Analysis of composite plates with variable stiffness using Galerkin Method

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ABSTRACT

A solution methodology is developed to solve plane stress problem of composite plates with variable stiffness by using Galerkin technique and polynomials as trial functions. In the solution process, analytical computation has been done wherever it is possible, and analytical-numerical type approach has been made for all problems. The methodology is applied to two known case problems, composite plate with variable fibre content and laminated plate with spatially varying fibre orientations. The formulation of these problems results into coupled partial differential equations (with variable coefficients). The solutions of these equations are obtained using the polynomials as trial functions in the Galerkin method. The results are compared to that of Ritz and collocation technique published elsewhere. The method is found to determine closely both the displacements and the stresses with a few number of terms and in good agreement with other approximating methods. Computations on some examples show that, the method with the help of a symbolic math package is simple and efficient for solving these types of problems in engineering applications.

1.0 INTRODUCTION

Due to their light weight and high stiffness, laminated composites have been used widely in many engineering applications, and many researchers have focused on the analysis of laminated composites. Recently, variable stiffness composite plates have been used in advanced structures[1]. For analysis of such laminates, numerical methods such as Rayleigh-Ritz, Galerkin, and Collocation techniques may be used.

The classical Rayleigh-Ritz method has been deployed by many researchers for static and dynamic responses of plates. This method utilises a functional that minimisation of it results into a set of algebraic equations. Coefficient matrix is nonsingular, and consequently the system of equations is uniquely solvable[2]. Here it should be noted that the trial functions must satisfy at least the geometric boundary conditions and must be mathematically complete for convergence[3].

Among the methods of weighted residuals, the Galerkin method is a powerful numerical solution technique to differential equations. This method involves the direct use of the differential equations (or variational form, i.e., virtual work), so it does not need the existence of a functional[4]. The trial functions do not need to satisfy all the boundary conditions, and this gives flexibility to the method, which can be applied to a broad class of problems. However, one deficiency must be pointed out that, a poor choice of the trial functions in the Galerkin method may produce an ill-conditioned coefficient matrix so that the equations may be difficult to solve within acceptable limits of accuracy[5].

Bhat[6] used orthogonal polynomials in Rayleigh-Ritz method to study natural frequencies of rectangular plates. Following this work, orthogonal polynomials have been used for the study of vibration and buckling analysis of plates[7]. Due to above mentioned deficiency of Galerkin method, some modifications have been made and powerful solution techniques (such as Bubnov-Galerkin, Petrov-Galerkin) are been developed for the analysis of engineering problems. De Sampaio[8] has used Petrov-Galerkin for the formulation of convection-diffusion problems, and Westerink and Shea[9] has applied higher degree Petrov-Galerkin to the transient solution of convection-diffusion equation. More recently, Dennis[10] used the Galerkin method to study the responses of numerous plate and shell, both linear and (geometrically) nonlinear. Saadatpour and Azhari[11] proposed a procedure based on Galerkin method to express the geometry of general shape plates. Tanriover and Senocak applied Galerkin method to study large deflection analysis of composite plates[12]. Storch and Strang[13] examine the boundary conditions in Ritz-Galerkin approach.

In the present work, the Galerkin method is applied plane elasticity problems successfully. Plane stress problem of composite plate with variable fibre content and the plane elasticity problem of laminated plate with spatially varying fibre orientations result into coupled partial differential equations with variable coefficients. The solutions of these equations are obtained using the polynomials as trial functions to

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approximate the two dimensional functions (displacements). The trial functions are constructed in a way that essential boundary conditions are satisfied. The method is found to determine closely both the displacements and the stresses with a few numbers of terms and in good agreement with other (Ritz, point collocation) approximating methods. A major advantage of the method presented here for the analysis of variable stiffness composite plates is that it produces the solutions easily and effectively with the help of a symbolic math package. Analytical-numerical type approach has been made for all problems solved.

2.0 GOVERNING EQUATIONS

Consider a laminated anisotropic rectangular plate of length a in the x-direction, width b in the y-direction subjected to external loading along the edges. The principle of virtual work yields the following in-plane equilibrium equations:

$$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left(N_{x,x}+N_{xy,y}\right)\delta u\,\mathrm{d}x\mathrm{d}y-\int_{-b/2}^{b/2}N_x\,\delta u\Big|_{-a/2}^{a/2}\mathrm{d}y - \int_{-a/2}^{a/2}N_{xy}\,\delta u\Big|_{-b/2}^{b/2}\mathrm{d}x=0 \qquad \ldots (1a)$$

$$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left(N_{y,y}+N_{xy,x}\right)\delta v\,\mathrm{d}x\mathrm{d}y-\int_{-a/2}^{a/2}N_y\,\delta v\Big|_{-b/2}^{b/2}\mathrm{d}x - \int_{-b/2}^{b/2}N_{xy}\,\delta v\Big|_{-a/2}^{a/2}\mathrm{d}y=0 \qquad \ldots (1b)$$

where N_x, N_y are the resultant forces per unit length; δu, δv are the arbitrary displacements in the x and y directions respectively, and N_{xy} is the shear force resultant per unit length in the xy plane.

The well known constitutive relation for a symmetrically laminated plate is given by,

$$\{N\}=[\mathbf{A}]\{\varepsilon\} \qquad \ldots (2)$$

where $\{N\}=\{N_x,\ N_y,\ N_{xy}\}^T$ represents the stress resultant forces, $\{\varepsilon\}=\{\varepsilon_x,\ \varepsilon_y,\ \varepsilon_{xy}\}^T$ denote the planar strains, and $\mathbf{A}$ is the extentional stiffness matrix, which may not have a constant value. The linear strain displacement relations for plane problems are

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \gamma_{xy}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \qquad \ldots (3)$$

where u and v are the displacements in the x and y directions respectively. Equations (2) and (3) are used in Equations (1) to obtain equilibrium equations in terms of displacements in the weak form.

