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SMALL-SAMPLE LIKELIHOOD-BASED INFERENCE IN THE ARFIMA MODEL

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The autoregressive fractionally integrated moving average (ARFIMA) model has become a popular approach for analyzing time series that exhibit long-range dependence. For the Gaussian case, there have been substantial advances in the area of likelihood-based inference, including development of the asymptotic properties of the maximum likelihood estimates and formulation of procedures for their computation. Small-sample inference, however, has not to date been studied. Here we investigate the small-sample behavior of the conventional and Bartlett-corrected likelihood ratio tests (LRT) for the fractional difference parameter. We derive an expression for the Bartlett correction factor. We investigate the asymptotic order of approximation of the Bartlett-corrected test. In addition, we present a small simulation study of the conventional and Bartlett-corrected LRT's. We find that for simple ARFIMA models both tests perform fairly well with a sample size of 40 but the Bartlett-corrected test generally provides an improvement over the conventional test with a sample size of 20.

1. INTRODUCTION

Granger and Joyeux (1980) and Hosking (1981) introduced fractional differencing and the general class of autoregressive fractionally integrated moving average (ARFIMA) models. This class of models appears to describe well the long memory characteristics of some economic and hydrological data. The fractional difference parameter is very closely linked to the self-similar parameter, defined earlier by Kolmogorov (1941). Many aspects of strong power law autocorrelation in various fields of science have been investigated over the years. Relatively recent surveys of the vast literature have been provided by, among others, Lawrance and Kottegoda (1977), Hipel and McLeod (1978), Taqqu (1986), and Robinson (1994). In connection with the ARFIMA model, Mandelbrot and Van Ness (1968) presented the fractional Gaussian noise model, and Mandelbrot (1969, 1972, 1973)

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promoted the relevance of nonsummable autocorrelation models to economics. The importance of Hurst's (1951) R/S statistic was emphasized in the latter paper. For a recent summary of financial and economic applications of models of strongly dependent series, the reader is referred to Sowell (1992, Sect. 1) and Robinson (1994).

Since the introduction of ARFIMA, a large body of literature has been dedicated to the estimation of the fractional difference parameter d. Aside from heuristic approaches, which were summarized by Beran (1994), there are essentially two dominant classes of estimation techniques: semiparametric and fully parametric. Within the first class, Geweke and Porter-Hudak's (1983) log periodogram estimator, $\bar{d}(0)$, is undoubtedly the most popular. It is appealing by virtue of its notable simplicity, the minimal computing power it requires, and the very few assumptions that it makes. In particular, the orders of the AR and the MA components of the series need not be specified. Standard least squares is all that is required for the computation of this estimator.

Several authors have commented on the inadequacy of Geweke and Porter-Hudak's estimator. In particular, Hurvich and Beltrao (1993) and Agakloglou, Newbold, and Wohar (1993) characterized its bias and attributed the bias to the neglect of a term in the periodogram regression, arising from the evaluation of the periodogram at a neighborhood of the origin. Robinson (1995a) generalized Geweke and Porter-Hudak's estimator to $\bar{d}(\ell)$, and Giraitis, Robinson, and Samarov (1997) showed that the modification is rate optimal. The estimator $\bar{d}(\ell)$ was, in fact, previously proposed by Künsch (1986), but Künch did not establish its asymptotic distribution. Large-sample theory for $\bar{d}(\ell)$ under various conditions was advanced by Kashyap and Eom (1988) and Hassler (1993) and appears to have been generalized and corrected by Robinson (1995a). See also Robinson's (1994) survey. Robinson (1995b) suggested a semiparametric Gaussian estimator that dominates $\bar{d}(\ell)$ by requiring much weaker assumptions and by being asymptotically more efficient. The estimator was suggested earlier by Künsch (1987), though without development of its statistical properties. In spite of its favorable properties, this estimator does not seem to be in much use. A possible explanation for this is that it does not have a closed form.

Within the class of parametric estimation procedures, the most prominent is the efficient Gaussian maximum likelihood estimator (MLE). It is well known that when the model is correctly specified, $\bar{d}(0)$ has a zero relative asymptotic efficiency compared with the Gaussian MLE. The MLE has been put on a rigorous theoretical basis by Fox and Taqqu (1986), Dahlhaus (1988, 1989), and Giraitis and Surgailis (1990). Sowell (1992) developed a method for computing the Gaussian MLE, deriving for this purpose an expression for the autocovariance function in terms of hypergeometric functions. The MLE is optimal in many respects, given a correct specification of the model. Its main drawback compared with the semiparametric procedures is the computational effort it requires. The software required is not widely available. In addition, if the model has been misspecified, the Gaussian MLE will be inconsistent.

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The existing literature on estimation for the ARFIMA model has focused on the asymptotic situation, though Sowell (1992) presented limited Monte Carlo evidence on the finite sample behavior of the Gaussian MLE. The focus on asymptotics is understandable given that ARFIMA models are aimed at characterizing long-range dependence, so that many of the applications of the model have been to long series. It is possible, however, within the framework of the ARFIMA model, to study long-range dependence even with a relatively short series of observations. To date, there does not seem to be any small-sample theory for any estimator of the fractional difference parameter.

In this paper, we take a first step toward advancing the small-sample theory of the Gaussian MLE of the fractional difference parameter d. Specifically, we derive the Bartlett correction for the likelihood ratio test (LRT) statistic for testing the hypothesis that d equals some given value. Though we work in a hypothesis testing framework, in principle our procedure can be inverted to produce a confidence interval for d. Also, though we focus on d as the parameter of interest, the theory we give applies equally well to inference on any given one of the ARFIMA parameters.

