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SCOTT SENTENCE COMPLEXITIES OF LINEAR ORDERINGS

DAVID GONZALEZ AND DINO ROSSEGGER

Abstract. We study possible Scott sentence complexities of linear orderings using two approaches. First, we investigate the effect of the Friedman–Stanley embedding on Scott sentence complexity and show that it only preserves Π^{in}_α complexities. We then take a more direct approach and exhibit linear orderings of all Scott sentence complexities except Σ^{in}_3 and $\Sigma^{\text{in}}_{\lambda+1}$ for λ a limit ordinal. We show that the former cannot be the Scott sentence complexity of a linear ordering. In the process we develop new techniques which appear to be helpful to calculate the Scott sentence complexities of structures.

§1. Introduction. In this article we give a comprehensive Scott analysis of the class of linear orderings using three definitions of rank. Given a structure $\mathcal{A}$, the *(parameterless) Scott rank* $SR(\mathcal{A})$ is the least α such that the automorphism orbits of all tuples in $\mathcal{A}$ are $\Sigma^{\text{in}}_\alpha$ definable; the *parameterized Scott rank* $pSR(\mathcal{A})$, is the least ordinal α such that there is a parameter $\bar{p} \in A^{<\omega}$ such that $SR((\mathcal{A}, \bar{p})) = \alpha$; and the *Scott sentence complexity* of $\mathcal{A}$, $SSC(\mathcal{A})$ is the least of the complexities Π^{in}_α, $\Sigma^{\text{in}}_\alpha$, $d\text{-}\Sigma^{\text{in}}_\alpha$ such that $\mathcal{A}$ has a Scott sentence of one of these complexities. The first two complexities were defined by Montalbán [19]. He showed that several notions of structural complexity coincide with these Scott ranks. The last notion was first formally investigated in [2]. While it does not assign an ordinal to structures, it was shown there that Scott sentence complexity is indeed a well-defined notion of rank. Table 1, taken from [22], shows the relationship between these invariants. As can be seen from the definitions of the two Scott ranks and Table 1, calculating the Scott sentence complexity of a structure comes down to calculating the complexity of the automorphism orbits of the structure's tuples in all but the limit ordinal cases. The limit ordinal case is robustly treated for the first time in this article and we discuss it more in Section 2.

By calculating the Scott sentence complexities occurring in a class of structures we obtain a detailed picture of the descriptive complexity of its isomorphism relation. By the Lopez-Escobar theorem, a structure's Scott sentence complexity corresponds to the Borel complexity of its isomorphism class. It is well known that the isomorphism relation on a class of structures is Borel if and only if the Scott ranks in the class are bounded below ω_1, see, for example, [12, Chapter 12].

The class of linear orderings is a particularly interesting object of study as it is Borel complete: For every class of structures $\mathfrak{K}$ there is a Borel function

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SSC	pSR	SR	Complexity of parameters
$\Sigma^{\text{in}}_{\alpha+2}$	α	$\alpha+2$	$\Pi^{\text{in}}_{\alpha+1}$
$d\text{-}\Sigma^{\text{in}}_{\alpha+1}$	α	$\alpha+1$	Π^{in}_{α}
$\Pi^{\text{in}}_{\alpha+1}$	α	α	None
		α limit	
$\Sigma^{\text{in}}_{\alpha+1}$	α	$\alpha+1$	Π^{in}_{α}
Π^{in}_{α}	α	α	None

TABLE 1. [22, Table 1]. Relationship of the different Scott invariants. The last column contains the complexity of the automorphism orbit of the parameters involved in the parameterized Scott rank.

$f : Mod(\mathfrak{K}) \to Mod(\text{LO})$ such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{K}$, $\mathcal{A} \cong \mathcal{B}$ if and only $f(\mathcal{A}) \cong f(\mathcal{B})$ [8]. But the class of linear orderings is not complete for stronger forms of reduction such as faithful Borel reducibility or effective bi-interpretability [11]. If one dives into the realm of computable structure theory, one quickly sees that linear orderings are far from being "structurally complete". For example, Richter's result that no linear ordering can code a non-computable subset of ω in its isomorphism type, shows that linear orderings are quite weak in terms of coding power [24]. However, if one turns to non-effective notions such as Scott rank and Scott sentence complexity, the situation becomes less clear. It is not at all difficult to produce linear orderings of Scott rank α for α any countable ordinal; in fact Ash shows that one can find such examples by considering well-orderings [5]. Even among computable linear orderings one can find every possible Scott rank; Harrison [15] and recently Calvert, Goncharov, and Knight [7] exhibited computable linear orderings of Scott rank $\omega_1^{\text{CK}} + 1$ and ω_1^{CK}, respectively. For the more fine-grained invariant of Scott sentence complexity, the possible Scott sentence complexities of computable structures with Scott rank $\omega_1^{\text{CK}} + 1$ or ω_1^{CK} were recently characterized in [1] but the examples given were not linear orderings. This beckons the question whether there are linear orderings of all possible high Scott sentence complexities.

There are two approaches one might consider to study the possible Scott sentence complexities in a specific class of structures. One is to study existing Borel reductions, such as the Friedman–Stanley embedding [8] and whether this embedding preserves Scott sentence complexities. We show that this approach fails in Section 2: The Friedman–Stanley embedding does not preserve Scott sentence complexity. In fact, for $\alpha > \omega$, if $SSC(\mathcal{A}) = \Sigma^{\text{in}}_{\alpha}$, then, $SSC(FS(\mathcal{A})) = \Pi^{\text{in}}_{\alpha+1}$ (This is true for finite α modulo a constant). The picture is different if $SSC(\mathcal{A}) = \Pi^{\text{in}}_{\alpha}$: Then $SSC(FS(\mathcal{A})) = \Pi^{\text{in}}_{\alpha}$ even in the case when α is a limit ordinal (again modulo a constant if α is finite). These results involve a deep analysis on the effect of parameters on the tree of tuples of a structure and a novel characterization of structures with Scott sentence complexity $\Pi^{\text{in}}_{\lambda}$ for λ a limit ordinal using what we call unstable λ-sequences.

In Section 3 we show that our choice to study the Friedman–Stanley embedding was not arbitrary; there are no better Borel embeddings into the class of linear orderings in terms of preservation of Scott sentence complexity. In order to show

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SSC	LO	Reference
Π^{in}_1	+	1-element l.o.
$d\text{-}\Sigma^{\text{in}}_1$	+	n-element l.o. $(n > 1)$
Π^{in}_2	+	$\mathbb{Q}$
Σ^{in}_3	-	Theorem 3.6
Successor α:		
$\Sigma^{\text{in}}_{\alpha+3}$	+	Corollary 4.10
$d\text{-}\Sigma^{\text{in}}_{\alpha}$	+	Corollary 4.10
Π^{in}_{α}	+	Corollary 4.10
λ a limit ordinal:		
$\Pi^{\text{in}}_{\lambda}$	+	Proposition 4.11
$\Sigma^{\text{in}}_{\lambda+1}$	?	Question 4.14
$\Sigma^{\text{in}}_{\lambda+2}$	+	Theorem 4.12
$\Sigma^{\text{in}}_{\lambda+3}$	+	Theorem 4.13

TABLE 2. Scott sentence complexities of linear orderings. Complexities not in the table are impossible for structures in general.

this, we develop a concept we call α-universality. This tool provides upper bounds for the number of structures with particular Scott sentence complexities.

In Sections 3.1 and 4 we take another more direct approach and try to find examples of linear orderings of given Scott sentence complexities. A summary of our findings can be found in the table above.

1.1. Remarks on effectivity. While this article's primary concern is not the effectiveness of the presented results, many of our results in this article can be effectivized without much effort. The interested reader can find further discussions on this matter in subsections at the end of Sections 2 and 4. Included are comments on potential difficulties and open questions that arise from these efforts.

1.2. Preliminaries. We assume that the reader is familiar with the basic techniques of Scott analysis and with linear orderings in general. For background we refer to Montalbán's upcoming book [22] and the standard reference by Ash and Knight [4].

We will make particular use of the following definition and lemma that are used regularly in Scott analysis.

DEFINITION 1.1 [4, Section 17.4]. We say that a tuple is α-*free* in the structure $\mathcal{A}$ if for every tuple $\bar{b} \in A$ and every $\beta < \alpha$, there are tuples $\bar{a}', \bar{b}'$ such that

$$\bar{a}\bar{b} \leq_\beta \bar{a}'\bar{b}'$$

$$\bar{a} \not\leq_\alpha \bar{a}.$$

LEMMA 1.2 [19]. *The unparameterized Scott rank of a structure is the least α such that no tuple is α-free.*

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Let us highlight the following lemmas concerning linear orderings which we will use as basic tools in our proofs without reference.

LEMMA 1.3 [4, Lemma 15.7]. *Suppose* $\mathcal{A}, \mathcal{B} \in \mathrm{LO}$, $\bar{a} \in \mathcal{A}^n$ *and* $\bar{b} \in \mathcal{B}^n$ *with* $\bar{a}$ *and* $\bar{b}$ *ordered. Then for every* $\alpha < \omega_1$

$$(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b}) \iff (-\infty, a_0) \leq_\alpha (-\infty, a_1) \wedge (a_0, a_1) \leq_\alpha (b_0, b_1) \wedge \\ \cdots \wedge (a_{n-1}, \infty) \leq_\alpha (b_{n-1}, \infty).$$

LEMMA 1.4 [4, Lemma 15.8]. *Suppose* $\mathcal{A}, \mathcal{B} \in \mathrm{LO}$. *Then* $\mathcal{A} \leq_1 \mathcal{B}$ *if and only if* $\mathcal{A}$ *is infinite or at least as large as* $\mathcal{B}$. *For* $\alpha > 1$, $\mathcal{A} \leq_\alpha \mathcal{B}$ *if and only if for every* $1 \leq \beta < \alpha$ *and every partition of* $\mathcal{B}$ *into intervals* $\mathcal{B}_0, \ldots, \mathcal{B}_n$, *with endpoints in* $\mathcal{B}$, *there is a partition of* $\mathcal{A}$ *into intervals* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *with endpoints in* $\mathcal{A}$, *such that* $\mathcal{B}_i \leq_\beta \mathcal{A}_i$.

§2. The Friedman–Stanley embedding and Scott sentence complexity. Friedman and Stanley [8] showed that the class of linear orderings is Borel complete by defining a computable operator Φ that takes a structure in a fixed vocabulary as input and outputs a linear ordering such that for all $\mathcal{A}, \mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$ if and only if $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$. This reduction is commonly referred to as the Friedman–Stanley embedding. It proceeds in two steps:

(1) Given a structure $\mathcal{A}$ it produces a labeled tree $\mathcal{T}_\mathcal{A}$, the tree of tuples of $\mathcal{A}$.
(2) From $\mathcal{T}_\mathcal{A}$ it produces a linear ordering $L_\mathcal{A} = \Phi(\mathcal{A})$.

The Friedman–Stanley embedding has been heavily studied both by descriptive set theorists and computable structure theorists. Closely connected to our investigation are Harrison-Trainor and Montalbán's study of the tree of tuples construction [16] and Knight, Soskova, and Vatev's analysis of possible reductions to linear orderings [17]. One result obtained in both articles is that one cannot embed graphs into linear orderings uniformly using $L_{\omega_1\omega}$ formulas. This raises the question whether the Friedman–Stanley embedding or any reduction reducing graphs to linear orderings can preserve Scott sentence complexity. We consider the first question in this section by analyzing the two steps of the reduction.

Using the fact that the class of graphs is complete for very strong reducibilities such as effective bi-interpretability (see [21]), we will only analyse the Friedman–Stanley embedding from the vocabulary containing one binary relation symbol to linear orderings. This has the advantage that we do not have to deal with technicalities resulting from infinite vocabularies in our proofs, but there is no effect on the strength of our results.

2.1. The tree of tuples and Scott sentence complexity.

DEFINITION 2.1. A *replicated labeled tree* consists of a tree $(T, \succeq)$ with a parent function and labeling function $l : T \to \omega$ such that for every $\sigma \in T$ with parent τ, there exist infinitely many children $\tilde{\sigma}$ of τ such that $\mathcal{T}_\sigma \cong \mathcal{T}_{\tilde{\sigma}}$, where $\mathcal{T}_\sigma$ denotes the subtree of $\mathcal{T}_\mathcal{A}$ rooted at σ.

Recall that the *atomic diagram of a tuple* $\bar{a}$ *in* $\mathcal{A}$, denoted by $D_\mathcal{A}(\bar{a})$ is the set of atomic formulas φ_i in variables $(x_i)_{i\in\omega}$ such that $(\mathcal{A}, \bar{a}) \models \varphi_i[x_j/a_j, j < |\bar{a}|]$. We use the convention that formulas mentioning variables x_j with $j \geq |\bar{a}|$ are not in $D_\mathcal{A}(\bar{a})$ and thus, assuming that $\mathcal{A}$ is a graph, $D_\mathcal{A}(\bar{a})$ will always be a finite set.

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DEFINITION 2.2. Given a graph $\mathcal{A}$ let $T_{\mathcal{A}}$ be the labeled tree consisting of all tuples from $\mathcal{A}$ ordered by inclusion where each tuple $\bar{a}$ is labeled by a natural number coding $D_{\mathcal{A}}(\bar{a})$ and the length of $\bar{a}$. The *tree of tuples* $\mathcal{T}_{\mathcal{A}}$ is obtained by replicating every branch in $T_{\mathcal{A}}$ infinitely many times.

Looking at the construction of $\mathcal{T}_{\mathcal{A}}$ we can see that the Friedman–Stanley embedding does not preserve much structure. Suppose that $\mathcal{A}$ is rigid (i.e., it has trivial automorphism group). The tree $\mathcal{T}_{\mathcal{A}}$ will have many automorphisms because it is replicated, and thus the automorphism groups of $\mathcal{A}$ and $\mathcal{T}_{\mathcal{A}}$ will behave quite differently.

On the other hand, the (unparameterized) Scott rank of the tree of tuples of a structure and the Scott rank of the structure are equal. In order to show this we need a few lemmas.

LEMMA 2.3. *Let $\mathcal{A}$ be a structure in vocabulary τ.*

(1) *Given Π^{in}_{α} τ-formula φ, there is a Π^{in}_{α} formula T_{φ} in the vocabulary of labeled trees such that for any $\bar{a} \in \mathcal{A}$ and every $\sigma \in \mathcal{T}_{\mathcal{A}}$ with $\sigma \succeq \bar{a}$*

$$\mathcal{A} \models \varphi(\bar{a}) \iff \mathcal{T}_{\mathcal{A}} \models T_{\varphi}(\sigma).$$

Moreover, T_{φ} can be taken to have the same quantifier complexity as φ.

(2) *Given a Π^{in}_{α} formula ψ in the vocabulary of labeled trees, there is a Π^{in}_{α} τ-formula ψ^* such that for any tuple $\bar{\sigma} = (\bar{a}_1, \dots, \bar{a}_n) \in \mathcal{T}_{\mathcal{A}}$*

$$\mathcal{T}_{\mathcal{A}} \models \psi(\bar{\sigma}) \iff \mathcal{A} \models \psi^*(\bar{a}, \dots, \bar{a}_n).$$

PROOF. We start by proving Item 1. To start suppose that φ is finite and quantifier-free and in disjunctive normal form. For a disjunct φ_i of φ, let T_{φ_i} be the formula expressing that $\varphi_i \subseteq l(x)$ and let $T_{\varphi} = \bigvee T_{\varphi_i}$. Since formulas with free variable indices exceeding the length of a tuple $\bar{a}$ are considered to be false in $\mathcal{A}$ and $l(\sigma) \subseteq l(\tau)$ if $\sigma \preceq \tau$, we have that $\mathcal{A} \models \varphi(\bar{a})$ if and only if for every $\sigma \succeq \bar{a}$, $\mathcal{T}_{\mathcal{A}} \models T_{\varphi}(\sigma)$, as required.

Now, say that we have defined T_{φ} given φ and that Item 1 holds for $\varphi(x)$. Define

$$T_{\exists x \varphi} = (\exists y \succeq \sigma) \bigvee_{i \in \omega} T_{\varphi[x/x_i]}(y),$$

$$T_{\forall x \varphi} = (\forall y \succeq \sigma) \bigwedge_{i \in \omega} T_{\varphi[x/x_i]}(y) \vee i > |\sigma|,$$

where $\varphi[x/x_i]$ denotes φ with all free occurrences of x replaced by x_i. The need for these replacements arises from our coding: We have that σ and τ in $\mathcal{T}_{\mathcal{A}}$ can code a witness a for x in different positions, say for example that $\sigma(i) = \tau(k) = a$ for $i \neq k$. Then σ satisfies $T_{\varphi[x/x_i]}$ but τ does not. To proceed we will need the following claim.