3.0 GALERKIN SOLUTION

In the Galerkin method, the displacements of the plate are approximated in the form shown below:

$$u=\sum_{m=0}^{M}\sum_{n=0}^{N}a_{mn}U_{mn} \qquad \ldots (4a)$$

$$v=\sum_{m=0}^{M}\sum_{n=0}^{N}b_{mn}V_{mn} \qquad \ldots (4b)$$

where a_{mn} and b_{mn} are unknown constants, U_{mn} and V_{mn} are the trial functions, and M and N are the number of terms in x and y directions, respectively. Herein, polynomials are used as trial function, which are chosen to satisfy the geometric boundary conditions, where as natural boundary conditions are not satisfied. In this case, simultaneous approximation is made to the solutions of differential equations and to the boundary conditions. Substituting Equations (4) into Equations (1) yields the following Galerkin equations[14].

$$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left(N_{x,x}+N_{xy,y}\right)U_{mn}\,\mathrm{d}x\mathrm{d}y-\int_{-b/2}^{b/2}N_x\,U_{mn}\Big|_{-a/2}^{a/2}\mathrm{d}y - \int_{-a/2}^{a/2}N_{xy}\,U_{mn}\Big|_{-b/2}^{b/2}\mathrm{d}x=0 \qquad \ldots (5a)$$

$$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left(N_{y,y}+N_{xy,x}\right)V_{mn}\,\mathrm{d}x\mathrm{d}y-\int_{-a/2}^{a/2}N_y\,V_{mn}\Big|_{-b/2}^{b/2}\mathrm{d}x - \int_{-b/2}^{b/2}N_{xy}\,V_{mn}\Big|_{-a/2}^{a/2}\mathrm{d}y=0 \qquad \ldots (5b)$$

Note that Equations (5) are defined in terms approximated displacements, and the unknown parameters a_{mn} and b_{mn} can be determined by solving this system of equations.

4.0 CHOICE OF TRIAL FUNCTIONS

We apply the Galerkin method to the solution of some plane elasticity problems and use polynomials as trial functions. The trial functions are chosen to satisfy the geometric boundary conditions, whereas natural boundary conditions are not satisfied. In this case, simultaneous approximation is made to the solutions of differential equations and to the boundary conditions. In order to keep track of the terms in a polynomial used for two-dimensional approximation it is convenient to make use of Pascal's triangle, which lists the terms in a systematic manner.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & x & y & & \\ & & & x^2 & xy & y^2 & \\ & & x^3 & x^2y & xy^2 & y^3 & \\ & x^4 & x^3y & x^2y^2 & xy^3 & y^4 & \\ x^5 & \ldots & & & & & \end{array} \qquad \ldots (6)$$

The trial functions are constructed by taking certain number of terms from Equation (6), and modified according to the boundary conditions of the problem to satisfy the geometric boundary conditions.

5.0 PLANAR LOADING OF COMPOSITE PLATES WITH VARIABLE FIBRE SPACING

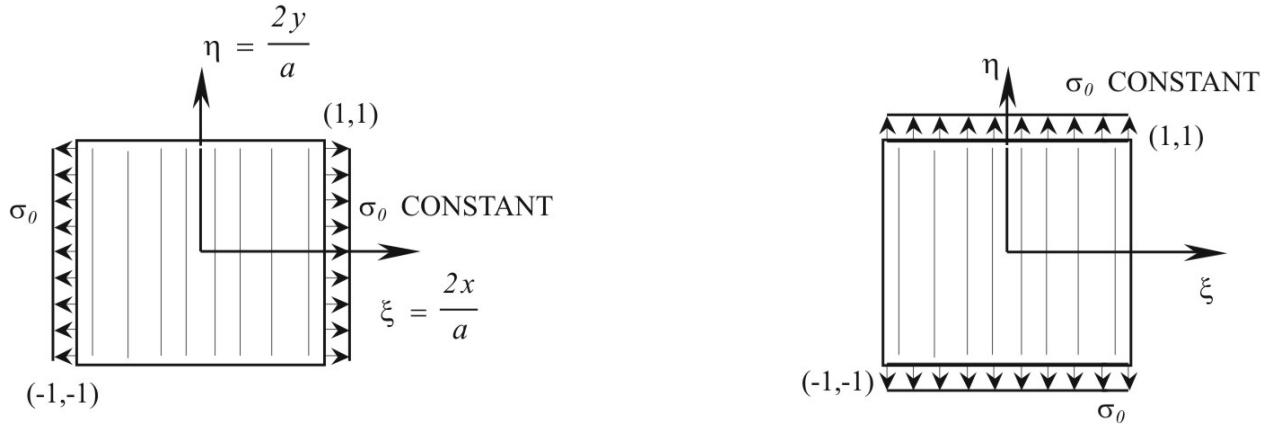
The structural performance of a lamina may be improved by varying fibre content with respect to in-plane location. The formulation of governing equations of such lamina results into a system of differential equations with variable coefficients. The derivation given here follows the lines of presentation in Ref. 15, and is given here for completeness. Aligning the y axis (called longitudinal) with the fibres the following expressions are obtained for a lamina.

$$E_x=\frac{E_f}{V_f+R_1(1-V_f)} \qquad \ldots (7a)$$

$$E_y=E_f\left(V_f+\frac{1-V_f}{R_1}\right) \qquad \ldots (7b)$$

$$\upsilon_{yx}=\upsilon_f\left(V_f+\frac{1-V_f}{R_2}\right) \qquad \ldots (7c)$$

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(a) Case 1 (b) Case 2

Figure 1. Loading of the plate.

$$\upsilon_{xy} = \frac{E_x}{E_y}\upsilon_{yx} \quad \ldots (7d)$$

$$G_{xy} = \frac{E_f}{2(1+\upsilon_f)(R_3(1-V_f)+V_f)} \quad \ldots (7e)$$

where $R_1 = E_f/E_m$, $R_2 = \nu_f/\nu_m$, and $(R_3) = (R_1/R_2)((R_2 + \nu_f)/(1 + \nu_f))$. Here, E_x, E_y are Young's modulus for the plate transverse and parallel to the fibres, respectively, and G_{xy} is shear modulus for the plate. E_f, E_m are Young's modulus, and ν_f, ν_m Poisson's ratios for the fiber and matrix, respectively. ν_{xy}, ν_{yx} denote minor and major Poisson's ratios for the plate, respectively. In Equations (7), V_f is volume fraction of the lamina occupied by the fibre, which is a function of ξ, where $\xi = 2x/a$ (called transverse direction). The non-dimensional η is defined as $\eta = 2y/a$.

