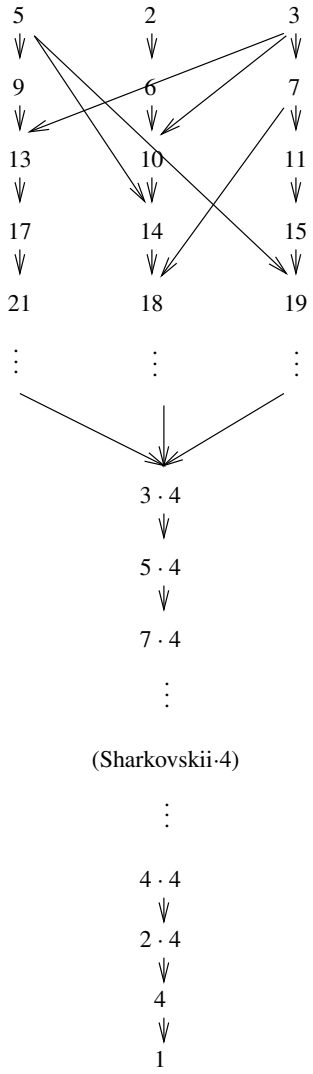


FIGURE 1. The ${}_4\geq$ ordering. The symbol $\rightarrow$ stands for ${}_4>$.

Given a map $f\colon X \to X$, the set of fixed points of f is denoted by $\mathrm{Fix}(f)$. The *set of periods of* f, denoted by $\mathrm{Per}(f)$, is the set of periods of all periodic orbits of f. A *tree* is a compact uniquely arcwise connected space which is a point or a union of a finite number of intervals (by an *interval* we mean any space homeomorphic to $[0, 1]$). Any continuous map from a tree into itself is called a *tree map*. If S is a tree and $x \in S$, we define the *valence* of x to be the number of connected components of $S \setminus \{x\}$. Each point of valence 1 is called an *endpoint* of S and the set of such points are denoted by $\mathrm{En}(S)$. A point of valence different from 2 is called a *vertex* of S and the set of vertices of S is denoted by $V(S)$. The closure of each connected component of $S \setminus V(S)$ is called an *edge* of S.

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Any tree which is a union of $r > 1$ intervals whose intersection is a unique point x of valence r is called a *r-star*, and x is called its *central point*.

Given any subset X of a topological space, we denote by $\mathrm{Int}(X)$ and $\mathrm{Cl}(X)$ the interior and the closure of X, respectively. For a finite set A we denote its cardinality by $|A|$. If $B \subset \mathbb{N}$ and $n \in \mathbb{N}$, nB stands for $\{nk : k \in B\}$. We take $\{1, 2, \ldots, n\}$ as the representatives of the classes of $\mathbb{Z}/n\mathbb{Z}$.

To state Theorem 1.1 we mainly use the Baldwin's orderings ${}_p\geq$ defined above. Let us introduce some more notation related to these orderings. Let $p \in \mathbb{N}$. For each $1 \leq j < p$, the set $\{n \in \mathbb{N} \setminus \{1\} : n \equiv j \pmod p\}$ is denoted by $\mathbb{N}_p^j$. Let $\mathcal{I}$ be an initial segment of the ${}_p\geq$ ordering. It is easy to see that $\mathcal{I}$ has at most $\max\{1, p-1\}$ ${}_p\geq$-maximal elements. The set of all $1 \leq j < p$ such that $\mathcal{I} \cap \mathbb{N}_p^j \neq \emptyset$ is denoted by $B(\mathcal{I})$. Observe that, in the diagram depicted in Figure 1, $B(\mathcal{I})$ can be thought of as the set of branches of non-multiples of p which intersect $\mathcal{I}$ (each branch is a set $\mathbb{N}_p^j$).

Let S and $\widetilde{S}$ be trees and let $p \geq 2$ be an integer. We write $S \sqsupset p\widetilde{S}$ to denote that S contains a subtree W with p endpoints such that $\widetilde{S}$ is homeomorphic to a connected component of $S \setminus \mathrm{Int}(W)$, and each connected component of $S \setminus \mathrm{Int}(W)$ has at least $|\mathrm{En}(\widetilde{S})|$ endpoints. Observe that, since there are exactly p connected components of $S \setminus \mathrm{Int}(W)$ and each of them has at most one endpoint which is not an endpoint of S (the common point with W), it follows that $|\mathrm{En}(S)| \geq p(|\mathrm{En}(\widetilde{S})| - 1)$.

We denote by Σ the set of all finite non-empty sequences of positive integers $(p_1, p_2, \ldots, p_m)$ such that $p_i \geq 2$ for $1 \leq i < m$. Given a tree S, we denote by Σ_S the set of all $(p_1, p_2, \ldots, p_m) \in \Sigma$ for which there exists a sequence of trees $(S_1, S_2, \ldots, S_m)$ satisfying the following conditions:

(S1) $S \supset S_1$, $S_i \sqsupset p_i S_{i+1}$ for $1 \leq i < m$ and $|\mathrm{En}(S_m)| \geq p_m$;

(S2) S_i is not a star for $1 \leq i < m$.

In Lemma 7.1 we show that conditions (S1) and (S2) imply the following conditions:

(S3) $m \leq 1 + \log_2(|\mathrm{En}(S)| - 2)$ whenever $m \geq 3$;

(S4) $p_1 \leq E_1 := |\mathrm{En}(S)|$ and $p_i \leq E_i := E_{i-1}/p_{i-1} + 1$ for $2 \leq i \leq m$.

Observe that, by (S3) and (S4), Σ_S is finite for each tree S.

For any finite non-empty sequence of positive integers $\underline{\mathtt{s}} = (p_1, p_2, \ldots, p_m)$ we set $\mathcal{K}_{\underline{\mathtt{s}}} = \{p_1, p_1p_2, \ldots, p_1p_2 \cdots p_{m-1}\}$ (which is empty whenever $m = 1$) and $\lceil \underline{\mathtt{s}} \rceil = p_1p_2 \cdots p_m \in \mathbb{N}$.

Now we are ready to state the main results of this paper.

THEOREM 1.1. *For each tree map $g\colon S \to S$ there exists a finite set $\mathsf{S} \subset \Sigma_S$ such that*

$$\mathrm{Per}(g) = \bigcup_{\underline{\mathtt{s}} \in \mathsf{S}} (\mathcal{K}_{\underline{\mathtt{s}}} \cup \mathcal{F}_{\underline{\mathtt{s}}} \cup (\mathcal{I}_{\underline{\mathtt{s}}} \setminus \lceil \underline{\mathtt{s}} \rceil \{2, 3, \ldots, \lambda_{\underline{\mathtt{s}}}\}))$$

where, for each $\underline{\mathtt{s}} = (p_1, p_2, \ldots, p_m) \in \mathsf{S}$, $\lambda_{\underline{\mathtt{s}}}$ is a non-negative integer (when $\lambda_{\underline{\mathtt{s}}} < 2$ we understand that $\lceil \underline{\mathtt{s}} \rceil \{2, 3, \ldots, \lambda_{\underline{\mathtt{s}}}\} = \emptyset$) and:

(a) *$\mathcal{I}_{\underline{\mathtt{s}}}$ is an initial segment of the ${}_{\lceil \underline{\mathtt{s}} \rceil}\geq$ ordering such that each ${}_{\lceil \underline{\mathtt{s}} \rceil}\geq$-maximal element of $\mathcal{I}_{\underline{\mathtt{s}}}$ belongs to $\{1\} \cup p_1p_2 \cdots p_{m-1}(\mathbb{N} \cup 2^\infty)$;*

(b) *if $\mathcal{I}_{\underline{\mathtt{s}}} \subsetneq \{1\} \cup \lceil \underline{\mathtt{s}} \rceil \mathbb{N}$, then $\lambda_{\underline{\mathtt{s}}} = 0$ and $\mathcal{F}_{\underline{\mathtt{s}}} = \emptyset$;*

(c) *$\mathcal{F}_{\underline{\mathtt{s}}}$ is disjoint from $\mathcal{K}_{\underline{\mathtt{s}}} \cup \mathcal{I}_{\underline{\mathtt{s}}} \setminus \lceil \underline{\mathtt{s}} \rceil \{2, 3, \ldots, \lambda_{\underline{\mathtt{s}}}\}$, and $\mathcal{F}_{\underline{\mathtt{s}}} \setminus \lceil \underline{\mathtt{s}} \rceil \mathbb{N} \subset \bigcup_{j \in B(\mathcal{I}_{\underline{\mathtt{s}}})} \mathbb{N}_{\lceil \underline{\mathtt{s}} \rceil}^j$;*

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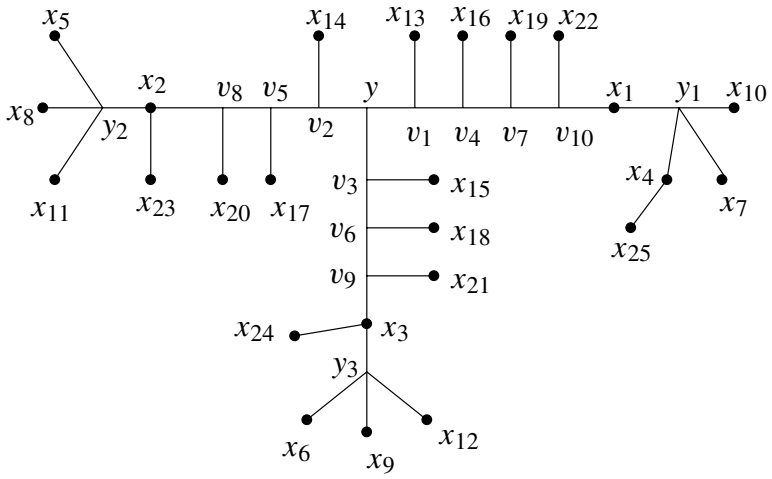


FIGURE 2. A piecewise monotone tree map $g\colon S \to S$ with $g(x_i) = x_{i+1 \bmod 25}$ for $1 \le i \le 25$, $g(y) = y$, $g(v_i) = v_{i+1}$ for $1 \le i \le 9$, $g(v_{10}) = x_2$ and $g(y_i) = y_{i+1 \bmod 3}$ for $1 \le i \le 3$.

(d) *$\mathcal{F}_{\underline{\mathsf{s}}}$ is finite (or empty); when $\mathcal{F}_{\underline{\mathsf{s}}} \neq \emptyset$, we have* $\min \mathcal{F}_{\underline{\mathsf{s}}} \ge \lambda_{\underline{\mathsf{s}}}\lceil \underline{\mathsf{s}} \rceil/2$ *and* $|\mathcal{F}_{\underline{\mathsf{s}}}| < |B(\mathcal{I}_{\underline{\mathsf{s}}})\|\mathrm{En}(S)| + (\lambda_{\underline{\mathsf{s}}} + 2)/2$.

Theorem 1.1 tells us that (up to an explicitly bounded finite set) the set of periods of any tree map is a finite union of initial segments of Baldwin's orderings ${}_p\!\ge$, and that the possible values of p are given in terms of the set of sequences Σ_S. This set, by definition, depends entirely on the combinatorial properties of the tree S. Let us see some examples (we use the above notation).

To illustrate Theorem 1.1 we start by calculating the set of periods of a particular tree map. Assume that S is the tree shown in Figure 2. Consider a map g acting over the set of vertices of S as follows: $g(x_i) = x_{i+1 \bmod 25}$ for $1 \le i \le 25$; $g(y) = y$, $g(v_i) = v_{i+1}$ for $1 \le i \le 9$; $g(v_{10}) = x_2$ and $g(y_i) = y_{i+1 \bmod 3}$ for $1 \le i \le 3$. Consider an extension of g to the whole S being monotone on each maximal interval with no points of $V(S) \cup \{x_i\}_{i=1}^{25}$ in its interior. By using standard techniques (Markov loops) it can be checked that $\mathrm{Per}(g) = \{25, 28, 31, 62, 65, 68\} \cup \{n \in \mathbb{N}\colon 34\ {}_3\!\ge\ n\} \setminus \{6, 9\}$. Hence, we obtain the statement of Theorem 1.1 for this example by taking $\underline{\mathsf{s}} = (3)$, $\mathsf{S} = \{\underline{\mathsf{s}}\}$, $\mathcal{F}_{\underline{\mathsf{s}}} = \{25, 28, 31, 62, 65, 68\}$, $\mathcal{I}_{\underline{\mathsf{s}}} = \{n \in \mathbb{N}\colon 34\ {}_3\!\ge\ n\}$ and $\lambda_{\underline{\mathsf{s}}} = 3$.

Let us compare Theorem 1.1 with Baldwin's theorem (stated above) when S is a n-star for some $n \ge 2$. In this case, $\Sigma_S = \{(1), (2), \ldots, (n)\}$. Indeed, let $\underline{\mathsf{s}} = (p_1, p_2, \ldots, p_m) \in \Sigma_S$. By (S1), S contains a tree S_1 which, by (S2), is not a star when $m > 1$. Hence, $m = 1$ and $\mathcal{K}_{\underline{\mathsf{s}}} = \emptyset$ for each $\underline{\mathsf{s}} \in \Sigma_S$. On the other hand, $p_1 \le n$ by (S4). Therefore, by Theorem 1.1, the set of periods of any continuous map from a n-star into itself is

$$\bigcup_{1 \le p \le n} (\mathcal{F}_p \cup (\mathcal{I}_p \setminus p\{2, 3, \ldots, \lambda_p\}))$$

where $\mathcal{I}_p$ is an initial segment of the ${}_p\!\ge$ ordering and $\mathcal{F}_p$ and λ_p satisfy Theorem 1.1(b)–(d). Although this statement differs (in a finite set) from that of Baldwin's

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theorem, from the proof of Theorem 1.1, one in fact gets that $\lambda_p = 1$ and $\mathcal{F}_p = \emptyset$ for all $1 \leq p \leq n$. This is due to the very special fact that if a star map g has a k-periodic orbit of *type* p (see [**11**] for a definition) with $k \notin p\mathbb{N}$, then g^p exhibits a *horseshoe* and thus $\mathrm{Per}(g) \supset p\mathbb{N}$. This is not true for a general tree map.

Let us see one more example illustrating the strong dependence between the set of sequences Σ_S and the combinatorial structure of the tree. Assume that S is the tree obtained as a union of a k-star T and k-many disjoint n-stars (with $k, n \geq 3$), each attached to T by identifying one of its endpoints with an endpoint of T. Observe that $|\mathrm{En}(S)| = k(n-1)$. In this case, by using (S1)–(S4), it is not difficult to check that

$$\Sigma_S = \{(p) : 1 \leq p \leq k(n-1)\} \cup \{(p_1, p_2) : 2 \leq p_1 \leq k, 1 \leq p_2 \leq n\} \\ \cup \{(p_1, p_2) : k < p_1 \leq k(n-1), 1 \leq p_2 \leq 2\}.$$

Thus, by Theorem 1.1 the set of periods of a self-map of S is (up to a finite set) a union of initial segments of the orderings ${}_p\geq$, for $1 \leq p \leq 2k(n-1) = 2|\mathrm{En}(S)|$.

The following natural question about the sharpness of Theorem 1.1 arises: for any set of integers $\mathcal{A}$ with the form given by the statement of the theorem, do there exist a tree S and a map defined on S whose set of periods coincides with $\mathcal{A}$? In this sense, the following converse of Theorem 1.1 holds.

