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Complementary variational principles for a class of differential equations

BY A. M. ARTHURS AND C. W. COLES
Department of Mathematics, University of York

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Abstract. Maximum and minimum principles for certain ordinary differential equations of order $2m$ are derived in a unified manner from the theory of complementary variational principles for multiple operator equations. The minimum principle is known in the literature, but the maximum principle appears to be new.

1. *Introduction.* The purpose of this paper is to use the theory of complementary variational principles to derive maximum and minimum principles associated with differential equations of the form

$$L\phi = f(x) \quad (a < x < b), \tag{1}$$

subject to

$$D^k\phi(a) = D^k\phi(b) = 0 \quad (k = 0, \ldots, m-1), \tag{2}$$

where D denotes differentiation with respect to x and

$$L\phi = \sum_{k=0}^{m} (-1)^k D^k\{p_k(x) D^k \phi\} \quad (m \geqslant 1), \tag{3}$$

with

$$p_0 > 0, \quad p_m > 0, \quad p_k \geqslant 0 \quad (k = 1, \ldots, m-1). \tag{4}$$

The minimum principle obtained here is already known in the literature (see (1)), but the maximum principle appears to be new and thus provides an alternative to the principle derived by Shampine (2).

2. *Canonical formalism.* The basic equations in the theory of complementary variational principles are the generalized canonical equations

$$T\phi = \frac{\partial H}{\partial u}, \tag{5}$$

$$T^*u = \frac{\partial H}{\partial \phi}, \tag{6}$$

where T is some linear operator and T^* is its adjoint (see (3)). To cast equation (1) into this form we introduce the operators

$$T_k = p_k^{\frac{1}{2}} D^k \tag{7}$$

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and their adjoints
$$T_k^* = (-1)^k D^k\{p_k^{\frac{1}{2}}\}, \tag{8}$$

where
$$\int_a^b \psi T_k \phi \, dx = \int_a^b (T_k^* \psi)\, \phi \, dx + \text{boundary terms} \tag{9}$$

for all suitable ψ and ϕ. These are now used to define the operators T and its adjoint T^* which are such that
$$T\phi = \begin{pmatrix} T_1\phi \\ T_2\phi \\ \vdots \\ T_m\phi \end{pmatrix} \tag{10}$$

for all suitable scalar functions ϕ, and
$$T^*u = T_1^* u_1 + T_2^* u_2 + \ldots + T_m^* u_m \tag{11}$$

for all suitable vector functions u with components $u_1, u_2, \ldots, u_m$. From these definitions it follows directly that the relation between T and T^* can be written
$$\int_a^b u^t T\phi \, dx = \int_a^b (T^*u)\, \phi \, dx + B(u, \phi), \tag{12}$$

where
$$B(u, \phi) = \left[\sum_{k=1}^{m} \sum_{r=0}^{k-1} (-1)^r D^r(p_k^{\frac{1}{2}} u_k)\, D^{k-1-r}\phi\right]_a^b, \tag{13}$$

and u^t denotes the transpose of the column vector u.

From (3), (7) and (8) we see that
$$L = \sum_{k=1}^{m} T_k^* T_k + p_0 \tag{14}$$
$$= T^*T + p_0 \tag{15}$$

by (10) and (11). Thus equation (1) takes the form
$$(T^*T + p_0)\, \phi = f(x), \tag{16}$$

subject to the boundary conditions (2). We can write (16) in the canonical form of (5) and (6) by taking
$$T\phi = u = \frac{\partial H}{\partial u}, \tag{17}$$
$$T^*u = f(x) - p_0\phi = \frac{\partial H}{\partial \phi}. \tag{18}$$

A suitable Hamiltonian H is therefore
$$H(u, \phi) = \tfrac{1}{2}u^t u + f\phi - \tfrac{1}{2}p_0\phi^2. \tag{19}$$

This completes the necessary canonical formalism.