CLAIM 1.3.1. *Suppose that φ is a formula where exactly one free variable x is not in the range $x_0, \dots, x_{|a|}$. Then for all b and $\bar{d}$ not containing b*

$$\mathcal{T}_{\mathcal{A}} \models T_{\varphi[x/x_{|\bar{a}|+1}]}(\bar{a}^\frown b) \iff \mathcal{T}_{\mathcal{A}} \models T_{\varphi[x/x_{|\bar{a}^\frown \bar{d}|+1}]}(\bar{a}^\frown \bar{d}^\frown b).$$

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PROOF. First note that if $\sigma = \pi(\tau)$ where π is a permutation of $|\sigma|$, then for any formula T_φ, $\mathcal{T}_\mathcal{A} \models T_\varphi(\sigma)$ if and only if $\mathcal{T}_\mathcal{A} \models T_\varphi[x_i/x_{\pi(i)}](\tau)$.[1]

Now, suppose that $\mathcal{T}_\mathcal{A} \models T_{\varphi[x/x_{|\bar{a}^\frown \bar{d}|+1}]}(\bar{a}^\frown \bar{d}^\frown b)$, then as T_φ does not contain variable symbols x_i for $i > |\bar{a}|$ except x, $\mathcal{T}_\mathcal{A} \models T_{\varphi[x/x_{|\bar{a}|+1}]}(\bar{a}^\frown b^\frown \bar{d})$ and in particular $\mathcal{T}_\mathcal{A} \models T_{\varphi[x/x_{|\bar{a}|+1}]}(\bar{a}^\frown b)$. The other implication is trivial. ⊣

We can now proceed with the induction step of Item 1. We have by the claim that for given $\bar{a}$ and b, $\mathcal{T}_\mathcal{A} \models T_\varphi(\bar{a}^\frown b)$ if and only if for every $\sigma \succeq \bar{a}$, $\mathcal{T}_\mathcal{A} \models T_{\varphi[x/x_{|\sigma|+1}]}(\sigma^\frown b)$ if $b \notin \sigma$ and $\mathcal{T}_\mathcal{A} \models T_{\varphi[x/x_i]}(\sigma)$ if $b = \sigma(i)$. Thus, we get that $\mathcal{A} \models \exists x\varphi(\bar{a})$ if and only if for every $\sigma \succeq \bar{a}$, $\mathcal{T}_\mathcal{A} \models T_{\exists x\varphi}(\sigma)$. For $T_{\forall x\varphi}$ note that $\mathcal{T}_\mathcal{A} \models T_{\forall x\varphi}(\bar{a})$ if and only if for every $\sigma \succeq \bar{a}$ and every $i \leq |\sigma|$, $\mathcal{A} \models \varphi(\bar{a}, \sigma(i))$, and, since $\bar{a} = \sigma \restriction |\bar{a}|$, in particular $\mathcal{A} \models \varphi(\sigma(0) \dots \sigma(|\bar{a}|), \sigma(i))$. Thus, $\mathcal{A} \models \forall x\varphi(\bar{a})$ if and only if $\mathcal{T}_\mathcal{A} \models T_{\forall x\varphi}$. We can extend our definitions to include infinite conjunctions and disjunctions by letting

$$T_{\bigvee \varphi_i} = \bigvee T_{\varphi_i} \quad \text{and } T_{\bigwedge \varphi_i} = \bigwedge T_{\varphi_i}.$$

It then follows that for every $\bar{a}$, $\mathcal{A} \models \varphi(\bar{a})$ if and only if $T_\mathcal{A} \models T_\varphi(\sigma)$ for all $\sigma \succeq \bar{a}$, and, by construction T_φ has the same quantifier complexity as φ.

We now prove Item 2 by defining for each $\bar{\sigma} = (\bar{a}_1, \dots, \bar{a}_n) \in \mathcal{T}_\mathcal{A}$ a Turing-computable embedding from the isomorphism class of $\mathcal{A}$ in the extended vocabulary with additional constant symbols $\bar{p} = \bar{p}_1 \dots, \bar{p}_n$ such that $|\bar{p}_i| = |\bar{a}_i|$ to the class of trees with constant symbols for σ. That is, the embedding Φ_σ will map $Iso(\mathcal{A}, \bar{p})$ to $Iso(\mathcal{T}_\mathcal{A}, \bar{\tau})$, where $\tau_i = \bar{p}_i$ for all $i < n$ and $\bar{p}_i \in A^{<\omega}$. Each of these embeddings will extend the Friedman–Stanley embedding that acts without parameters.

Fix $\sigma = (\bar{a}_1, \dots, \bar{a}_n) \in \mathcal{T}_\mathcal{A}$ and consider a structure $(\mathcal{B}, \bar{p})$ in the extended vocabulary such that $\mathcal{B} \cong \mathcal{A}$. The Turing computable embedding Φ_σ computes the standard tree of structures for $\mathcal{B}$ and picks the lexicographically least tuple $\bar{\tau}$ from $\mathcal{T}_\mathcal{B}$ such that each τ_i codes $\bar{p}_i$ and the finite subtree described by $\bar{\tau}$ in $\mathcal{T}_\mathcal{B}$ is isomorphic to the finite subtree described by $\bar{\sigma}$ (notice that this last condition is necessary since $\mathcal{T}_\mathcal{A}$ is replicated. While $\bar{a}_i$ and $\bar{a}_j$ could agree on an initial segment, it could happen that the subtrees generated by the elements coding these tuples in $\mathcal{T}_\mathcal{A}$ are disjoint). It is not difficult to check that $\Phi_\sigma : (\mathcal{B}, \bar{p}) \mapsto (\mathcal{T}_\mathcal{B}, \bar{\tau})$ is a Turing computable embedding mapping $(\mathcal{A}, \bar{a}_1, \dots, \bar{a}_n)$ to $(\mathcal{T}_\mathcal{A}, \bar{\sigma})$. Now, for any formula ψ in the language of labeled trees and parameters $\sigma = (\bar{a}_1, \dots, \bar{a}_n)$ by the pull-back theorem for $\Psi_{\bar{\sigma}}$ there is ψ^* of the same complexity as ψ such that $\mathcal{A} \models \psi^*(\bar{a}_1 \dots \bar{a}_n)$ if and only if $\mathcal{T}_\mathcal{A} \models \psi(\bar{\sigma})$. This proves Item 2.

Lemma 2.3 might suggest that the tree of tuples reduction preserves Scott sentence complexity. However, as we will see, it does not. The reason for this is the following. Let K be a class of structures and look at $T(K)$, the set of trees-of-tuples of the elements of K, then within $T(K)$, Scott sentences are preserved. However, in general $T(K)$ is not a Borel subset of the class of labeled trees and thus the Scott sentence complexity of $\mathcal{T}_\mathcal{A}$ is not a priori connected to the Scott sentence complexity of $\mathcal{A}$.

[1] One should not be misled to think that the two nodes σ and τ are automorphic, in particular, their labeling functions might be differing.

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PROPOSITION 2.4. *Given* $\bar{\sigma} = (\bar{a}_1, \dots, \bar{a}_n), \bar{\tau} = (\bar{b}_1, \dots, \bar{b}_n) \in \mathcal{T}_{\mathcal{A}}$, *if* $\bar{\sigma} \leq_\alpha \bar{\tau}$, *then for each* $i < n$, $\bar{a}_i \leq_\alpha \bar{b}_i$.

PROOF. We prove this by contraposition. Assume that for some i, $\bar{a}_i \not\leq_\alpha \bar{b}_i$. This is the same thing as saying that there is a Π^{in}_α formula φ with $\mathcal{A} \models \varphi(\bar{a}_i)$ and $\mathcal{A} \models \neg\varphi(\bar{b}_i)$. From Item 1 of Lemma 2.3 we get that $\mathcal{T}_{\mathcal{A}} \models T_\varphi(\bar{a}_i)$ and $\mathcal{T}_{\mathcal{A}} \models \neg T_\varphi(\bar{b}_i)$. Then, since T_φ is Π^{in}_α, $\sigma \not\leq_\alpha \tau$, as desired. ⊣

LEMMA 2.5. *Given a tree of tuples* $\mathcal{T}_{\mathcal{A}}$, *an element* $\sigma \in \mathcal{T}_{\mathcal{A}}$, *and a set of finitely many children* $\tau_1, \dots, \tau_n$ *of* σ,

$$\mathcal{T}_\sigma \cong \mathcal{T}_\sigma \setminus (\mathcal{T}_{\tau_1} \cup \cdots \cup \mathcal{T}_{\tau_n}).$$

PROOF. Since $\mathcal{T}_{\mathcal{A}}$ is replicated, for every child ρ of σ there are infinitely many children ρ' of σ such that $\mathcal{T}'_\rho \cong \mathcal{T}_\rho$. Hence, $\mathcal{T}_\sigma \cong \mathcal{T}_\sigma \setminus \mathcal{T}_\rho$. The full lemma follows by a trivial induction. ⊣

THEOREM 2.6. *For any structure* $\mathcal{A}$, $SR(\mathcal{A}) = SR(\mathcal{T}_{\mathcal{A}})$.

PROOF. Let $\mathcal{A}$ be a structure such that $SR(\mathcal{T}_{\mathcal{A}}) = \gamma$. Consider a tuple $\bar{a} \in \mathcal{A}$. Let $\psi \in \Sigma^{\text{in}}_\gamma$ define the orbit of $\bar{a} \in \mathcal{T}_{\mathcal{A}}$. By Item 2 in Lemma 2.3 we have for all $\bar{b} \in \mathcal{A}$

$$\mathcal{A} \models \psi^*(\bar{b}) \iff \mathcal{T}_{\mathcal{A}} \models \psi(\bar{b}) \iff \bar{b} \cong \bar{a}.$$

Therefore, ψ^* defines the orbit of $\bar{a}$ as desired and hence, $SR(\mathcal{A}) \leq SR(\mathcal{T}_{\mathcal{A}})$.

We now show that $SR(\mathcal{T}_{\mathcal{A}}) \leq SR(\mathcal{A}) = \gamma$. Let $\bar{\sigma} = \bar{a}_1 \dots \bar{a}_n$ be a tuple in $\mathcal{T}_{\mathcal{A}}$. The automorphism orbit of each $\bar{a}_i$ and each of its prefixes is $\Sigma^{\text{in}}_\gamma$ definable in $\mathcal{A}$. Thus, by Item 1 in Lemma 2.3, for every k there is a $\Sigma^{\text{in}}_\gamma$ formula, $\chi_{i,k}$, true only of elements in $\tau \in \mathcal{T}_{\mathcal{A}}$ where τ codes a tuple with length k prefix automorphic to the length k prefix of $\bar{a}_i$. In particular, $\mathcal{T}_{\tau \restriction k} \cong \mathcal{T}_{\bar{a}_i \restriction k}$. Now, let ψ describe the finite subtree containing precisely the elements of σ as its leaves and let φ be the conjunction of ψ together with the $\chi_{i,k}$. Note that φ can be taken to have free variables corresponding to σ. Because the labels encode the height of the labeled element in the tree, ψ can be taken Σ^{in}_1.

Let $\bar{\tau} = \bar{b}_1 \dots \bar{b}_n$ satisfy φ and consider the subtrees $\mathcal{T}_\sigma$ and $\mathcal{T}_\tau$ containing elements of $\mathcal{T}_{\mathcal{A}}$ comparable with σ, respectively τ. These two subtrees are isomorphic by Lemma 2.5 and as $\mathcal{T}_{\bar{a}_i \restriction k} \cong \mathcal{T}_{\bar{b}_i \restriction k}$ for all $i < n$ and $k < |\bar{a}_i|$. Fix $h(\tau)$ such that $\mathcal{T}_\tau \cong \mathcal{T}_{h(\tau)}$ yet the tree $\mathcal{T}_{h(\tau)}$ is disjoint from both $\mathcal{T}_\sigma$ and $\mathcal{T}_\tau$. Since $\mathcal{T}$ is replicated we can find such an $h(\tau)$ and furthermore find an automorphism h such that τ is switched with $h(\tau)$ and all points outside of $\mathcal{T}_\tau$ and $\mathcal{T}_{h(\tau)}$ are fixed. Similarly, as $\mathcal{T}_\sigma \cong \mathcal{T}_{h(\tau)}$ there is an automorphism f that switches σ with $h(\tau)$ and all points outside of $\mathcal{T}_\sigma$ and $\mathcal{T}_{h(\tau)}$ are fixed. Composing these automorphisms appropriately, we obtain that $h^{-1}(f(\sigma)) = \tau$, so τ is in the automorphism orbit of σ. Thus, φ gives a $\Sigma^{\text{in}}_\gamma$ definition of the automorphism orbit of σ as required. ⊣

Theorem 2.6 can be viewed as a strengthening of a result of Gao. He showed, using a different definition of Scott rank, that the ranks of $\mathcal{A}$ and $\mathcal{T}_{\mathcal{A}}$ can be at most ω^2 apart [11].

We now turn our attention to the parameterized Scott rank. Unlike the unparameterized case, parameterized Scott rank will not be preserved. To see this, we first need the following definitions. Given a tree T with root r and

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a tuple $\bar{c} = (c_1, \ldots, c_n) \in T$, we say that $y \in T$ is *somewhat comparable* to $\bar{c}$ if $\exists z, i\ z \neq r \wedge c_i \succeq z \wedge y \succeq z$. A tuple is somewhat comparable to $\bar{c}$ if all of its points are. Similarly, we say that a point is *entirely incomparable* to $\bar{c}$ if it is not somewhat comparable to $\bar{c}$ and that a tuple is entirely incomparable to $\bar{c}$ if all of its points are entirely incomparable to $\bar{c}$. Note that if two elements are somewhat comparable, they must have a length 1 predecessor in common.

LEMMA 2.7. *For $\mathcal{T}_\mathcal{A}$ a tree of tuples, every $\alpha \in \omega_1$ and tuples $\bar{a}$ and $\bar{b}$ that are entirely incomparable to $\bar{p}$,*

$$\bar{a}\,\bar{p} \leq_\alpha \bar{b}\,\bar{p} \iff \bar{a} \leq_\alpha \bar{b}.$$

PROOF. We go by transfinite induction on α. The $\alpha = 0$ case is immediate, as is the limit case. Assume the result for $\alpha = \beta$. If $\bar{a}\,\bar{p} \leq_{\beta+1} \bar{b}\,\bar{p}$, then $\bar{a} \leq_{\beta+1} \bar{b}$ follows immediately. Now assume that $\bar{a} \leq_{\beta+1} \bar{b}$ and consider an arbitrary tuple $\bar{c}$. We can split $\bar{c}$ into the points somewhat comparable to $\bar{p}$, named $\bar{c}_p$, and the points entirely incomparable to $\bar{p}$, named $\bar{c}_i$ (one of these may be the empty tuple). Because $\bar{a} \leq_{\beta+1} \bar{b}$, there is a tuple $\bar{d}_i$ such that $\bar{a}\bar{d}_i \geq_\beta \bar{b}\bar{c}_i$. We show that we can find a $\bar{d}_i'$ that is entirely incomparable to $\bar{p}$ and has $\bar{a}\bar{d}_i' \geq_\beta \bar{b}\bar{c}_i$ by modifying the given $\bar{d}_i$. Let $p_1 \ldots p_n$ enumerate the length 1 points below any element in the tuple $\bar{p}$, and let $a_1 \ldots a_n$ be similar for $\bar{a}$. Note that $p_1 \ldots p_n$ and $a_1 \ldots a_n$ do not intersect as $\bar{a}$ is entirely incomparable to $\bar{p}$. We let $d_1 \ldots d_n$ be similar for $\bar{d}$, but we remove the points that appear among $a_1 \ldots a_n$. For every d_i let d_i' be an element with $\mathcal{T}_{d_i} \cong \mathcal{T}_{d_i'}$ that does not appear among the $p_1 \ldots p_n$ and $a_1 \ldots a_n$. This is possible because $\mathcal{T}_\mathcal{A}$ is an infinitely replicated tree. Consider the automorphism Φ of $\mathcal{T}_\mathcal{A}$ that exchanges $\mathcal{T}_{d_i}$ and $\mathcal{T}_{d_i'}$ but fixes all elements not among these subsets. We define $\bar{d}_i' = \Phi(\bar{d})$ and claim that this tuple has the desired properties. Φ was constructed to fix $\bar{a}$; this means that $\bar{d}$ is automorphic to $\bar{d}'$ over $\bar{a}$ and therefore $\bar{a}\bar{d}_i' \geq_\beta \bar{a}\bar{d}_i \geq_\beta \bar{b}\bar{c}_i$. Furthermore, none of the d_i' are among the p_i by construction so $\bar{d}'$ and $\bar{p}$ are entirely incomparable. We can now note that $\bar{a}\bar{d}_i'$ and $\bar{b}\bar{c}_i$ are entirely incomparable with $\bar{p}\bar{c}_p$ as somewhat comparability is an equivalence relation. In particular, the induction hypothesis gives that

$$\bar{a}\bar{d}_i'\bar{p}\bar{c}_p \geq_\beta \bar{b}\bar{c}_i\bar{p}\bar{c}_p.$$

Rearranging the tuples gives that

$$\bar{a}\,\bar{p}\bar{d}_i'\bar{c}_p \geq_\beta \bar{b}\,\bar{p}\bar{c}_i\bar{c}_p$$

which shows that $\bar{a}\,\bar{p} \leq_{\beta+1} \bar{b}\,\bar{p}$, as desired. ⊣

LEMMA 2.8. *For every tree of tuples $\mathcal{T}_\mathcal{A}$, $SR(\mathcal{T}_\mathcal{A}) = pSR(\mathcal{T}_\mathcal{A})$.*

PROOF. Consider a tuple $\bar{a}$ that is α-free in $\mathcal{T}_\mathcal{A}$. Without loss of generality (by picking parameters automorphic to $\bar{p}$) we have that $\bar{a}$ is entirely incomparable with $\bar{p}$. We demonstrate that $\bar{a}$ is also α-free in $(\mathcal{T}_\mathcal{A}, \bar{p})$.