For a square plate (with dimensions *axa*) and unit thickness, force resultants are given by Ref. 15,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \ldots (8)$$

where,

$$C_{11} = \frac{E_x}{1-\upsilon_{yx}\upsilon_{xy}} \quad \ldots (9a)$$

$$C_{12} = \upsilon_{yx}C_{11} \quad \ldots (9b)$$

$$C_{22} = \frac{E_y}{1-\upsilon_{yx}\upsilon_{xy}} \quad \ldots (9c)$$

$$C_{66} = G_{xy} \quad \ldots (9d)$$

In the case of symmetry, some terms may be excluded and the size of the problem can be reduced. For a doubly symmetric problem, a quarter of the plate may be considered and the following polynomials are used to represent the displacement field.

$$u = a_1\xi + a_2\xi^3 + a_3\xi\eta^2 + a_4\xi^5 + a_5\xi^3\eta^2 + a_6\xi\eta^4 + a_7\xi^7 + ... \quad \ldots (10a)$$

$$v = b_1\eta + b_2\eta\xi^2 + b_3\eta^3 + b_4\eta\xi^4 + b_5\eta^3\xi^2 + b_6\eta^5 + b_7\eta\xi^6 + ... \quad \ldots (10b)$$

In the above equations, the terms are taken from Equation (6), ($\xi = 2x/a$, $\eta = 2y/a$) and the displacement field is determined. For example, u is constructed by taking the terms from the Pascal's triangle that contain odd powers of ξ and even powers of η. Similarly, v is constructed with odd powers of η and even powers of ξ, so that the essential boundary conditions are satisfied.

The approximated displacements of Equation (10) are substituted into Equation (5) in conjunction with Equations (8), and the resulting system of equations is solved. The resulting equations are shown in Appendix, Equation A(2). The symbolic math package, Mathematica™[16] performed the extensive algebra.

Taking $V_f = 1 - \xi^2$, $R_1 = 10$, $R_2 = 1$ and $\nu_f = 0{\cdot}25$, the cases of uniformly applied longitudinal stress or uniformly applied transverse normal stress are investigated.

Case 1 (transversely applied stress, shown in Fig. 1(a):

$\sigma_x = \sigma_0$, $\tau_{xy} = 0$ on the edges $\xi = \pm 1$
$\sigma_y = \tau_{xy} = 0$ on the edges $\eta = \pm 1$

Case 2 (longitudinally applied stress, shown in Fig. 1(b):

$\sigma_x = \tau_{xy} = 0$ on the edges $\xi = \pm 1$
$\sigma_y = \sigma_0$, $\tau_{xy} = 0$ on the edges $\eta = \pm 1$

The displacements and stresses are determined and the results are compared to that of Ref. 15, which utilises the Ritz approach for the solution of this problem. Equilibrium equations are solved exactly for the boundary conditions of $\sigma_x = \tau_{xy} = 0$ on the edges $\xi = \pm 1$, $v = 0$, $\tau_{xy} = 0$ on the edges $\eta = \pm 1$, and closed-form solution is given in Ref. 15. This case is also solved by the Galerkin method, (see Equation A(3)) and the displacements are shown in Fig. 2. We have solved Case 1 and Case 2, and computed the stresses from the displacements. The displacements are given by Tables 1-4, and the variation of displacement v is shown for Case 1, by Fig. 3. The stresses are calculated from the displacements, and Figs 4 and 5 show their behaviour. When the results of the Galerkin method is compared to that of Ritz[15], a very close agreement is observed.

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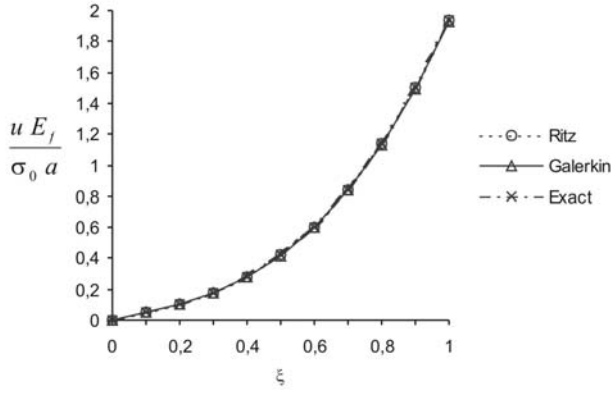


Figure 2. Comparison of the results.

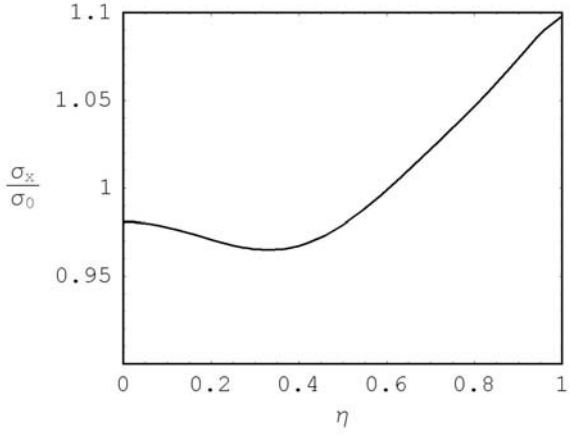


Figure 4. Transverse stress at $\xi = 0$ for Case 1.