THEOREM 1.2. *Given a finite set* $\mathsf{S} \subset \Sigma$*, a family* $\{\mathcal{F}_{\underline{\mathsf{s}}}, \mathcal{I}_{\underline{\mathsf{s}}}, \lambda_{\underline{\mathsf{s}}}\}_{\underline{\mathsf{s}} \in \mathsf{S}}$ *and* $N \in \mathbb{N}$ *such that, for each* $\underline{\mathsf{s}} \in \mathsf{S}$*, properties (a)–(d) of the statement of Theorem 1.1 hold with* $|\mathrm{En}(S)|$ *replaced by* N*, there exist a tree* S *and a tree map* $g\colon S \to S$ *such that* $|\mathrm{En}(S)| \geq N$*,* $\mathsf{S} \subset \Sigma_S$ *and*

$$\mathrm{Per}(g) = \bigcup_{\underline{\mathsf{s}} \in \mathsf{S}} (\mathcal{K}_{\underline{\mathsf{s}}} \cup \mathcal{F}_{\underline{\mathsf{s}}} \cup (\mathcal{I}_{\underline{\mathsf{s}}} \setminus \lceil \underline{\mathsf{s}} \rceil \{2, 3, \ldots, \lambda_{\underline{\mathsf{s}}}\})).$$

Remark 1.1. Theorem 1.2 determines a tree S which is *compatible* with a given family $\{\mathcal{F}_{\underline{\mathsf{s}}}, \mathcal{I}_{\underline{\mathsf{s}}}, \lambda_{\underline{\mathsf{s}}}\}_{\underline{\mathsf{s}} \in \mathsf{S}}$ according to the following definition:

(i) property (d) of Theorem 1.1 holds;

(ii) $\mathsf{S} \subset \Sigma_S$.

It is easy to see that any tree containing a subtree homeomorphic to S is also compatible with $\{\mathcal{F}_{\underline{\mathsf{s}}}, \mathcal{I}_{\underline{\mathsf{s}}}, \lambda_{\underline{\mathsf{s}}}\}_{\underline{\mathsf{s}} \in \mathsf{S}}$. The natural question of determining, for a given family $\{\mathcal{F}_{\underline{\mathsf{s}}}, \mathcal{I}_{\underline{\mathsf{s}}}, \lambda_{\underline{\mathsf{s}}}\}_{\underline{\mathsf{s}} \in \mathsf{S}}$, the minimum number N^* of endpoints of a compatible tree is a difficult open problem. From the proof of Theorem 1.2 one can easily obtain an upper bound for N^*. However, this estimate has not been made explicit in the statement for simplicity, since this is not a central issue in the characterization of the set of periods for tree maps. □

Theorems 1.1 and 1.2 give a characterization of the *tree-realizable sets of periods*, in the sense that they describe which sort of subsets of $\mathbb{N}$ can be sets of periods of tree maps. This goes in the direction of Theorem 3.1 of [**16**], which states that a subset of $\mathbb{N}$ is a graph-realizable set of periods if and only if it coincides, up to a finite set, with a finite union of sets of the form $l\mathbb{N}$ and $\{2^n k : n \geq 0\}$. An analogous result for tree maps can be found in [**10**, Theorem E]. However, both characterizations depend on a finite set which is not described.

This paper is organized as follows. In §2 we recall the notions and results from [**2**] and [**4**] concerning the minimality properties of monotone models, which are strongly used

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in the rest of the paper. In §3 we discuss our approach to the problem and propose a strategy for proving Theorem 1.1 consisting of four consecutive steps, which we perform respectively in §§4, 5, 6 and 7. Additionally, in §7 we prove Theorem 1.1. Finally in §8 we prove Theorem 1.2.

2. *Monotone models*

The aim of this section is to recall the minimality properties of the dynamics of monotone models studied in [**2**] and [**4**]. We start by introducing some basic definitions which are used throughout this paper.

A triplet (T, A, f) is called a *model* if $f: T \to T$ is a tree map and $A \subset T$ is a finite f-invariant set. In particular, if A is a periodic orbit of f then (T, A, f) is called a *periodic model*. Given $P \subset T$ we define the *convex hull* of P, denoted by $\langle P\rangle_T$ or simply by $\langle P\rangle$, as the smallest closed connected subset of T containing P. When $P = \{x, y\}$, we write $\langle x, y\rangle$ or $[x, y]$ to denote $\langle P\rangle$. The notations (x, y), $(x, y]$ and $[x, y)$ are understood in the natural way.

Let T be a tree and let $A \subset T$ be a finite subset of T. The pair (T, A) is called a *pointed tree*. A set $Q \subset A$ is said to be a *discrete component* of (T, A) if either $|Q| > 1$ and there is a connected component C of $T \setminus A$ such that $Q = \mathrm{Cl}(C) \cap A$, or $|Q| = 1$ and $Q = A$. We say that two pointed trees (T, A) and (T', A') are *equivalent* if there exists a bijection $\phi: A \to A'$ which preserves discrete components. The equivalence class of a pointed tree (T, A) is denoted by $[T, A]$.

Now we are ready to introduce the notion of *pattern*, that is crucial in the study of the set of periods of tree maps. Let (T, A) and (T', A') be equivalent pointed trees, and let $\theta: A \to A$ and $\theta': A' \to A'$ be maps. We say that θ and θ' are *equivalent* if $\theta' = \varphi \circ \theta \circ \varphi^{-1}$ for a bijection $\varphi: A \to A'$ which preserves discrete components. The equivalence class of θ by this relation is denoted by $[\theta]$. If $[T, A]$ is an equivalence class of pointed trees and $[\theta]$ is an equivalence class of maps then the pair $([T, A], [\theta])$ is called a *pattern*. We say that a model (T, A, f) *exhibits* a pattern $(\mathcal{T}, \Theta)$ if $\mathcal{T} = [T, A]$ and $\Theta = [f|_A]$. This pattern is denoted by $[T, A, f]$.

The simplest models exhibiting a given pattern are the monotone ones, according to the following definition. Let $f: T \to T$ be a tree map. Given $a, b \in T$ we say that $f|_{[a,b]}$ is *monotone* if either $f([a, b])$ is a point or it is an interval and, given two homeomorphisms $\phi: [0, 1] \to [a, b]$ and $\varphi: g([a, b]) \to [0, 1]$, then $\varphi \circ f \circ \phi : [0, 1] \to [0, 1]$ is monotone (as a real function). If $P \subset T$ is a finite f-invariant set such that $\mathrm{En}(T) \subset P$, we say that f is *P-monotone* if $f([a, b]) = [f(a), f(b)]$ and $f|_{[a,b]}$ is monotone whenever $[a, b] \cap P = \{a, b\}$. In this case we say that the model (T, P, f) is *monotone*.

Remark 2.1. When (T, P, f) is a monotone model, it is shown in Proposition 4.2 of [**2**] that the image of each vertex z is uniquely determined and is either a vertex or belongs to P. In fact, if we take $a, b, c \in P$ in such a way that $z \in [a, b] \cap [a, c] \cap [b, c]$ and $\langle\{a, b, c\}\rangle_T \setminus P$ is connected, then it can be easily seen that $f(z)$ is the only point contained in $f([a, b]) \cap f([a, c]) \cap f([b, c])$.

The topological entropy (see [**1**]) is a well-known quantitative measure of the dynamical complexity of a model. It is an important topological invariant which is defined for

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continuous maps on compact metric spaces. The next theorem tells us that each pattern has a minimal model both from the point of view of its combinatorial simplicity (monotonicity) and its dynamical complexity (minimization of topological entropy).

THEOREM 2.1. (Theorem A of [**2**]) *Let* $(\mathcal{T}, \Theta)$ *be a pattern. Then the following statements hold.*

(a) *There exists a monotone model* (T, A, f) *exhibiting the pattern* $(\mathcal{T}, \Theta)$.

(b) *The topological entropy of* f *is the minimum within the class of models which exhibit the pattern* $(\mathcal{T}, \Theta)$.

The dynamics of monotone models is also minimal in a sense different from that given by Theorem 2.1(b). To be more precise, we recall some more notions and results from [**2**].

Let $f: T \to T$ be a tree map and let $x, y \in T$ be fixed points of f^n for some $n \in \mathbb{N}$. We say that x and y are *f-monotone equivalent* if either $x = y$ or $f^n|_{\langle x,y\rangle}$ is monotone. Observe that two f-monotone equivalent points must be both periodic and the number n appearing in the definition can be any common multiple of their periods. Given a model (T, A, f), we say that a periodic point of f is *A-significant* if it is not f-monotone equivalent to any element of $A \cup V(T)$ and its period is minimal within its f-monotone equivalence class.

Remark 2.2. It is easy to see that if x and y are f-monotone equivalent then $f^i(x)$ and $f^i(y)$ are also f-monotone equivalent, for each $i \geq 0$.

Let (T, A, f) be a model exhibiting a pattern $(\mathcal{T}, [\theta])$. Any (unordered) binary subset of a discrete component is called a *basic path* of (T, A). The $(\mathcal{T}, [\theta])$*-path graph* is the oriented graph whose vertices are in one-to-one correspondence with the basic paths of (T, A) and there is an arrow from the vertex corresponding to the basic path π_i to the vertex corresponding to the basic path π_j if and only if $\pi_j \subset \langle\theta(\pi_i)\rangle$. We say that a loop $\pi_0 \to \pi_1 \to \cdots \to \pi_{n-1} \to \pi_0$ of the $(\mathcal{T}, [\theta])$-path graph and a point $x \in T$ are *associated* if $f^i(x) \in \langle\pi_{i \bmod n}\rangle$ for each $i \geq 0$. We recall that a loop is called *simple* if it is not an n-repetition ($n \geq 2$) of any other loop.

Now we are ready to describe in which sense the monotone models have minimal dynamics. This is done in the next two results from [**2**].

THEOREM 2.2. (Theorem C of [**2**]) *Let* (T, A, f) *be a monotone model exhibiting a pattern* $(\mathcal{T}, \Theta)$. *Then the following statements hold.*

(a) *For each A-significant point x of f of period n there exists a unique simple loop β of length n in the $(\mathcal{T}, \Theta)$-path graph such that x and β are associated.*

(b) *Each simple loop β of length n in the $(\mathcal{T}, \Theta)$-path graph is associated either with an A-significant point of f of period n or to a periodic point which is f-monotone equivalent to a point of $A \cup V(T)$ and whose period is a divisor of n. In both cases, the point associated to β is unique up to f-monotone equivalence.*

THEOREM 2.3. (Theorem D of [**2**]) *Let* (T, A, f) *be a model exhibiting a pattern* $(\mathcal{T}, \Theta)$ *and let β be a simple loop of length n of the $(\mathcal{T}, \Theta)$-path graph. Then there exists a fixed point x of f^{2n} such that β and x are associated.*

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Let us see why Theorems 2.2 and 2.3 imply that the dynamics of monotone models is minimal from the point of view of the set of periods. Observe that the $(\mathcal{T}, \Theta)$-path graph is a combinatorial object uniquely associated with the pattern $(\mathcal{T}, \Theta)$. That is, it can be constructed by using only combinatorial data of the pattern, and is independent of the particular choice of the representatives of $\mathcal{T}$ and Θ. By Theorem 2.2, there is essentially a one-to-one correspondence between the loops of the $(\mathcal{T}, \Theta)$-path graph and the significant periodic points of a monotone model exhibiting $(\mathcal{T}, \Theta)$. On the other hand, Theorem 2.3 tells us that each loop of length n in the $(\mathcal{T}, \Theta)$-path graph gives rise to a periodic point (whose period is a divisor of $2n$) on each model exhibiting the pattern $(\mathcal{T}, \Theta)$. Therefore, the set of periods of a monotone model is essentially contained (up to f-monotone equivalence and period-doubling) in the set of periods of each model of the same pattern.

In fact, for patterns of periodic orbits the following result, which is an immediate corollary of the main theorem of [**4**], holds. It improves Theorem 2.3 and is essential in proving Theorem 1.1.

THEOREM 2.4. *Let $g: S \to S$ be a tree map, let P be a periodic orbit of g and let (T, A, f) be a monotone model of the pattern $[S, P, g]$. If f has a significant n-periodic point then $n \in \mathrm{Per}(g)$.*

3. *How to calculate the set of periods of a tree map*

In this section we give a scheme of the strategy (divided into four steps) which we use to calculate the set of periods of a tree map $g: S \to S$. In fact, the approach we propose is an extension of the strategy used in the proofs of Sharkovskii and Baldwin theorems. In §1 of [**3**] one can find a complete and detailed discussion on this matter. In the first step of the strategy, we use the notion of a *twist* model. Roughly speaking, a model (S, P, g) is twist when g exhibits a certain rotational behavior around a fixed point (this notion is defined precisely in §4).

Step 1. Given a periodic orbit P of g, construct a sequence $\{(S_i, P_i, g_i), y_i, p_i\}_{i=1}^{m}$ such that:

(i) $S_1 = \langle P \rangle_S$, $P_1 = P$ and $g_1 = r \circ g$, where $r: S \to \langle P \rangle_S$ is the natural retraction—it is easy to see that $\mathrm{Per}(g_1) \subset \mathrm{Per}(g)$;

(ii) (S_i, P_i, g_i) is a periodic model with $|P_i| = p_i |P_{i+1}|$ for $1 \le i \le m$;

(iii) y_i is a fixed point of g_i and (S_i, P_i, g_i) is twist around y_i for $1 \le i < m$;

(iv) $\mathrm{Per}(g_i) \supset p_i \,\mathrm{Per}(g_{i+1})$ for $1 \le i < m$;

(v) (S_m, P_m, g_m) is non-twist.

Then, $\mathrm{Per}(g) \supset \{1, p_1, p_1 p_2, \ldots, p_1 p_2 \cdots p_{m-1}\} \cup p_1 p_2 \cdots p_{m-1} \,\mathrm{Per}(g_m)$.

Step 2. Consider a monotone model (T, A, f) of the pattern $[S_m, P_m, g_m]$ and estimate a subset Γ_{P_m} of the set of periods of (T, A, f), as large as possible.

Step 3. Prove that Γ_{P_m} is contained in the set of periods of each model having the pattern $[T, A, f] = [S_m, P_m, g_m]$. In particular, $\Gamma_{P_m} \subset \mathrm{Per}(g_m)$. Thus, from Step 1, $\mathrm{Per}(g) \supset \Lambda_P$, where is defined to be

$$\Lambda_P := \{1, p_1, p_1 p_2, \ldots, p_1 p_2 \cdots p_{m-1}\} \cup p_1 p_2 \cdots p_{m-1} \Gamma_{P_m}.$$

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Step 4. Consider each orbit P of g and its associated set Λ_P, and obtain a finite structure of the set $\mathrm{Per}(g)$ by arranging adequately the (perhaps uncountable) union of all sets Λ_P.

To perform Step 2, we use the results of [**3**], a paper by the present authors where the sets of periods of monotone models were studied. It follows from [**3**] that estimating the set of periods of a monotone model is much easier when this model is non-twist. This fact justifies the need of the reduction process proposed in Step 1. Finally, the convenience of taking monotone models and the viability of Step 3 are suggested by the dynamic minimality properties exhibited by this sort of model (see Theorems 2.2, 2.3 and 2.4).