Consider $\bar{b} \in \mathcal{T}_\mathcal{A}$ and $\beta < \alpha$. We wish to show that there are $\bar{a}', \bar{b}'$ such that $\bar{a}\bar{b}\,\bar{p} \leq_\beta \bar{a}'\bar{b}'\,\bar{p}$ and $\bar{a}\,\bar{p} \not\leq_\alpha \bar{a}'\,\bar{p}$.

We can split $\bar{b}$ into the points somewhat comparable to $\bar{p}$, named $\bar{b}_p$, and the points entirely incomparable to $\bar{p}$, named $\bar{b}_i$ (one of these may be the empty tuple).

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Using the α-freeness of $\bar{a}$ in $\mathcal{T}_\mathcal{A}$, we may find $\bar{a}'$ and $\bar{b}_i'$ (entirely incomparable to p via an automorphism fixing $\bar{a}$) such that $\bar{a}\bar{b}_i \leq_\beta \bar{a}'\bar{b}_i'$ and $\bar{a} \not\leq_\alpha \bar{a}'$.

It follows from the previous lemma that $\bar{a}\bar{b}_i\bar{p}\bar{b}_p \leq_\beta \bar{a}'\bar{b}_i'\bar{p}\bar{b}_p$ and $\bar{a}\bar{p} \not\leq_\alpha \bar{a}'\bar{p}$. Rearranging the tuples we get that $\bar{a}\bar{b}\bar{p} \leq_\beta \bar{a}'\bar{b}'\bar{p}$ and $\bar{a}\bar{p} \not\leq_\alpha \bar{a}'\bar{p}$. Therefore, $\bar{a}$ is also α-free in $(\mathcal{T}_\mathcal{A}, \bar{p})$.

In particular, if there are no α-free tuples in $(\mathcal{T}_\mathcal{A}, \bar{p})$ then there are no α-free tuples in $\mathcal{T}_\mathcal{A}$. Hence, $SR(\mathcal{T}_\mathcal{A}) = pSR(\mathcal{T}_\mathcal{A})$. ⊣

The previous two results can be used to understand the effect of the Friedman–Stanley embedding on Scott sentence complexity in all but the limit cases. We use the following new technique to deal with these cases.

Definition 2.9. For a structure $\mathcal{A}$ and a limit ordinal λ, a *λ-sequence* in $\mathcal{A}$ is a set of tuples $\bar{y}_i \in \mathcal{A}$ for $i \in \omega$ such that $\bar{y}_i \equiv_{\alpha_i} \bar{y}_{i+1}$ for some fundamental sequence $(\alpha_i)_{i\in\omega}$ for λ. We say that a λ-sequence is *unstable* if $y_i \not\equiv_{\alpha_{i+1}} y_{i+1}$.

Lemma 2.10. *Consider a structure $\mathcal{A}$ with $SR(\mathcal{A}) = \lambda$ for λ a limit ordinal. The Scott sentence complexity of $\mathcal{A}$ is Π^{in}_λ if and only if there are no unstable λ-sequences in $\mathcal{A}$.*

Proof. We start with the right to left implication and argue by contraposition. Assume $SR(\mathcal{A}) = \lambda$, yet $\mathcal{A}$ does not have a Π^{in}_λ Scott sentence. Then there is a structure $\mathcal{B}$ such that $\mathcal{A} \equiv_\lambda \mathcal{B}$ yet $\mathcal{A} \not\cong \mathcal{B}$. Because $\mathcal{A}$ has Scott rank λ this is equivalent to saying that $\mathcal{A} \equiv_\lambda \mathcal{B}$ yet $\mathcal{A} \not\leq_{\lambda+1} \mathcal{B}$. In other words, there is a tuple $\bar{x} \in \mathcal{B}$ such that for any $\bar{z} \in \mathcal{A}$, $\bar{z} \not\equiv_\lambda \bar{x}$. However, as $\mathcal{A} \equiv_\lambda \mathcal{B}$, given a fundamental sequence $(\alpha_i)_{i\in\omega}$ for λ, for every i there is some $\bar{y}_i \in \mathcal{A}$ such that $\bar{y}_i \equiv_{\alpha_i} \bar{x}$. It follows that the $\bar{y}_i$ are a λ-sequence. Furthermore, given any i, there must be some $k > i$ such that $\bar{y}_i \not\equiv_{\alpha_k} \bar{y}_k$. Otherwise, $\bar{y}_i \equiv_\lambda \bar{x}$, a contradiction to the choice of $\bar{x}$. The above k can be taken to equal $i + 1$ if we thin out the sequence by removing intermediate elements between i and k. In particular, the sequence $(\bar{y}_i)_{i\in\omega}$ can be taken to be unstable.

We now prove the left to right implication, again by contraposition. Consider a structure $\mathcal{A}$ with an unstable λ-sequence $(\bar{y}_i)_{i\in\omega}$. Consider the sequence of structures given by $(\mathcal{A}, \bar{y}_i)_{i\in\omega}$. By [21, Lemma XII.6], there is a structure $\mathcal{B}$ and tuple $\bar{z} \in \mathcal{B}$ such that for each i, $(\mathcal{B}, \bar{z}) \equiv_{\alpha_i} (\mathcal{A}, \bar{y}_i)$. Note that, in particular, this means that $\mathcal{B} \equiv_\lambda \mathcal{A}$. Furthermore, for each i there is some $k > i$ such that $\alpha_k \geq 2\alpha_i + 3$, as the difference between $2\alpha_i + 3$ and α_i is finite. For this k, we can observe that

$$(\mathcal{A}, \bar{y}_k) \models \exists \bar{x}\ \bar{x} \not\equiv_{\alpha_{i+1}} \bar{y}_k \wedge \bar{x} \equiv_{\alpha_i} \bar{y}_k.$$

By [21, Lemma VI.14] this is a $\Sigma^{\text{in}}_{2\alpha_i+3}$ sentence that is computable in $\mathcal{A}$. As $(\mathcal{B}, \bar{z}) \equiv_{\alpha_k} (\mathcal{A}, \bar{y}_k)$, by choice of k,

$$(\mathcal{B}, \bar{z}) \models \exists \bar{x}\ \bar{x} \not\equiv_{\alpha_{i+1}} \bar{z} \wedge \bar{x} \equiv_{\alpha_i} \bar{z}.$$

Now consider the automorphism orbit of $\bar{z}$ inside $\mathcal{B}$. It cannot be defined by a Σ^{in}_β formula for $\beta < \lambda$, as if we take $\alpha_i > \beta$, by the above observation, there is a tuple that is not in the automorphism orbit of $\bar{z}$ that is α_i-equivalent to $\bar{z}$. This means that $SR(\mathcal{B}) > \lambda$ so $\mathcal{B} \not\cong \mathcal{A}$. As $\mathcal{B} \equiv_\lambda \mathcal{A}$, there is no Π^{in}_λ Scott sentence for A. ⊣

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The former lemma can be better understood as an internal compactness property that is present in structures with Scott sentence complexity Π^{in}_λ. For the sake of completeness we include this equivalent formulation here.

COROLLARY 2.11. *Consider a structure $\mathcal{A}$ with $SR(\mathcal{A}) = \lambda$ for λ a limit ordinal. Then the following are equivalent.*

(1) *$\mathcal{A}$ has Scott sentence complexity Π^{in}_λ.*
(2) *For every fundamental sequence $(\delta_n)_{n\in\omega}$ for λ and complete δ_n-types $p_n(\bar{x})$ with m free variables for some m and $p_n \subset p_{n+1}$, every type p_n is realized in $\mathcal{A}$ if and only if there is $\bar{a} \in A^m$ realizing every type p_n.*

PROOF. Say $\mathcal{A}$ has Scott sentence complexity Π^{in}_λ and fix $m \in \omega$. If $\mathcal{A}$ realizes all the δ_n-types $p_n(\bar{x})$ with m free variables and $p_n \subset p_{n+1}$, then any sequence of elements $(\bar{y}_n)_n \in \omega$ such that $\bar{y}_n$ realizes the type $p_n(\bar{x})$ forms a λ-sequence. If $\bar{y}_n \not\equiv_{\delta_k} \bar{y}_k$ for $k > n$ on a cofinal subset, then we can obtain an unstable sequence by thinning out the sequence, a contradiction. Therefore, there is some K for which $\bar{y}_n \equiv_{\delta_k} \bar{y}_k$ for all $k > n \geq K$. Then $\bar{y}_K$ realizes every type p_n.

On the other hand, if $\mathcal{A}$ does not have Scott sentence complexity Π^{in}_λ it must have some unstable sequence, $(\bar{z}_n)_{n\in\omega}$. If we take the δ_n-type of $\bar{z}_n$ to be p_n, then clearly every type p_n is realized in $\mathcal{A}$. However, if there is $\bar{a}$ that realizes all of the types p_n, then we have that $\bar{a} \equiv_{\delta_n} \bar{z}_n$ yet $\bar{a} \not\equiv_{\delta_{n+1}} \bar{z}_{n+1}$. Therefore the orbit of $\bar{a}$ is not $\Sigma^{\text{in}}_\lambda$ definable, a contradiction as $SR(\mathcal{A}) = \lambda$. ⊣

Note that this cannot be upgraded from complete types to general formulas. In other words, there may be a sequence of properly $\Pi^{\text{in}}_{\delta_n}$ formulas φ_n such that $\varphi_{n+1} \to \varphi_n$ and $\mathcal{A} \models \bigwedge_n \exists \bar{x}\ \varphi_n(\bar{x})$ yet $\mathcal{A} \models \neg\exists \bar{x} \bigwedge_n \varphi_n(\bar{x})$. To see this, we need not look past our example of a Π^{in}_λ linear order for λ a limit given in Proposition 4.11. In particular the following formulas are a counterexample satisfied by this ordering:

$$\varphi_n(z) := \exists \bar{x}, \bar{y} \bigwedge_{i<n} (x_i, y_i) \cong \mathbb{Z}^{\delta_i} \wedge y_i < z.$$

We now note that unstable sequences transfer between a structure and its tree of tuples. This will allow us to resolve the final Scott sentence complexity ambiguity. In order to do this, we first need the following technical lemma.

LEMMA 2.12. *Let $S = (x_1, \dots, x_n)$ and $T = (y_1, \dots, y_n)$ be (finite) isomorphic downward closed subsets of $\mathcal{T}_\mathcal{A}$. $S \leq_\alpha T$ if and only if for all i, $x_i \leq_\alpha y_i$.*

PROOF. If $S \leq_\alpha T$, $x_i \leq_\alpha y_i$ for any i follows immediately from the monotonicity of $\leq_\alpha$.

Say that for all i, $x_i \leq_\alpha y_i$. Note that for all i, $\mathcal{T}_{x_i} \leq_\alpha \mathcal{T}_{y_i}$. This holds because any Π^{in}_α formula about $\mathcal{T}_{x_i}$ can be transformed into a Π^{in}_α formula about x_i by restricting all quantifiers to being above x_i. Let

$$\mathcal{T}^*_{x_i} := \mathcal{T}_{x_i} - \left(\bigcup_{j:x_j \text{ is a child of } x_i} \mathcal{T}_{x_j} \right) \quad \text{and} \quad \mathcal{T}^*_{y_i} := \mathcal{T}_{y_i} - \left(\bigcup_{j:y_j \text{ is a child of } y_i} \mathcal{T}_{y_j} \right).$$

Note that $\mathcal{T}_\mathcal{A} = \sqcup_i \mathcal{T}^*_{x_i} = \sqcup_i \mathcal{T}^*_{y_i}$ as sets. By Lemma 2.5 we can observe that $\mathcal{T}^*_{x_i} \cong \mathcal{T}_{x_i}$ and $\mathcal{T}^*_{y_i} \cong \mathcal{T}_{y_i}$. In particular, this means that for all i, $\mathcal{T}^*_{x_i} \leq_\alpha \mathcal{T}^*_{y_i}$.

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We have a sequence of strategies σ_i where each σ_i wins the back-and-forth game for the $\exists$-player demonstrating that $\mathcal{T}^*_{x_i} \leq_\alpha \mathcal{T}^*_{y_i}$. We use these strategies to describe a winning strategy in the back-and-forth game for the $\exists$-player demonstrating that $S \leq_\alpha T$. At any given even stage of the game the $\forall$-player will have played a tuple $(S, \bar{d})$. Write $\bar{d} = (\bar{d}_1 \ldots \bar{d}_n)$ where $\bar{d}_i$ is the (possibly empty) subset of $\bar{d}$ that is in $\mathcal{T}^*_{y_i}$. The $\exists$-player will play the move $(T, \sigma_1(\bar{d}_1), \ldots, \sigma_n(\bar{d}_n))$ in response. The analogous strategy on the $\mathcal{T}^*_{x_i}$ is executed on the odd stages. In sum, the $\exists$-player will play a winning strategy on each part of the partition $\mathcal{T}_\mathcal{A} = \sqcup_i \mathcal{T}^*_{x_i} = \sqcup_i \mathcal{T}^*_{y_i}$ completely agnostic of what is being played on the other parts of the partition.

Consider tuples $(T, \bar{a})$ and $(S, \bar{b})$ that are produced at the end of a play where the $\exists$-player plays according to the above strategy. To confirm that the strategy is winning, we need to show that these tuples have the same atomic diagrams. Write $\bar{a} = (\bar{a}_1 \ldots \bar{a}_n)$ and $\bar{b} = (\bar{b}_1 \ldots \bar{b}_n)$ where $\bar{a}_i$ is the (possibly empty) subset of $\bar{a}$ that is in $\mathcal{T}^*_{y_i}$ and $\bar{b}_i$ is the (possibly empty) subset of $\bar{b}$ that is in $\mathcal{T}^*_{x_i}$. Because we played a winning strategy on each of the parts of the partition $\mathcal{T}_\mathcal{A} = \sqcup_i \mathcal{T}^*_{x_i} = \sqcup_i \mathcal{T}^*_{y_i}$, we know that for all i, $\bar{a}_i$ and $\bar{b}_i$ have the same atomic diagram. We must only check that the binary relations between elements in different tuples $\bar{a}_i$ and $\bar{a}_j$ are mirrored in the binary relations between $\bar{b}_i$ and $\bar{b}_j$ and that the binary relations between $\bar{a}_i$ and T are mirrored in the binary relations between $\bar{b}_i$ and S. Note that no binary relations hold between elements of distinct $\bar{a}_i$ and $\bar{a}_j$ as these elements are not comparable; the same is true for distinct $\bar{b}_i$ and $\bar{b}_j$. Given $y_k \in T$, it is less than all elements of $\bar{a}_i$ if $y_k \leq y_i$ and incomparable otherwise; the same is true for $x_k \in S$ and the elements of $\bar{b}_i$. This confirms that the binary relations between different parts of the tuples $(T, \bar{a})$ and $(S, \bar{b})$ are the same in both cases. This means that the strategy is winning for the $\exists$-player as claimed and $S \leq_\alpha T$ as desired. ⊣

LEMMA 2.13. *A structure $\mathcal{A}$ has an unstable λ-sequence if and only if its tree of tuples, $\mathcal{T}_\mathcal{A}$ does.*

PROOF. Consider an unstable λ-sequence in $\mathcal{A}$ given by $(\bar{y}_k)_{k\in\omega}$. Say each tuple is coded by the element $m_k \in \mathcal{T}_\mathcal{A}$, then Lemma 2.3 gives us at once that m_k is an unstable λ-sequence.