6.0 IN-PLANE RESPONSE OF LAMINATES WITH SPATIALLY VARYING FIBER ORIENTATIONS

In this section, a solution for a symmetrically laminated composite plate with spatially varying fibre orientations has been obtained. The derivation of the governing differential equation of equilibrium given in Ref. 17, as follows: The well known constitutive relation for a symmetrically laminated plate is given by,

$$N = \mathbf{A}\varepsilon \qquad \ldots (11)$$

where N represents the stress resultant forces, $\mathbf{A}$ denote the midplane strains, and $\mathbf{A} = \mathbf{A}(x,y)$ is the extentional stiffness matrix, which does not have a constant value. Using the material invariants[17] for an orthotropic lamina, U_1, U_2, ... U_5 the following is obtained for the matrix $\mathbf{A}$.

$$\mathbf{A} = h\begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & U_5 \end{bmatrix} + h\begin{bmatrix} U_2 & 0 & 0 \\ 0 & -U_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathrm{Cos}(2\theta) + h\begin{bmatrix} U_3 & -U_3 & 0 \\ -U_3 & U_3 & 0 \\ 0 & 0 & -U_3 \end{bmatrix}\mathrm{Cos}(4\theta) \qquad \ldots (12)$$

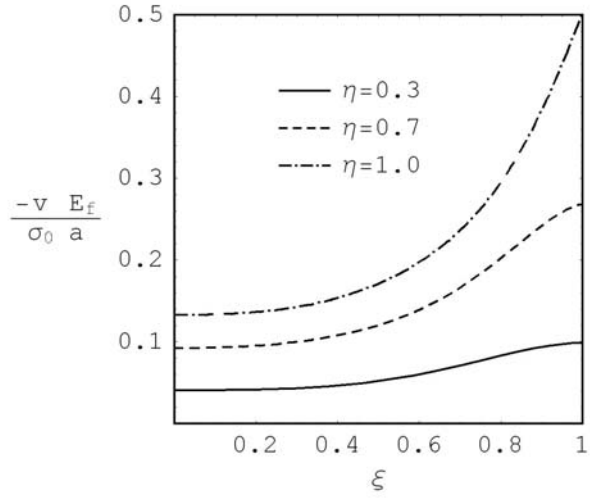


Figure 3. Displacements in the η direction for Case 1.

where, h is the total thickness of the laminate, and θ is the fiber angle. For a balanced symmetric laminate in-plane constitutive relation is the following.

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \ldots (13)$$

Equation (13) is substituted into Equation (5) and the related Galerkin equations `are obtained. Herein, the fibre angle is a function of the co-ordinate x.

There are several ways to vary the fibre angle. A simple fibre pattern may be parallel fibres to one another, the fibre orientation is made to vary only along one of the co-ordinates. A linear variation of the fibre orientation along the x co-ordinate is obtained if the fibre orientation is defined by

$$\theta(x) = \frac{2(T_1 - T_0)}{a}x + T_0 \qquad \ldots (14)$$

where T_0 is the fibre angle at the plate mid-length $x = 0$, and reaches a value of T_1 at the plate end $x = a/2$. Introducing Equation (14) into Equation (12), and solving Equation (5) by using Equation (13) under certain boundary conditions gives the results. The polynomial trial functions have been used to approximate the solution for two different sets of boundary conditions. In both cases one quarter of the plate is modeled, and the boundary conditions at $x = 0$ and $y = 0$ represent symmetry conditions ($T_0 = 45°$, $T_1 = 0°$).

Case 3: Transverse edges are traction free. (see Fig. 6(a))

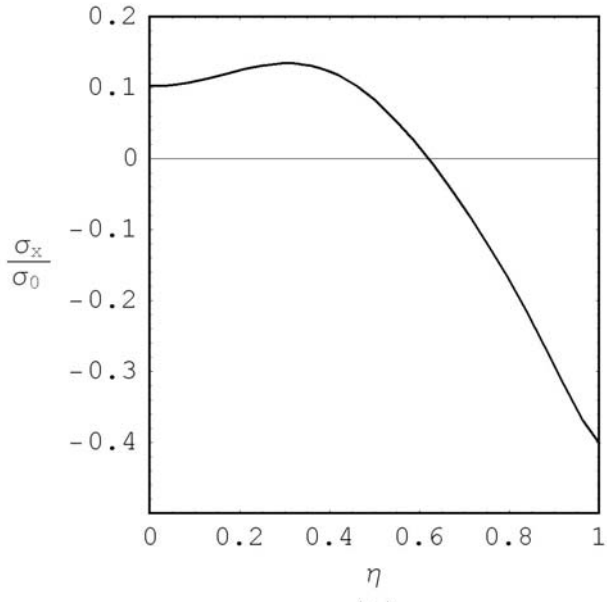
$u = u_0$, $N_{xy} = 0$ at $x = a/2$

$N_{xy} = N_y = 0$ at $y = b/2$

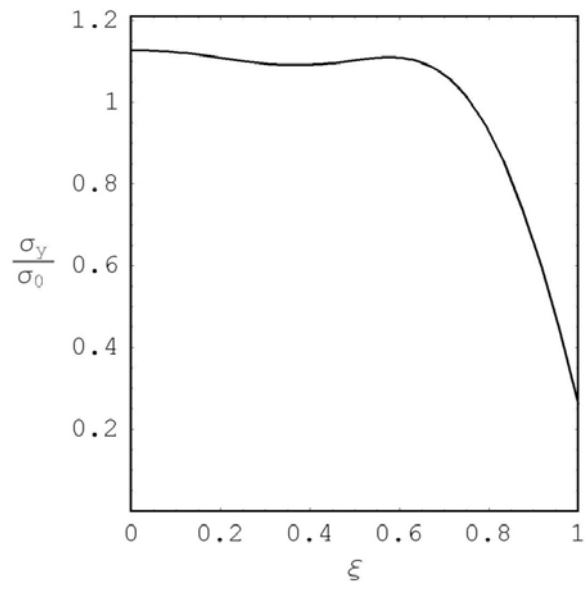
The following trial functions are obtained from Equation (6). The terms that contain are excluded from the Pascal's triangle to approximate u, and similarly the terms containing x are omitted to approximate v.