The basic idea behind Bartlett adjustment is to multiply the LRT statistic by a correction factor such that the corrected statistic more closely follows the asymptotic chi-square distribution. Bartlett (1937) originated the idea, whereas Lawley (1956) developed a correction formula for the general maximum likelihood setting. Taniguchi (1986, 1988) presented the Bartlett correction factor for the classical autoregressive moving average (ARMA) model in the simple case in which there is one parameter of interest and no nuisance parameters, as obtains when interest is focused on one ARMA parameter and the remaining ARMA parameters are regarded as known. Our development is closely related to his. Here, however, in developing a Bartlett correction for the LRT for the parameter d, we allow the remaining model parameters to be estimated. In addition, we deal with spectral density functions not covered by the class of functions considered by Taniguchi (1988), in that the spectral density of the ARFIMA process with $0 < d < \frac{1}{2}$ has a pole at the origin.

Barndorff-Nielsen and Hall (1988) have shown, under general conditions, that the error of the asymptotic chi-square distribution of the corrected LRT is of order $O(n^{-2})$, compared with an error of order $O(n^{-1})$ only for the conventional statistic, where n is the sample size. We present an argument indicating that the error rate of $O(n^{-2})$ for the Bartlett-corrected LRT holds in our ARFIMA setting.

The plan of the paper is as follows. In Section 2, we introduce the model and present the LRT statistic. The Bartlett correction is derived in Section 3. In Section 4 we investigate the order of the approximation error. In Section 5 we provide a method for the calculation of the terms in the correction. In Section 6 we provide a numerical analysis on the adequacy of the Bartlett correction. We show that the correction often yields a more accurate and reliable approximation than the asymptotic distribution of the LRT in small samples. Section 7 summarizes our results.

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2. THE MODEL AND THE LRT STATISTIC

We consider the stationary and invertible ARFIMA (p,d,q) model. Let L denote the lag operator: $LX_t = X_{t-1}$. Then the model is written as

$$\phi(L)\Delta^d X_t = \psi(L)\varepsilon_t,$$

where $\phi(L) = 1 + \sum_{j=1}^{p}\phi_j L^j$, $\psi(L) = 1 + \sum_{j=1}^{q}\psi_j L^j$, $\Delta^d X_t = (1-L)^d X_t$, $d \in (-0.5,0.5)$, and $\varepsilon_t \sim \text{NID}(0,\sigma_\varepsilon^2)$. Here $(1-L)^d$ is defined by the binomial formula. For stationarity and invertibility we need the additional assumptions that the roots of $\phi(z)$ and $\psi(z)$ are outside the unit circle. In this model, interest lies in testing the hypothesis $H_0: d = d_0$ and, in particular, that $d = 0$. Under the null hypothesis $d = 0$, X_t is a short memory process, whereas for $d > 0$, X_t is said to have long-range dependence. See, for instance, Robinson (1995a). Asymptotic inference in this model can be conducted with standard likelihood-based tests, such as the LRT. Under Gaussianity, the log-likelihood based on the sample $x = (X_1,\ldots,X_n)'$ is given by

$$\ell = -\frac{n}{2}\log 2\pi - \frac{1}{2}\log\det\Sigma - \frac{1}{2}\operatorname{tr} xx'\Sigma^{-1}$$

with $\Sigma = E(xx') = (\gamma(|i-j|))_{1\le i,j\le n}$, where γ denotes the autocovariance function. The exact form of γ for any given p, d, and q is given in Sowell (1992, equation (8)). Write $\theta = (\theta_i,\theta_2,\ldots,\theta_{p+q+2})$, with $\theta_1 = d$ and $(\theta_2,\ldots,\theta_{p+q+2}) = (\sigma_\varepsilon^2,\phi_1,\ldots,\phi_p,\psi_1,\ldots,\psi_q)$. Then the LRT statistic for the hypothesis $H_0: d = d_0$ is given by $W(d_0)$, where

$$W(d) = 2\{\ell(\hat\theta) - \ell(\hat\theta(d))\},$$

with $\hat\theta$ denoting the unrestricted MLE of θ and $\hat\theta(d)$ denoting the MLE of θ with θ_1 fixed at a specified value d. It is well known that under the null hypothesis $W(d_0)$ is asymptotically distributed as χ_1^2. The order of the error of this approximation is quite generally $O(n^{-1})$. As noted in the introduction, Bartlett correction reduces the order of the error to $O(n^{-2})$. We shall derive the Bartlett correction for the LRT for the hypothesis of interest and show that in small samples the modification yields an improvement to the asymptotic distribution.

3. BARTLETT CORRECTION TO THE LRT OF $H_0: d = d_0$

Bartlett correction involves expressing the expectation of the LRT statistic in an expansion $E[W(d)] = 1 + H(\theta) + O(n^{-3/2})$, where $H(\theta)$ is $O(n^{-1})$, and then forming a corrected LRT statistic by either $\{1 + H(\theta)\}^{-1}W(d_0)$ or $\{1 - H(\theta)\}W(d_0)$. As argued in Section 4, both versions of the corrected statistic are distributed as χ_1^2 under H_0 to order $O(n^{-2})$. In practice, $H(\theta)$ has to be estimated. Possible options are to substitute $\hat\theta$ or $\hat\theta(d_0)$. Arguments as in Barndorff-Nielsen and Hall (1988) show that with either option the error rate is maintained at order $O(n^{-2})$.

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Lawley (1956) derived a general formula for $H(\theta)$ in the general maximum likelihood setting. We use the equivalent formula given by DiCiccio and Stern (1994, equation (14)). Our development follows the following outline: (1) we recall the DiCiccio–Stern formula for $H(\theta)$; (2) we compute the necessary expected log-likelihood derivatives; and (3) we compute the derivatives of the autocovariance function that appear in the expected log-likelihood derivatives.