4. *Step 1. A reduction process*

In this section we perform Step 1 of the strategy described in §3. We start by introducing the notion of p-block reduction and studying its properties. This is the key tool of the reduction process.

Let S be a tree and let $\widetilde{S}$ be a subtree of S (note that, for each connected component K of $S \setminus \widetilde{S}$, the set $\mathrm{Cl}(K) \cap \widetilde{S}$ consists of a single point). The *natural retraction* $r\colon S \to \widetilde{S}$ is defined as follows. For each $x \in \widetilde{S}$ we set $r(x) = x$, and for each $x \in S \setminus \widetilde{S}$ we define $r(x)$ to be the unique point of $\mathrm{Cl}(K) \cap \widetilde{S}$, where K is the connected component of $S \setminus \widetilde{S}$ containing x. It is not difficult to see that r is well defined and continuous.

Let (S, P, g) be a periodic model. For $p \geq 2$, we say that (S, P, g) has a *p-block structure* (or simply a *block structure*) if there exists a partition $P = P_1 \cup P_2 \cup \cdots \cup P_p$ such that $\langle P_i \rangle_S \cap \langle P_j \rangle_S = \emptyset$ for $i \neq j$ and $f(P_i) = P_{i+1 \bmod p}$ for $1 \leq i \leq p$. To a model (S, P, g) having a p-block structure we can associate a model $(\widetilde{S}, \widetilde{P}, \widetilde{g})$, which is called a *$p$-block reduction of* (S, P, g), as follows. Take $i \in \{1, 2, \ldots, p\}$ such that $|\mathrm{En}(\langle P_i \rangle_S)| = \min\{|\mathrm{En}(\langle P_j \rangle_S)|\}_{j=1}^{p}$. Then we set $\widetilde{S} = \langle P_i \rangle_S$, $\widetilde{P} = P_i$ and

$$\widetilde{g} = r_1 \circ g|_{K_p} \circ r_p \circ g|_{K_{p-1}} \circ \cdots \circ r_3 \circ g|_{K_2} \circ r_2 \circ g|_{K_1},$$

where, for each $1 \leq j \leq p$, K_j stands for $\langle P_{i-j+1 \bmod p} \rangle_S$ and $r_j\colon S \to K_j$ denotes the natural retraction. Observe that for a fixed partition $P = P_1 \cup P_2 \cup \cdots \cup P_p$ there are possibly several different p-block reductions of (S, P, g), since there can be several choices for i.

In order to unify the notation and simplify the writing, given a monotone model (S, P, g) we say that the model $(\langle P \rangle_S, P, r \circ g)$, where $r\colon S \to \langle P \rangle_S$ is the natural retraction, is a *1-block reduction* of (S, P, g).

In the literature one can find several kinds of block structures and related notions for periodic orbits. In the interval case, Sharkovskii's *square root construction* (see [**24**] or [**6**]) is an earlier example of a block structure. Also the notion of *extension*, which first appeared in [**13**], gives rise to some particular cases of block structures for interval periodic orbits. This notion was subsequently used by several authors to characterize the cyclic permutations exhibiting some kind of minimality property (the terms *simple*, *minimal* and *primary* are used depending on that property: see, for instance, [**19**], [**20**] or [**7**]). Finally, the notion of *division*, introduced in [**23**] for interval periodic orbits and generalized in [**9**], [**8**] and [**10**] to 3-star, n-star and tree maps, respectively, has been used in a number of papers to study the set of periods and the topological entropy.

The following lemma studies some basic properties of a p-block reduction.

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LEMMA 4.1. *Let $(\widetilde{S}, \widetilde{P}, \widetilde{g})$ be a p-block reduction of a model (S, P, g), given by a partition $P = P_1 \cup P_2 \cup \cdots \cup P_p$. The following statements hold.*

(a) $\mathrm{En}(\widetilde{S}) \subset \widetilde{P}$.

(b) *$(\widetilde{S}, \widetilde{P}, \widetilde{g})$ is a periodic model and $|\widetilde{P}| = |P|/p$.*

(c) *Let $\mathcal{Q}$ be the set of periodic points x of g such that $g^i(x) \in \langle P_{i+1 \bmod p}\rangle_S$ for $i \geq 0$. If $x \in \mathcal{Q}$, then x is an n-periodic point of $\widetilde{g}$ and an np-periodic point of g, for some $n \in \mathbb{N}$. Therefore, $\mathrm{Per}(g) = p\,\mathrm{Per}(\widetilde{g}) \cup R$, where R is the set of periods of all periodic points of g which do not belong to $\mathcal{Q}$.*

Proof. By assumption, $\langle P_i\rangle_S \cap \langle P_j\rangle_S = \emptyset$ for $i \neq j$ and $f(P_i) = P_{i+1 \bmod p}$ for $1 \leq i \leq p$. Set $K_i = \langle P_i\rangle_S$. Without loss of generality we can assume that $\widetilde{S} = K_1$ and $\widetilde{P} = P_1$. Statement (a) follows directly from the definition of $\widetilde{S}$ and $\widetilde{P}$. Since P is a periodic orbit and $f(P_i) = P_{i+1 \bmod p}$ for $1 \leq i \leq p$, statement (b) follows from the fact that the sets $\langle P_i\rangle_S$ are pairwise disjoint. Now we prove (c). Let $r_i : S \to K_i$ be the natural retraction from S onto K_i for each $1 \leq i \leq p$ (we recall that $\widetilde{g} = r_1 \circ g|_{K_p} \circ r_p \circ g|_{K_{p-1}} \circ \cdots \circ r_3 \circ g|_{K_2} \circ r_2 \circ g|_{K_1}$). From the definition of a natural retraction and the fact that $\mathrm{En}(K_i) \subset P$ for $1 \leq i \leq p$, it follows that for each $z \in \widetilde{S}$ we have

$$\begin{cases} \mathrm{Orb}_{g^p}(z) = \mathrm{Orb}_{\widetilde{g}}(z) & \text{if } g^i(z) \in K_{i+1 \bmod p} \text{ for } i \geq 0; \text{ or} \\ \widetilde{g}^j(z) \in \widetilde{P} \text{ for some } j \geq 0 & \text{otherwise.} \end{cases} \tag{1}$$

Now let $x \in \mathcal{Q}$. In particular, x is a periodic point of g and $x \in K_1 = \widetilde{S}$. Since $g^i(x) \in K_{i+1 \bmod p}$ for $i \geq 0$ and $K_i \cap K_j = \emptyset$ for $i \neq j$, it follows that $|\mathrm{Orb}_g(x)| = np$ for some $n \in \mathbb{N}$. Moreover, from (1) we get that $\mathrm{Orb}_{g^p}(x) = \mathrm{Orb}_{\widetilde{g}}(x)$. Therefore, x is also a periodic point of $\widetilde{g}$, and $|\mathrm{Orb}_{\widetilde{g}}(x)| = n$. □

Note that when $p = 1$, Lemma 4.1(c) states that $\mathrm{Per}(r \circ g) \subset \mathrm{Per}(g)$, where $r : S \to \langle P\rangle_S$ is the natural retraction. This is well known, even when one replaces $\langle P\rangle_S$ by any subtree of S (Corollary 4.2 of [**11**]).

When we are able to calculate the set of periods of a p-block reduction $\widetilde{g}$, Lemma 4.1(c) gives $p\,\mathrm{Per}(\widetilde{g})$ as an estimate of $\mathrm{Per}(g)$, the set of periods of the non-reduced model (S, P, g). However, this estimate can be very bad. In fact, as the following example shows, the set R of Lemma 4.1(c) can be infinite even when (S, P, g) is monotone. Let $S = [0, 1] \subset \mathbb{R}$ and set $P = \{x_i\}_{i=1}^6$ where $x_1 < x_4 < x_2 < x_5 < x_3 < x_6$. Let g be a P-monotone map satisfying $g(x_i) = x_{i+1 \bmod 6}$ for $1 \leq i \leq 6$ (in the terminology of [**6**], the orbit P is a *2-extension* of the pattern $(1, 2, 3)$). It is not difficult to see that g has a 3-periodic orbit. Thus, from Sharkovskii's theorem it follows that $\mathrm{Per}(g) = \mathbb{N}$. On the other hand, (S, P, g) has a 3-block structure defined by the partition $P = P_1 \cup P_2 \cup P_3$ where $P_1 = \{x_1, x_4\}$, $P_2 = \{x_2, x_5\}$ and $P_3 = \{x_3, x_6\}$. Thus (S, P, g) admits a 3-block reduction $(\widetilde{S}, \widetilde{P}, \widetilde{g})$ with $\widetilde{P} = P_1$ and $\widetilde{S} = [x_1, x_4]$. Moreover $\widetilde{P}$ is a 2-periodic orbit of $\widetilde{g}$ and $\widetilde{g}$ is $\widetilde{P}$-monotone. It follows that $\mathrm{Per}(\widetilde{g}) = \{1, 2\}$ and, hence, $p\,\mathrm{Per}(\widetilde{g}) = \{3, 6\}$.

To overcome the problem pointed out in the above example, we introduce the notion of a *twist orbit*. A twist orbit is a particular class of block structure for which it can be shown that, with the notation of Lemma 4.1, if (S, P, g) is monotone and P is a twist orbit then

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R reduces to $\{1\}$ and thus $\mathrm{Per}(g) = p\,\mathrm{Per}(\widetilde{g}) \cup \{1\}$. The notion of a twist orbit is closely related to the idea of a division (see [**10**]). In fact, the name *twist orbit* was used in that paper to denote a notion that is very similar to that which we propose here.

Let S be a tree. Consider a finite set $Q \subset S$ and a point $y \in S$. The connected component Z of $(S \setminus Q) \cup \{y\}$ containing y is called the *central* (Q, y)*-component*, and the connected components of $S \setminus Z$ are called (Q, y)*-branches* (compare with the notions *central point* and *branches* of a star). Observe that if $Q \setminus \{y\}$ is non-empty then the set of (Q, y)-branches is non-empty.

Let (S, P, g) be a periodic model, let $y \in \langle P \rangle \setminus P$ and let $Z_1, Z_2, \ldots, Z_p$ be the (P, y)-branches. We say that (S, P, g) and P are *twist around* y (or simply *twist*) if either $|P| = 1$ or (S, P, g) has a p-block structure defined by the partition $P = \bigcup_{i=1}^{p} (P \cap Z_i)$. If $|P| > 1$ and there is no point y such that (S, P, g) is twist around y then (S, P, g) and P are called *non-twist*.

We remark that if (S, P, g) is a twist model then P has a division (according to the definition given in [**10**]), but the converse is true only when S is an interval.

The following lemma shows that a twist model always admits a p-block reduction. Hence, in view of Lemma 4.1(c) the calculation of its set of periods can be reduced to the calculation of the (simpler) set of periods of the reduced model. This is what motivates the use of a reduction process of a model performed by a sequence of block reductions, as a first stage in the study of its set of periods. We recall that the notation $S \sqsupset p\widetilde{S}$ (where S and $\widetilde{S}$ are trees and $p \geq 2$ is an integer) has been introduced in §1.

LEMMA 4.2. *Let (S, P, g) be a twist model such that $|P| > 1$ and $\mathrm{En}(S) \subset P$. Then there exist an integer $p > 1$ and a p-block reduction $(\widetilde{S}, \widetilde{P}, \widetilde{g})$ of (S, P, g) such that $S \sqsupset p\widetilde{S}$ (hence, in particular, $|\mathrm{En}(\widetilde{S})| \leq |\mathrm{En}(S)|/p + 1$).*

Proof. Note that $S = \langle P \rangle$ and let $y \in S \setminus P$ be a point such that (S, P, g) is twist around y. Let $Z_1, Z_2, \ldots, Z_p$ be the (P, y)-branches and let W be the closure of the central (P, y)-component. Since $|P| > 1$ and $\mathrm{En}(S) \subset P$, it follows that $p > 1$ and $|\mathrm{En}(W)| = p$.

Since (S, P, g) is twist around y, the partition $\bigcup_{i=1}^{p} (P \cap Z_i)$ defines a block structure for (S, P, g). Thus, there exists a p-block reduction $(\widetilde{S}, \widetilde{P}, \widetilde{g})$ of (S, P, g). In particular, $|\mathrm{En}(\widetilde{S})| = \min\{|\mathrm{En}(Z_i)|\}_{i=1}^{p}$. Hence, $S \sqsupset p\widetilde{S}$ holds since Z_i are the connected components of $S \setminus \mathrm{Int}(W)$, $|\mathrm{En}(W)| = p$ and Z_i has at least $|\mathrm{En}(\widetilde{S})|$ endpoints for $1 \leq i \leq p$. □

Before introducing the notion of a *sequence of block reductions* (which is the central tool of this section), we need a few notions borrowed from [**3**].

Let (T, Q, f) be a model. Take $y \in \mathrm{Fix}(f)$, let Z be the central (Q, y)-component and let $Z_1, Z_2, \ldots, Z_n$ be the (Q, y)-branches. Observe that the set $\mathrm{Cl}(Z) \cap Z_i$ consists of a single point x_i which belongs to Q. Set $X = \{x_i\}_{i=1}^{n}$. We say that Q is y*-typifiable* if $f(X) \cap Z = \emptyset$ (observe that if $y \notin Q$, then Q is y-typifiable, and if $y \in Q$, then Q is y-typifiable if and only if $f(x_i) \neq y$ for $1 \leq i \leq n$). When Q is y-typifiable, it makes sense to define the *type* of Q, a generalization of the notion of *type of a periodic orbit*, first introduced in [**11**] for star maps. Consider the map $\Phi\colon X \to X$ defined by

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$\Phi(x_i) = x_j$ if and only if $f(x_i) \in Z_j$. Observe that Φ is well defined and, since it acts on a finite set, it has periodic orbits. The period of a periodic orbit of Φ is called a *type of* Q (note that the type of a y-typifiable set is not necessarily unique).

Let (T, A, f) be a monotone model. Since f is A-monotone and, by Remark 2.1, $A \cup V(T)$ is f-invariant, it is not difficult to see that f is also $(A \cup V(T))$-monotone. We say that (T, A, f) is *y-expansive* for $y \in \mathrm{Fix}(f) \setminus A$ if $\mathrm{Orb}_f(v)$ is not contained in the central (A, y)-component for every $v \in V(T) \setminus \{y\}$. This property plays an important role in this paper. For y-expansive models, the type of some 'natural' f-invariant sets is well defined: it can be shown (Lemma 4.3 of [**3**]) that if $P \subset T$ is a finite (or empty) f-invariant set such that $y \notin P$ then the sets A, $A \cup \{y\}$, P, $P \cup \{y\}$, $A \cup V(T) \cup P$ and $A \cup V(T) \cup P \cup \{y\}$ are y-typifiable.

In this paper we deal frequently with a particular kind of monotone model (see, for instance, Step 2 of the strategy described in §3, where a monotone model of a given pattern has to be chosen), which is called a *canonical model*. This notion was first introduced in [**2**] to be a special kind of 'minimal' monotone model. Given a monotone model (S, B, g), a canonical model (T, A, f) can be constructed from (S, B, g) by collapsing all maximal (closed) invariant forests which do not intersect B. It easily follows that (T, A, f) is also monotone and the patterns $[T, A, f]$ and $[S, B, g]$ coincide. Thus, from Theorem 2.1(a) it follows that *each pattern admits a canonical model* (Theorem B of [**2**]). Moreover, it can be shown that this canonical model is unique (up to homeomorphisms). If, in addition, A is a periodic orbit, in [**3**] it is shown that there exists $y \in \mathrm{Fix}(f)$ such that (T, A, f) is y-expansive.