$$u = 2x + (x - 0{\cdot}5)x(a_1 + a_2x + a_3x^2 + a_4y^2 + a_5x^3 + a_6xy^2 + a_7x^4 + ...) \qquad \ldots (15a)$$

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(a) Transverse stress at $\xi = 0$



(b) Longitudinal stress at $\eta = 0$.

Figure 5. Stresses for Case 2.

Table 1
Nondimensionalised displacement in the ξ direction for Case 1, ($uE_f/\sigma_0 a$), (at η = 0)

	Number of terms				
ξ	3	8	12	16	24
0·0	0·00000	0·00000	0·00000	0·00000	0·00000
0·1	0·04789	0·05089	0·04992	0·05018	0·05083
0·2	0·10471	0·11036	0·10865	0·10907	0·11010
0·3	0·17936	0·18710	0·18500	0·18553	0·18669
0·4	0·28078	0·29006	0·28783	0·28862	0·28987
0·5	0·41788	0·42842	0·42619	0·42750	0·42875
0·6	0·59957	0·61156	0·60953	0·61132	0·61211
0·7	0·83478	0·84878	0·84766	0·84922	0·84916
0·8	1·13244	1·14891	1·15029	1·15048	1·15009
0·9	1·50146	1·51965	1·52539	1·52332	1·52343
1·0	1·95075	1·96668	1·97574	1·96952	1·96811

Table 2
Nondimensionalised displacement in the η direction for Case 1, ($-vE_f/\sigma_0 a$), (at ξ = 0)

	Number of terms				
η	3	8	12	16	24
0·0	0·00000	0·00000	0·00000	0·00000	0·00000
0·1	0·01163	0·01295	0·01342	0·01362	0·01381
0·2	0·02331	0·02589	0·02679	0·02716	0·02738
0·3	0·03505	0·03883	0·04006	0·04054	0·04060
0·4	0·04691	0·05176	0·05318	0·05373	0·05349
0·5	0·05892	0·06472	0·06615	0·06670	0·06622
0·6	0·07111	0·07774	0·07895	0·07949	0·07896
0·7	0·08352	0·09091	0·09163	0·09218	0·09118
0·8	0·09619	0·10432	0·10428	0·10493	0·10503
0·9	0·10916	0·11812	0·11707	0·11799	0·11853
1·0	0·12246	0·13250	0·13027	0·13174	0·13257

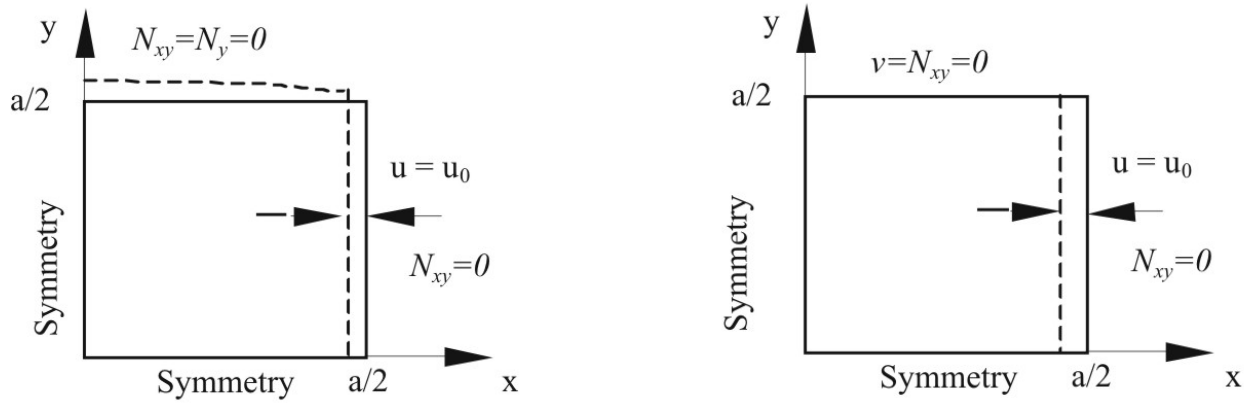
Table 3
Nondimensionalised displacement in the direction for Case 2, ($-uE_f/\sigma_0 a$), (at η = 1/2)

	Number of terms				
ξ	3	8	12	16	24
0·0	0·00000	0·00000	0·00000	0·00000	0·00000
0·1	0·00822	0·00709	0·01007	0·00933	0·00873
0·2	0·01695	0·01404	0·01972	0·01806	0·01679
0·3	0·02671	0·02083	0·02859	0·02581	0·02439
0·4	0·03799	0·02773	0·03658	0·03263	0·03235
0·5	0·05134	0·03534	0·04387	0·03901	0·04089
0·6	0·06724	0·04472	0·05117	0·04629	0·04959
0·7	0·08622	0·05736	0·05976	0·05631	0·05961
0·8	0·10878	0·07511	0·07145	0·07205	0·07577
0·9	0·13545	0·10009	0·08839	0·09745	0·10328
1·0	0·16672	0·13443	0·11227	0·13636	0·14619

Table 4
Nondimensionalised displacement in the direction for Case 2, ($vE_f/\sigma_0 a$), (at ξ =0).