On the other hand, consider an unstable λ-sequence in $\mathcal{T}_\mathcal{A}$ given by the fundamental sequence $(\delta_k)_{k\in\omega}$ and associated tuples $(\bar{z}_k)_{i\in\omega}$. Let S_k denote the finite tree of elements less than or equal to some point in the tuple, $\bar{z}_k$. Note that the isomorphism type of S_k is fixed so long as we have without loss of generality that $\delta_k > 1$ for all k. By the definition of unstable λ-sequences we have that

$$\bar{z}_k \equiv_{\delta_k} \bar{z}_{k+1} \text{ and } \bar{z}_k \not\equiv_{\delta_{k+1}} \bar{z}_{k+1}.$$

By picking all of the elements in S_k or S_{k+1} as the first move of a back-and-forth game, we obtain that

$$S_k \equiv_{\delta_{k-1}} S_{k+1} \text{ and } S_k \not\equiv_{\delta_{k+1}} S_{k+1}.$$

It follows from Lemma 2.12 that by reducing to some reindexed fundamental subsequence of $(\delta_k)_{k\in\omega}$, say $(\gamma_k)_{k\in\omega}$, we have that for some sequence of points $x_k \in S_k$

$$x_k \equiv_{\gamma_k} x_{k+1} \text{ and } x_k \not\equiv_{\gamma_{k+1}} x_{k+1}.$$

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By Lemma 2.3, the $\bar{y}_k$ coded by the x_k, satisfy

$$\bar{y}_k \equiv_{\gamma_k} \bar{y}_{k+1} \text{ and } \bar{y}_k \not\equiv_{\gamma_{k+1}} \bar{y}_{k+1}.$$

Therefore, there is an unstable λ-sequence in $\mathcal{A}$. Together, we obtain the desired result. ⊣

COROLLARY 2.14. *$\mathcal{A}$ has Scott sentence complexity Π^{in}_λ if and only if $T_\mathcal{A}$ has Scott sentence complexity Π^{in}_λ.*

COROLLARY 2.15. *If $\mathcal{A}$ has Scott sentence complexity Γ^{in}_β, $\mathcal{T}_\mathcal{A}$ has Scott sentence complexity Π^{in}_α where $\alpha = \beta$ if $\Gamma = \Pi$ and $\alpha = \beta + 1$ otherwise.*

PROOF. If $SR(\mathcal{A}) = SR(\mathcal{T}_\mathcal{A})$ is not a limit ordinal, then it follows from $SR(\mathcal{T}_\mathcal{A}) = pSR(\mathcal{T}_\mathcal{A})$ that $\mathcal{T}_\mathcal{A}$ has $\Pi^{\text{in}}_{SR(\mathcal{A})+1}$ Scott sentence complexity.

If $SR(\mathcal{A}) = SR(\mathcal{T}_\mathcal{A}) = \lambda$ is a limit ordinal, then by Corollary 2.14 $SSC(\mathcal{A}) = \Pi^{\text{in}}_\lambda$ if and only if $SSC(\mathcal{T}_\mathcal{A}) = \Pi^{\text{in}}_\lambda$. ⊣

2.2. From tree of structures to linear orderings. We now turn our attention to the second step of the Friedman–Stanley embedding. Following Friedman and Stanley [8] we define the linear ordering $L(T)$ given a labeled tree as follows.

DEFINITION 2.16. Let T be a labeled tree with labeling function $l_T : T \to \omega$ and take the linear ordering $\mathbb{Q}^{<\omega}$, given by the lexicographic order on finite strings of elements in $\mathbb{Q}$. We first define a map $f : \mathbb{Q}^{<\omega} \to T$ by recursion as follows: Map the empty string to the root of T, then assuming we have defined $f(\sigma)$ we define f on elements of the form $\sigma^\frown q$ for $q \in \mathbb{Q}$ to be a map from $\mathbb{Q}$ to $\{\tau : \exists i\ \tau = \sigma^\frown i\}$ such that $f^{-1}\tau$ is dense in $\{\sigma^\frown q : q \in \mathbb{Q}\}$. Then, for every $\sigma \in T$, if $l(\sigma) = n$, then for every $x \in f^{-1}(\sigma)$, replace x by the finite linear ordering of size $n + 2$. If T is the tree of structures $\mathcal{T}_\mathcal{A}$ of $\mathcal{A}$, then we refer to $L(T)$ as $L_\mathcal{A}$.

LEMMA 2.17. *For every α and ordered $\bar{a}, \bar{b} \in L_\mathcal{A}$ such that if a_i, or b_i are from a labeled block, then every element in this block is in $\bar{a}$, respectively $\bar{b}$, $\bar{a} \leq_{3+\alpha} \bar{b}$ if and only if $f(\bar{a}) \leq_{1+\alpha} f(\bar{b})$.*

PROOF. The proof is by transfinite induction with the only interesting case being the base case. For this note that the label and predecessor relation of $\mathcal{T}_\mathcal{A}$ in $L_\mathcal{A}$ are both Σ^{in}_3 and Π^{in}_3 definable in $L_{\mathcal{T}_\mathcal{A}}$. Let φ be a Σ^{in}_1 formula true of $\bar{a}$, then this formula can be translated into a Σ^{in}_3 formula true of $f(\bar{a})$ and thus $\bar{a} \leq_3 \bar{b}$ implies $f(\bar{a}) \leq_1 f(\bar{b})$.

On the other hand, assume that $f(\bar{a}) \leq_1 f(\bar{b})$ and without loss of generality that $\bar{a}$ and $\bar{b}$ contain elements from exactly two different blocks. Let us consider the back-and-forth game where the $\forall$-player plays elements $\bar{b}^1$ on their first turn. For every b^1_i the $\exists$-player looks for elements a^1_i such that $l(f(a^1_i)) = l(f(b^1_i))$, a^1_i is in the same position in its block as b^1_i and $\bar{a}\bar{a}^1$ and $\bar{b}\bar{b}^1$ are isomorphic in the tree-ordering. Note that they will find a suitable $\bar{a}^1$ because $f(\bar{a}) \leq_1 f(\bar{b})$. Now on their last turn the $\forall$-player plays a tuple $\bar{a}^2$ and the $\exists$-player has to respond playing $\bar{b}^2$ such that $\bar{a}\bar{a}^1\bar{a}^2 \leq_1 \bar{b}\bar{b}^1\bar{b}^2$. All they need to do is ensure that the intervals in $\bar{a}\bar{a}^1\bar{a}^2$ are at least as large as the intervals in $\bar{b}\bar{b}^1\bar{b}^2$. Between any two elements in different blocks in $L_\mathcal{A}$, there are blocks coding paths in $\mathcal{T}_\mathcal{A}$. Thus, between any two elements

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in different blocks there are blocks of arbitrary large size and thus the $\exists$-player can find elements satisfying this requirement. ⊣

LEMMA 2.18. *For any structure $\mathcal{A}$, $2 + SR(\mathcal{T}_\mathcal{A}) = SR(L_\mathcal{A})$. In particular, $\mathcal{T}_\mathcal{A}$ has a $\Pi^{\text{in}}_{1+\alpha}$ Scott sentence if and only if $L_\mathcal{A}$ has a $\Pi^{\text{in}}_{3+\alpha}$ Scott sentence.*

PROOF. For successor ordinals it is sufficient to note that Lemma 2.17 implies that a tuple $\bar{a}$ in $L_\mathcal{A}$ is $(3+\alpha)$-free if and only if $f(\bar{a})$ is $(1+\alpha)$-free in $\mathcal{T}_\mathcal{A}$. Likewise, a tuple $\bar{a}$ is $(1+\alpha)$-free in $\mathcal{T}_\mathcal{A}$ if and only if $f^{-1}(\bar{a})$ is $(3+\alpha)$-free in $L_\mathcal{A}$.

For α a limit, note that by Lemma 2.17 an α-sequence for $L_\mathcal{A}$ is unstable if and only if the pull-back along f is unstable in $\mathcal{T}_\mathcal{A}$. ⊣

Combining everything developed in this section we obtain the following.

THEOREM 2.19. *For $\alpha > \omega$, the Friedman–Stanley embedding preserves parameterized Scott rank, but it does not preserve parameterless Scott rank. In particular, for any given structure $\mathcal{A}$ and any ordinal α:*

(1) *If $SSC(\mathcal{A}) = \Pi^{\text{in}}_{1+\alpha}$, then $SSC(L_\mathcal{A}) = \Pi^{\text{in}}_{3+\alpha}$.*
(2) *If $SSC(\mathcal{A}) = \Sigma^{\text{in}}_{1+\alpha}$ or $SSC(\mathcal{A}) = d\text{-}\Sigma^{\text{in}}_{1+\alpha}$, then $SSC(L_\mathcal{A}) = \Pi^{\text{in}}_{3+\alpha+1}$.*

Another useful characterization of structures with Scott sentence complexity Π^{in}_λ is that they are precisely those structures whose Π^{in}_λ infinitary theory is $\aleph_0$-categorical. Our results thus show the following.

COROLLARY 2.20. *For any limit ordinal λ and any structure $\mathcal{A}$, the Π^{in}_λ infinitary theory of $\mathcal{A}$ is $\aleph_0$-categorical if and only if the Π^{in}_λ theory of $L_\mathcal{A}$ is $\aleph_0$-categorical.*

The above corollary is related to a result by Calvert, Goncharov, and Knight who showed that there is a linear ordering of Scott rank ω_1^{CK} by proving a "rank-preservation property" for computable embeddings [7]. Their arguments show that the Friedman–Stanley embedding preserves $\aleph_0$-categoricity of the computable infinitary theory of structures. As a structure's computable infinitary theory is $\aleph_0$-categorical if and only if it has Scott rank ω_1^{CK} this shows that the Friedman–Stanley embedding preserves Scott rank at that level (see [1]).

2.3. Remarks on effectivity. Some Borel reductions maintain Scott rank externally in the sense that they take sufficiently complex Borel isomorphism classes and push them forward to classes of equal complexity in the image space. In this case, Scott sentence complexity is preserved for sufficiently high complexities.

PROPOSITION 2.21. *If $\Phi : Mod(\sigma) \to Mod(\tau)$ is a Borel reduction such that $[\Phi(X)]_\cong$ is Borel and that eventually satisfies push-forward, i.e., there is δ such that for $\gamma \geq \delta$ and $\psi \in \Pi^{\text{in}}_\gamma$ there is $\psi^* \in \Pi^{\text{in}}_\gamma$ such that*

$$\mathcal{A} \models \psi \iff \Phi(A) \models \psi^*,$$

then there is an ordinal α such that $SR(\mathcal{A}) \geq \alpha$ implies that

$$SSC(\mathcal{A}) = SSC(\Phi(\mathcal{A})).$$

PROOF. Say Φ is $\mathbf{\Sigma}^0_\lambda$-measurable and the image of Φ is $\mathbf{\Pi}^0_\beta$. Let $\chi \in \Pi^{\text{in}}_\beta$ describe the image of Φ in $Mod(\tau)$. Let $\mathcal{A}$ have a $\Gamma^{\text{in}}_\gamma$ Scott sentence for $\gamma > \delta$, say φ. Then $\varphi^* \wedge \chi$ is a $\Gamma^{\text{in}}_\gamma$ Scott sentence for $\Phi(\mathcal{A})$. On the other hand say $\Phi(\mathcal{A})$ has a $\Gamma^{\text{in}}_\gamma$ Scott

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sentence for any γ, then by the pullback theorem $\mathcal{A}$ has a $\Gamma^{\text{in}}_{\lambda+\gamma}$ Scott sentence. So for any $\mathcal{A}$ with $SR(\mathcal{A}) > \max(\lambda \cdot \omega, \delta, \beta)$, Scott sentence complexity is preserved. ⊣

Note that this proof has a straightforward effectivization. In particular, if Φ is hyperarithmetic and ψ computable implies that ψ^* is computable, then the complexity of the simplest computable formula that describes the isomorphism type of $\mathcal{A}$ is also preserved.

However, as we have seen in Theorem 2.19, the Friedman–Stanley embedding does not have the property that Scott sentence complexity is eventually preserved. We thus get the following corollary.

COROLLARY 2.22. *The image of the Friedman–Stanley embedding is not a Borel subset of the class of linear orderings.*

It should be noted that we can observe that the class of trees of structures is not Borel among the class of labeled trees directly. In particular, consider $\mathcal{T}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{B}}$ for structures with $\mathcal{A} \equiv_{\alpha+1} \mathcal{B}$ yet $\mathcal{A} \not\cong \mathcal{B}$. We can now define $T_{\mathcal{A},\mathcal{B}} := \mathcal{T}_{\mathcal{A}} \sqcup_r \mathcal{T}_{\mathcal{B}}$, the tree formed by identifying the roots of $\mathcal{T}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{B}}$ and adding no other relations between them. Given our established criteria for $\equiv_\alpha$, it is not difficult to show that $T_{\mathcal{A},\mathcal{B}} \equiv_\alpha \mathcal{T}_{\mathcal{A}} \equiv_\alpha \mathcal{T}_{\mathcal{B}}$. However, $T_{\mathcal{A},\mathcal{B}}$ is not a tree of structures, as it lacks the property that comeager many paths represent the same structure. By taking α arbitrarily large we can observe that there can be no $L_{\omega_1\omega}$ formula defining the class of tree of structures within the class of labeled trees.

This justifies why the proof of preservation of Scott rank proceeded as it did above. In particular, Scott rank was treated "internally" by looking at the definitions of automorphism orbits inside of the structure. Montalbán's theorem on the robustness of the Scott rank [19] and the work of Alvir, Greenberg, Harrison-Trainor, and Turetsky [1] allowed us to take this approach, as it provides a correspondence between the internal characterization of Scott sentence complexity and the external one. However, no such correspondence exists for computable formulas. Therefore, the proof technique used does not make it clear if there is a preservation of computable Scott rank.

The closest result to an effective version of Montalbán's theorem is that of Alvir, Knight, and McCoy [2]. They prove the following.

THEOREM 2.23. *Let $\mathcal{A}$ be a computable structure and consider the following three properties*:

(1) *There is a computable function f taking each tuple $\bar{a}$ to a computable $\Sigma^{\text{in}}_\alpha$ formula that defines its automorphism orbit.*
(2) *$\mathcal{A}$ has a computable $\Pi^{\text{in}}_{\alpha+1}$ Scott sentence.*
(3) *Every tuple in $\mathcal{A}$ has a computable $\Sigma^{\text{in}}_\alpha$-definable automorphism orbit.*

Item 1 *implies Item* 2 *implies Item* 3.

Effectivizing our results in the previous section, it is not difficult to see that the Friedman–Stanley embedding preserves Item 1 and Item 3. However, preservation of Item 2 may need a different technique, or not be true at all. Alvir, Knight, and McCoy [2] ask to give a necessary and sufficient condition for a structure $\mathcal{A}$ to have a computable $\Pi^{\text{in}}_{\alpha+1}$ Scott sentence. We offer the following variation closer connected to our studies.

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QUESTION 2.24. *Does $\mathcal{A}$ have a computable $\Pi^{\text{in}}_{\alpha+1}$ Scott sentence if and only if $\mathcal{T}_{\mathcal{A}}$ has a computable $\Pi^{\text{in}}_{\alpha+1}$ Scott sentence*?

If this question has a positive answer, we suspect that the technique involved in proving this could be used to answer the question of Alvir, Knight, and McCoy.

§3. α-universality and linear orderings. The main purpose of this section is to show that there is no linear ordering of Scott sentence complexity Σ^{in}_3 and that there is no embedding from any class of structures into linear orderings that is simpler than the Friedman–Stanley embedding in the sense that it increases the Scott sentence complexity less.

In order to do this we introduce and study the following notion for the class of linear orderings.