The quantity $H(\theta)$ is given as follows. Put $\ell_r = \partial\ell/\partial\theta_r$, $\ell_{rs} = \partial^2\ell/\partial\theta_r\partial\theta_s$, $\ell_{rst} = \partial^3\ell/\partial\theta_r\partial\theta_s\partial\theta_t$, $\lambda_{rs} = E(\ell_{rs})$, and $\lambda_{rst} = E(\ell_{rst})$. Let (λ^{rs}) be the $(p+q+2)\times(p+q+2)$ matrix inverse of (λ_{rs}). Put $\nu^{rs} = \lambda^{rs} - \lambda^{r1}\lambda^{s1}\sigma_{11}$, with $\sigma_{11} = (\lambda^{11})^{-1}$, i.e., σ_{11} is the reciprocal of the upper left element in the matrix $(\lambda^{rs})_{1\le r,s\le(p+q+2)}$. Finally, let $(\lambda_{rs})_t = \partial\lambda_{rs}/\partial\theta_t$, $(\lambda_{rs})_{tu} = \partial^2\lambda_{rs}/\partial\theta_t\partial\theta_u$, and $(\lambda_{rst})_u = \partial\lambda_{rst}/\partial\theta_u$. Then, using the summation convention, $H(\theta)$ is given as

$$\begin{aligned} H(\theta) = {} & (\lambda^{ru}\lambda^{st} - \nu^{ru}\nu^{st})\{\tfrac{1}{4}\lambda_{rstu} - (\lambda_{rst})_u + (\lambda_{rs})_{tu}\} \\ & - (\lambda^{ru}\lambda^{st}\lambda^{vw} - \nu^{ru}\nu^{st}\nu^{vw})\{\tfrac{1}{4}\lambda_{rst}\lambda_{uvw} - \lambda_{rst}(\lambda_{uv})_w + (\lambda_{rs})_t(\lambda_{uv})_w\} \\ & - (\lambda^{ru}\lambda^{sw}\lambda^{tv} - \nu^{ru}\nu^{sw}\nu^{tv})\{\tfrac{1}{6}\lambda_{rst}\lambda_{uvw} - \lambda_{rst}(\lambda_{uv})_w + (\lambda_{rs})_t(\lambda_{uv})_w\}. \end{aligned}$$

All the terms required for the calculation of the Bartlett correction are provided in Appendix A.

4. ERROR ANALYSIS

For the independent data case, Barndorff-Nielsen and Hall (1988) proved that

$$\Pr((1+H(\theta))^{-1}W(d_0) \le c) = \Pr(\chi_1^2 \le c) + R_n^B$$

with $R_n^B = O(n^{-2})$. Their proof hinges on the assumption that the joint distribution of the components of the log-likelihood derivatives, the MLE, and the estimated Bartlett correction term admit a multivariate Edgeworth expansion. Both the Bartlett-corrected LRT and the Edgeworth expansion are functions of the cumulants of the log-likelihood derivatives. These cumulants are $O(n)$ in the independent data case. One of the main features of strongly dependent data is that the autocovariance function is not absolutely summable. Because the cumulants of the log-likelihood derivatives are functions of the autocovariances and their derivatives, it is not immediately obvious that the order of magnitude of these cumulants is as in the case of independent data. We show subsequently that in our case the cumulants in question are still of order $O(n)$.

Let $\ell_\nu, \nu = (r_1 \ldots r_k)$ represent a given log-likelihood derivative of degree k. The quantity ℓ_ν is composed of terms of the following form:

$$a_n^{(\nu)} = -\frac{1}{2}\operatorname{tr}\prod_{j=1}^{p(\nu)}\Sigma^{-1}\Sigma^*(g_j)$$

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and

$$b_n^{(\nu)} = -\frac{1}{2}\, x' \left(\prod_{j=1}^{p'(\nu)} \Sigma^{-1}\Sigma^*(g_j) \right) \Sigma^{-1} x,$$

where $\Sigma^*(g_j)$ are matrices of derivatives of the autocovariance function of degree less than or equal to k and g_j are functions of the spectral density. Because ℓ_ν is a quadratic form in normal variables, its mth cumulant, $\kappa_m^{(\nu)}$, is a finite sum of terms of the form $2^{m-1}(m-1)!\,\mathrm{tr}\prod_{j=1}^{p'(\nu)}\Sigma^{-1}\Sigma^*(g_j)$. Now Σ and its derivatives fulfill Assumptions (A0)–(A9) of Dahlhaus (1989, pp. 1750–1751). Therefore, it follows from Theorem 5.1 of that paper that $\kappa_m^{(\nu)} = O(n)$, for all $k, m \geq 1$, except $\nu = r$ and $m = 1$, in which case $\kappa_1^{(r)} = 0$. We deduce that all the terms in the Edgeworth expansion and in the Bartlett correction behave as in the independent data case and hence $R_n^B = O(n^{-2})$. We remark that this error analysis is not restricted to fractional ARMA models but is also valid for other strongly dependent Gaussian processes satisfying (A0)–(A9) of Dahlhaus (1989).

5. DERIVATIVES OF THE AUTOCOVARIANCE MATRIX

The implementation of the Bartlett correction requires the evaluation of up to third order derivatives of Σ. Sowell (1992) expressed the autocovariance function in terms of hypergeometric functions, and it is clear that in this form the evaluation and computation of its higher order derivatives present a very formidable task. To make the Bartlett correction feasible for our problem, we work with an alternative representation for the autocovariance function.