	Number of terms				
η	3	8	12	16	24
0·0	0·00000	0·00000	0·00000	0·00000	0·00000
0·1	0·04655	0·05181	0·05369	0·05447	0·05494
0·2	0·09324	0·10360	0·10717	0·10862	0·10909
0·3	0·14024	0·15534	0·16024	0·16217	0·16204
0·4	0·18767	0·20707	0·21775	0·21493	0·21383
0·5	0·23569	0·25889	0·26460	0·26682	0·26497
0·6	0·28446	0·31099	0·31581	0·31797	0·31605
0·7	0·33411	0·36365	0·36653	0·36873	0·36757
0·8	0·39479	0·41730	0·41714	0·41973	0·41991
0·9	0·43665	0·47250	0·46830	0·47198	0·47369
1·0	0·48985	0·53001	0·52109	0·52695	0·52982

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(a) Case 3

(b) Case 4

Figure 6. Quarter plate boundary conditions.

$$v = y(b_1 + b_2 y + b_3 x^2 + b_4 y^2 + b_5 x^3 + b_6 x^2 y + b_7 y^3 + b_8 x^4 + ...) \quad \ldots (15b)$$

Case 4: The transverse edges are fixed (Fig. 6 (b)).

$u = u_0$, $N_{xy} = 0$ at $x = a/2$

$v = u_0$, $N_{xy} = 0$ $y = b/2$

For this case the displacements field is chosen to be,

$$u = 2x + (x - 0{\cdot}5)x(a_1 + a_2 x + a_3 x^2 + a_4 y^2 + a_5 x^3 + a_6 xy^2 + a_7 x^4 + ...) \quad \ldots (16a)$$

$$v = y(y - 0{\cdot}5)(b_1 + b_2 y + b_3 x^2 + b_4 y^2 + b_5 x^3 + b_6 x^2 y + b_7 y^3 + b_8 x^4 + ...) \quad \ldots (16b)$$

Equations (5) are solved for the boundary conditions defined by Case 3 and Case 4, and the results are compared to that of the point collocation technique given in Ref. 17. The Galerkin solution is obtained from Equation A(4) (see Appendix) for Case 3, and from equation A(5) for Case 4. The displacements for Case 3 are given by Tables 5 and 6, and the displacement for Case 4 in the direction is given by Table 7. The v displacement for Case 4, is found to be zero in the domain, which is the exact solution for this case. The displacements in the x direction, are shown for Case 3 and 4 by Fig. 7. Figure 8 shows the stresses computed from the displacements.

7.0 REMARKS

The use of symbolic math package permits the easy development of efficient analytical-numerical approach to the problems. In the Galerkin method used here, the evaluations of integrals are symbolically computed. If M and N terms are considered for the functions u and v, $(M + N)^2$ integrals should be computed to implement the method. This is easily done with the choice of polynomials.

Table 5
Nondimensionalised displacement in the *x* direction for Case 3, u/u_0 (at $y = b/2$)

$\frac{x}{a}$	Number of terms 6	8	12	16
0·0	0·00000	0·00000	0·00000	0·00000
0·1	0·36611	0·34549	0·34302	0·34069
0·2	0·60195	0·56726	0·56125	0·55823
0·3	0·76447	0·72586	0·72042	0·71624
0·4	0·88981	0·86153	0·86024	0·85690
0·5	1·00000	1·00000	1·00000	1·00000

Table 6
Nondimensionalised displacement in the direction for Case 3, $-v/u_0$ (at $y = b/2$)

$\frac{x}{a}$	Number of terms 6	8	12	16
0·0	1·51685	1·55128	1·58874	1·59486
0·1	1·47785	1·50719	1·52325	1·52468
0·2	1·37750	1·38566	1·36799	1·36445
0·3	1·24078	1·21162	1·17111	1·16793
0·4	1·09269	1·02180	0·98648	0·98509
0·5	0·95822	0·86467	0·90348	0·89353

Table 7
Nondimensionalised displacement in the direction for Case 4, u/u_0 (at $y = b/2$)

$\frac{x}{a}$	Number of terms 4	6	8	12	Ref. 17
0·0	0·00000	0·00000	0·00000	0·00000	0·00000
0·1	0·33113	0·33130	0·33119	0·33118	0·33796
0·2	0·55814	0·55729	0·55730	0·55732	0·56504
0·3	0·72756	0·72802	0·72810	0·72808	0·73415
0·4	0·86997	0·87029	0·87019	0·87020	0·87346
0·5	1·00000	1·00000	1·00000	1·00000	1·00000

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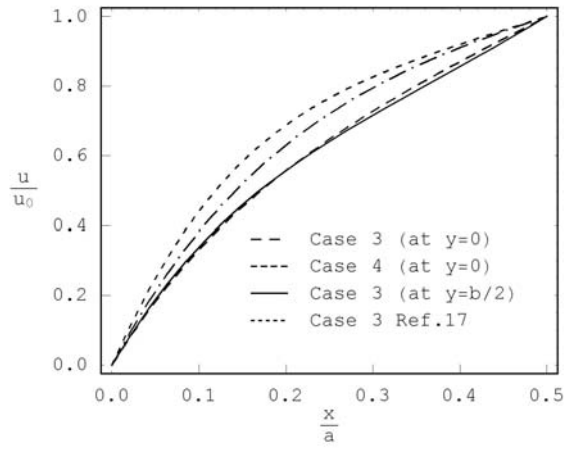


Figure 7. Displacements in the *x* direction.

For Case 3 and Case 4, closed-form solutions are given in Ref 15. Norms of errors for u and v are computed and nondimensionalised by u_0a^2. $\varepsilon(u) = \| u - u_n \| / u_0 a$ and) $\varepsilon(v) = \| v - v_n \| / u_0 a$ show the errors for u and v, respectively, where $\| \ \|$ denotes a norm (u_n and v_n are the results of Galerkin solution). Convergence in the mean implies that and should be less than prescribed values[18]. The computed errors are given by Table 8.