DEFINITION 3.1. Let $K_0 \subseteq K_1$ be isomorphism invariant classes of structures. The class K_0 is *α-universal* for K_1 if for every $\mathcal{A} \in K_1$ there is $\mathcal{B} \in K_0$ such that $\mathcal{A} \leq_\alpha \mathcal{B}$.

LEMMA 3.2. *If K_0 is α-universal for K_1, then K_0 contains all structures in K_1 with Π^{in}_α Scott sentences.*

PROOF. Consider $\mathcal{A} \in K_1$ with a Π^{in}_α Scott sentence. By α-universality, there is $\mathcal{B} \in K_0$ with $\mathcal{A} \leq_\alpha \mathcal{B}$. But then $\mathcal{A} \cong \mathcal{B}$ and thus $\mathcal{A} \in K_0$, as K_0 is isomorphism invariant. ⊣

COROLLARY 3.3. *If K_0 is α-universal for K_1, then K_0 contains all structures in K_1 of parameterless Scott rank less than α.*

LEMMA 3.4. *The class $K = \mathbb{N} \cup \{\mathbb{Q}\}$ is 2-universal for* LO. *In particular, $\{\mathbb{Q}\}$ is 2-universal for infinite linear orderings.*

PROOF. Let $\mathcal{A} \in$ LO. If $\mathcal{A}$ is finite then $\mathcal{A} \in K$. If $\mathcal{A}$ is infinite then we claim that $\mathcal{A} \leq_2 \mathbb{Q}$. As any interval I in $\mathbb{Q}$ is infinite we have that $I \leq_1 \mathcal{A}_0$ for any interval $\mathcal{A}_0$ of $\mathcal{A}$. Thus, $\mathcal{A} \leq_2 \mathbb{Q}$ and, hence, K is 2-universal. ⊣

3.1. Σ^{in}_3 is not the Scott sentence complexity of a linear ordering. Alvir, Greenberg, Harrison-Trainor, and Turetsky [1] showed that no countable structure has Scott sentence complexity Σ^{in}_2 and that no infinite countable structure has a d-Σ^{in}_1 Scott sentence. Thus, the first Σ Scott sentence complexity attainable by a countable structure is Σ^{in}_3.

Remmel [23] showed that the computably categorical linear orderings are precisely those with finite adjacency relation. It is well-known that the computably categorical linear orderings and the relatively computably categorical linear orderings coincide [10]. Both of these results relativize and thus one gets that the linear orderings with Σ^{in}_3 Scott sentences are precisely those with finite adjacency relation. For the convenience of the reader we give a simple proof using 2-universality.

THEOREM 3.5. *A linear ordering has a Σ^{in}_3 Scott sentence if and only if its adjacency relation is finite.*

PROOF. Say $\mathcal{A} \in$ LO has a Σ^{in}_3 Scott sentence, then by Table 1 there is a parameter $\bar{p} \in A^{<\omega}$ such that $(\mathcal{A}, \bar{p})$ has a Π^{in}_2 Scott sentence. By the 2-universality of $\mathbb{N} \cup \{\mathbb{Q}\}$

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every interval induced by $\bar{p}$ is isomorphic to either $\mathbb{Q}$ or n. Hence, $\mathcal{A}$ has at most finitely many adjacencies.

On the other hand, say that $Adj^{\mathcal{A}}$ is finite and let $\bar{p}$ be the ordered tuple of elements in the field of $Adj^{\mathcal{A}}$. Then every interval induced by $\bar{p}$ is either empty or isomorphic to $\mathbb{Q}$. As $\mathbb{Q}$ has a Π^{in}_2 Scott sentence, $(\mathcal{A}, \bar{p})$ has parameterless Scott rank 1, $\mathcal{A}$ has parameterized Scott rank 2, and, thus a Σ^{in}_3 Scott sentence. ⊣

THEOREM 3.6. *No linear ordering has Scott sentence complexity* Σ^{in}_3.

PROOF. The only candidates for linear orderings with Scott sentence complexity Σ^{in}_3 are by Theorem 3.5 those with finite adjacency relation. Now, these linear orderings are either finite, isomorphic to $\mathbb{Q}$, $1+\mathbb{Q}$, $\mathbb{Q}+1$, $1+\mathbb{Q}+1$, or isomorphic to a linear ordering obtained by replacing finitely many points in one of these four orderings by finite intervals. All but the last case are well-known to have Π^{in}_3 Scott sentences and thus can not have Scott sentence complexity Σ^{in}_3. Consider an ordering whose order type falls in the last case, and that, without loss of generality, its order type is $n_1+\mathbb{Q}+n_2+\mathbb{Q}+n_3+\mathbb{Q}+\cdots+n_k$. Let $N=\sum_{i\leq k} n_i$ and assume that x is the jth element in the ordered tuple containing all the successor chains. Then its orbit is defined by

$$\exists x_1 \ldots x_N \forall y \left(y \geq x_1 \wedge x_j = x \wedge \bigwedge_{i<j, i\notin\{n_1, n_1+n_2, \ldots, N\}} S(x_i, x_{i+1}) \right),$$

where S is the Π^0_1 definable successor relation. The formula is Σ^{in}_2 and defines the orbit of the jth element in the successor chains as required. Now, let x be an element in the jth $\mathbb{Q}$ copy where $i=\sum_{l\leq j} n_l$. Then its automorphism orbit is defined by

$$\exists x_1 \ldots x_N \left(\bigwedge_{j<N, j\notin\{n_1, n_1+n_2, \ldots, N\}} S(x_j, x_{j+1}) \wedge x_i < x < x_{i+1} \right)$$

and this definition is again Σ^{in}_2. It is easy to see that the defining formula of the automorphism orbit of any ordered tuple is just the conjunction of the defining formulas of the automorphism orbits of its elements. Thus, $n_1+\mathbb{Q}+n_2+\mathbb{Q}+\cdots+n_k$ has a Π^{in}_3 Scott sentence. ⊣

3.2. 3-universality and optimality of the Friedman–Stanley embedding.

THEOREM 3.7. *The following class is* 3*-universal for the class of linear orderings* $K=\{\omega+k, k+\omega^*, k+\mathbb{Z}+k' : k,k'\in\mathbb{N}\}\cup\{\omega+\omega^*\}\cup\{(\sum_{i\in k} n_i + m_i\cdot\mathbb{Q})+n_{k+1} : n_i, m_i\in\mathbb{N}, n_i > m_{i-1}, m_{i+1}\}\cup\{k : k\in\mathbb{N}\}$. *There is no* 3*-universal class* $J\subset K$.

PROOF. Given a linear ordering $\mathcal{L}$ we can write $\mathcal{L}=W+\mathcal{K}+R$ with W well ordered or empty, R reverse well ordered or empty, and $\mathcal{K}$ empty or without a greatest or least element.

The element in K that $\mathcal{L}$ is $\leq_3$-below will depend on the sizes of W and R, and the order type of $\mathcal{K}$. Let us consider the different cases.

Case 1. W is infinite and R is finite. In this case, $W=\omega+\alpha$ for some ordinal α, so $\mathcal{L}=\omega+\mathcal{K}'+k$ for some $k\in\mathbb{N}$ and $\mathcal{K}'$ is either empty or without greatest

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element. We claim that $\mathcal{L} \leq_3 \omega + k$. It is sufficient to show that $\omega + \mathcal{K}' \leq_3 \omega$. The $\exists$-player always wins the corresponding back-and-forth game by playing the following strategy. In the first round they play isomorphically on the initial ω. Because of that they only need to win the back-and-forth game for $\omega \leq_2 \omega + \mathcal{K}'$. Say the $\forall$-player picks elements such that $\omega + \mathcal{K}'$ is partitioned into n intervals $\mathcal{A}_i$, each of cardinality m_i for $i < n$ where m_i might be infinite. The $\exists$-player must pick a partition of ω such that for all intervals $\mathcal{B}_i$, each of cardinality k_i, $\mathcal{A}_i \leq_1 \mathcal{B}_i$. This inequality holds if and only if $m_i \geq k_i$. Given this inequality, the intervals formed by picking the first n elements of ω clearly form the desired partition.

Case 2. W is finite and R is infinite. This case is analogous to the first case.

Case 3. Both W and R are infinite. In this case, we can write $\mathcal{L} = \omega + \mathcal{K}' + \omega^*$. We claim that $\mathcal{L} \leq_3 \omega + \omega^*$. The winning strategy for the $\exists$-player is similar to the one for Case 1. In the first play they play isomorphically on the initial ω and final ω^* to reduce the game to showing that $\omega + \omega^* \leq_2 \omega + \mathcal{K}' + \omega^*$. They can then win this game by picking partitions with smaller intervals, exactly as above.

Case 4. Both W and R are finite. There are two subcases with different behavior here. In both we assume that $\mathcal{K}$ has no greatest and least element, as if it was empty, then $\mathcal{L}$ would be finite and thus isomorphic to a structure in K.

Subcase 1. There are arbitrarily large successor chains in $\mathcal{K}$. Here we claim that $\mathcal{K} \leq_3 \mathbb{Z}$, which gives that $\mathcal{L} \leq_3 k + \mathbb{Z} + k'$ for $k, k' \in \mathbb{N}$. The $\exists$-player always wins the corresponding game by playing the following strategy. Assume the $\forall$-player plays an ordered tuple with the distance between least and greatest element n. Then the $\exists$-player plays a successor chain of size n in $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}_1 + n + \mathcal{K}_2$ where $\mathcal{K}_1$ has no first element and $\mathcal{K}_2$ has no last element. It is then sufficient to show that $\omega^* \leq_2 \mathcal{K}_1$ and $\omega \leq_2 \mathcal{K}_2$. In these games, similar to the cases above, the $\exists$-player just needs to ensure that they play partitions of smaller size then the $\forall$-player and this is possible by the same strategy.

Subcase 2. There is an $n \in \mathbb{N}$ such that no element in $\mathcal{K}$ has n successors. In this case, we consider the block relation $\sim$ given by $a \sim b$ if and only if $[a, b]$ and $[b, a]$ are finite. By assumption every block is a linear ordering of size n or less. As no block can be infinite, no two blocks can be successors in the quotient $\mathcal{K}/\sim$. Furthermore, as there is no least or greatest element, there are no least or greatest blocks in $\mathcal{K}/\sim$. Therefore, $\mathcal{K}/\sim = \mathbb{Q}$. In total, this means that $\mathcal{K} = \sum_{q \in \mathbb{Q}} i_n(q)$ where $i_n : \mathbb{Q} \to \{1, \ldots, n\}$. If there are only finitely many blocks of size n, we may write $\mathcal{K}$ as a sum $\mathcal{K}_1 + n + \cdots + n + \mathcal{K}_l$ where no block of size n occurs in any of the $\mathcal{K}_i$. Repeating this process as necessary, we can write $\mathcal{K}$ as $\mathcal{K}_1 + n_1 + \mathcal{K}_2 + n_2 + \ldots \mathcal{K}_l$ where each $\mathcal{K}_i$ is given by $\mathcal{K}_i = \sum_{q \in (b_i, t_i)} i_n(q)$ where t_i is the smallest element of n_i and b_i is the largest element of n_{i-1} or $-\infty$ if $i = 1$ such that $\{q \in (b_i, t_i) : i_n(q) = m_i\}$ is infinite for $m_i = \max \operatorname{range}(i_n \restriction (b_i, t_i))$. Furthermore, we have that $m_i < n_i$ and if $i > 1$, $m_i < n_{i-1}$.

We now show that $\mathcal{L} = k + \mathcal{K}_1 + n_1 + \ldots \mathcal{K}_l + k' \leq_3 k + m_1 \cdot \mathbb{Q} + n_1 \ldots m_l \cdot \mathbb{Q} + k'$. It is enough to show that $\mathcal{K}_i = \sum_{q \in (b_i, t_i)} i_n(q) \leq_3 m_i \cdot \mathbb{Q}$. The $\exists$-player always wins this game by playing the following strategy. Every element the $\forall$-player plays in the first play is in a block of size m_i. The $\exists$-player can respond by playing elements in the same position in blocks of size m_i. The resulting intervals in $\mathcal{K}_i$ are of the form $l_1 + \sum_{q \in (b_i^j, t_i^j)} i_n(q) + l_2$ where $b_i^j > b_i$ and $t_i^j < t_i$. The corresponding intervals

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in $m_i \cdot \mathbb{Q}$ are of the form $l_1 + m_i \cdot \mathbb{Q} + l_2$. Winning the game thus comes down to finding a winning strategy for the game $m_i \cdot \mathbb{Q} \leq_2 \sum_{q \in (b_i^j, t_i^j)} i_n(q)$. Given a partition in $\sum_{q \in (b_i^j, t_i^j)} i_n(q)$, the $\exists$-player needs to find a partition in $m_i \cdot \mathbb{Q}$ such that all intervals are smaller than the intervals in the partition picked by the $\forall$-player. They can do that using the fact that the finite blocks are densely ordered and always of maximum size.

This shows that K is 3-universal. Standard arguments show that each element of K has a Π^{in}_3 Scott sentence. Therefore, by Lemma 3.2, there is no $J \subset K$ such that J is 3-universal. ⊣

Note that McCoy [18] obtained a similar characterization for the linear orderings that are Δ^0_2 categorical. Theorem 3.7 can be used to show a boldface version of this result.

COROLLARY 3.8. *If $\mathcal{L}$ is a linear ordering with a Π^{in}_3 Scott sentence, then $\mathcal{L}$ is in K.*

Since by Table 1 and Lemma 1.4 every linear ordering with a Σ^{in}_4 Scott sentence is sum of linear orderings with Π^{in}_3 Scott sentences we obtain the following.

COROLLARY 3.9. *If $\mathcal{L}$ is a linear ordering with a Σ^{in}_4 Scott sentence, then $\mathcal{L}$ is a finite sum of linear orderings in K.*

That the Friedman–Stanley embedding is optimal for Scott sentence complexity now follows easily from Corollary 3.9 by cardinality considerations.

THEOREM 3.10. *There is no Turing computable embedding $\Phi : Mod(\tau) \to \text{LO}$ such that for all $\mathcal{A} \in Mod(\tau)$, $SSC(\mathcal{A}) = \Pi^{\text{in}}_2$ if and only if $SSC(\Phi(\mathcal{A})) \in \{\Sigma^{\text{in}}_4, d\text{-}\Sigma^{\text{in}}_3, \Pi^{\text{in}}_3\}$ for any countable vocabulary τ.*

PROOF. We may assume that τ contains only one binary relation symbol, i.e., that the τ-structures are graphs. Notice that there are continuum many graphs with Scott sentence complexity Π^{in}_2: For any $X \subseteq \omega$ the graph consisting of a cycle of size $n+2$ for every $n \in X$ has a Π^{in}_2 Scott sentence. To see this just notice that the automorphism orbit of any element x in a cycle of size c_x is definable by the Σ^0_1-formula

$$\exists y_1, \dots, y_{c_x} \left(y_1 = x \wedge y_{c_x} \; E \; y_1 \wedge \bigwedge_{i < c_x} y_i \; E \; y_{i+1} \right).$$

However, by Corollary 3.9 there are only countably many linear orderings with a Σ^{in}_4 Scott sentence. As any potential embedding Φ needs to respect isomorphism, it cannot map every graph with a Π^{in}_2 Scott sentence to a linear ordering with a Scott sentence of complexity in $\{\Sigma^{\text{in}}_4, d\text{-}\Sigma^{\text{in}}_3, \Pi^{\text{in}}_3\}$. ⊣

PROPOSITION 3.11. *There is no countable class of 4-universal linear orderings.*

PROOF. As mentioned in the proof of Theorem 3.10 there are uncountably many graphs with Scott sentence complexity Π^{in}_2. The Friedman–Stanley embedding transforms those into linear orderings of Scott sentence complexity Π^{in}_4. Thus, by Lemma 3.2 there can not be a countable 4-universal set of linear orderings. ⊣

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§4. Possible Scott sentence complexities of linear orderings. We now exhibit possible Scott sentence complexities of linear orderings. Many of our examples are constructed by producing shuffle sums of linear orderings.

DEFINITION 4.1. Given a set S of countable linear orderings, the *shuffle sum* $Sh(S)$ of S is obtained by partitioning $\mathbb{Q}$ into $|S|$ dense sets K_i and replacing each point in K_i with a copy of S_i.