Specifically, we recall that if the spectral density of a process, $f_X(\xi)$, exists, then the autocovariance $\gamma(s)$ of order s may be written as

$$\gamma(s) = 2\int_0^\pi f_X(\xi)\cos(\xi s)\, d\xi. \tag{1}$$

For the ARFIMA (p, d, q) process, $f_X(\xi)$ is given by

$$\begin{aligned} f_X(\xi) &= \frac{\sigma_\varepsilon^2}{2\pi}\,|1 - e^{-i\xi}|^{-2d}\,\frac{|\psi(e^{-i\xi})|^2}{|\phi(e^{-i\xi})|^2} \\ &= \frac{\sigma_\varepsilon^2}{2\pi}\,e^{-d\log[2(1-\cos\xi)]}\,\frac{|\psi(e^{-i\xi})|^2}{|\phi(e^{-i\xi})|^2} \end{aligned}$$

(see Brockwell and Davis, 1991, equation (13.2.18)). Sowell (1992) essentially used the representation (1) of $\gamma(s)$ as the starting point for deriving his expression for $\gamma(s)$. Our approach here is to leave (1) as it is and differentiate w.r.t. the ARFIMA model parameters under the integral sign. This operation is justified by an argument given at the end of this section. This strategy works nicely because the integrand in (1) is a relatively simple function composed of four factors, one

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involving only d, one involving only σ_ε^2, one involving only the AR parameters, and one involving only the MA parameters.

To proceed, we define

$$A(\xi) \equiv |\psi(e^{-i\xi})|^2 = \left(\sum_{j=0}^{q} \psi_j \cos j\xi\right)^2 + \left(\sum_{j=0}^{q} \psi_j \sin j\xi\right)^2.$$

Then

$$\alpha_k \equiv \frac{\partial A}{\partial \psi_k} = 2\cos k\xi \sum_{j=0}^{q} \psi_j \cos j\xi + 2\sin k\xi \sum_{j=0}^{q} \psi_j \sin j\xi,$$

and

$$\begin{aligned}\frac{\partial^2 A}{\partial \psi_k \partial \psi_\ell} &= 2\cos k\xi \cos \ell\xi + 2\sin k\xi \sin \ell\xi \\ &= 2\cos(k-\ell)\xi.\end{aligned}$$

All other derivatives of $A(\xi)$ of any order w.r.t. the ARFIMA model parameters are equal to zero.

Similarly,

$$B(\xi) \equiv |\phi(e^{-i\xi})|^2 = \left(\sum_{j=0}^{p} \phi_j \cos j\xi\right)^2 + \left(\sum_{j=0}^{p} \phi_j \sin j\xi\right)^2.$$

Thus,

$$\beta_t = \frac{\partial B}{\partial \phi_t} = 2\cos t\xi \sum_{j=0}^{p} \phi_j \cos j\xi + 2\sin t\xi \sum_{j=0}^{p} \phi_j \sin j\xi,$$

$$\frac{\partial^2 B}{\partial \phi_t \partial \phi_u} = 2\cos(t-u)\xi.$$

All other derivatives of $B(\xi)$ of any order w.r.t. the ARFIMA model parameters are equal to zero.

We can write

$$\gamma(s) = \frac{\omega}{\pi} \int_0^\pi e^{-dC(\xi)} \frac{A(\xi)}{B(\xi)} \cos \xi s \, d\xi,$$

where $C(\xi) = \log[2(1-\cos\xi)]$ and $\omega = \sigma_\varepsilon^2$.

The required derivatives of $\gamma(s)$ with respect to the model parameters may be computed with the aid of the foregoing results on the derivatives of $A(\xi)$ and $B(\xi)$. The results are given in Appendix B.

Differentiation beneath the integral sign may be justified by an argument based on the Lebesgue dominated convergence theorem, provided that the derivative of

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each integrand in the integrals being differentiated is dominated by an integrable function (see Ferguson, 1996, p. 124). This condition may be verified as follows. First, it is clear that A, α_k, and β_t are all bounded functions of ξ, and the reciprocal of B is also bounded by virtue of the assumption that $\psi(z)$ has no roots on the unit circle. It follows that the derivatives of all the integrands in question are dominated by a function of the form $h(\xi) = Me^{-dC(\xi)}|C(\xi)|^r$ where M is some constant and r equals 0, 1, 2, or 3. For any positive η, $C(\xi)$ is clearly bounded on $[\eta, \pi]$, so integrability over this interval is no problem. As $\xi \to 0$, $e^{-C(\xi)} \sim \xi^{-2}$, from which it may be seen that, under the restriction $d < \frac{1}{2}$, the function $h(\xi)$ is integrable over $[0, \eta]$ also.

6. SIMULATION STUDY

In this section we present a simulation study assessing the small-sample accuracy of the conventional LRT and the Bartlett-corrected LRT. Fourteen different cases are considered. Six of these correspond to the hypothesis $H_0: d = 0$ with the models ARFIMA $(0,d,0)$, ARFIMA $(0,d,1)$ with MA parameter equal to 0.5 and ARFIMA $(1,d,0)$ with an AR parameter equal to -0.2, under sample sizes $n = 20$ and $n = 40$. The remaining eight cases correspond to the hypotheses $H_0: d = 0.2$, 0.1, -0.1, -0.2, with $n = 20$, 40 under the ARFIMA $(0,d,1)$ model with an MA parameter equal to 0.5. In the ARFIMA $(0,d,0)$ model the nuisance parameter is the error variance, and in the ARFIMA $(0,d,1)$ and ARFIMA $(1,d,0)$ models the nuisance parameters are the MA and AR parameters, respectively, with the error variance set to unity for simplicity.

We wrote a MATHEMATICA program to calculate the MLE's and the Bartlett corrections. The program was run with MATHEMATICA version 3.0 on a SPARCstation 10 computer. We used the MATHEMATICA numerical integration command NINTEGRATE, which handles the singularity of the spectral density function at the origin, to evaluate numerically the integrals appearing in the derivatives of the autocovariance function. Where possible, we verified the numerical integrations by running suitable MATHEMATICA programs to evaluate the log-likelihood derivatives directly from Sowell's hypergeometric function–based likelihood expression. We encountered difficulties with such direct calculation for quantities involving differentiation with respect to d twice or more; the occurrence of these difficulties further motivates our numerical integration approach.