8.0 CONCLUSIONS

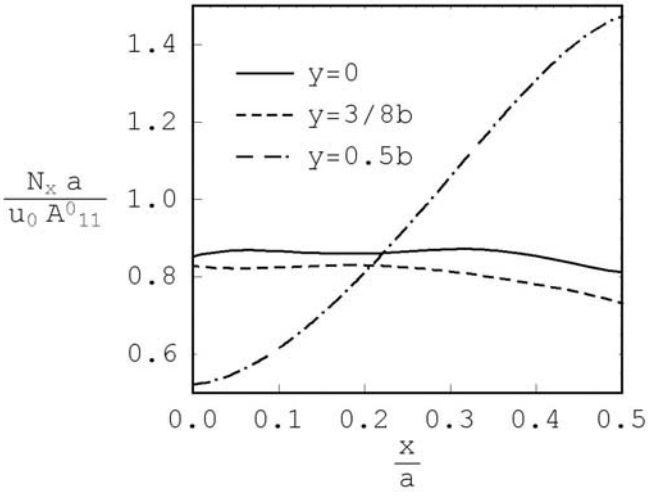
It has been established that the Galerkin method performs well for the plane elasticity problem of composite plates with variable stiffness. The presented solution methodology is applied to two known case problems, and observed that the results are in good agreement. If the essential boundary conditions are satisfied, the polynomials can be used effectively as trial functions. The present method proves to be simple, efficient, and easy to implement. The present solution methodology may be used to solve variable stiffness plane stress problems of composite plates in an easy and effective way with the help of a symbolic math package.

Table 8
Norms of errors for *u* and *v*

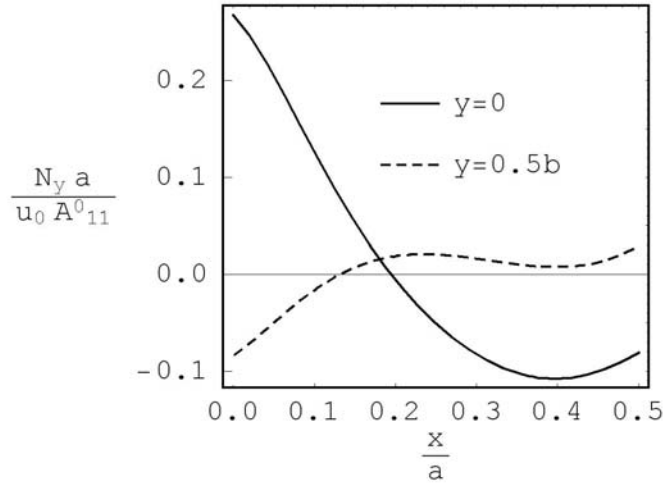
Number of terms	Case 3		Case 4	
	$\varepsilon(u)$	$\varepsilon(v)$	$\varepsilon(v)$	$\varepsilon(v)$
4	–	–	0·0012689	3e–16
6	0·006871	0·072620	0·0012544	6e–15
8	0·006555	0·070802	0·0012537	1e–14
12	0·006080	0·071099	0·0012537	8e–12
16	0·006017	0·070900	–	–

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(a) Axial stress resultant



(b) Transverse stress resultant

Figure 8. Stress resultants for Case 3. (A^0_{11} is the value of A_{11} at $\theta = 0$).

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APPENDIX

For the Galerkin solution of Case 1 and Case 2, the displacement field defined by Equation (10) is rewritten as follows.

$$u = \sum_{i=1}^{M} a_i \phi_{ui} = a_1\xi + a_2\xi^3 + a_3\xi\eta^2 + a_4\xi^5 + a_5\xi^3\eta^2 + a_6\xi\eta^4 + a_7\xi^7 + ... \quad \text{. . . (A1)}$$

$$v = \sum_{j=1}^{N} b_j \phi_{vj} = b_1\eta + b_2\eta\xi^2 + b_3\eta^3 + b_4\eta\xi^4 + b_5\eta^3\xi^2 + b_5\eta^3\xi^2 + b_6\eta^5 + b_7\eta\xi^6 + ...$$

The weight functions are taken to be the trial functions of the displacements, and introducing Equation (10) and Equation (8) into Equation (5), the following integral equations are obtained. Here, i and j are positive integers ranging from 1 to $M = N$, where M and N represent the number of terms considered for the displacements u and v, respectively. ϕ_{ui} and ϕ_{vi} are the weight functions for the first and the second equation of Equation A(2) respectively, and they are obtained from the displacements; e.g., $\phi_{u1} = \xi$, $\phi_{v1} = \eta$, $\phi_{u2} = \xi^3$, $\phi_{v2} = \eta\xi^2$, for i, j = 1, 2. Following this, $(M + N)$ simultaneous algebraic equations are obtained to implement the Galerkin method.

$$\int_0^1\int_0^1 \phi_{ui}\left(\frac{\partial}{\partial\xi}\left(C_{11}\frac{\partial u}{\partial\xi} + C_{12}\frac{\partial v}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left[C_{66}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right)\right]\right)d\xi\,d\eta$$

$$-a\int_0^1 \phi_{ui}\left(C_{11}\frac{\partial u}{\partial\xi} + C_{12}\frac{\partial v}{\partial\eta} - \sigma_0\right)\Big|_{\xi=1} d\eta$$

$$-a\int_0^1 \phi_{ui}\, C_{66}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right)\Big|_{\eta=1} d\xi = 0 \quad \text{. . . (A2)}$$

$$i = 1,2, ..., M$$

$$\int_0^1\int_0^1 \phi_{vj}\left(\frac{\partial}{\partial\eta}\left(C_{12}\frac{\partial u}{\partial\xi} + C_{22}\frac{\partial v}{\partial\eta}\right) + \frac{\partial}{\partial\xi}\left[C_{66}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right)\right]\right)d\xi\,d\eta$$

$$-a\int_0^1 \phi_{vj}\left(C_{12}\frac{\partial u}{\partial\xi} + C_{22}\frac{\partial v}{\partial\eta}\right)\Big|_{\eta=1} d\xi$$

$$-a\int_0^1 \phi_{vj}\, C_{66}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right)\Big|_{\xi=1} d\eta = 0$$

$$j = 1,2, ..., N$$

For Case 2, the same displacement field given by equation A(1) is utilised. Note that the difference appears in the boundary residue terms.