Shuffle sums are a very common construction that allows the coding of information into linear orderings. For example, if one considers a set $A \subseteq \omega$, we can consider it as a set of finite order types and take the shuffle sum $Sh(A)$, effectively coding the set. For a computability theoretic analysis of the coding strength of linear orderings that quite heavily uses shuffle sums see, for example, [9]. The crucial feature of shuffle sums is that if we partition a shuffle sum $Sh(S)$ into intervals P_i that are closed under the orderings in S (the partition does not split any subordering of type $S_i \in S$), then $P_i \cong Sh(S)$ for all P_i. This immediately follows from the fact that every S_i is dense in $Sh(S)$. Note that a linear ordering of the form $\mathcal{L} \cdot \mathbb{Q}$ is a special case of a shuffle sum. We begin by proving a couple of lemmas about the back-and-forth relations of shuffle sums.

LEMMA 4.2. *Let $\mathcal{A}$ be a shuffle sum and $\mathcal{B} = \mathcal{L} \cdot \mathbb{Q}$ for some ordering $\mathcal{L}$. If $\mathcal{A} \leq_\alpha \mathcal{B}$ then $\mathcal{A} + \mathcal{L} + \mathcal{B} \leq_{\alpha+2} \mathcal{A} + \mathcal{B}$.*

PROOF. To show this, we describe a winning strategy for the $\exists$-player in the $\alpha + 2$-back-and-forth game. On the first turn, the $\exists$-player copies the selected tuple of the $\forall$-player using an isomorphism of the initial $\mathcal{A}$s and the final $\mathcal{B}$s. The only possible non-isomorphic interval is the one between the smallest element of the tuple b that is in $\mathcal{B}$ and the largest element of the tuple a that is in $\mathcal{A}$. In other words, we need to show that

$$\mathcal{A}_{>a} + \mathcal{L} + \mathcal{B}_{<b} \geq_{\alpha+1} \mathcal{A}_{>a} + \mathcal{B}_{<b}.$$

Because $\mathcal{A}$ and $\mathcal{B}$ are shuffle sums, $\mathcal{A}_{>a} \cong N + \mathcal{A}$ and $\mathcal{B}_{<b} \cong \mathcal{B} + K$ for some linear orderings N and K. This means that we only need to show that $\mathcal{A} + \mathcal{L} + \mathcal{B} \geq_{\alpha+1} \mathcal{A} + \mathcal{B}$ if we play isomorphically on N and K.

On the next turn, the $\forall$-player plays on $\mathcal{A} + \mathcal{L} + \mathcal{B}$, and the $\exists$-player responds on $\mathcal{A} + \mathcal{B}$ using an isomorphism between the initial $\mathcal{A}$s and an isomorphism of $\mathcal{L} + \mathcal{B}$ to a final segment of $\mathcal{B}$. Assume without loss of generality that c is the smallest element in $\mathcal{L}$ played by the $\forall$-player in the distinguished copy of $\mathcal{L}$ within $\mathcal{A} + \mathcal{L} + \mathcal{B}$. The only possible non-isomorphic interval is between c and the largest element a selected within $\mathcal{A}$. In other words, we need to show that

$$\mathcal{A}_{>a} + \mathcal{L}_{<c} \leq_\alpha \mathcal{A}_{>a} + \mathcal{B} + \mathcal{L}_{<c}.$$

As $\mathcal{A}_{>a} \cong M + \mathcal{A}$ and the $\exists$-player can play the isomorphism between the copies of $\mathcal{L}_{<c}$ and M, it is sufficient to show that $\mathcal{A} \leq_\alpha \mathcal{A} + \mathcal{B}$. As $\mathcal{A}$ is a shuffle sum $\mathcal{A} \cong \mathcal{A} + \mathcal{A}$. Therefore, it is enough to show that $\mathcal{A} \leq_\alpha \mathcal{B}$, as desired. ⊣

COROLLARY 4.3. *Let $\mathcal{A}$ be a shuffle sum and $\mathcal{B} = \mathcal{L} \cdot \mathbb{Q}$ for some ordering $\mathcal{L}$. If $\mathcal{A} + \mathcal{L} + \mathcal{B} \not\cong \mathcal{A} + \mathcal{B}$ and $\mathcal{A} \leq_\alpha \mathcal{B}$, then $\mathcal{A} + \mathcal{L} + \mathcal{B}$ has no $\Pi^{\text{in}}_{\alpha+2}$ Scott sentence.*

We are now ready to provide the constructions of example linear orderings.

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THEOREM 4.4. *There is a linear ordering with Scott sentence complexity* Σ^{in}_4.

PROOF. Consider the linear ordering $\mathcal{L} = 2 \cdot \mathbb{Q} + 1 + \mathbb{Q}$ and call the 1 in the middle c. We claim that $SR(\mathcal{L}, c) = \max\{SR(2 \cdot \mathbb{Q}), SR(\mathbb{Q})\} \leq 2$ (the first equality follows from [14, Lemma 11]). The inequality follows from the fact that in $2 \cdot \mathbb{Q}$ the orbits of singletons are defined by the Σ^{in}_2-formulas $\exists y\ S(x) = y$ and $\exists y\ S(y) = x$. This definition extends to tuples by adding in the order of the tuple to the definition along with any successor relations that hold.

We now show that $\mathcal{L}$ has Scott sentence complexity Σ^{in}_4. This will be done by showing that it does not have a Π^{in}_4 Scott sentence. In order to do this, we appeal to Corollary 4.3 which implies that all we need to show is that $2 \cdot \mathbb{Q} \leq_2 \mathbb{Q}$. However, this follows immediately from the 2-universality of $\mathbb{Q}$ among infinite linear orderings (Lemma 3.4). ⊣

Note that the above proof also implies that $2 \cdot \mathbb{Q} + 1 + \mathbb{Q} \leq_4 2 \cdot \mathbb{Q} + \mathbb{Q}$.

THEOREM 4.5. *There is a linear ordering with Scott sentence complexity* Σ^{in}_5.

PROOF. Let $\mathcal{A} = Sh(\{1, \omega\})$, $\mathcal{B} = \omega \cdot \mathbb{Q}$, $\mathcal{L} = \mathcal{A} + \omega + \mathcal{B}$ and denote the first element of the ω in the middle by c. We claim that $\mathcal{L}$ has Scott sentence complexity Σ^{in}_5.

First, we show that $SR(\mathcal{L}, c) = \max\{SR(\mathcal{A}), SR(\mathcal{B})\} \leq 3$ (the first equality follows from [14, Lemma 11]). Define the following family of formulas that denote if x is the top of an successor chain of length $n > 1$:

$$S_n(x): \qquad \exists x_1, \ldots, x_{n-1} \bigwedge_{i<n-1} S(x_i, x_{i+1}) \wedge S(x_{n-1}, x).$$

Note that $S_n(x)$ is Σ^{in}_2. It is not difficult to see that the automorphism orbits of elements in $\mathcal{B}$ are given by the formulas

$$\forall y \neg S(y, x) \text{ and } S_n(x) \wedge \neg S_{n+1}(x) \text{ for } n \in \omega.$$

These formulas are all Σ^{in}_3 and the automorphism orbits of tuples can be constructed by taking conjunctions over them along with adding any needed successor relations between the elements. Thus, $SR(\mathcal{B}) \leq 3$. Similarly, within $\mathcal{A}$ the automorphism orbits of elements are given by the formulas

$$\forall y\ \neg S(x, y),\ \forall y\ \neg S(y, x) \wedge \exists y\ S(x, y) \text{ and } S_n(x) \wedge \neg S_{n+1}(x) \text{ for all } n \in \omega.$$

To extend these definitions to tuples we only need to add the successor relations that hold between the elements and the order of the elements. All of these definitions are Σ^{in}_3 and therefore $SR(\mathcal{A}) \leq 3$.

It remains to show that $\mathcal{L}$ does not have a Π^{in}_5 Scott sentence. By Corollary 4.3 it is enough to show that $\omega \cdot \mathbb{Q} \geq_3 \mathcal{A}$. Towards this let $\bar{p} \in \omega \cdot \mathbb{Q}$ be an ordered tuple. We want to find $\bar{q} \in \mathcal{A}$ such that $(\omega \cdot \mathbb{Q}, \bar{p}) \leq_2 (\mathcal{A}, \bar{q})$. To do this, take an ordered tuple $\bar{q} \in \mathcal{A}$ such that every q_i is in an ω-block and in the same position within that block as p_i. Furthermore, assure that for each pair $i < j$, p_i and p_j are in the same ω-block if and only if q_i and q_j are. In particular, finite intervals formed by $\bar{p}$ within an ω-block are isomorphic to the corresponding finite intervals formed by $\bar{q}$ within an ω-block. Note that, for each i not accounted for in the above analysis, there

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is some $k_i \in \omega$, such that $(p_i, p_{i+1}) \cong \omega + \mathcal{A} + k_i$ and $(q_i, q_{i+1}) \cong \omega + \omega \cdot \mathbb{Q} + k_i$. Therefore, it is enough to show that

$$\mathcal{A} \geq_2 \omega \cdot \mathbb{Q}.$$

Repeating the style of argument from above we fix a tuple $\bar{p} \in \mathcal{A}$ and find a suitable $\bar{q} \in \omega \cdot \mathbb{Q}$ such that $(\mathcal{A}, \bar{p}) \leq_1 (\omega \cdot \mathbb{Q}, \bar{q})$. Recall that $\mathcal{A} \leq_1 \mathcal{B}$ if and only if $|\mathcal{B}| \leq |\mathcal{A}|$. Assuming that $\bar{p}$ is ordered, (p_i, p_{i+1}) has cardinality $n \in \omega$ or $\aleph_0$ with the first and last intervals always having cardinality $\aleph_0$. Any combination of gaps of size n and $\aleph_0$ can also be found in $\omega \cdot \mathbb{Q}$ and we can thus find a suitable tuple $\bar{q}$ to finish the proof. ⊣

Given these base case examples of linear orderings of small Scott sentence complexity, we now devise a method of systematically using these examples to fill in most of the larger Scott sentence complexities. The examples are quite simple; we consider orderings of the form $\mathbb{Z}^\alpha \cdot \mathcal{L}$ where $\mathcal{L}$ is one of our previously constructed examples. The more difficult work is determining exactly of what Scott sentence complexity these orderings are.

This will be proven by demonstrating an upper and lower bound on the complexity of the Scott sentence. We begin with proving the lower bound. In the proof of the below lemma we use ζ_α to denote the unique up to isomorphism initial segment of $\mathbb{Z}^\alpha$. It is easy to check that ζ_α^* is the unique end segment of $\mathbb{Z}^\alpha$ and that $\zeta_{\alpha+1}^* = \zeta_\alpha + \mathbb{Z}^\alpha \cdot \omega$.

LEMMA 4.6. *For the sake of organization, consider the following ordinal indexed propositions.*

(A_α) *For any* $\mathcal{K}$ *and* $\mathcal{L}$, $\mathbb{Z}^\alpha \cdot \mathcal{K} \leq_{2\alpha} \mathbb{Z}^\alpha \cdot \mathcal{L}$.
(B_α) *For any* $\mathcal{K}$ *and* $\mathcal{L}$ *with* $|\mathcal{K}| \geq |\mathcal{L}|$, $\mathbb{Z}^\alpha \cdot \mathcal{K} \leq_{2\alpha+1} \mathbb{Z}^\alpha \cdot \mathcal{L}$.
(C_α) *For any* $\mathcal{K}$ *and any* $\mathcal{L}$ *without a last element*, $\mathbb{Z}^\alpha \cdot (\omega + \mathcal{K}) \leq_{2\alpha+2} \mathbb{Z}^\alpha \cdot (\omega + \mathcal{L})$.

For any countable ordinal α, A_α, B_α, *and* C_α *are true.*

PROOF. The proof is by induction with the statements A_0 and B_0 trivially true. We start by showing that B_α implies $A_{\alpha+1}$.

Select a tuple $a_0 \ldots a_n$ in $\mathbb{Z}^{\alpha+1} \cdot \mathcal{L} = \mathbb{Z}^\alpha \cdot \mathbb{Z} \cdot \mathcal{L}$. We will find a tuple $b_0 \ldots b_n$ in $\mathbb{Z}^{\alpha+1} \cdot \mathcal{K} = \mathbb{Z}^\alpha \cdot \mathbb{Z} \cdot \mathcal{K}$ such that $(\mathbb{Z}^{\alpha+1} \cdot \mathcal{L}, \bar{a}) \leq_{2\alpha+1} (\mathbb{Z}^{\alpha+1} \cdot \mathcal{K}, \bar{b})$. Pick a point b_0 arbitrarily. Given b_j, we pick b_{j+1} as follows. If a_j and a_{j+1} are in the same copy of $\mathbb{Z}^{\alpha+1}$, we can pick b_{j+1} such that $(a_j, a_{j+1}) \cong (b_j, b_{j+1})$. If not, we pick b_{j+1} arbitrarily in the copy of $\mathbb{Z}^\alpha$ that is the successor of the successor of the copy of $\mathbb{Z}^\alpha$ containing b_j. In other words, we pick b_{j+1} with $(b_j, b_{j+1}) \cong \zeta_\alpha^* + \mathbb{Z}^\alpha + \zeta_\alpha$.

To show that our choice of $\bar{b}$ is correct, we have to confirm the following relations between non-isomorphic intervals.

(1) $(b_j, b_{j+1}) \cong \zeta_\alpha^* + \mathbb{Z}^\alpha + \zeta_\alpha \geq_{2\alpha+1} \zeta_\alpha^* + \mathbb{Z}^\alpha \cdot \mathcal{L}_1 + \zeta_\alpha$, where $\mathcal{L}_1$ is an infinite order,
(2) $(-\infty, b_0) \cong \mathbb{Z}^\alpha \cdot \mathcal{K}_1 + \zeta_\alpha \geq_{2\alpha+1} \mathbb{Z}^\alpha \cdot \mathcal{L}_1 + \zeta_\alpha$, where $\mathcal{L}_1$ and $\mathcal{K}_1$ are some initial segments of $\mathbb{Z} \cdot \mathcal{L}$ and $\mathbb{Z} \cdot \mathcal{K}$, and therefore infinite, or
(3) $(b_n, \infty) \cong \zeta_\alpha^* + \mathbb{Z}^\alpha \cdot \mathcal{L}_1 \geq_{2\alpha+1} \zeta_\alpha^* + \mathbb{Z}^\alpha \cdot \mathcal{K}_1$, where $\mathcal{L}_1$ and $\mathcal{K}_1$ are some final segments of $\mathbb{Z} \cdot \mathcal{L}$ and $\mathbb{Z} \cdot \mathcal{K}$, and therefore infinite.

It follows immediately from B_α that all of these back-and-forth relations hold.

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We now show that B_α implies C_α. So assume that $\mathcal{K}$ and $\mathcal{L}$ have no last element and that B_α holds. Select a tuple $a_0 \dots a_n$ in $\mathbb{Z}^\alpha \cdot (\omega + \mathcal{L})$. Without loss of generality, a_0 is in the first copy of $\mathbb{Z}^\alpha$. We will find a tuple $b_0 \dots b_n$ in $\mathbb{Z}^\alpha \cdot (\omega + \mathcal{K})$ such that $(\mathbb{Z}^\alpha \cdot (\omega + \mathcal{L}), \bar{a}) \leq_{2\alpha+1} (\mathbb{Z}^\alpha \cdot (\omega + \mathcal{K}), \bar{b})$. For the a_j in the initial $\mathbb{Z}^\alpha \cdot \omega$, the b_j are picked isomorphically. We now define the rest of the b_j by induction so that all of them are in the initial $\mathbb{Z}^\alpha \cdot \omega$. If a_j is in the same $\mathbb{Z}^\alpha$ block as a_{j+1} or a_{j+1} is in the successor of a_j's block, define b_{j+1} so that $(b_j, b_{j+1}) \cong (a_j, a_{j+1})$. Otherwise, define b_{j+1} as being some element in the successor of the successor of the $\mathbb{Z}^\alpha$ block containing b_j. Not including isomorphic intervals, we only need to check the following cases.

(1) $(b_j, b_{j+1}) \cong \zeta_\alpha^* + \mathbb{Z}^\alpha + \zeta_\alpha \geq_{2\alpha+1} \zeta_\alpha^* + \mathbb{Z}^\alpha \cdot \mathcal{L}_1 + \zeta_\alpha$ where $\mathcal{L}_1$ is infinite,
(2) $(b_n, \infty) \cong \zeta_\alpha^* + \mathbb{Z}^\alpha \cdot (\omega + \mathcal{K}) \geq_{2\alpha+1} \zeta_\alpha^* + \mathbb{Z}^\alpha \cdot \mathcal{L}_1$, where $\mathcal{L}_1$ is some final segment of $\omega + \mathcal{L}$, and thus infinite.

All of these relations follow from B_α, as desired.