Our optimization routine employed grid search to find a global maximum of the log-likelihood over the relevant ranges of d and the nuisance parameter jointly, rather than merely a local maximum. It avoids the approximation error induced by truncating the polynomial $(1-L)^d X_t$ in the manner indicated by Sowell (1992, p. 176). Although this search procedure is slower than Sowell's, it guarantees the reliability of the simulations. For each case, we conducted 1,000 replications.

The results are reported in Tables 1 and 2 in terms of Type I error levels. Selected Q–Q plots are given in Figures 1–5. The 45° line in the Q–Q plots

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TABLE 1. Type I error levels of conventional and Bartlett-corrected LRT's

			Nominal Type I Error Level			
Model	n	LRT form	0.500	0.100	0.050	0.010
ARFIMA $(0,d,0)$	20	Conventional	0.509	0.107	0.052	0.005
		Bartlett	0.505	0.105	0.045	0.005
	40	Conventional	0.464	0.106	0.044	0.006
		Bartlett	0.463	0.106	0.044	0.006
ARFIMA $(0,d,1)$	20	Conventional	0.465	0.066	0.027	0.003
		Bartlett	0.500	0.071	0.037	0.009
	40	Conventional	0.487	0.098	0.051	0.007
		Bartlett	0.498	0.107	0.052	0.011
ARFIMA $(1,d,0)$	20	Conventional	0.485	0.064	0.017	0.002
		Bartlett	0.490	0.076	0.026	0.002
	40	Conventional	0.495	0.102	0.050	0.012
		Bartlett	0.490	0.094	0.046	0.012

TABLE 2. Type I error levels of conventional and Bartlett-corrected LRT's in the model $\Delta^d X_t = (1 + 0.5L)\,\varepsilon_t$

			Nominal Type I Error Level			
Hypothesis	n	LRT form	0.500	0.100	0.050	0.010
$H_0: d = -0.2$	20	Conventional	0.487	0.051	0.020	0.002
		Bartlett	0.515	0.075	0.030	0.005
	40	Conventional	0.495	0.093	0.040	0.007
		Bartlett	0.508	0.107	0.049	0.011
$H_0: d = -0.1$	20	Conventional	0.465	0.068	0.029	0.002
		Bartlett	0.500	0.092	0.050	0.006
	40	Conventional	0.474	0.087	0.035	0.006
		Bartlett	0.483	0.093	0.042	0.008
$H_0: d = 0.1$	20	Conventional	0.442	0.066	0.025	0.001
		Bartlett	0.469	0.100	0.034	0.005
	40	Conventional	0.482	0.092	0.035	0.003
		Bartlett	0.494	0.106	0.044	0.009
$H_0: d = 0.2$	20	Conventional	0.467	0.081	0.035	0.002
		Bartlett	0.504	0.090	0.044	0.008
	40	Conventional	0.478	0.092	0.042	0.008
		Bartlett	0.498	0.102	0.049	0.011

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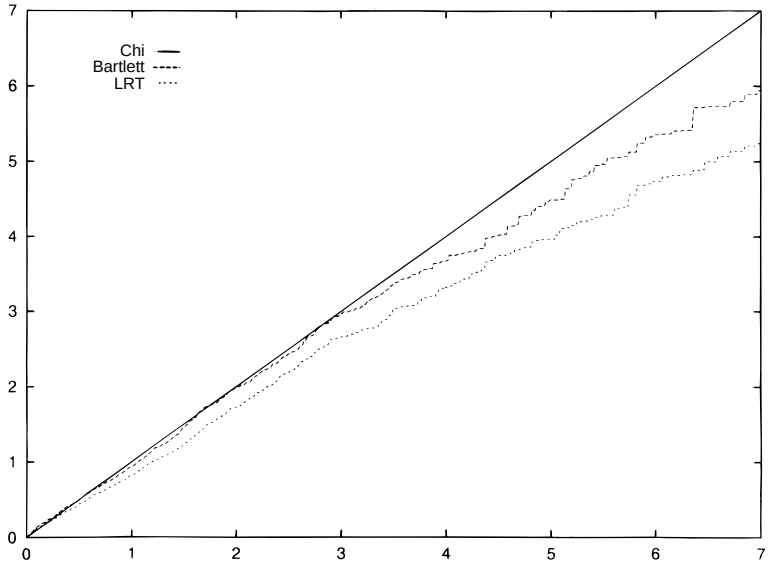


FIGURE 1. Q–Q plot for Bartlett corrections in the ARFIMA model: $\Delta^d X_t = (1 + 0.5L)\varepsilon_t$, $\varepsilon_t \sim N(0,1)$, where $n = 20$ and $H_0: d = 0$.