3. *Complementary variational principles.* To obtain variational principles associated with the boundary problem in (1) and (2) we introduce the functional (see (3))
$$I(U, \Phi) = \int_a^b \{U^t T\Phi - H(U, \Phi)\}\, dx - B(U, \Phi), \tag{20}$$
$$= \int_a^b \{(T^*U)\, \Phi - H(U, \Phi)\}\, dx, \tag{21}$$

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where we have used (12). The term $B(U, \Phi)$ is introduced in (20) to ensure that $I(U, \Phi)$ leads to the required boundary conditions (2) (see (4) for a general discussion of boundary terms). We also define functionals $J(\Phi)$ and $G(U)$ by

$$J(\Phi) \equiv I(T\Phi, \Phi) = \int_a^b \{(T\Phi)^t T\Phi - H(T\Phi, \Phi)\}\, dx - B(T\Phi, \Phi), \tag{22}$$

and

$$G(U) \equiv I(U, p_0^{-1}(f - T^*U)) = \int_a^b \{(T^*U)\, p_0^{-1}(f - T^*U) - H(U, p_0^{-1}(f - T^*U))\}\, dx. \tag{23}$$

Let u and ϕ denote the exact solution pair of (17), (18) and (2). Then we obtain the following results:

3·1. *Variational principles.*
(*a*) $I(U, \Phi)$ is stationary at (u, ϕ).
(*b*) $J(\Phi)$ is stationary at ϕ.
(*c*) $G(U)$ is stationary at u.

3·2. *Minimum principle.* For admissible functions Φ which satisfy the exact boundary conditions (2), the minimum principle

$$I(u, \phi) \leqslant J(\Phi) \tag{24}$$

holds.

3·3. *Maximum principle.* For admissible functions U, the maximum principle

$$G(U) \leqslant I(u, \phi) \tag{25}$$

holds.

Since the exact function u satisfies $u = T\phi$ by (17), it is convenient to choose $U = T\Psi$, where Ψ is an approximation to the exact function ϕ. Then

$$G(T\Psi) \leqslant I(u, \phi) \tag{26}$$

for all admissible functions Ψ.

3·4. *Complementary variational principles.* Combining the results in (24) and (26) we obtain the complementary variational principles

$$G(T\Psi) \leqslant I(u, \phi) \leqslant J(\Phi), \tag{27}$$

where Φ satisfies the boundary conditions (2) as essential conditions, but Ψ has no essential conditions imposed on it.

From (22) and the definitions of T, H and B in (10), (19) and (13), we find that the functional J is

$$J(\Phi) = \int_a^b \left\{\frac{1}{2}\sum_{k=0}^{m} p_k(x)\, (D^k\Phi)^2 - f\Phi\right\} dx, \tag{28}$$

where Φ satisfies the conditions

$$D^k\Phi(a) = D^k\Phi(b) = 0 \quad (k = 0, \ldots, m-1), \tag{29}$$

and from (23) we obtain the functional G

$$G(T\Psi) = -\frac{1}{2}\int_a^b \left[\sum_{k=1}^{m} p_k(x)\, (D^k\Psi)^2 + p_0^{-1}(f - L\Psi + p_0\Psi)^2\right] dx. \tag{30}$$

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The functional $J(\Phi)$ in (28), which gives the minimum principle (24), is already known in the literature (see (1)), while the functional $G(T\Psi)$ in (30), which gives the maximum principle (26), appears to be new.

The exact functional $I(u, \phi)$ bounded by G and J is readily seen to be

$$I(u, \phi) = -\frac{1}{2}\int_a^b f\phi \, dx. \tag{31}$$

In addition to providing upper and lower bounds for $I(u, \phi)$, the functionals J and G can give a measure of the accuracy of the variational function Φ. This follows from the error bound result (2, 5)

$$\|\Phi - \phi\|_{L^2} \leqslant [2\Lambda^{-1}\{J(\Phi) - G(T\Psi)\}]^{\frac{1}{2}}, \tag{32}$$

where $\|\psi\|_{L^2}$ is the usual L^2 norm $\left\{\int_a^b \psi^2 \, dx\right\}^{\frac{1}{2}}$, and Λ is a lower bound to the lowest eigenvalue of the equation $L\theta = \lambda\theta \, (a < x < b)$, with

$$D^k\theta(a) = D^k\theta(b) = 0 \quad (k = 0, \ldots, m-1).$$

An error bound for $\sup|\Phi - \phi|$ can also be given in terms of $J - G$ (2, 5).

Finally, we remark that the multiple operator formalism employed in this paper can be applied to other equations, such as non-linear equations of the form $L\phi = f(\phi)$, various partial differential equations, and certain integrodifferential equations. It is hoped to discuss these in a subsequent communication.

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