In what follows, when saying that (T, A, f) *is a y-expansive canonical model* we mean that (T, A, f) is a canonical model and y is a fixed point of f which has been chosen in such a way that (T, A, f) is y-expansive. We stress the fact that, from now on, all the considered monotone models are, in addition, y-expansive canonical models.

If (T, A, f) is a y-expansive model, then the set $A \cup V(T) \cup \{y\}$ is denoted by A^y (we recall that A^y is f-invariant and y-typifiable).

We are now ready to define in detail the notion of a *sequence of block reductions*, that will allow us to perform the reduction of a twist model to a model whose set of periods can be computed in an easier way. This notion is close to the notions *z-tower* and *snowflake* introduced by Blokh in [**18**].

Let (S, P, g) be a periodic model. A sequence $\{(S_i, P_i, g_i), y_i, p_i\}_{i=1}^m$ is called a *sequence of block reductions* of (S, P, g) if and only if:

(BR1) (S_1, P_1, g_1) is a 1-block reduction of (S, P, g);

(BR2) $(S_{i+1}, P_{i+1}, g_{i+1})$ is a p_i-block reduction of (S_i, P_i, g_i) such that $S_i \supset p_i S_{i+1}$ (thus $|\mathrm{En}(S_{i+1})| \leq |\mathrm{En}(S_i)|/p_i + 1$) and $p_i \geq 2$ for $1 \leq i < m$;

(BR3) P_i is twist around y_i and S_i is not a star (in particular, $3 < |\mathrm{En}(S_i)| \leq |P_i|$) for $1 \leq i < m$;

(BR4) either P_m is non-twist or S_m is a star; in both cases, one of the following two statements holds:

(BR4.1) $|P_m| = 1$, $p_m = 1$ and y_m is the unique element of P_m; or

(BR4.2) (T, A, f) is y_m-expansive and p_m is a type of A^{y_m}, for some canonical model (T, A, f) of the pattern $[S_m, P_m, g_m]$.

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The following result studies the set of periods of a map which admits a sequence of block reductions.

PROPOSITION 4.1. *Let* $\{(S_i, P_i, g_i), y_i, p_i\}_{i=1}^m$ *be a sequence of block reductions of* (S, P, g). *Then* $\mathrm{En}(S_i) \subset P_i$ *and* P_i *is a periodic orbit of* g_i *for each* $1 \leq i \leq m$. *Moreover,* $|P| = p_1 p_2 \cdots p_{m-1} |P_m|$, $\mathrm{Per}(g) \supset \mathrm{Per}(g_1)$ *and* $\mathrm{Per}(g_i) \supset p_i \mathrm{Per}(g_{i+1}) \cup \{1\}$ *for* $1 \leq i < m$. *Thus,* $\mathrm{Per}(g) \supset \{1, p_1, p_1 p_2, \ldots, p_1 p_2 \cdots p_{m-1}\} \cup p_1 p_2 \cdots p_{m-1} \mathrm{Per}(g_m)$.

Proof. It follows immediately from (BR1) and (BR2) and Lemma 4.1. □

The following proposition tells us that the reduction process consisting of a sequence of block reductions is always possible, thus completing Step 1 of the programme described in §3.

PROPOSITION 4.2. *Each periodic model admits a sequence of block reductions.*

Proof. Let (S, P, g) be a periodic model. We formally denote $\{(S_i, P_i, g_i), y_i, p_i\}_{i=1}^k$ by $\mathcal{B}^k$ for any $k \geq 0$ (by convention we set $\mathcal{B}^0 = \emptyset$).

We start by defining (S_1, P_1, g_1) to be a 1-block reduction of (S, P, g). By Lemma 4.1(a), $\mathrm{En}(S_1) \subset P_1$. Moreover, (BR1)–(BR3) trivially hold with 1 instead of m. Now we proceed by induction on k.

Let $k \geq 1$ and assume that we have constructed a sequence $\mathcal{B}^{k-1}$ and a model (S_k, P_k, g_k) such that:

(a) P_k is a periodic orbit with $\mathrm{En}(S_k) \subset P_k$;

(b) (BR1)–(BR3) hold with k instead of m.

If, in addition, there exist y_k and p_k such that (BR4) holds with k instead of m, then $\mathcal{B}^k$ is a sequence of block reductions and we are done. So, we must define y_k and p_k and then decide whether $\mathcal{B}^k$ is a sequence of block reductions (in this case we stop by setting $m = k$), or we construct a model $(S_{k+1}, P_{k+1}, g_{k+1})$ such that $\mathcal{B}^k$ and $(S_{k+1}, P_{k+1}, g_{k+1})$ satisfy (a) and (b) with $k+1$ instead of k and we continue the induction procedure.

When $|P_k| = 1$, we set $p_k = 1$ and define y_k to be the unique element of P_k. Then $\mathcal{B}^k$ satisfies (BR4.1) and thus it is a sequence of block reductions.

When $|P_k| > 1$ we consider two cases.

Case 1. Either P_k *is non-twist or* S_k *is a star.* Let (T, A, f) be a canonical model of $[S_k, P_k, g_k]$. Since A is a periodic orbit by (a), there exists $y_k \in \mathrm{Fix}(f)$ such that (T, A, f) is y_k-expansive. Let p_k be a type of A^{y_k}. Then $\mathcal{B}^k$ satisfies (BR4.2) and thus it is a sequence of block reductions.

Case 2. P_k *is twist and* S_k *is not a star.* Clearly, $|P_k| > 1$. We take y_k so that P_k is twist around y_k. Since (a) holds, by Lemma 4.2 there exist a model $(S_{k+1}, P_{k+1}, g_{k+1})$ and an integer $p_k \geq 2$ such that $(S_{k+1}, P_{k+1}, g_{k+1})$ is a p_k-block reduction of (S_k, P_k, g_k) and $S_k \sqsupset p_k S_{k+1}$. By Lemma 4.1, $\mathrm{En}(S_{k+1}) \subset P_{k+1}$ and P_{k+1} is a periodic orbit of g_{k+1}. Therefore, $\mathcal{B}^k$ and $(S_{k+1}, P_{k+1}, g_{k+1})$ satisfy (a) and (b) with $k+1$ instead of k. Moreover, from Lemma 4.1(b) we have that $|P_{k+1}| = |P_k|/p_k < |P_k|$. Therefore, by the finiteness of P, this iterative construction stops after finitely many steps. □

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5. *Step 2. Computing sets of periods of non-twist canonical models*

The aim of this section is to perform Step 2 of the strategy described in §3 by estimating the set of periods of a canonical model exhibiting the pattern of a given non-twist periodic model. This has essentially been done in [**3**], but here, for completeness, we outline the necessary definitions and the results from that paper which we use (Theorem 5.1). The following lemma ensures that the canonical model of a non-twist model is also non-twist.

LEMMA 5.1. *Let* (S, P, g) *be a periodic model and let* (T, A, f) *be a canonical model of* $[S, P, g]$. *Then* (T, A, f) *is non-twist if and only if* (S, P, g) *is non-twist.*

Proof. Assume that (T, A, f) is twist around a point $y \in T \setminus A$. Since (T, A, f) is a monotone model, $\mathrm{En}(T) \subset A$. Let Z be the central (A, y)-component and let $Z_1, Z_2, \ldots, Z_p$ be the (A, y)-branches. We have $f(A \cap Z_i) = A \cap Z_{i+1 \bmod p}$ for $1 \leq i \leq p$. Note that there exists a discrete component X of (T, A) such that $X = \mathrm{Cl}(Z) \setminus Z$. Moreover, $A \cap Z_i$ is a union of discrete components of (T, A) for $1 \leq i \leq p$. Let $\theta : A \to P$ be a bijection which preserves discrete components and $g|_P = \theta \circ f \circ \theta^{-1}|_P$ (such a bijection exists because (T, A, f) exhibits the pattern $[S, P, g]$). For $1 \leq i \leq p$, set $P_i = \theta(A \cap Z_i) \subset P$ and let W_i be the closure of the connected component of $S \setminus \langle \theta(X) \rangle_S$ which contains P_i. Observe that $W_1, W_2, \ldots, W_p$ are the (P, w)-branches for each point $w \in \mathrm{Int}(\langle \theta(X) \rangle_S)$. We have $g(P \cap W_i) = \theta(f(\theta^{-1}(P \cap W_i))) = \theta(f(\theta^{-1}(P_i))) = \theta(f(A \cap Z_i)) \subset \theta(A \cap Z_{i+1 \bmod p}) = P_{i+1 \bmod p} = P \cap W_{i+1 \bmod p}$ for $1 \leq i \leq p$. Therefore, (S, P, g) is twist around any point from $\mathrm{Int}(\langle \theta(X) \rangle_S)$.

The proof of the 'only if' part of the lemma is analogous. □

Studying the loops of a Markov graph associated to a finite invariant set is the standard way to obtain periodic orbits of a monotone map. Let us introduce the necessary notions to do it. A monotone model (T, Q, f) is called *Markov* if $V(T) \subset Q$. An interval of T is called *Q-basic* if it is the closure of a connected component of $T \setminus Q$. Given $K, L \subset T$, we say that K *f-covers* L, denoted by $K \to L$, if $f(K) \supset L$. In this setting, it makes sense to consider the *(Markov) f-graph of Q*, whose vertices are Q-basic intervals and, if I, J are Q-basic intervals, there is an arrow from I to J if and only if I f-covers J. It is well known that there is a certain correspondence between periodic points of f and *loops* in the f-graph of Q. We use the usual notions (see [**15**] or Chapter 1 of [**6**]): the *concatenation* of two loops α and β is denoted by $\alpha\beta$, and $\alpha^n = \alpha\alpha \ldots \alpha$ (n times) is called an *n-repetition of α*. We recall that a loop is *simple* if it is not an n-repetition of any other loop with $n \geq 2$. The length of a loop α is denoted by $|\alpha|$. If $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$ is a loop α and $x \in \mathrm{Fix}(f^n)$, we say that x (and $\mathrm{Orb}(x)$) and α are *associated* if $f^i(x) \in J_i$ for $0 \leq i < n$. In this case, the period of x can be a strict divisor of $|\alpha|$.

Let (T, A, f) be a y-expansive model. Let Z be the (A^y, y)-central component and let $Z_1, Z_2, \ldots, Z_n$ be the (A^y, y)-branches. Denote by x_i the unique point of $\mathrm{Cl}(Z) \cap Z_i$. Take a type p of A^y, and assume without loss of generality that the (A^y, y)-branches are indexed in such a way that $f(x_i) \in Z_{i+1 \bmod p}$ for $1 \leq i \leq p$. Denote by I_i the A^y-basic intervals $[y, x_i] \subset \mathrm{Cl}(Z)$ for $1 \leq i \leq p$, which are called *typical intervals*. The f-graph of A^y contains the loops $I_{i \bmod p} \to I_{i+1 \bmod p} \to \cdots \to I_{i+p \bmod p}$, which are called *typical loops*.

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Theorem 5.1 is the main result of this section and completes Step 2. It is a particular instance of Theorem A of [**3**] which allows us to compute periodic orbits of a non-twist canonical model (T, A, f). These periods are computed by linking the typical loops with some special loops in the Markov f-graph of A^y. Before stating Theorem 5.1, we need to introduce a notation to deal with a special type of initial segments of the ${}_p\geq$ orderings. If $p \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{p2^\infty\}$, we define $\mathcal{S}_p(r) = \{k \in \mathbb{N} : r \ {}_p\geq k\}$. Note that if $r \in p\mathbb{N}$ then $\mathcal{S}_p(r) = \{1\} \cup p\{k \in \mathbb{N} : r/p \trianglerighteq k\}$ and if $r \notin p\mathbb{N}$ then $\mathcal{S}_p(r) = \{1, r\} \cup \{ri + pj : i \geq 0, j \geq 1\}$. Given $p, r \in \mathbb{N}$, we define

$$\mathcal{S}_p^*(r) = \begin{cases} \mathcal{S}_p(r) & \text{if } r \notin p\mathbb{N}, \\ \mathcal{S}_p(3p) & \text{if } r \in p\mathbb{N}. \end{cases}$$

Observe that if $r \in p\mathbb{N}$, then $\mathcal{S}_p^*(r) = p\mathbb{N} \cup \{1\} \supset \mathcal{S}_p(r)$.

Remark 5.1. Let k, p, r be natural numbers. Then we have that $\{1\} \cup k\mathcal{S}_p^*(r) = \{k\} \cup \mathcal{S}_{kp}^*(kr)$. Indeed, if $r \notin p\mathbb{N}$ then $\{1\} \cup k\mathcal{S}_p^*(r) = \{1\} \cup k(\{1, r\} \cup \{ri + pj : i \geq 0, j \geq 1\}) = \{1, k, kr\} \cup \{kri + kpj : i \geq 0, j \geq 1\} = \{k\} \cup \mathcal{S}_{kp}^*(kr)$. On the other hand, when $r \in p\mathbb{N}$ we get $\{1\} \cup k\mathcal{S}_p^*(r) = \{1\} \cup k(p\mathbb{N} \cup \{1\}) = \{1, k\} \cup kp\mathbb{N} = \{k\} \cup \mathcal{S}_{kp}^*(kr)$.

THEOREM 5.1. *Let (T, A, f) be a non-twist y-expansive canonical model and let p be a type of A^y. Then* $\mathrm{Per}(f) \supset \mathcal{S}_p^*(|A| + lp) \setminus \{p, 2p, 3p, \ldots, \lambda p\}$ *for some integers $\lambda, l \geq 0$ (which are not uniquely determined) such that $lp \leq |\mathrm{En}(T)| - p$ and $\lambda p \leq 2|A|$. Moreover, each of these periods (except 1) corresponds to a periodic orbit which does not intersect A^y and is associated with a simple loop in the Markov f-graph of A^y containing strictly a typical loop.*

Proof. It follows from Propositions 6.5 and 8.1, Lemma 4.6 and Theorem 4.7 of [**3**]. See also [**22**] for a detailed proof. □

6. *Step 3 and inclusion of periods*

In this section we prove Theorem 6.1 that allows us to compare the periods of the models (S_m, P_m, g_m) and (T, A, f) (which exhibits the pattern $[S_m, P_m, g_m]$ by the assumptions of Step 2). This result tells us that when (T, A, f) is a non-twist y-expansive canonical model and satisfies a piecewise linearity condition, then all of the periods of f computed in Theorem 5.1 are also periods of g_m. To do this we use Theorem 2.4 as a key tool. In the second part of this section we prove Theorem 6.2 which, given a sequence of block reductions, describes the set of periods forced by a periodic orbit of a tree map. Theorem 6.2, whose proof uses Theorems 5.1 and 6.1, is the final outcome of Steps 1, 2 and 3.