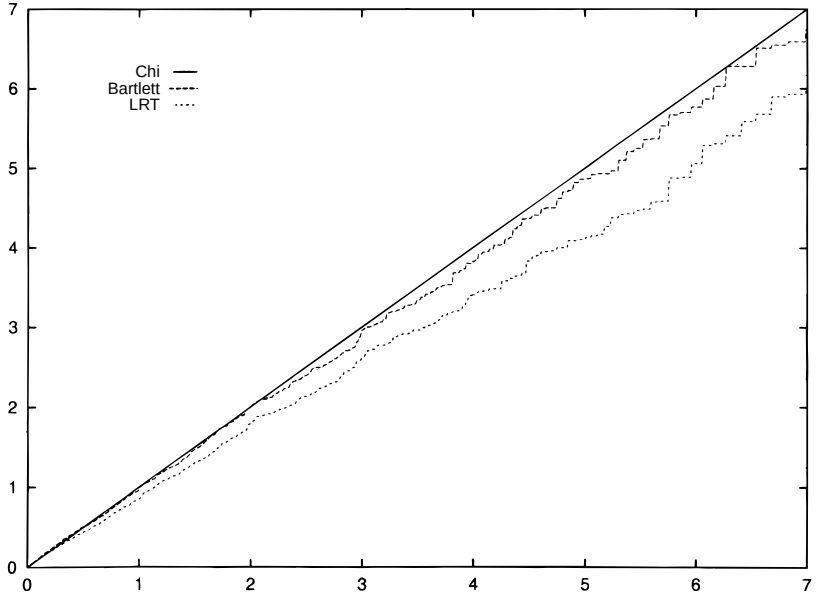


FIGURE 2. Q–Q plot for Bartlett corrections in the ARFIMA model: $\Delta^d X_t = (1 + 0.5L)\varepsilon_t$, $\varepsilon_t \sim N(0,1)$, where $n = 20$ and $H_0: d = 0.2$.

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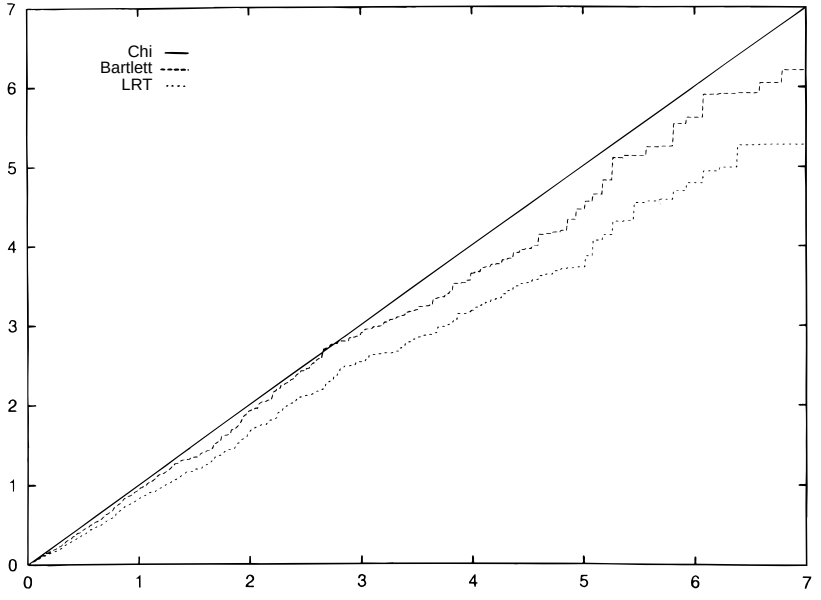


FIGURE 3. Q–Q plot for Bartlett corrections in the ARFIMA model: $\Delta^d X_t = (1 + 0.5L)\varepsilon_t$, $\varepsilon_t \sim N(0,1)$, where $n = 20$ and $H_0: d = 0.1$.

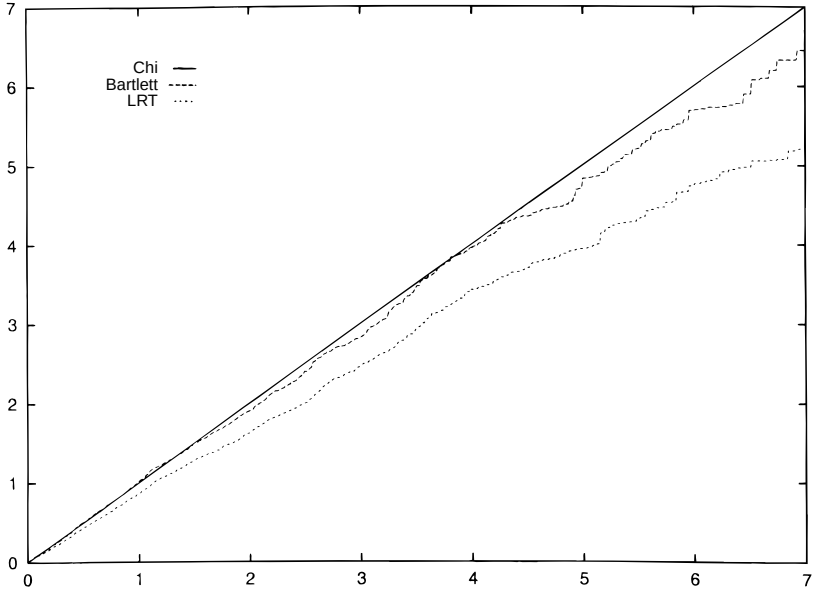


FIGURE 4. Q–Q plot for Bartlett corrections in the ARFIMA model: $\Delta^d X_t = (1 + 0.5L)\varepsilon_t$, $\varepsilon_t \sim N(0,1)$, where $n = 20$ and $H_0: d = -0.1$.

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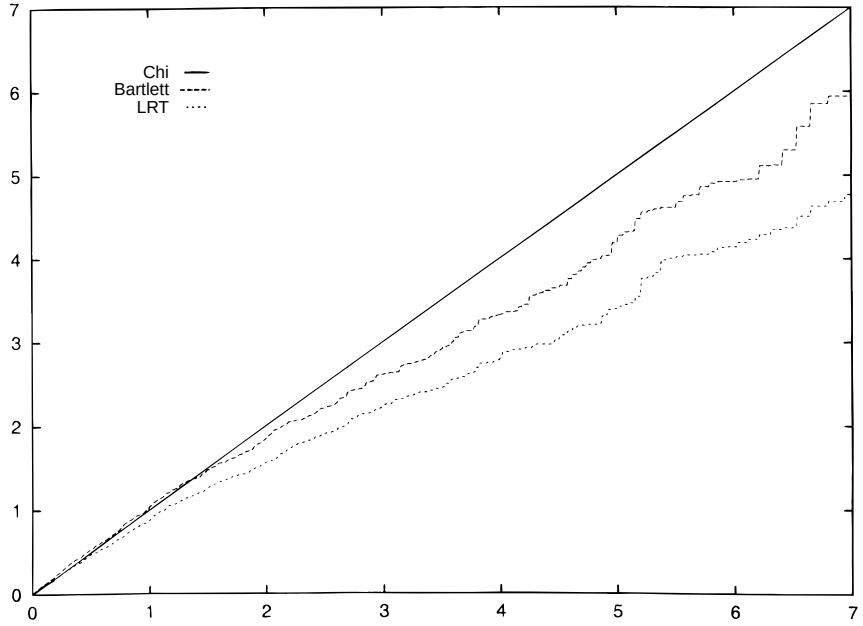