To complete the proof of the successor step of the induction, we show that C_α and $A_{\alpha+1}$ imply $B_{\alpha+1}$.

Select a tuple $a_0 \dots a_n$ in $\mathbb{Z}^{\alpha+1} \cdot \mathcal{L}$. These points belong to at most n distinct copies of $\mathbb{Z}^{\alpha+1}$. Because $|\mathcal{K}| \geq |\mathcal{L}|$, we can match each chosen copy of $\mathbb{Z}^{\alpha+1}$ to one in $\mathbb{Z}^{\alpha+1} \cdot \mathcal{K}$. Furthermore, if there is a copy of $\mathbb{Z}^{\alpha+1}$ between two of the chosen copies in $\mathbb{Z}^{\alpha+1} \cdot \mathcal{L}$, we can assure that the corresponding copies in $\mathbb{Z}^{\alpha+1} \cdot \mathcal{K}$ are also not successors. Also, if a_n is not in the last copy of $\mathbb{Z}^{\alpha+1}$, we can assure that neither is b_n and, similarly for a_0 and b_0 in the first copy. In particular, not including the isomorphic intervals, we only need to check the following cases.

(1) $(b_j, b_{j+1}) \cong \zeta_{\alpha+1}^* + \mathbb{Z}^{\alpha+1} \cdot \mathcal{K}_1 + \zeta_{\alpha+1} \geq_{2\alpha+2} \zeta_{\alpha+1}^* + \mathbb{Z}^{\alpha+1} \cdot \mathcal{L}_1 + \zeta_{\alpha+1}$ for some orderings $\mathcal{K}_1$ and $\mathcal{L}_1$,
(2) $(b_n, \infty) \cong \zeta_{\alpha+1}^* + \mathbb{Z}^{\alpha+1} \cdot \mathcal{K}_1 \geq_{2\alpha+2} \zeta_{\alpha+1}^* + \mathbb{Z}^{\alpha+1} \cdot \mathcal{L}_1$ for some orderings $\mathcal{K}_1$ and $\mathcal{L}_1$,
(3) $(-\infty, b_0) \cong \mathbb{Z}^{\alpha+1} \cdot \mathcal{K}_1 + \zeta_{\alpha+1} \geq_{2\alpha+2} \mathbb{Z}^{\alpha+1} \cdot \mathcal{L}_1 + \zeta_{\alpha+1}$ for some orderings $\mathcal{K}_1$ and $\mathcal{L}_1$,
(4) $(-\infty, b_0) \cong \zeta_{\alpha+1} \geq_{2\alpha+2} \mathbb{Z}^{\alpha+1} \cdot \mathcal{L}_1 + \zeta_{\alpha+1}$ for some order $\mathcal{L}_1$,
(5) $(b_n, \infty) \cong \zeta_{\alpha+1}^* + \mathbb{Z}^{\alpha+1} \cdot \mathcal{K}_1 \geq_{2\alpha+2} \zeta_{\alpha+1}^*$ for some order $\mathcal{K}_1$.

The first three cases are handled immediately by $A_{\alpha+1}$. By symmetry it is sufficient to show case (5). However, this is the same as showing that

$$\zeta_\alpha^* + \mathbb{Z}^\alpha \cdot \omega \leq_{2\alpha+2} \zeta_\alpha^* + \mathbb{Z}^\alpha \cdot (\omega + \mathbb{Z} \cdot \mathcal{K}_1).$$

This follows directly from C_α.

At last we consider the remaining limit levels. We already have seen that B_λ implies C_λ. The statement A_λ follows immediately from A_γ for $\gamma < \lambda$. A bit more nuanced is the issue of B_λ. It follows from A_λ and C_γ for $\gamma < \lambda$. Analogous to the proof in the successor case, we analyze the possibilities:

(1) $\zeta_\lambda^* + \mathbb{Z}^\lambda \cdot \mathcal{K}_1 + \zeta_\lambda \geq_\lambda \zeta_\lambda^* + \mathbb{Z}^\lambda \cdot \mathcal{L}_1 + \zeta_\lambda$ for some orderings $\mathcal{L}_1$ and $\mathcal{K}_1$,
(2) $\zeta_\lambda^* + \mathbb{Z}^\lambda \cdot \mathcal{K}_1 \geq_\lambda \zeta_\lambda^* + \mathbb{Z}^\lambda \cdot \mathcal{L}_1$ for some orderings $\mathcal{L}_1$ and $\mathcal{K}_1$,
(3) $\mathbb{Z}^\lambda \cdot \mathcal{K}_1 + \zeta_\lambda \geq_\lambda \mathbb{Z}^\lambda \cdot \mathcal{L}_1 + \zeta_\lambda$ for some orderings $\mathcal{L}_1$ and $\mathcal{K}_1$,
(4) $\mathbb{Z}^\lambda \cdot \mathcal{K}_1 + \zeta_\lambda \geq_\lambda \zeta_\lambda$ for some order $\mathcal{K}_1$,
(5) $\zeta_\lambda^* + \mathbb{Z}^\lambda \cdot \mathcal{K}_1 \geq_\lambda \zeta_\lambda^*$ for some order $\mathcal{K}_1$.

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The first three cases are handled immediately by A_λ. By symmetry it is clear that it is enough to show case 5. However, this is the same as showing that for all $\gamma < \lambda$,

$$\zeta_\gamma^* + \mathbb{Z}^\gamma \cdot (\omega + \mathbb{Z} \cdot \zeta_\lambda^*) \leq_\gamma \zeta_\gamma^* + \mathbb{Z}^\gamma \cdot (\mathbb{Z} \cdot \zeta_\lambda^* + \mathbb{Z}^\lambda \cdot \mathcal{K}_1),$$

which follows from the C_γ for $\gamma < \lambda$.

Therefore, for any countable ordinal α, A_α, B_α, and C_α all hold. ⊣

The proposition B_α is analogous to Lemma II.38 in [22]. However, it has the notable advantage of being more universal, cleaner to state and easier to apply for our purposes. It is also independently interesting as it furthers our understanding of the behavior of powers of $\mathbb{Z}$, which have been studied in [13] and [6]. For our purpose, what is important is that the proposition B_α serves as the base case for the following lemma, which generalizes Lemmas 7.2 and 7.3 from [13] and Proposition 4.8 from [6].

LEMMA 4.7. *For any $\mathcal{L}$ and $\mathcal{K}$ $\mathcal{L} \leq_\beta \mathcal{K} \implies \mathbb{Z}^\alpha \cdot \mathcal{L} \leq_{2\alpha+\beta} \mathbb{Z}^\alpha \cdot \mathcal{K}$.*

PROOF. We demonstrate this by induction on β. Notice that the base case is given by the above lemma. The limit case follows immediately by the definition of the back-and-forth relations along with the observation that for any linear ordering $\mathcal{N}$, $\mathcal{L} \leq_\beta \mathcal{K} \implies \mathcal{N} \cdot \mathcal{L} \leq_\beta \mathcal{N} \cdot \mathcal{K}$. Thus, we need only look at the successor case.

Let $\beta = \gamma + 1$ and assume $\mathcal{L} \leq_{\gamma+1} \mathcal{K}$. We need to show that $\mathbb{Z}^\alpha \cdot \mathcal{L} \leq_{2\alpha+\gamma+1} \mathbb{Z}^\alpha \cdot \mathcal{K}$. We view $\mathbb{Z}^\alpha \cdot K$ as a product ordering with elements of the form (δ, b) where $b \in K$ and $\delta \in \mathbb{Z}^\alpha$. Consider a play of the game where the $\forall$-player plays a tuple $(\delta_i, b_i)_{i \in k}$ from $\mathbb{Z}^\alpha \cdot \mathcal{K}$. If s is a winning strategy for demonstrating that $\mathcal{L} \leq_{\gamma+1} \mathcal{K}$, the $\exists$-player may play $(\delta_i, s(b_i))_{i \in k}$ in response. Let $b_{-1} = s(b_{-1}) = -\infty$ and $b_k = s(b_k) = \infty$ by convention. We need only show that

$$[(\delta_i, b_i), (\delta_{i+1}, b_{i+1})] \leq_{2\alpha+\gamma} [(\delta_i, s(b_i)), (\delta_{i+1}, s(b_{i+1}))].$$

Note that the left-hand side is isomorphic to $\zeta_\alpha + \mathbb{Z}^\alpha \cdot (b_i, b_{i+1}) + \zeta_\alpha^*$ and the right-hand side is isomorphic to $\zeta_\alpha + \mathbb{Z}^\alpha \cdot (s(b_i), s(b_{i+1})) + \zeta_\alpha^*$. As s is a winning strategy, by induction,

$$\mathbb{Z}^\alpha \cdot (b_i, b_{i+1}) \leq_{2\alpha+\gamma} \mathbb{Z}^\alpha \cdot (s(b_i), s(b_{i+1})),$$

which demonstrates the claim. ⊣

The following corollary follows immediately from Lemma 4.7 and the fact that multiplying by $\mathbb{Z}^\alpha$ is an injective map on linear orderings.

COROLLARY 4.8. *For any linear ordering $\mathcal{L}$, if there is a $\mathcal{K}$ with $\mathcal{L} \leq_\beta \mathcal{K}$ yet $\mathcal{L} \not\cong \mathcal{K}$ then $\mathbb{Z}^\alpha \cdot \mathcal{L}$ has no $\Pi^{\text{in}}_{2\alpha+\beta}$ Scott sentence. For any linear ordering $\mathcal{L}$, if there is a $\mathcal{K}$ with $\mathcal{L} \geq_\beta \mathcal{K}$ yet $\mathcal{L} \not\cong \mathcal{K}$ then $\mathbb{Z}^\alpha \cdot \mathcal{L}$ has no $\Sigma^{\text{in}}_{2\alpha+\beta}$ Scott sentence.*

This allows to transfer lower bounds for the Scott sentence complexity of $\mathcal{L}$ to $\mathbb{Z}^\alpha \cdot \mathcal{L}$.

We now move to proving the upper bound. In particular, we must demonstrate that multiplying by a power of $\mathbb{Z}$ does not make the Scott rank too high. For this we use the notion of the block relation $\sim_\alpha$. We follow the definitions in [3].

LEMMA 4.9. *If $\mathcal{L}$ has a Γ^{in}_β Scott sentence for $\Gamma \in \{\Sigma, \Pi, d\text{-}\Sigma\}$, then the linear ordering $\mathbb{Z}^\alpha \cdot \mathcal{L}$ has a $\Gamma^{\text{in}}_{2\alpha+\beta}$ Scott sentence.*

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PROOF. Given a formula φ in the language of linear orderings we define φ_α as being the same as φ except for the fact that instances of $x < y$ are replaced by $x < y \wedge x \not\sim_\alpha y$ and instances of $x = y$ are replaced by $x \sim_\alpha y$. Note that if φ is Γ^{in}_β, then φ_α is at worst $\Gamma^{\text{in}}_{2\alpha+\beta}$ by the fact that $\sim_\alpha$ is $\Sigma^{\text{in}}_{2\alpha}$ definable (see [3, Proposition 4]). It is also not difficult to see that if φ is a Scott sentence for $\mathcal{L}$, then $\mathcal{N} \models \varphi_\alpha$ guarantees that $\mathcal{N}/\sim_\alpha \cong \mathcal{L}$. Finally, we define

$$\psi = \forall x \bigwedge\!\!\!\bigwedge_{\delta<\alpha} \bigwedge\!\!\!\bigwedge_{n\in\omega} (\exists y\ S^n_\delta(x) = y) \wedge (\exists y\ S^n_\delta(y) = x).$$

Here, $S^n_\delta(x) = y$ is shorthand for saying that y is the nth δ-successor of x. In other words,

$$\exists z_0 \dots z_n\ x = z_0 \wedge y = z_n \wedge \bigwedge_{i<n} z_i \not\sim_\delta z_{i+1} \wedge \forall w\ z_i < w < z_{i+1} \to (w \sim_\delta z_i \vee w \sim_\delta z_{i+1}).$$

Overall, this is $\Sigma^{\text{in}}_{2\delta+2}$. This gives that ψ is, at worst, $\Pi^{\text{in}}_{2\alpha+1}$. Furthermore, ψ guarantees that every $\sim_\alpha$ equivalence class is isomorphic to $\mathbb{Z}^\alpha$. Therefore, $\varphi_\alpha \wedge \psi$ gives the desired Scott sentence for $\mathbb{Z}^\alpha \cdot \mathcal{L}$. ⊣

This result along with the previous constructions gives that there are linear orderings with any Scott sentence complexity that is not too close to a limit ordinal.

COROLLARY 4.10. *There are linear orderings of the following Scott sentence complexities*:

(1) $\Sigma^{\text{in}}_{\alpha+n}$ *for any countable ordinal* α *and* $n \geq 4$,
(2) $d\text{-}\Sigma^{\text{in}}_{\alpha+n}$ *for any countable ordinal* α *and* $n \geq 1$,
(3) $\Pi^{\text{in}}_{\alpha+n}$ *for any countable ordinal* α *and* $n \geq 1$.

PROOF. Let $\mathcal{L}_4$ be the constructed example of Scott sentence complexity Σ^{in}_4 and $\mathcal{K}_4$ such that $\mathcal{L}_4 \leq_4 \mathcal{K}_4$. Similarly define $\mathcal{L}_5$ and $\mathcal{K}_5$. For any α we know that $\mathbb{Z}^\alpha \cdot \mathcal{L}_4$ has a $\Sigma^{\text{in}}_{2\alpha+4}$ Scott sentence, yet it has no $\Pi^{\text{in}}_{2\alpha+4}$ Scott sentence as $\mathbb{Z}^\alpha \cdot \mathcal{L}_4 \leq_{2\alpha+4} \mathbb{Z}^\alpha \cdot \mathcal{K}_4$. Thus, $\mathbb{Z}^\alpha \cdot \mathcal{L}_4$ has Scott sentence complexity $\Sigma^{\text{in}}_{2\alpha+4}$. Similarly, $\mathbb{Z}^\alpha \cdot \mathcal{L}_5$ has Scott sentence complexity $\Sigma^{\text{in}}_{2\alpha+5}$. These constructions provide examples for all of the claimed cases of the form $\Sigma^{\text{in}}_{\alpha+n}$.

Let $\mathcal{L}_2 := \mathbb{Q} + 2 + \mathbb{Q}$. It is not difficult to see that this order has a $d\text{-}\Sigma^{\text{in}}_2$ Scott sentence and that $\mathbb{Q} + 3 + \mathbb{Q} \leq_2 \mathcal{L}_2 \leq_2 \mathbb{Q}$, so this is indeed optimal. Using the same reasoning as above, we see that $\mathbb{Z}^\alpha \cdot \mathcal{L}_2$ has Scott sentence complexity $d\text{-}\Sigma^{\text{in}}_{2\alpha+2}$. Ash [5] analyzed the back-and-forth relations of well-orderings and it follows from his results that $\omega^\alpha \cdot 2$ has Scott sentence complexity $d\text{-}\Sigma^{\text{in}}_{2\alpha+1}$ (see [3, proof of Proposition 19]). These constructions provide examples for all of the claimed cases of the form $d\text{-}\Sigma^{\text{in}}_{\alpha+n}$.

It is a basic exercise to show that $\mathbb{Q}$ has Scott sentence complexity Π^{in}_2. Using the same reasoning as above, we see that $\mathbb{Z}^\alpha \cdot \mathbb{Q}$ has Scott sentence complexity $\Pi^{\text{in}}_{2\alpha+2}$. Furthermore, it again follows from results of Ash [5] that ω^α has Scott sentence complexity $\Pi^{\text{in}}_{2\alpha+1}$. These constructions provide examples for all of the claimed cases of the form $\Pi^{\text{in}}_{\alpha+n}$. ⊣

This covers every case except for some possibilities that lie close to limit ordinals. The following results fill most of these gaps.

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PROPOSITION 4.11. *For any limit ordinal λ, there is a scattered linear ordering of Scott sentence complexity $\Pi^{\text{in}}_{\lambda}$.*

PROOF. Let $(\delta_n)_{n\in\omega}$ be a fundamental sequence for λ. We show that

$$\mathcal{L}_\lambda := \sum_{i\in\omega} i + \mathbb{Z}^{\delta_n}$$

is of the desired complexity.

First note that it indeed does have a $\Pi^{\text{in}}_{\lambda}$ Scott sentence. The sentence states the following:

(1) There is exactly one 1-block isomorphic to each natural number.
(2) These 1-blocks are ordered like the natural numbers.
(3) The order between n and $n+1$ is isomorphic to $\mathbb{Z}^{\delta^n}$.