Let us introduce the piecewise linearity condition mentioned above. It is well known that a tree T admits a *taxicab metric* $d : T \to T$ which, by definition, satisfies $d(x, y) = d(x, z) + d(z, y)$ for each $x, y \in T$ and $z \in [x, y]$. Let I be a closed interval and let $f : I \to T$ be continuous. We say that f is *linear* if either $f(I)$ reduces to a point, or there exist isometries $\theta : [0, 1] \to I$ and $\phi : f(I) \to [0, 1]$ given by the Lebesgue measure on $[0, 1]$ and a taxicab metric on T such that $\phi \circ f \circ \theta$ is an affine map. A y-expansive

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canonical model (T, A, f) is called *linear* if f is linear on each A^y-basic interval. Without loss of generality, every y-expansive canonical model can be assumed to be linear.

The following three technical lemmas are used in the proof of Theorem 6.1. They aim at showing that the periodic points given by Theorem 5.1 satisfy the conditions of Theorem 2.4 (that is, they are significant).

LEMMA 6.1. *Let (T, A, f) be a model and let x be a periodic point of f whose f-monotone equivalence class contains no points of $A \cup V(T)$. Then, for each $i \geq 0$, the f-monotone equivalence class of $f^i(x)$ contains no points of $A \cup V(T)$. Moreover, if $z \in T$ is f-monotone equivalent to x, then $[f^i(x), f^i(z)] \cap (A \cup V(T)) = \emptyset$ for $i \geq 0$.*

Proof. Assume that there exist $y \in A \cup V(T)$ and $i \geq 0$ such that $f^i(x)$ and y are f-monotone equivalent. Let $n \in \mathbb{N}$ be such that $f^n(x) = x$. Then $x = f^n(x) = f^{n-(i \bmod n)}(f^{i \bmod n}(x)) = f^{n-(i \bmod n)}(f^i(x))$. Thus, by Remark 2.2, x is f-monotone equivalent to $f^{n-(i \bmod n)}(y)$ which, by Remark 2.1, belongs to $A \cup V(T)$. This contradiction proves the first assertion of the lemma. In particular, $f^i(x)$ does not belong to $A \cup V(T)$ for each $i \geq 0$.

Now we must prove that $[f^i(x), f^i(z)] \cap (A \cup V(T)) = \emptyset$ for each $i \geq 0$. This is obvious when $x = z$. So we assume that $x \neq z$. Let $i \geq 0$. By Remark 2.2, $f^i(x)$ and $f^i(z)$ are f-monotone equivalent. Hence, there exists $m \in \mathbb{N}$ such that $f^m(f^i(x)) = f^i(x)$, $f^m(f^i(z)) = f^i(z)$ and $f^m|_{[f^i(x), f^i(z)]}$ is monotone. Assume that there exists $w \in (f^i(x), f^i(z)) \cap (A \cup V(T))$. Let $<$ be the orientation of $[f^i(x), f^i(z)]$ such that $f^i(x) < f^i(z)$. Since $f^m|_{[f^i(x), f^i(z)]}$ is monotone and $f^i(x)$ and $f^i(z)$ are fixed points of f^m, it follows that $f^m([f^i(x), f^i(z)]) = [f^i(x), f^i(z)]$ and f^m is increasing (with respect to the ordering chosen above). Therefore, $\mathrm{Orb}_{f^m}(w) \subset [f^i(x), f^i(z)]$. By Remark 2.1, $\mathrm{Orb}_{f^m}(w) \subset A \cup V(T)$. Then, from the finiteness of $A \cup V(T)$ and the monotonicity of f^m it follows that there is a fixed point $w' \in (A \cup V(T)) \cap (f^i(x), f^i(z))$ of f^m. Therefore, w' and $f^i(x)$ are f-monotone equivalent, in contradiction with the first assertion of the lemma. □

LEMMA 6.2. *Let (T, Q, f) be a Markov model and let x be a periodic point associated with a simple loop β in the Markov f-graph of Q. If there are no points of Q in the f-monotone equivalence class of x, then x is Q-significant.*

Proof. We only have to prove that the period of x is minimal within this f-monotone equivalence class. In fact, we see that the period of any point in the class coincides with the period of x. Let $z \in T$ be f-monotone equivalent to x. By assumption, $x, z \notin Q$ and, by Lemma 6.1, for each $k \geq 0$ there exists a Q-basic interval J_k such that $f^k(x), f^k(z) \in \mathrm{Int}(J_k)$. Hence, x and z are associated with the same loop β. Since β is simple and x and z belong to $\mathrm{Int}(J_0)$, by standard arguments we have $|\mathrm{Orb}(x)| = |\mathrm{Orb}(z)| = |\beta|$. □

We recall that the notion of a *typical interval* has been defined in §5. We also use the related notation introduced in that page.

LEMMA 6.3. *Let (T, A, f) be a y-expansive model, let p be a type of A^y and let $[y, x_i]$ be a typical interval for each $1 \leq i \leq p$. Then, for each $n \in \mathbb{N}$, there exists $w \in (y, x_i]$ such that $f^n(w) = x_{n+i \bmod p}$ and $(y, w) \cap f^{-n}(A^y) = \emptyset$.*

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Proof. Recall that, by convention, the points x_i are indexed in such a way that $x_{i+1 \bmod p} \in (y, f(x_i)]$ for each $1 \leq i \leq p$. We proceed by induction on n. Since $f(y) = y$, $x_{i+1 \bmod p} \in (y, f(x_i)]$ and $f|_{[y,x_i]}$ is monotone, the lemma holds for $n = 1$.

Let $n > 1$ and assume that the lemma holds for $n - 1$. Let $w' \in (y, x_i]$ be such that $f^{n-1}(w') = x_{n-1+i \bmod p}$ and $(y, w') \cap f^{-(n-1)}(A^y) = \emptyset$. Since $f^n(y) = y$, $f^n(w') = f(f^{n-1}(w')) = f(x_{n-1+i \bmod p})$, $x_{n+i \bmod p} \in (y, f(x_{n-1+i \bmod p})]$ and $f|_{[y,x_{n-1+i \bmod p}]}$ is monotone, it follows that there exists $w \in (y, w'] \subset (y, x_i]$ such that $f^n(w) = x_{n+i \bmod p}$ and $(y, w) \cap f^{-n}(A^y) = \emptyset$. This completes the induction. □

Now we are ready to state and prove Theorem 6.1, which is the main result of Step 3.

THEOREM 6.1. *Let (S, P, g) be a non-twist model and let (T, A, f) be a y-expansive linear canonical model of $[S, P, g]$. Assume that there exists an n-periodic orbit of f which does not intersect A^y and is associated with a simple loop in the Markov f-graph of A^y containing strictly a typical loop. Then $n \in \mathrm{Per}(g)$.*

Proof. Let p be a type of A^y. Let x be an n-periodic point of f whose orbit does not intersect A^y and is associated with a simple loop β which strictly contains a typical loop. By relabeling if necessary, β contains a path of the form $I_1 \to I_2 \to \cdots \to I_p \to I_1 \to J$, where $I_1, I_2, \ldots, I_p$ are typical intervals and J is not typical. Since f is monotone on I_1 and $J \neq I_2$, it easily follows that $\mathrm{Int}(J) \cap I_2 = \emptyset$. Therefore, I_1 contains at least two points of the orbit of x. Set $I_i = [y, x_i]$ for $1 \leq i \leq p$. By definition of a type we have $x_{i+1 \bmod p} \in (y, f(x_i)]$ for each $1 \leq i \leq p$. By Theorem 2.4, it is enough to show that x is A-significant.

For each $k \in \mathbb{N}$ we define $C^{-k} = f^{-k}(A^y)$. Note that C^{-k} has finitely many connected components and $C^{-k} \supset A^y \supset V(T)$. Since $x \notin A^y$ and x is periodic, $x \notin C^{-k}$ for $k \geq 1$. Observe that if $(u, v) \cap C^{-k} = \emptyset$, then $f^k|_{[u,v]}$ is linear. By Remark 2.2, it is enough to prove that some point in the orbit of x is A-significant. Since $I_1 \in \beta$, we can choose a point of $\mathrm{Orb}_f(x)$ which belongs to (y, x_1). Moreover, we can take it to be the point of $\mathrm{Orb}_f(x) \cap (y, x_1)$ closest to y. Thus, from now on we assume that $x \in (y, x_1)$ and $(y, x) \cap \mathrm{Orb}_f(x) = \emptyset$. Since β is simple, by Lemma 6.2 we only need to prove that the f-monotone equivalence class of x does not contain points of A^y. Assume in contrast that there exists $z \in A^y$ being f-monotone equivalent to x. Hence, there exists $m \in n\mathbb{N}$ such that $f^m(x) = x$, $f^m(z) = z$ and $f^m|_{[x,z]}$ is monotone (in fact, piecewise linear). It follows that $f^m([x, z]) = [x, z]$. Summarizing,

$$f^m|_{[x,z]} \text{ is monotone and } f^m([x, z]) = [x, z]. \tag{2}$$

By Lemma 6.3, there exists $w \in (y, x_1]$ such that $f^m(w) = x_{m+1 \bmod p}$ (so $w \in C^{-m}$) and $(y, w) \cap C^{-m} = \emptyset$. Therefore, f^m is linear on $[y, w]$. We claim that

$$w \in (y, x). \tag{3}$$

Otherwise, $x \in (y, w]$ and, since f^m is linear on $[y, w]$ and has y and x as fixed points, it follows that f^m is the identity map on $[y, w]$. In particular, $w = f^m(w) \in A^y$. This implies that $w = x_1$, and this is equivalent to the fact that $f^i(x_1) = x_{1+i \bmod p}$ for each $i \geq 0$. Therefore, $P := \{x_i\}_{i=1}^p$ is a periodic orbit of f, and either $P \subset V(T)$,

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contradicting the fact that (T, A, f) is y-expansive, or $P = A$, contradicting the fact that (T, A, f) is non-twist. Thus, the claim follows.

Since $V(T) \cap (y, x_1) = \emptyset$, it follows that $T \setminus (y, x_1)$ consists of two connected components and z belongs to one of them because $z \in A^y$ and $A^y \cap (y, x_1) = \emptyset$. Observe that z cannot belong to the connected component which contains y. Otherwise, $y \in [z, x)$ and, from (2) and the fact that y and x are fixed points of f^m, we get $f^m([y, x]) = [y, x]$, in contradiction with (3) since $f^m(w) = x_{m+1 \bmod p} \notin [y, x]$.

Therefore, z belongs to the connected component of $T \setminus (y, x_1)$ which contains x_1. Take an orientation $\leq$ on $[y, z]$ such that $y < z$. By (3), $y < w < x < x_1 \leq z$.

Since $x \notin C^{-m}$, there exist $a, b \in C^{-m}$ such that $(a, b) \cap C^{-m} = \emptyset$ and $w \leq a < x < b \leq x_1$. Hence, f^m is linear on $[a, b]$. Recall that $|\mathrm{Orb}_f(x) \cap I_1| \geq 2$. Let x' be a point of $\mathrm{Orb}_f(x) \cap (y, x_1)$ different from x. Then $x < x' < x_1 \leq z$ by our choice of x. Clearly, x' is a fixed point of f^m because $m \in n\mathbb{N}$. Since $f^m|_{[x,x']}$ is monotone and x and x' are fixed points of f^m, it follows that $f^m([x, x']) = [x, x']$. Thus $x' < b$, because if $x < b < x'$ then $f^m(b) \in [x, x'] \cap A^y \subset (y, x_1) \cap A^y$, a contradiction. Hence, we have $a < x < x' < b \leq x_1 \leq z$. However, f^m is linear on $[a, b]$, and x and x' are fixed points of f^m. It follows that f^m is the identity map on $[a, b]$. In particular, $f^m(a) = a$, a contradiction since $f^m(a) \in A^y$ and $A^y \cap (y, x_1) = \emptyset$. □

The next goal of this section is to summarize Steps 1, 2 and 3 by describing the set of periods forced by a periodic orbit of a tree map. To this end we need to introduce some more notation.

Let $\mathcal{B} = \{(S_i, P_i, g_i), y_i, p_i\}_{i=1}^m$ be a sequence of block reductions of a model (S, P, g), where $g : S \to S$ is a tree map and P is a periodic orbit of g. To such a sequence of block reductions we associate a 4-tuple $(\underline{\mathtt{s}}, \tau, l, \lambda) = (\underline{\mathtt{s}}(\mathcal{B}), \tau(\mathcal{B}), l(\mathcal{B}), \lambda(\mathcal{B}))$, called a *characteristic of* $\mathcal{B}$, which is defined as follows.

- $\underline{\mathtt{s}} = \underline{\mathtt{s}}(\mathcal{B})$ denotes the sequence $(p_1, p_2, \ldots, p_m)$.
- $\tau = \tau(\mathcal{B})$ denotes the formal symbol ϕ if (S_m, P_m, g_m) is twist around y_m, and the symbol $*$ otherwise.
- The non-negative integers $l = l(\mathcal{B})$ and $\lambda = \lambda(\mathcal{B})$ are set to 0 when $\tau = \phi$. In the case $\tau = *$, let (T, A, f) be the y_m-expansive canonical model of $[S_m, P_m, g_m]$ given by (BR4.2) (note that, since $\tau = *$, P_m is non-twist; in particular, $|P_m| > 1$ and thus (BR4.1) does not hold). Then, the numbers l and λ are those provided by Theorem 5.1 for the model (T, A, f). Observe that the assumptions of Theorem 5.1 hold (with $p = p_m$) because, by Lemma 5.1, we know that (T, A, f) is non-twist.

Given a sequence of block reductions $\mathcal{B}$, the sequence $\underline{\mathtt{s}}(\mathcal{B})$ and the symbol $\tau(\mathcal{B})$ are uniquely determined, while from Theorem 5.1 it follows that for a fixed type p_m there can be several choices for the numbers $l(\mathcal{B})$ and $\lambda(\mathcal{B})$. Therefore, a characteristic of a sequence of block reductions is not unique.

In what follows, $\mathcal{S}_p^\tau(n)$ denotes $\mathcal{S}_p^*(n)$ when $\tau = *$ and $\mathcal{S}_p(n)$ when $\tau = \phi$. Then, given a characteristic $(\underline{\mathtt{s}}, \tau, l, \lambda)$ of a sequence of block reductions of a periodic model (S, P, g) we set

$$\Lambda(\underline{\mathtt{s}}, \tau, l, \lambda) := \{|P|\} \cup \mathcal{K}_{\underline{\mathtt{s}}} \cup (\mathcal{S}_{\lceil \underline{\mathtt{s}} \rceil}^\tau(|P| + l\lceil \underline{\mathtt{s}} \rceil) \setminus \lceil \underline{\mathtt{s}} \rceil\{2, 3, \ldots, \lambda\}).$$

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THEOREM 6.2. (Set of periods forced by a periodic orbit) *Let $g\colon S \to S$ be a tree map, let P be a periodic orbit of g and let $(\underline{\mathtt{S}}, \tau, l, \lambda)$ be a characteristic of a sequence of block reductions of (S, P, g). Then,* $\mathrm{Per}(g) \supset \Lambda(\underline{\mathtt{S}}, \tau, l, \lambda)$.