FIGURE 5. Q–Q plot for Bartlett corrections in the ARFIMA model: $\Delta^d X_t = (1 + 0.5L)\varepsilon_t$, $\varepsilon_t \sim N(0,1)$, where $n = 20$ and $H_0: d = -0.2$.

corresponds to the asymptotic χ_1^2 distribution, and the closeness of the other curves to it is indicative of the accuracy of the asymptotic distribution.

The simulations show that for each of the models considered, the accuracy of both the standard LRT and the Bartlett-corrected LRT is higher for larger n, supporting the asymptotic tendencies of both statistics to a χ_1^2 variate. In fact, for the models with $d_0 = 0$, both statistics are quite accurate for $n = 40$, except in the upper tail, where their behavior deteriorates.

At the same time, the simulations show that in many cases, though not all, the Bartlett-corrected statistic has noticeably better accuracy than the conventional statistic. With $n = 20$ under the ARFIMA $(0, d, 1)$ model, under all hypotheses examined, Figures 1–5 show vividly that the Bartlett-corrected LRT is an improvement over the conventional LRT. There is not much difference between the statistics in the fractional Gaussian noise case. With $n = 20$ under the ARFIMA $(1, d, 0)$ model, the Bartlett-corrected LRT is better behaved than the conventional LRT, except at the far tail, where they both deteriorate. With $n = 40$ under the ARFIMA $(0, d, 1)$ model with $d_0 \neq 0$ at the nominal Type I error level of 0.05, the accuracy of the conventional LRT is somewhat unsatisfactory, whereas that of the Bartlett-corrected statistic is fairly good. Overall, Bartlett correction improves the accuracy of the asymptotic approximation in most cases studied, with the improvement being significant in some cases and more modest in others.

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7. DISCUSSION

The ARFIMA model has become a popular approach for analyzing time series that exhibit long-range dependence. For the Gaussian case, there have been substantial advances in the area of likelihood-based inference. Sowell (1992) has presented a procedure for computing the maximum likelihood estimates (MLE's). Several investigators have analyzed the statistical properties of the MLE's in an asymptotic setting.

In principle, the ARFIMA model can be used to study long-range dependence even with a relatively short series. However, there has been little or no previous work on small-sample statistical inference for the ARFIMA model. In this paper, we have investigated the small-sample behavior of the conventional and Bartlett-corrected LRT's for the fractional difference parameter. We have derived an expression for the Bartlett correction factor. We have presented results on the asymptotic order of approximation of the Bartlett-corrected test. Finally, we have presented a small simulation study of the conventional and Bartlett-corrected LRT's. Though we focused here on the fractional difference parameter, the same general methodology can be used for the other ARFIMA parameters.

In developing the Bartlett correction formula, we expressed the log-likelihood and its derivatives with respect to the parameters in terms of integrals amenable to fairly straightforward numerical evaluation. The availability of the derivatives of the log-likelihood function in easily computable form opens the way to implementing a variety of established optimization algorithms for computing the MLE's. It is possible that an approach along these lines would be computationally more efficient than Sowell's algorithm based on hypergeometric functions. We have not attempted here, however, to explore computational efficiency issues. Our interest focused on the statistical properties of the procedures studied rather than on efficient computation, and so we opted for an extremely simple and conservative optimization algorithm.

Our simulation study showed that for simple ARFIMA models with $d_0 = 0$, both tests perform fairly well in terms of Type I error with a sample size of 40, whereas the Bartlett-corrected test is generally better than the conventional test with a sample size of 20 and in cases with $d_0 \neq 0$ even with a sample size of 40. The results observed support the notion that inference can be carried out for the ARFIMA model with reasonable success even in small samples. We conjecture that the Bartlett correction may have more impact in more complex ARFIMA models.

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APPENDIX A: THE TERMS IN $H(\theta)$

To present the necessary expected log-likelihood derivatives for our problem, we introduce the following notation:

$$
\begin{aligned}
(\Sigma^{-1}\Sigma^*)_r &= \Sigma^{-1}\dot{\Sigma}_r,\\
(\Sigma^{-1}\Sigma^*)_{r,s} &= \Sigma^{-1}\dot{\Sigma}_r\Sigma^{-1}\dot{\Sigma}_s,\\
(\Sigma^{-1}\Sigma^*)_{r,s,t} &= \Sigma^{-1}\dot{\Sigma}_r\Sigma^{-1}\dot{\Sigma}_s\Sigma^{-1}\dot{\Sigma}_t,\\
(\Sigma^{-1}\Sigma^*)_{s,tu} &= \Sigma^{-1}\dot{\Sigma}_s\Sigma^{-1}\ddot{\Sigma}_{tu},\\
(\Sigma^{-1}\Sigma^*)_{r,st,u} &= \Sigma^{-1}\dot{\Sigma}_r\Sigma^{-1}\ddot{\Sigma}_{st}\Sigma^{-1}\dot{\Sigma}_u,\\
(\Sigma^{-1}\Sigma^*)_{(rt,s+r,st)} &= (\Sigma^{-1}\Sigma^*)_{rt,s} + (\Sigma^{-1}\Sigma^*)_{r,st},
\end{aligned}
$$