This is $\Pi^{\text{in}}_{\lambda}$ because each of the items is $\Pi^{\text{in}}_{<\lambda}$. The first two statements can be expressed using finitely many alternations of quantifiers. The final statement can be expressed by the Scott sentence of $\mathbb{Z}^{\delta_n}$ relativized to the specific interval, this sentence has Scott sentence complexity below λ by [3], as the Hausdorff rank of the interval is below λ and one can relativize the Scott sentence to the interval using finite quantifier rank by specifying that the elements are between the corresponding finite 1-blocks.

By Lemma 4.6, the $\mathbb{Z}^{\delta_n}$ have unbounded Scott rank below λ. As these are all Δ^{in}_0-definable over parameters in $\mathcal{L}_\lambda$, by [20, Lemma 4.3], we have that for each n, $SR(\mathcal{L}_\lambda) \geq \delta_n$, so $\Pi^{\text{in}}_{\lambda}$ is indeed the optimal Scott sentence complexity. ⊣

We can use a variation of this construction to get a structure of Scott sentence complexity $\Sigma^{\text{in}}_{\lambda+2}$. Unlike some of the previous constructions, this construction is not a sum of shuffle sums, so Corollary 4.3 will not apply.

THEOREM 4.12. *For any limit ordinal λ, there is a linear ordering of Scott sentence complexity $\Sigma^{\text{in}}_{\lambda+2}$.*

PROOF. Let $(\delta_n)_{n\in\omega}$ be a fundamental sequence for λ and let

$$\mathcal{L}_\lambda := \sum_{i\in\omega} i + \mathbb{Z}^{\delta_n}.$$

Furthermore, define for every n, the linear orderings:

$$\mathcal{L}_{\lambda,n} := \sum_{i<n} i + \mathbb{Z}^{\delta_i} + \sum_{i\geq n} i + \mathbb{Z}^{\delta^n}.$$

Note that by Lemma 4.6, $\mathcal{L}_\lambda$ is approximated by the sequence of these orderings in the sense that $\mathcal{L}_\lambda \equiv_{2\delta_n} \mathcal{L}_{\lambda,n}$.

Let $\mathcal{B} = \mathcal{L}_\lambda \cdot (1+\mathbb{Q})$ and $\mathcal{A}$ be a copy of $\mathbb{Q}$ where an unbounded, increasing sequence of points c_n is replaced by $\mathcal{L}_{\lambda,n}$ and each other point is replaced by $\mathcal{L}_\lambda$. We now claim that $\mathcal{K} = \mathcal{A} + \mathcal{B}$ is the desired order. Note that over the parameter of the initial element of $\mathcal{B}$, $pSR(\mathcal{K}) = \max(SR(\mathcal{A}), SR(\mathcal{B}))$ (this equality follows from [14, Lemma 11]). We now demonstrate that this quantity is exactly λ.

To define the automorphism orbits of elements within $\mathcal{B}$ we have to distinguish elements in the initial copy of $\mathcal{L}_\lambda$. For an element $b \in \mathcal{B}$, let n be the size of the finite block to the left of b. We can write a formula of finite rank saying that there is

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precisely one finite block of size n to the left of b and thus use this type of formula to distinguish elements in the first $\mathcal{L}_\lambda$ copy. If b is in a finite block we conjunct this formula or its negation with its position in this finite block; if b is in a copy of $\mathbb{Z}^{\delta_n}$, then we relativize the formula defining its automorphism orbit in this ordering to the interval restricted by block of size n to the left and a block of size $n+1$ to the right, just as in the proof of Proposition 4.11. Conjuncting this with the formula deciding whether b is in the initial copy of $\mathcal{L}_\lambda$ yields the defining formula for its automorphism orbit. To extend this definition to tuples we take the conjunction of the formulas defining orbits for elements and a formula specifying the order of the tuple.

For $\mathcal{A}$ the definitions are a little more subtle, but generally follow the same plan. In particular, if the point in $\mathbb{Q}$ that x lies in between c_n and c_{n+1}, then the following describes the automorphism orbit of x:

(1) Let z be the greatest element below x with no successor or predecessor.
(2) Let z_n be the initial element with no successor or predecessor within c_n; i.e. say that it is the greatest element with no successor or predecessor below an $n+1$ and $n+2$ block that bound a structure isomorphic to $\mathbb{Z}^{\delta_n}$.
(3) Define z_{n+1} analogously to z_n, expect it is in c_{n+1}.
(4) Say that $z_n \leq x < z_{n+1}$.
(5) Check if $z = z_n$ or not.
(6) In an analogous manner to the case of $\mathcal{A}$, define the automorphism orbit of x within its copy of $\mathbb{Z}^{\delta_k}$ or the finite block that it is in.

This describes the automorphism orbit of a point and is of complexity less than $\Sigma^{\text{in}}_\lambda$. In order to extend this definition to tuples the only additional detail needed is the order of the tuple and whether the tuples lie in the same point in $\mathbb{Q}$ or not. This can be done by comparing the various values of "z" that emerge for each point in the tuple. To obtain defining formulas for the automorphism orbits of tuples in $\mathcal{K} = \mathcal{A} + \mathcal{B}$, we add the first element of $\mathcal{B}$ as a parameter p and restrict the formulas for tuples from $\mathcal{B}$ and $\mathcal{A}$ to elements greater or less than p. As there is no automorphism mapping elements from $\mathcal{A}$ into elements in $\mathcal{B}$, defining formulas for mixed tuples can be obtained by combining the formulas for tuples in $\mathcal{A}$ and $\mathcal{B}$.

In order to finish the proof, all that remains is to show that $SR(\mathcal{K}) = \lambda + 2$, or in other words, that it has an orbit that is not $\Sigma^{\text{in}}_{\lambda+1}$-definable. Let c be the initial point in $\mathcal{B}$ and b be some other point in $\mathcal{B}$ in a block of size 1. We will show that $(\mathcal{K}, b) \leq_{\lambda+1} (\mathcal{K}, c)$, so the orbit of c is not $\Sigma^{\text{in}}_{\lambda+1}$ definable in $\mathcal{K}$.

Note that $\mathcal{K}_{>c} \cong \mathcal{K}_{>b}$, so it is sufficient to show that $\mathcal{K}_{<b} \leq_{\lambda+1} \mathcal{K}_{<c}$. As $\mathcal{K}_{<c}$ is an initial segment of $\mathcal{K}_{<b}$, for the first turn of the game, the $\exists$-player can pick whatever points the $\forall$-player picked in $\mathcal{K}_{<c}$ according to the canonical embedding of $\mathcal{K}_{<c}$ in $\mathcal{K}_{<b}$. Note that each interval is isomorphic except for the one between the greatest chosen points (call them a_k) and c or b respectively. a_k may be greater than some finite number of the c_n. Therefore, up to a finite difference in the choice of sequence δ_n used to define $\mathcal{L}_{\lambda,n}$, (a_k, c) is just $\mathcal{M} + \mathcal{A}$ where $\mathcal{M}$ is $\mathcal{L}_{\lambda,>c}$ or $\mathcal{L}_{\lambda,n,>c}$ depending on which type of order a_k lives in and (a_k, b) is just $\mathcal{M} + \mathcal{K}$. In short, it is enough to demonstrate that

$$\mathcal{A} \geq_\lambda \mathcal{K},$$

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to show the claim. To see this, choose $\alpha < \lambda$ and show that $\mathcal{A} \geq_\alpha \mathcal{K}$. Take $\delta_n > \alpha$. It is indeed the case that for $k \geq n$, $\mathcal{L}_\lambda \equiv_\alpha \mathcal{L}_{\lambda,k}$. For some point d in a block of size 1 between c_n and c_{n+1} we have that

$$\mathcal{A} = \mathcal{A}_{<d} + \sum_{q \in \mathbb{Q}} \mathcal{A}_q \quad \text{and} \quad \mathcal{K} = \mathcal{A}_{<d} + \sum_{q \in \mathbb{Q}} \mathcal{B}_q,$$

where $\mathcal{A}_q \equiv_\alpha \mathcal{L}_\lambda \equiv_\alpha \mathcal{B}_q$. This proves the claim. ⊣

We need a different construction to deal with the $\Sigma^{\text{in}}_{\lambda+3}$ case. However, it is arguably simpler than the construction we used for the example of Scott sentence complexity $\Sigma^{\text{in}}_{\lambda+2}$, as it is a sum of shuffle sums.

THEOREM 4.13. *For any limit ordinal λ, there is a linear ordering of Scott sentence complexity $\Sigma^{\text{in}}_{\lambda+3}$.*

PROOF. We let $\mathcal{L}_1 := 1 + \mathbb{Z} + \omega^\lambda$ and $\mathcal{L}_2 := 1 + \mathbb{Z} + \omega^\lambda \cdot 2$. One important initial observation is that $\mathcal{L}_1 \leq_{\lambda+1} \mathcal{L}_2$ as $\omega^\lambda \leq_{\lambda+1} \omega^\lambda \cdot 2$. As alluded to above, we take $\mathcal{A} = \mathcal{L}_1 \cdot \mathbb{Q}$ and $\mathcal{B} = \mathcal{L}_2 \cdot \mathbb{Q}$. Finally we let $\mathcal{L} = \mathcal{A} + \mathcal{L}_2 + \mathcal{B}$ and claim that $\mathcal{L}$ has the desired complexity. Let us begin with showing that $SR(\mathcal{A}) \leq \lambda$. We define several helping predicates from which the claim will easily follow.

(1) Let $pt(x)$ if x is in a block of size 1.
(2) Let $Z(x)$ say $(\exists z < x) pt(z) \wedge \forall y \ z < y < x \to \bigvee_n S^n(y) = x$. Note that $Z(x)$ holds if and only if x is inside one of the copies of $\mathbb{Z}$.
(3) Let $init(x)$ if $(\exists z < x) pt(z) \wedge (z, x) \cong \mathbb{Z}$. Note that $init(x)$ if and only if x is the first element in a copy of ω^λ.
(4) For every $\gamma < \omega^\lambda$ let $\gamma(x)$ if $(\exists z < x) init(z) \wedge [z, x) \cong \gamma$. Note that this definition is of complexity lower than $\Sigma^{\text{in}}_\lambda$ as λ is a limit ordinal. It describes the points that are of rank γ within a copy of ω^λ.

Using these predicates one can define the automorphism orbits of any tuple in $\mathcal{L}_1$, and thus in $\mathcal{A}$, by formulas of complexity less than $\Sigma^{\text{in}}_{\lambda+1}$.

All of these predicates have the same meaning within $\mathcal{B}$, however they leave points in the second copy of ω^λ undefined. But it is still the case that $SR(B) \leq \lambda + 1$. To see this we need to define the additional relations:

(1) Let

$$sec(x, z) \iff init(z) \wedge z < x \wedge \neg \exists y \left(z < y < x \wedge pt(y) \right) \wedge \bigwedge_{\delta < \lambda} (\forall y < x) y \not\sim_\delta x.$$

This Π^{in}_λ predicate holds only if z is the initial element in the second copy of ω^λ above the copy that z is initial in.
(2) For every $\gamma < \omega^\lambda$ let $\gamma_2(x)$ if $\exists y, z \ sec(y, z) \wedge [y, x) \cong \gamma$. Note that this definition is of complexity $\Sigma^{\text{in}}_{\lambda+1}$ as λ is a limit ordinal. It describes the points that are of rank γ within a second copy of ω^λ.

Using these relations and those defined above, one can define the automorphism orbits of tuples in $\mathcal{L}_2$, and thus in $\mathcal{B}$.

To get defining formulas for tuples in $\mathcal{A} + \mathcal{L}_2 + \mathcal{B}$ one fixes as parameter the first point p in $\mathcal{L}_2$, relativizes the defining formulas for tuples in $\mathcal{A}$ to elements less than p, the defining formulas for tuples in $\mathcal{L}_2$ to elements such that there is no point between

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the tuple and p, and the defining formulas for elements of $\mathcal{B}$ by saying that there is at least one more point between p and the left-most point in the tuple. Using these relativized definitions one can then obtain defining formulas for the automorphism orbits for any tuples in $\mathcal{A} + \mathcal{L}_2 + \mathcal{B}$.

In order to finish the proof of the claim, we appeal to Corollary 4.3 to show that there is no $\Pi^{\text{in}}_{\lambda+3}$ Scott sentence. This follows immediately as $\mathcal{A} \leq_{\lambda+1} \mathcal{B}$ is a direct consequence of $\mathcal{L}_1 \leq_{\lambda+1} \mathcal{L}_2$. ⊣

This leaves open the case of $\Sigma^{\text{in}}_{\lambda+1}$. It is unclear to us if such a linear ordering exists.

QUESTION 4.14. *Is there a linear ordering of Scott sentence complexity $\Sigma^{\text{in}}_{\lambda+1}$ for λ a limit ordinal*?

All of the new examples of Scott sentence complexities given in this section except Proposition 4.11 are linear orderings that contain $\mathbb{Q}$ as a subordering and are thus non-scattered. While it is not hard to adapt the proof of Theorem 3.6 to obtain that no scattered linear ordering can have Scott sentence complexity Σ^{in}_4, it seems difficult to look beyond that. An analysis akin to the one presented in this article, but for Scott sentence complexities of scattered linear orderings seems like it could prove interesting.

QUESTION 4.15. *Which Scott sentence complexities are realized by scattered linear orderings*? *In particular, for a given successor ordinal $\alpha > 3$, is there a scattered linear ordering of Scott sentence complexity $\Sigma^{\text{in}}_\alpha$*?

4.1. Remarks on effectivity. The examples we presented in this section are homogeneous enough so that if α is a computable ordinal, then the examples witnessing Scott sentence complexity Π^{in}_α, $\Sigma^{\text{in}}_\alpha$, or d-$\Sigma^{\text{in}}_\alpha$ have computable copies. Furthermore, these structures have c.e. Scott families, that is there is a computable enumeration of codes for the formulas defining the automorphism orbits of the tuples (potentially over a parameter). Thus, our examples give computable linear orderings having computable Scott sentences.

The situation is a bit more tricky for examples of high Scott rank, that is computable linear orderings with Scott rank ω_1^{CK} or $\omega_1^{\text{CK}} + 1$. Our examples do not give such structures. However, it is not hard to show that such structures exist from existing examples in the literature and our results in Section 2. We note that Calvert, Goncharov, and Knight [7] already showed that there are computable linear orderings of Scott rank ω_1^{CK} and $\omega_1^{\text{CK}} + 1$. The following result is merely a refinement.

THEOREM 4.16. *There are computable linear orderings with Scott sentence complexities $\Pi^{\text{in}}_{\omega_1^{\text{CK}}}$, $\Pi^{\text{in}}_{\omega_1^{\text{CK}}+1}$, and $\Pi^{\text{in}}_{\omega_1^{\text{CK}}+2}$.*

PROOF. As is remarked in [1], the Harrison linear ordering $\omega_1^{\text{CK}} \cdot (1 + \mathbb{Q})$ has Scott sentence complexity $\Pi^{\text{in}}_{\omega_1^{\text{CK}}+2}$. The existence of structures of linear orderings of Scott sentence complexity $\Pi^{\text{in}}_{\omega_1^{\text{CK}}+1}$ and $\Pi^{\text{in}}_{\omega_1^{\text{CK}}}$ follows from Theorem 2.19 and the existence of a computable graph with these Scott sentence complexities. ⊣

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As for general limit ordinals we do not know whether there are computable linear orderings of Scott sentence complexity $\Sigma^{\text{in}}_{\omega_1^{\text{CK}}+1}$. Additionally, we do not know whether there is a computable linear of Scott rank $d\text{-}\Sigma^{\text{in}}_{\omega_1^{\text{CK}}+1}$.

QUESTION 4.17. *Is there a computable linear ordering of Scott sentence complexity* $\Sigma^{\text{in}}_{\omega_1^{\text{CK}}+1}$?

QUESTION 4.18. *Is there a computable linear ordering of Scott sentence complexity* $d\text{-}\Sigma^{\text{in}}_{\omega_1^{\text{CK}}+1}$?

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, 970 EVANS HALL
BERKELEY, CA 94720
USA
E-mail: david_gonzalez@berkeley.edu

INSTITUTE OF DISCRETE MATHEMATICS AND GEOMETRY
TECHNISCHE UNIVERSITÄT WIEN
WIEDNER HAUPTSTRAßE
8–10/104 1040 WIEN
AUSTRIA
E-mail: dino.rossegger@tuwien.ac.at

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