Proof. Let $\{(S_i, P_i, g_i), y_i, p_i\}_{i=1}^m$ be a sequence of block reductions of (S, P, g) which has $(\underline{\mathtt{S}}, \tau, l, \lambda)$ as characteristic, and set $p = p_m$ and $k = \max \mathcal{K}_{\underline{\mathtt{S}}} = \lceil \underline{\mathtt{S}} \rceil / p$. By Proposition 4.1,

$$|P| = k|P_m| \quad \text{and} \quad \mathrm{Per}(g) \supset \{|P|, 1\} \cup \mathcal{K}_{\underline{\mathtt{S}}} \cup k\,\mathrm{Per}(g_m). \tag{4}$$

First we consider the case $\tau = \phi$ (that is, P_m is twist). Then, S_m is a star by (BR4) and, by Baldwin's theorem (Theorem 2.12 of [**11**]), $\mathrm{Per}(g_m) \supset \mathcal{S}_p(|P_m|)$. From the definitions it is not difficult to see that $\{1\} \cup k\mathcal{S}_p(|P_m|) = \{k\} \cup \mathcal{S}_{kp}(k|P_m|)$ Therefore, from (4) it follows that

$$\begin{aligned}\mathrm{Per}(g) &\supset \{|P|, 1\} \cup \mathcal{K}_{\underline{\mathtt{S}}} \cup k\mathcal{S}_p(|P_m|) = \{|P|\} \cup \mathcal{K}_{\underline{\mathtt{S}}} \cup \mathcal{S}_{kp}(k|P_m|) \\ &= \{|P|\} \cup \mathcal{K}_{\underline{\mathtt{S}}} \cup (\mathcal{S}^{\tau}_{\lceil \underline{\mathtt{S}} \rceil}(|P| + l\lceil \underline{\mathtt{S}} \rceil) \setminus \lceil \underline{\mathtt{S}} \rceil\{2, 3, \ldots, \lambda\}).\end{aligned}$$

Now we consider the case $\tau = *$. Since P_m is non-twist, $|P_m| > 1$. Let (T, A, f), y_m and p be given by (BR4.2). Then, (T, A, f) is a y_m-expansive canonical model of $[S_m, P_m, g_m]$ and p is a type of A^{y_m}. Moreover, as it has been said before, the model (T, A, f) can be assumed to be linear without loss of generality, and it is non-twist by Lemma 5.1.

By Theorem 5.1, $\mathrm{Per}(f) \supset \mathcal{S}_p^*(|P_m| + lp) \setminus p\{1, 2, 3, \ldots, \lambda\}$. Moreover, each of these periods corresponds to an orbit that does not intersect A^y and is associated with a simple loop containing strictly a typical loop. Thus, by Theorem 6.1, all of these periods are also periods of g_m. In consequence, $\mathrm{Per}(g_m) \supset \mathcal{S}_p^*(|P_m| + lp) \setminus p\{1, 2, 3, \ldots, \lambda\}$. Suppose we also had that $p \in \mathrm{Per}(g_m)$. Then, $\mathrm{Per}(g_m) \supset \mathcal{S}_p^*(|P_m| + lp) \setminus p\{2, 3, \ldots, \lambda\}$ and the theorem would follow as above by using (4) and Remark 5.1.

Thus, to end the proof of the theorem we only need to show that $p \in \mathrm{Per}(g_m)$. To prove it, we claim that A has type p. Let us prove the claim. For any pair of points $z, w \in T$, let us write $z \prec w$ if and only if $z \in [y_m, w)$. Let us denote by x_i the unique point of the ith (A^{y_m}, y_m)-branch contained in the closure of the central (A^{y_m}, y_m)-component, and assume that the (A^{y_m}, y_m)-branches are indexed in such a way that $x_{i+1 \bmod p} \preceq f(x_i)$ for $1 \le i \le p$ (recall that A^{y_m} has type p). Note that there can be more than p points x_i, but in what follows we only use $x_1, x_2, \ldots, x_p$ which correspond to the typical branches. In particular, for the rest of the proof of all the subindices of the points x_i are considered modulo p.

Analogously, each (A, y_m)-branch has a unique point contained in the closure of the (A, y_m)-central component. Let us denote the set of such points by X. To prove that A has type p, we have to show that there exists a p-periodic orbit of the map $\Phi\colon X \to X$ defined by $\Phi(a) = b$ if and only if $b \preceq f(a)$. Take $z \in X$ and $j \in \{1, 2, \ldots, p\}$ with $x_j \preceq z$ (recall that $X \subset A \subset A^{y_m}$ and that, since (T, A, f) is a monotone model, $\mathrm{En}(T) \subset A$). Since f is $(A \cup \{y_m\})$-monotone and $(y_m, z) \cap A = \emptyset$, it follows that there exists $z' \in X$ such that $y_m \prec x_{j+1} \preceq z' \preceq f(z)$. In particular we have $\Phi(z) = z'$. Since A is finite, by iterating this argument we obtain a periodic orbit $\{a^0, a^1, \ldots, a^{pr-1}\} \subset X$ of Φ with $r \ge 1$ such that $a^i \succeq x_{k+i}$ for some $k \in \{1, 2, \ldots, p\}$ and $0 \le i \le pr - 1$. In particular, A has type pr.

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Suppose that $a^i = x_{k+i}$ for some $0 \leq i \leq pr - 1$. Since p is a type of A^{y_m}, it follows that $a^{i+p-1 \bmod pr} \succeq x_{k+i+p-1} = x_{k+i-1}$. Therefore, since f is $(A \cup \{y_m\})$-monotone, it follows that $a^i = x_{k+i} \preceq f(x_{k+i-1}) \preceq f(a^{i+p-1 \bmod pr})$. That is, $a^{i+p \bmod pr} = \Phi(a^{i+p-1 \bmod pr}) = a^i$ and, hence, A has type p. This ends the proof of the claim in this case.

Now consider the case $x_{k+i} \prec a^i$ for all $0 \leq i \leq pr - 1$ and suppose that $r > 1$. Then $x_k \prec a^0$, $x_k \prec a^p$ and $T^0 := \langle\{y_m, a^0, a^p\}\rangle$ is a 3-star whose central point is x_k. Since f is $(A \cup \{y_m\})$-monotone, $f(T^0) = \langle\{y_m, f(a^0), f(a^p)\}\rangle$. Furthermore, from Remark 2.1 it follows that $f(x_k) = x_{k+1}$. Since $a^1 \preceq f(a^0)$ and $a^{p+1} \preceq f(a^p)$, we have that $T^1 := \langle\{y_m, a^1, a^{p+1}\}\rangle$ is a 3-star contained in $f(T^0)$ whose central point is x_{k+1}. By iterating p times this argument we get that $\{x_1, x_2, \ldots, x_p\}$ is a periodic orbit of f contained in the central (A, y_m)-component, in contradiction with the fact that (T, A, f) is y_m-expansive. This ends the proof of the claim.

Finally we prove that $p \in \mathrm{Per}(g_m)$. Observe that X is a discrete component of (T, A). Since $[T, A, f] = [S_m, P_m, g_m]$, there exists a bijection $\theta\colon A \to P_m$ which sends discrete components of (T, A) to discrete components of (S_m, P_m). Let $Z = \mathrm{Int}(\langle\theta(X)\rangle_{S_m})$ and let $w \in Z$. Since $g_m \circ \theta|_A = \theta \circ f|_A$ and, by the claim above, A has type p, it easily follows that $\theta(a^{i+1}) \in (w, g_m(\theta(a^i))]$ for $0 \leq i < p - 1$ and $\theta(a^0) \in (w, g_m(\theta(a^{p-1}))]$.

Let $r : S_m \to \mathrm{Cl}(Z)$ be the natural retraction. Then $r \circ g_m(\theta(a^i)) = \theta(a^{i+1})$ for $0 \leq i < p - 1$ and $r \circ g_m(\theta(a^{p-1})) = \theta(a^0)$. Hence $\theta(a^0)$ is a p-periodic point of $r \circ g_m$ and thus $p \in \mathrm{Per}(r \circ g_m)$. Then, the theorem follows because $\mathrm{Per}(r \circ g_m) \subset \mathrm{Per}(g_m)$ (see, for instance, Corollary 4.2 of [**11**]). □

7. *Step 4. Proof of Theorem 1.1*

The goal of this section is to perform Step 4 of the strategy described in §3, while proving Theorem 1.1. In doing this we use two technical lemmas. The first one shows that the first element of any characteristic of a model (S, P, g) belongs to Σ_S, and the second studies the unions of the sets $\Lambda(\underline{\mathtt{s}}, \tau, l, \lambda)$ corresponding to a fixed sequence $\underline{\mathtt{s}}$. We recall that conditions (S1)–(S4) and (BR1)–(BR4) have been stated above.

LEMMA 7.1. *Let $g : S \to S$ be a tree map, let P be a periodic orbit of g and let $(\underline{\mathtt{s}}, \tau, l, \lambda)$ be a characteristic of a sequence of block reductions of (S, P, g). Then:*

(a) $\underline{\mathtt{s}} \in \Sigma_S$;

(b) *(S3) and (S4) hold;*

(c) $\lambda\lceil\underline{\mathtt{s}}\rceil \leq 2|P|$ *when* $\tau = *$;

(d) $l < |\mathrm{En}(S)|$.

Proof. We use the notation from properties (BR1)–(BR4) and from the definition of a characteristic. In particular, $\underline{\mathtt{s}} = (p_1, p_2, \ldots, p_m)$.

Let us prove (a). From (BR2) it follows that, for $1 \leq i < m$, $p_i \in \mathbb{N} \setminus \{1\}$. Moreover, p_m is also positive by (BR4). Therefore, $\underline{\mathtt{s}} \in \Sigma$. On the other hand, (S2) follows from (BR3), and by (BR1) and (BR2) we have $S \supset S_1$ and $S_i \sqsupset p_i S_{i+1}$ for $1 \leq i < m$. Thus, to prove (S1) we only have to see that $p_m \leq |\mathrm{En}(S_m)|$. This inequality trivially holds when $p_m = 1$. Hence, by (BR4), we may assume that p_m is a type of A^{y_m}, where (T, A, f) is a y_m-expansive canonical model of $[S_m, P_m, g_m]$. Therefore, $p_m \leq |\mathrm{En}(T)|$.

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On the other hand, $\mathrm{En}(T) \subset A$ because (T, A, f) is a monotone model. Since $\mathrm{En}(S_m) \subset P_m$ by Proposition 4.1 and (T, A, f) and (S_m, P_m, g_m) have the same pattern, it follows that $|\mathrm{En}(T)| = |\mathrm{En}(S_m)|$.

Let us prove (b). Set $e_i = |\mathrm{En}(S_i)|$ for $1 \le i \le m$. Since (S1) holds, $e_{i+1} \le e_i/p_i + 1$ for $1 \le i \le m-1$. Consequently, when $m \ge 3$ we have

$$\begin{aligned} e_{m-1} &\le \frac{e_1}{p_1 p_2 \cdots p_{m-2}} + \frac{1}{p_2 p_3 \cdots p_{m-2}} + \cdots + \frac{1}{p_{m-2}} + 1 \\ &\le \frac{e_1}{2^{m-2}} + \frac{1}{2^{m-3}} + \cdots + \frac{1}{2} + 1 \\ &= \frac{e_1}{2^{m-2}} + 2\left(1 - \frac{1}{2^{m-2}}\right) \le \frac{1}{2^{m-2}}(|\mathrm{En}(S)| - 2) + 2. \end{aligned}$$

Thus, (S3) follows from the fact that $e_{m-1} \ge 4$ by (S2).

Now we prove (S4). Since $e_1 \le E_1$, it follows from the definition of E_i and a simple inductive argument using (S1) that $e_i \le E_i$ for $1 \le i \le m$. Consequently, we only have to show that $p_i \le e_i$ for $1 \le i \le m$. For $1 \le i \le m-1$ this follows directly from the fact that $(S_{i+1}, P_{i+1}, g_{i+1})$ is a p_i-block reduction of (S_i, P_i, g_i) (see (BR2)), and $p_m \le e_m$ holds by (S1). This ends the proof of (S4) and (b).

Now let us prove (c). When $\tau = *$, the model (S_m, P_m, g_m) is non-twist. Hence, by Lemma 5.1 (T, A, f) is also non-twist. From Theorem 5.1 we get that $\lambda p_m \le 2|P_m|$. By multiplying this inequality on both sides by $p_1 p_2 \cdots p_{m-1}$, we have that $\lambda \lceil \underline{\mathtt{s}} \rceil \le 2 p_1 p_2 \cdots p_{m-1} |P_m|$. Then (c) holds since, by Proposition 4.1, $|P| = p_1 p_2 \cdots p_{m-1} |P_m|$.

Finally we prove (d), which obviously holds when $l = 0$. Assume that $l > 0$ (in particular, $\tau = *$). As above, this implies that (T, A, f) is a non-twist model and from Theorem 5.1 we have $l p_m \le |\mathrm{En}(T)| - p_m = |\mathrm{En}(S_m)| - p_m \le |\mathrm{En}(S)| - p_m$. Multiplying on both sides by $p_1 p_2 \cdots p_{m-1}$ we get $l \lceil \underline{\mathtt{s}} \rceil \le |\mathrm{En}(S)| p_1 p_2 \cdots p_{m-1} - \lceil \underline{\mathtt{s}} \rceil$. Since $p_1 p_2 \cdots p_{m-1} \le \lceil \underline{\mathtt{s}} \rceil$, we get $l \lceil \underline{\mathtt{s}} \rceil \le |\mathrm{En}(S)| \lceil \underline{\mathtt{s}} \rceil - \lceil \underline{\mathtt{s}} \rceil$ and thus (d) follows. □

LEMMA 7.2. *Let $g : S \to S$ be a tree map and let $\{(\underline{\mathtt{s}}, \tau_i, l_i, \lambda_i)\}_{i \in R}$ be a family of characteristics of sequences of block reductions of periodic models (S, P_i, g). Then, if $\underline{\mathtt{s}} = (p_1, p_2, \ldots, p_m)$,*

$$\bigcup_{i \in R} \Lambda(\underline{\mathtt{s}}, \tau_i, l_i, \lambda_i) = \mathcal{K}_{\underline{\mathtt{s}}} \cup \mathcal{F}_{\underline{\mathtt{s}}} \cup \left(\mathcal{I}_{\underline{\mathtt{s}}} \setminus \lceil \underline{\mathtt{s}} \rceil \{2, 3, \ldots, \lambda_{\underline{\mathtt{s}}}\}\right)$$

where $\lambda_{\underline{\mathtt{s}}}$ is a non-negative integer and:

(a) *$\mathcal{I}_{\underline{\mathtt{s}}}$ is an initial segment of the $\lceil \underline{\mathtt{s}} \rceil \ge$ ordering such that each $\lceil \underline{\mathtt{s}} \rceil \ge$-maximal element of $\mathcal{I}_{\underline{\mathtt{s}}}$ belongs to $\{1\} \cup p_1 p_2 \cdots p_{m-1}(\mathbb{N} \cup 2^\infty)$;*

(b) *if $\mathcal{I}_{\underline{\mathtt{s}}} \subsetneq \{1\} \cup \lceil \underline{\mathtt{s}} \rceil \mathbb{N}$, then $\lambda_{\underline{\mathtt{s}}} = 0$ and $\mathcal{F}_{\underline{\mathtt{s}}} = \emptyset$;*