$$\int_0^1\int_0^1 \phi_{ui}\left(\frac{\partial}{\partial\xi}\left(C_{11}\frac{\partial u}{\partial\xi} + C_{12}\frac{\partial v}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left[C_{66}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right)\right]\right)d\xi\,d\eta$$

$$-a\int_0^1 \phi_{ui}\left(C_{11}\frac{\partial u}{\partial\xi} + C_{12}\frac{\partial v}{\partial\eta}\right)\Big|_{\xi=1} d\eta$$

$$-a\int_0^1 \phi_{ui}\, C_{66}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right)\Big|_{\eta=1} d\xi = 0 \quad \text{. . . (A3)}$$

$$\int_0^1\int_0^1 \phi_{vj}\left(\frac{\partial}{\partial\eta}\left(C_{12}\frac{\partial u}{\partial\xi} + C_{22}\frac{\partial v}{\partial\eta}\right) + \frac{\partial}{\partial\xi}\left[C_{66}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right)\right]\right)d\xi\,d\eta$$

$$-a\int_0^1 \phi_{vj}\left(C_{12}\frac{\partial u}{\partial\xi} + C_{22}\frac{\partial v}{\partial\eta} - \sigma_0\right)\Big|_{\xi=1} d\xi$$

$$-a\int_0^1 \phi_{vj}\, C_{66}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right)\Big|_{\eta=1} d\eta = 0$$

In Case 3, Equation (15) is introduced into Equation (13), and from Equation (5) the following system of equations is determined. The weight functions are obtained from Equation (15). For example; $\phi_{u1} = (x - 0{\cdot}5)x$, $\phi_{v1} = y$, $\phi_{u2} = (x - 0{\cdot}5)x^2$, $\phi_{v2} = y^2$.

$$\int_0^{b/2}\int_0^{a/2} \phi_{uk}\left(A_{11}(x)\frac{\partial^2 u}{\partial x^2} + A_{66}(x)\frac{\partial^2 u}{\partial y^2} + \frac{\partial A_{11}(x)}{\partial x}\frac{\partial u}{\partial x} + \left[A_{12}(x) + A_{66}(x)\right]\frac{\partial^2 v}{\partial x\partial y} - \frac{\partial A_{12}(x)}{\partial x}\frac{\partial v}{\partial y}\right)dx\,dy$$

$$-\int_0^{a/2} \phi_{uk} A_{66}(x)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\Big|_{y=b/2} dx = 0$$

$$k = 1,2, ..., N \quad \text{. . . (A4)}$$

$$\int_0^{b/2}\int_0^{a/2} \phi_{vl}\left(A_{66}(x)\frac{\partial^2 v}{\partial x^2} + A_{22}(x)\frac{\partial^2 v}{\partial y^2} + \frac{\partial A_{66}(x)}{\partial x}\frac{\partial v}{\partial x} + \left[A_{12}(x) + A_{66}(x)\right]\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial A_{66}(x)}{\partial x}\frac{\partial u}{\partial y}\right)dx\,dy$$

$$-\int_0^{a/2} \phi_{vi} A_{66}(x)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\Big|_{x=b/2} dy$$

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$$-\int_0^{a/2}\phi_{vi}\left(A_{12}(x)\frac{\partial u}{\partial x}+A_{22}(x)\frac{\partial v}{\partial y}\right)\Bigg|_{y=b/2}dx=0$$

$$l=1,2,\ldots.,N$$

Similarly, for Case 4, from Equation (16)

$$\int_0^{b/2}\int_0^{a/2}\psi_{up}\left(A_{11}(x)\frac{\partial^2 u}{\partial x^2}+A_{66}(x)\frac{\partial^2 u}{\partial y^2}+\frac{\partial A_{11}(x)}{\partial x}\frac{\partial u}{\partial x}+\right.$$

$$\left.+\left[A_{12}(x)+A_{66}(x)\right]\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial A_{12}(x)}{\partial x}\frac{\partial v}{\partial y}\right)dxdy$$

$$-\int_0^{a/2}\psi_{up}A_{66}(x)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Bigg|_{y=b/2}dx=0$$

$$p=1,2,\ldots.,N \qquad \ldots \text{(A5)}$$

$$\int_0^{b/2}\int_0^{a/2}\phi_{vq}\left(A_{66}(x)\frac{\partial^2 v}{\partial x^2}+A_{22}(x)\frac{\partial^2 v}{\partial y^2}+\frac{\partial A_{66}(x)}{\partial x}\frac{\partial v}{\partial x}+\right.$$

$$\left.+\left[A_{12}(x)+A_{66}(x)\right]\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial A_{66}(x)}{\partial x}\frac{\partial u}{\partial y}\right)dxdy$$

$$-\int_0^{b/2}\phi_{vq}\,A_{66}(x)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\Bigg|_{x=a/2}dy=0$$

$$q=1,2,\ldots.,N$$

is obtained. $\Psi_{u1}=(x-0{\cdot}5)x$, $\Psi_{v1}=y(y-0{\cdot}5)$, $\Psi_{u2}=(x-0{\cdot}5)x^2$, $\Psi_{v2}=(y-0{\cdot}5)y^2$.

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