(c) *$\mathcal{F}_{\underline{\mathtt{s}}}$ is disjoint from $\mathcal{K}_{\underline{\mathtt{s}}} \cup \mathcal{I}_{\underline{\mathtt{s}}} \setminus \lceil \underline{\mathtt{s}} \rceil \{2, 3, \ldots, \lambda_{\underline{\mathtt{s}}}\}$ and $\mathcal{F}_{\underline{\mathtt{s}}} \setminus \lceil \underline{\mathtt{s}} \rceil \mathbb{N} \subset \bigcup_{j \in B(\mathcal{I}_{\underline{\mathtt{s}}})} \mathbb{N}^j_{\lceil \underline{\mathtt{s}} \rceil}$;*

(d) *$\mathcal{F}_{\underline{\mathtt{s}}}$ is finite (or empty)—when $\mathcal{F}_{\underline{\mathtt{s}}} \ne \emptyset$, we have $\min \mathcal{F}_{\underline{\mathtt{s}}} \ge \lambda_{\underline{\mathtt{s}}} \lceil \underline{\mathtt{s}} \rceil / 2$ and $|\mathcal{F}_{\underline{\mathtt{s}}}| < |B(\mathcal{I}_{\underline{\mathtt{s}}})\|\mathrm{En}(S)| + (\lambda_{\underline{\mathtt{s}}} + 2)/2$.*

Proof. Set

$$\mathcal{I}_{\underline{\mathtt{s}}} := \bigcup_{i \in R} \mathcal{S}^{\tau_i}_{\lceil \underline{\mathtt{s}} \rceil}(|P_i| + l_i \lceil \underline{\mathtt{s}} \rceil).$$

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Observe that $\mathcal{S}^{\tau_i}_{\lceil \underline{\mathbf{s}} \rceil}(|P_i| + l_i \lceil \underline{\mathbf{s}} \rceil)$ is an initial segment of the ordering ${}_{\lceil \underline{\mathbf{s}} \rceil}\geq$ for each $i \in R$, and recall that, by definition, the union of initial segments of a given ordering is an initial segment of the same ordering. By definition, $\lceil \underline{\mathbf{s}} \rceil \in p_1 p_2 \cdots p_{m-1}\mathbb{N}$. Moreover, by Proposition 4.1, $|P_i| \in p_1 p_2 \cdots p_{m-1}\mathbb{N}$ for each $i \in R$. Consequently, each ${}_{\lceil \underline{\mathbf{s}} \rceil}\geq$-maximal element of $\mathcal{I}$ belongs to $\{1\} \cup p_1 p_2 \cdots p_{m-1}(\mathbb{N} \cup 2^\infty)$ and (a) holds.

Set

$$\lambda_{\underline{\mathbf{s}}} := \min\{\lambda_i : i \in R\} \in \mathbb{N} \cup \{0\} \quad \text{and} \quad R^* := \{i \in R : \tau_i = *\}.$$

For each $i \in R \setminus R^*$ we have $\tau_i = \not{0}$. Thus, $l_i = \lambda_i = 0$ by definition. Moreover,

$$\mathcal{S}^{\tau_i}_{\lceil \underline{\mathbf{s}} \rceil}(|P_i| + l_i \lceil \underline{\mathbf{s}} \rceil) \setminus \lceil \underline{\mathbf{s}} \rceil\{2, 3, \ldots, \lambda_i\} = \mathcal{S}_{\lceil \underline{\mathbf{s}} \rceil}(|P_i|) \ni |P_i|.$$

Consequently,

$$\begin{aligned}\bigcup_{i\in R} \Lambda(\underline{\mathbf{s}}, \tau_i, l_i, \lambda_i) &= \bigcup_{i\in R}(\{|P_i|\} \cup \mathcal{K}_{\underline{\mathbf{s}}} \cup (\mathcal{S}^{\tau_i}_{\lceil \underline{\mathbf{s}} \rceil}(|P_i| + l_i \lceil \underline{\mathbf{s}} \rceil) \setminus \lceil \underline{\mathbf{s}} \rceil\{2, 3, \ldots, \lambda_i\})) \\ &= \mathcal{K}_{\underline{\mathbf{s}}} \cup \left(\bigcup_{i\in R^*} \{|P_i|\} \right) \cup (\mathcal{I}_{\underline{\mathbf{s}}} \setminus \lceil \underline{\mathbf{s}} \rceil\{2, 3, \ldots, \lambda_{\underline{\mathbf{s}}}\}) \\ &= \mathcal{K}_{\underline{\mathbf{s}}} \cup \mathcal{F}_{\underline{\mathbf{s}}} \cup (\mathcal{I}_{\underline{\mathbf{s}}} \setminus \lceil \underline{\mathbf{s}} \rceil\{2, 3, \ldots, \lambda_{\underline{\mathbf{s}}}\}),\end{aligned}$$

where

$$\mathcal{F}_{\underline{\mathbf{s}}} := \left(\bigcup_{i\in R^*} \{|P_i|\} \right) \Big\setminus (\mathcal{K}_{\underline{\mathbf{s}}} \cup (\mathcal{I}_{\underline{\mathbf{s}}} \setminus \lceil \underline{\mathbf{s}} \rceil\{2, 3, \ldots, \lambda_{\underline{\mathbf{s}}}\})) \subset \mathbb{N}.$$

Now observe that for each $i \in R^*$ it follows that $\{1\} \cup \lceil \underline{\mathbf{s}} \rceil\mathbb{N} \subset \mathcal{S}^{\tau_i}_{\lceil \underline{\mathbf{s}} \rceil}(|P_i| + l_i \lceil \underline{\mathbf{s}} \rceil) \subset \mathcal{I}_{\underline{\mathbf{s}}}$. Consequently, $R^* = \emptyset$ whenever $\mathcal{I}_{\underline{\mathbf{s}}} \subsetneq \{1\} \cup \lceil \underline{\mathbf{s}} \rceil\mathbb{N}$. Hence, (b) follows from the definitions of $\lambda_{\underline{\mathbf{s}}}$ and $\mathcal{F}_{\underline{\mathbf{s}}}$.

Now we prove (c). The set $\mathcal{F}_{\underline{\mathbf{s}}}$ is disjoint from $\mathcal{I}_{\underline{\mathbf{s}}} \setminus \lceil \underline{\mathbf{s}} \rceil\{2, 3, \ldots, \lambda_{\underline{\mathbf{s}}}\}$ and $\mathcal{K}_{\underline{\mathbf{s}}}$ by definition. Since $\mathcal{F}_{\underline{\mathbf{s}}} \setminus \lceil \underline{\mathbf{s}} \rceil\mathbb{N} \subset \bigcup_{j=1}^{\lceil \underline{\mathbf{s}} \rceil - 1} \mathbb{N}^j_{\lceil \underline{\mathbf{s}} \rceil}$, it is enough to show that if $(\mathcal{F}_{\underline{\mathbf{s}}} \setminus \lceil \underline{\mathbf{s}} \rceil\mathbb{N}) \cap \mathbb{N}^j_{\lceil \underline{\mathbf{s}} \rceil} \neq \emptyset$ for some $1 \leq j < \lceil \underline{\mathbf{s}} \rceil$, then $j \in B(\mathcal{I}_{\underline{\mathbf{s}}})$. Assume that for some $i \in R^*$ and $1 \leq j < \lceil \underline{\mathbf{s}} \rceil$ we have $|P_i| \in \mathbb{N}^j_{\lceil \underline{\mathbf{s}} \rceil}$. From the definitions we obtain

$$\mathcal{I}_{\underline{\mathbf{s}}} \supset \mathcal{S}^{\tau_i}_{\lceil \underline{\mathbf{s}} \rceil}(|P_i| + l_i \lceil \underline{\mathbf{s}} \rceil) = \mathcal{S}_{\lceil \underline{\mathbf{s}} \rceil}(|P_i| + l_i \lceil \underline{\mathbf{s}} \rceil) \supset \{|P_i| + t \lceil \underline{\mathbf{s}} \rceil : t \geq l_i\}.$$

Consequently, $\mathcal{I}_{\underline{\mathbf{s}}} \cap \mathbb{N}^j_{\lceil \underline{\mathbf{s}} \rceil} \neq \emptyset$ because $(|P_i| + t \lceil \underline{\mathbf{s}} \rceil) \equiv j \pmod{\lceil \underline{\mathbf{s}} \rceil}$ for all t. Thus $j \in B(\mathcal{I}_{\underline{\mathbf{s}}})$ and (c) follows.

Finally we prove (d). Assume that $\mathcal{F}_{\underline{\mathbf{s}}} \neq \emptyset$. By definition, we have

$$\mathcal{F}_{\underline{\mathbf{s}}} \subset \mathbb{N} \setminus (\mathcal{K}_{\underline{\mathbf{s}}} \cup (\mathcal{I}_{\underline{\mathbf{s}}} \setminus \lceil \underline{\mathbf{s}} \rceil\{2, 3, \ldots, \lambda_{\underline{\mathbf{s}}}\})) \subset (\mathbb{N} \setminus \mathcal{I}_{\underline{\mathbf{s}}}) \cup \lceil \underline{\mathbf{s}} \rceil\{2, 3, \ldots, \lambda_{\underline{\mathbf{s}}}\}.$$

By Lemma 7.1(c), for each $i \in R^*$ we have $|P_i| \geq \lambda_i \lceil \underline{\mathbf{s}} \rceil/2 \geq \lambda_{\underline{\mathbf{s}}} \lceil \underline{\mathbf{s}} \rceil/2$. Hence, $\min \mathcal{F}_{\underline{\mathbf{s}}} \geq \lambda_{\underline{\mathbf{s}}} \lceil \underline{\mathbf{s}} \rceil/2$ and

$$\mathcal{F}_{\underline{\mathbf{s}}} \subset (\mathbb{N} \setminus \mathcal{I}_{\underline{\mathbf{s}}}) \cup \{n \lceil \underline{\mathbf{s}} \rceil : n \geq 2 \text{ and } \lambda_{\underline{\mathbf{s}}}/2 \leq n \leq \lambda_{\underline{\mathbf{s}}}\}.$$

Since $\mathcal{F}_{\underline{\mathbf{s}}} \neq \emptyset$, from (b) it follows that $\mathcal{I}_{\underline{\mathbf{s}}} \supset \{1\} \cup \lceil \underline{\mathbf{s}} \rceil\mathbb{N}$. From this and (c) we get that

$$\mathcal{F}_{\underline{\mathbf{s}}} \subset \bigcup_{j\in B(\mathcal{I}_{\underline{\mathbf{s}}})} (\mathcal{F}_{\underline{\mathbf{s}}} \cap \mathbb{N}^j_{\lceil \underline{\mathbf{s}} \rceil}) \cup \{n \lceil \underline{\mathbf{s}} \rceil : n \geq 2 \text{ and } \lambda_{\underline{\mathbf{s}}}/2 \leq n \leq \lambda_{\underline{\mathbf{s}}}\}.$$

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Since $|\{n\lceil \underline{s}\rceil : n \geq 2 \text{ and } \lambda_{\underline{s}}/2 \leq n \leq \lambda_{\underline{s}}\}| \leq (\lambda_{\underline{s}}+2)/2$, to end the proof of (d) it is enough to show that $|\mathcal{F}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil}| \leq |\mathrm{En}(S)|$ for each $j \in B(\mathcal{I}_{\underline{s}})$.

Fix $j \in B(\mathcal{I}_{\underline{s}})$. From (c), $\mathcal{F}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil} \subset \mathbb{N}^j_{\lceil \underline{s}\rceil} \setminus \mathcal{I}_{\underline{s}}$. Set

$$s_j = \max_{\lceil \underline{s}\rceil \geq} (\mathcal{I}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil}).$$

Since $\mathcal{I}_{\underline{s}}$ is an initial segment of the ordering ${}_{\lceil \underline{s}\rceil}\geq$, it follows that

$$\mathcal{I}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil} = \{s_j + t\lceil \underline{s}\rceil : t \geq 0\}.$$

Consequently, $\{j, j+\lceil \underline{s}\rceil, j+2\lceil \underline{s}\rceil, \ldots, s_j - \lceil \underline{s}\rceil\} \supset \mathbb{N}^j_{\lceil \underline{s}\rceil} \setminus \mathcal{I}_{\underline{s}} \supset \mathcal{F}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil}$.

On the other hand, take $i \in R^*$ such that $|P_i| \in \mathcal{F}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil}$ and

$$|P_i| = \max_{\lceil \underline{s}\rceil \geq} (\mathcal{F}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil}).$$

Then $|P_i|\ {}_{\lceil \underline{s}\rceil}\geq s_j - \lceil \underline{s}\rceil$ and $\mathcal{F}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil} \subset A_j := \{|P_i|, |P_i|+\lceil \underline{s}\rceil, |P_i|+2\lceil \underline{s}\rceil, \ldots, s_j - \lceil \underline{s}\rceil\}$. Since $|P_i| + l_i\lceil \underline{s}\rceil \in \mathcal{I}_{\underline{s}}$, it follows that $s_j - \lceil \underline{s}\rceil\ {}_{\lceil \underline{s}\rceil}> |P_i| + l_i\lceil \underline{s}\rceil$. Hence,

$$A_j \subset \{|P_i|, |P_i| + \lceil \underline{s}\rceil, |P_i| + 2\lceil \underline{s}\rceil, \ldots, |P_i| + (l_i - 1)\lceil \underline{s}\rceil\}.$$

Consequently, $|\mathcal{F}_{\underline{s}} \cap \mathbb{N}^j_{\lceil \underline{s}\rceil}| \leq |A_j| \leq l_i$. Now the lemma follows since $l_i < |\mathrm{En}(S)|$ by Lemma 7.1(d). □

Finally we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{(\underline{s}_i, \tau_i, l_i, \lambda_i)\}_{i\in R}$ be the family of all characteristics of sequences of block reductions of periodic orbits of g. For each $i \in R$, denote by P_i the orbit which has generated the characteristic $(\underline{s}_i, \tau_i, l_i, \lambda_i)$. By definition, $|P_i| \in \Lambda(\underline{s}_i, \tau_i, l_i, \lambda_i)$ for each i. Therefore,

$$\bigcup_{i\in R} \Lambda(\underline{s}_i, \tau_i, l_i, \lambda_i) = \mathrm{Per}(g)$$

by Theorem 6.2.

Set $\mathsf{S} = \bigcup_{i\in R} \underline{s}_i$. By Lemma 7.1 it follows that $\mathsf{S} \subset \Sigma_S$. For each $\underline{s} \in \mathsf{S}$, we set $R_{\underline{s}} = \{i \in R : \underline{s}_i = \underline{s}\}$. Thus,

$$\mathrm{Per}(g) = \bigcup_{i\in R} \Lambda(\underline{s}_i, \tau_i, l_i, \lambda_i) = \bigcup_{\underline{s}\in\mathsf{S}} \left(\bigcup_{i\in R_{\underline{s}}} \Lambda(\underline{s}, \tau_i, l_i, \lambda_i) \right),$$

and the theorem holds by Lemma 7.2. □

8. *Proof of Theorem 1.2*

This section is devoted to proving Theorem 1.2. We start with four technical lemmas, which are used to construct tree maps with a prescribed set of periods.

LEMMA 8.1. *Let $f : T \to T$ be a tree map and let $\mathcal{F} \subset \mathbb{N}$ be finite. Then there exist a tree $S \supset T$ and a tree map $g : S \to S$ such that* $\mathrm{Per}(g) = \mathrm{Per}(f) \cup \mathcal{F}$.

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Proof. It is enough to prove that, for any $n \in \mathbb{N}$, there exists a tree $S \supset T$ and a tree map $g\colon S \to S$ such that $\mathrm{Per}(g) = \mathrm{Per}(f) \cup \{n\}$.

Let $n \in \mathbb{N}$ and let $y \in \mathrm{Fix}(f)$. Consider a tree X containing T such that $\mathrm{Cl}(X \setminus T)$ is a n-star whose central point is y, and denote its branches by $[y, b_i]$ for $1 \le i \le n$. Now consider a map $g\colon S \to S$ such that $g|_T = f$, $g(b_i) = b_{i+1 \bmod n}$ and g maps $[y, b_i]$ monotonically onto $[y, b_{i+1 \bmod n}]$ for each $1 \le i \le n$. It easily follows that $\mathrm{Per}(g) = \mathrm{Per}(f) \cup \{n\}$. □

We recall that the converse part of Baldwin's theorem states that, given any union $\mathcal{A}$ of finitely many initial segments of the ${}_p{\geq}$ orderings for $1 \le p \le r$, there exists a r-star map whose set of periods is $\mathcal{A}$.

LEMMA 8.2. *Let $p \in \mathbb{N}$, $\lambda \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N} \cup \{p2^\infty\}$. Assume that $\lambda = 0$ when $\mathcal{S}_p(n) \subsetneq \{1\} \cup p\mathbb{N}$. Then there exist a tree S with $|\mathrm{En}(S)| \ge \max\{3, p\}$ and a tree map $g\colon S \to S$ such that $\mathrm{Per}(g) = \mathcal{S}_p(n) \setminus p\{2, 3, \ldots, \lambda\}$.*

Proof. Assume first that $\lambda \le 1$. In this case, $p\{2, 3, \ldots, \lambda\} = \emptyset$. By Baldwin's theorem, there exists a $(\max\{3, p\})$-star map g such that $\mathrm{Per}(g) = \mathcal{S}_p(n)$ and the lemma follows in this case.

Now we consider the case $\lambda > 1$. Hence, $\mathcal{S}_p(n) \supset \{1\} \cup p\mathbb{N}$. Since $\mathcal{S}_p(n)$ is an initial segment of the ${}_p{\geq}$ ordering, we have either:

(i) $\mathcal{S}_p(n) = \{1\} \cup p\mathbb{N}$ (then $n = 3p$); or

(ii) $\{1\} \cup p\mathbb{N} \subsetneq \mathcal{S}_p(n)$ (then $n \notin p\mathbb{N}$).

If $p = 1$, then (ii) cannot occur. Therefore, we must construct a tree map $g\colon S \to S$ such that $\mathrm{Per}(g) = \mathcal{S}_1(3) \setminus \{2, 3, \ldots, \lambda\} = \mathbb{N} \setminus \{2, 3, \ldots, \lambda\}$. From the definition of Baldwin's orderings, it is not difficult to see that

$$\mathbb{N} \setminus \{2, 3, \ldots, \lambda\} = \bigcup_{1 \le i \le \lambda} \mathcal{S}_{\lambda+1}(\lambda + 1 + i).$$

Then we choose S to be a $(\lambda + 1)$-star and, by Baldwin's theorem, there exists a tree map $g\colon S \to S$ such that $\mathrm{Per}(g) = \bigcup_{1 \le i \le \lambda} \mathcal{S}_{\lambda+1}(\lambda + 1 + i)$. Moreover, since $\lambda > 1$ we have $|\mathrm{En}(S)| \ge 3 = \max\{3, p\}$ and the lemma follows in this case.

From now on we assume that $p > 1$. Let l be the least non-negative integer such that $r := n + lp > (\lambda + 1)p$. By Case 2 of Proposition 9.3 of [**3**], there exist a tree T and a tree map $f\colon T \to T$ such that $\mathrm{Per}(f) = \mathcal{S}_p^*(r) \setminus p\{2, 3, \ldots, \lambda\}$ and $|\mathrm{En}(T)| = r - p$ (in the terminology of [**3**], f is called an (r, p, q)-*model* with $q = 0$). Note that $|\mathrm{En}(T)| \ge 2p + 1 > \max\{3, p\}$.

When (i) occurs, $r = 3p + lp \in p\mathbb{N}$ and therefore $\mathcal{S}_p^*(r) = \{1\} \cup p\mathbb{N} = \mathcal{S}_p(n)$. Then the lemma holds in this case. On the other hand, if (ii) holds then $\mathcal{S}_p^*(r) = \mathcal{S}_p(r)$, and it is not difficult to see that $\mathcal{F} := \mathcal{S}_p(n) \setminus \mathcal{S}_p^*(r) = \{n\} \cup (\{in + jp : 1 \le i < p/\gcd(p, n), 1 \le j \le li\} \setminus \{r\})$. Since $\mathcal{F}$ is finite or empty, by Lemma 8.1 there exist a tree $S \supset T$ and a tree map $g\colon S \to S$ such that $\mathrm{Per}(g) = \mathrm{Per}(f) \cup \mathcal{F} = \mathcal{S}_p^*(r) \cup \mathcal{F} = \mathcal{S}_p(n)$. □

LEMMA 8.3. *Let $g\colon S \to S$ be a tree map and let $p \ge 2$ be an integer. Then there exist a tree T and a tree map $f\colon T \to T$ such that $T \sqsupset pS$ and $\mathrm{Per}(f) = \{1\} \cup p\,\mathrm{Per}(g)$.*

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Proof. Let X be a p-star and let y be the central point of X. Choose one endpoint e of S. Let T be the tree obtained by attaching p-many disjoint copies of S to X, each copy attached by identifying e with an endpoint of X. Clearly, $T \supset pS$. It is easy to define a map $f: T \to T$ such that $f(y) = y$ and $\mathrm{Per}(f) = \{1\} \cup p\,\mathrm{Per}(g)$. The detailed construction has been done—when g is piecewise monotone—in the proof of Proposition 9.2 of [**3**], and it can be directly extended to the case of a continuous map. □

LEMMA 8.4. *Let $\{f_i: T_i \to T_i\}_{i\in R}$ be a finite set of tree maps. Then there exist a tree T and a tree map $f: T \to T$ such that T contains a subtree homeomorphic to T_i for each $i \in R$ and* $\mathrm{Per}(f) = \bigcup_{i\in R} \mathrm{Per}(f_i)$.

Proof. For each $i \in R$, we choose $y_i \in \mathrm{Fix}(f_i)$. Clearly, there exists a tree T such that, for some $y \in T$;

(i) $T = \bigcup_{i\in R} \widetilde{T}_i$;

(ii) $\widetilde{T}_i \cap \widetilde{T}_j = \{y\}$ for all $i \neq j$;

(iii) each $\widetilde{T}_i$ is homeomorphic to T_i by a homeomorphism ϕ_i that maps y on y_i.

Now let $f: T \to T$ be the map such that $f|_{\widetilde{T}_i} = \phi_i^{-1} \circ f_i \circ \phi_i$ for each $i \in R$. Observe that f is continuous and well defined. Moreover, $f(y) = y$ and $\mathrm{Per}(f) = \bigcup_{i\in R} \mathrm{Per}(f_i)$. □

Now we are ready to prove Theorem 1.2. We recall that conditions (S1)–(S4) have been stated in §1.

Proof of Theorem 1.2. First we prove that, by using Lemma 8.4, it is enough to show that there exist a family of trees $\{T_{\underline{\mathtt{s}}}\}_{\underline{\mathtt{s}}\in \mathsf{S}}$ and, for each $\underline{\mathtt{s}} \in \mathsf{S}$, a tree map $f_{\underline{\mathtt{s}}}: T_{\underline{\mathtt{s}}} \to T_{\underline{\mathtt{s}}}$ satisfying:

(I) $\underline{\mathtt{s}} \in \Sigma_{T_{\underline{\mathtt{s}}}}$; and

(II) $\mathrm{Per}(f_{\underline{\mathtt{s}}}) = \mathcal{K}_{\underline{\mathtt{s}}} \cup \mathcal{F}_{\underline{\mathtt{s}}} \cup (\mathcal{I}_{\underline{\mathtt{s}}} \setminus \lceil \underline{\mathtt{s}} \rceil\{2, 3, \ldots, \lambda_{\underline{\mathtt{s}}}\})$.

Afterwards, we prove that such a family of trees and tree maps exists.

Since S is finite, by Lemma 8.4 there exists a tree map $g: S \to S$ such that

$$\mathrm{Per}(g) = \bigcup_{\underline{\mathtt{s}}\in \mathsf{S}} \mathrm{Per}(f_{\underline{\mathtt{s}}}) = \bigcup_{\underline{\mathtt{s}}\in \mathsf{S}} (\mathcal{K}_{\underline{\mathtt{s}}} \cup \mathcal{F}_{\underline{\mathtt{s}}} \cup (\mathcal{I}_{\underline{\mathtt{s}}} \setminus \lceil \underline{\mathtt{s}} \rceil\{2, 3, \ldots, \lambda_{\underline{\mathtt{s}}}\}))$$

and S contains a tree $\widetilde{T}_{\underline{\mathtt{s}}}$ homeomorphic to $T_{\underline{\mathtt{s}}}$ for each $\underline{\mathtt{s}} \in \mathsf{S}$. By (I), $\underline{\mathtt{s}}$ satisfies (S1) and (S2) (with $\widetilde{T}_{\underline{\mathtt{s}}}$ instead of S) for each $\underline{\mathtt{s}} \in \mathsf{S}$. Since $S \supset \widetilde{T}_{\underline{\mathtt{s}}}$, (S1) (and obviously (S2)) remains true when one replaces $\widetilde{T}_{\underline{\mathtt{s}}}$ by S. Therefore, $\underline{\mathtt{s}} \in \Sigma_S$ for each $\underline{\mathtt{s}} \in \mathsf{S}$. Since $\Sigma_S \subset \Sigma_{S'}$ whenever S' is any tree containing S, without loss of generality we can assume $|\mathrm{En}(S)| \geq N$, and thus the theorem follows.

Now we show that for each $\underline{\mathtt{s}} = (p_1, p_2, \ldots, p_m) \in \mathsf{S}$ there exist a tree $T_{\underline{\mathtt{s}}}$ and a tree map $f_{\underline{\mathtt{s}}}: T_{\underline{\mathtt{s}}} \to T_{\underline{\mathtt{s}}}$ satisfying (I) and (II). We have $p_i \geq 2$ for $1 \leq i < m$ and $\mathcal{K}_{\underline{\mathtt{s}}} = \{p_1, p_1p_2, \ldots, p_1p_2\cdots p_{m-1}\}$. Set $k = p_1p_2\cdots p_{m-1}$ (or $k = 1$ when $m = 1$) and $p = p_m$.

Let M be the set of $_{\lceil \underline{\mathtt{s}} \rceil}\geq$-maximal elements n of $\mathcal{I}_{\underline{\mathtt{s}}}$ such that $n \neq 1$. Observe that M is either empty or finite, and $M = \emptyset$ if and only if $\mathcal{I}_{\underline{\mathtt{s}}} = \{1\}$. We have

$$\mathcal{I}_{\underline{\mathtt{s}}} = \{1\} \cup \bigcup_{n\in M} \mathcal{S}_{\lceil \underline{\mathtt{s}} \rceil}(n). \tag{5}$$

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Moreover, from the definition of the $_{\lceil \underline{s} \rceil}\geq$ ordering and the fact that, by hypotheses, properties (a) and (b) of the statement of Theorem 1.1 hold, we easily get that:

(i) $n \in k(\mathbb{N} \cup 2^\infty)$ for each $n \in M$;

(ii) if $n \in M$ it follows that if $\mathcal{S}_{\lceil \underline{s} \rceil}(n) \subsetneq \{1\} \cup \lceil \underline{s} \rceil \mathbb{N}$ then $\lambda_{\underline{s}} = 0$.

Since $\mathcal{S}_{\lceil \underline{s} \rceil}(n) = \{1\} \cup k\mathcal{S}_p(n/k)$ for each $n \in k\mathbb{N}$ (see, for instance, Remark 5.1, where this fact is proved when $n/k \notin p\mathbb{N}$), from (ii) we get:

(ii′) if $n \in M$ it follows that if $\mathcal{S}_p(n/k) \subsetneq \{1\} \cup p\mathbb{N}$ then $\lambda_{\underline{s}} = 0$.

Moreover, from (5) we easily get that

$$\mathcal{K}_{\underline{s}} \cup (\mathcal{I}_{\underline{s}} \setminus \lceil \underline{s} \rceil \{2, 3, \ldots, \lambda_{\underline{s}}\}) = \{1\} \cup \mathcal{K}_{\underline{s}} \cup k \bigcup_{n \in M} (\mathcal{S}_p(n/k) \setminus p\{2, 3, \ldots, \lambda_{\underline{s}}\}). \tag{6}$$

Now assume that we have constructed a tree, which we denote by S_m, such that:

(p1) $\max\{3, p\} \leq |\mathrm{En}(S_m)|$;

and a tree map $g_m \colon S_m \to S_m$ such that

$$\mathrm{Per}(g_m) = \{1\} \bigcup_{n \in M} \mathcal{S}_p(n/k) \setminus p\{2, 3, \ldots, \lambda_{\underline{s}}\}. \tag{7}$$

Then, since $p_i \geq 2$ for $1 \leq i < m$, by using $m - 1$ times Lemma 8.3 by backwards induction we obtain, for $1 \leq i < m$, tree maps $g_i \colon S_i \to S_i$ such that:

(p2) $S_i \sqsupset p_i S_{i+1}$;

(p3) $\mathrm{Per}(g_i) = \{1\} \cup p_i \,\mathrm{Per}(g_{i+1})$.

Since S_m is not an interval, from (p2) it easily follows that:

(p4) S_i is not a star for $1 \leq i < m$.

Then we define $T_{\underline{s}} := S_1$ and $f_{\underline{s}} := g_1$. From (p3) we easily get that $\mathrm{Per}(f_{\underline{s}}) = \{1\} \cup \mathcal{K}_{\underline{s}} \cup k \,\mathrm{Per}(g_m)$. Consequently, (II) follows from (6), (7) and Lemma 8.1. On the other hand, (I) holds by (p1), (p2) and (p4). Thus the theorem follows.

Hence, to end the proof of the theorem it is enough to show that there exist a tree S_m and a tree map $g_m \colon S_m \to S_m$ satisfying (p1) and (7).

When $M = \emptyset$, clearly we can take S_m to be a $\max\{p, 3\}$-star and g_m to be the identity map on S_m. Assume that $M \neq \emptyset$. For any $n \in M$, since $n \neq 1$ and (ii′) holds, we can use Lemma 8.2 to obtain a tree K_n with $|\mathrm{En}(K_n)| \geq \max\{3, p\}$ and a tree map $h_n \colon K_n \to K_n$ such that $\mathrm{Per}(h_n) = \mathcal{S}_p(n/k) \setminus p\{2, 3, \ldots, \lambda_{\underline{s}}\}$. Since M is finite, by Lemma 8.4 there exist a tree, which we define to be S_m, containing a subtree homeomorphic to K_n for each $n \in M$, and a tree map $g_m \colon S_m \to S_m$ verifying (7). Moreover, (p1) holds since $\max\{3, p\} \leq |\mathrm{En}(K_n)|$ for each $n \in M$. □

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