and so forth. In the preceding notation, $\dot{\Sigma}_r \equiv \partial\Sigma/\partial\theta_r$, $\ddot{\Sigma}_{rs} \equiv \partial^2\Sigma/\partial\theta_r\partial\theta_s$, and so on. The first four log-likelihood derivatives are available on request from the authors. With this notation, the expected log-likelihood derivatives are as follows:

$$\lambda_{rs} = -\tfrac{1}{2}\,\mathrm{tr}(\Sigma^{-1}\Sigma^*)_{r,s}, \tag{A.1}$$

$$\lambda_{rst} = -\tfrac{1}{2}\,\mathrm{tr}\{-4(\Sigma^{-1}\Sigma^*)_{r,s,t} + (\Sigma^{-1}\Sigma^*)_{(rt,s+r,st+t,rs)}\}, \tag{A.2}$$

$$
\begin{aligned}
\lambda_{rstu} = -\tfrac{1}{2}\,\mathrm{tr}\{&6(\Sigma^{-1}\Sigma^*)_{(u,t,r,s+u,r,s,t+u,s,t,r)}\\
&- 4(\Sigma^{-1}\Sigma^*)_{(tu,r,s+t,ru,s+t,r,su+rt,u,s+r,u,st+t,u,rs)}\\
&+ (\Sigma^{-1}\Sigma^*)_{(rtu,s+rt,su+ru,st+r,stu+rsu,t+rs,tu+u,rst)}\}.
\end{aligned} \tag{A.3}
$$

We remark that the fourth order derivatives of the covariance matrix do not appear in λ_{rstu}.

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Further,

$$(\lambda_{rs})_t = -\tfrac{1}{2}\,\mathrm{tr}\{-2(\Sigma^{-1}\Sigma^*)_{(r,s,t)} + (\Sigma^{-1}\Sigma^*)_{(rt,s+r,st)}\}, \tag{A.4}$$

$$\begin{aligned}(\lambda_{rs})_{tu} = -\tfrac{1}{2}\,\mathrm{tr}\{&2(\Sigma^{-1}\Sigma^*)_{(u,r,s,t+r,u,s,t+r,s,u,t)}\\ &- 2(\Sigma^{-1}\Sigma^*)_{(ru,s,t+r,su,t+r,s,tu+u,rt,s+u,r,st)}\\ &+ (\Sigma^{-1}\Sigma^*)_{(rtu,s+rt,su+ru,st+r,stu)}\},\end{aligned} \tag{A.5}$$

$$\begin{aligned}(\lambda_{rst})_u = -\tfrac{1}{2}\,\mathrm{tr}\{&4(\Sigma^{-1}\Sigma^*)_{(u,r,s,t+r,u,s,t+r,s,u,t)}\\ &- 4(\Sigma^{-1}\Sigma^*)_{(ru,s,t+r,su,t+r,s,tu)}\\ &- 2(\Sigma^{-1}\Sigma^*)_{(u,rt,s+u,r,st+u,t,rs)}\\ &+ (\Sigma^{-1}\Sigma^*)_{(rtu,s+rt,su+ru,st+r,stu+tu,rs+t,rsu)}\}.\end{aligned} \tag{A.6}$$

Equations (A.1)–(A.6) are all the terms required for the calculation of the Bartlett correction.

APPENDIX B: DERIVATIVES OF $\gamma(s)$

$$\frac{\partial\gamma(s)}{\partial d} = -\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A}{B}\,C(\xi)\cos\xi s\,d\xi,$$

$$\frac{\partial\gamma(s)}{\partial\psi_k} = \frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\alpha_k}{B}\cos\xi s\,d\xi,$$

$$\frac{\partial\gamma(s)}{\partial\phi_t} = -\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A\beta_t}{B^2}\cos\xi_s\,d\xi,$$

$$\frac{\partial\gamma(s)}{\partial\omega} = \frac{1}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A}{B}\cos\xi s\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial d^2} = \frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A}{B}\,C(\xi)^2\cos\xi s\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial d\partial\psi_k} = \frac{-\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\alpha_k}{B}\,C(\xi)\cos\xi s\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial d\partial\phi_t} = \frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A\beta_t}{B^2}\,C(\xi)\cos\xi s\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial d\partial\omega} = -\frac{1}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A}{B}\,C(\xi)\cos\xi s\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial\psi_k\partial\psi_\ell} = 2\,\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}B^{-1}\cos(k-\ell)\xi\cos\xi s\,d\xi,$$

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$$\frac{\partial^2\gamma(s)}{\partial\psi_k\partial\phi_t} = -\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\alpha_k\beta_t}{B^2}\cos\xi s\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial\psi_k\partial\omega} = \frac{1}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\alpha_k}{B}\cos\xi s\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial\phi_t\partial\phi_u} = 2\,\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}A\cos\xi s[-B^{-2}\cos(t-u)\xi + B^{-3}\beta_t\beta_u]\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial\phi_t\partial\omega} = -\frac{1}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A\beta_t}{B^2}\cos\xi s\,d\xi,$$

$$\frac{\partial^2\gamma(s)}{\partial\omega^2} = 0,$$

$$\frac{\partial^3\gamma(s)}{\partial d^3} = -\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A}{B}\,C(\xi)^3\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d^2\partial\psi_k} = \frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\alpha_k}{B}\,C(\xi)^2\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d^2\partial\phi_t} = -\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A\beta_t}{B^2}\,C(\xi)^2\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d^2\partial\omega} = \frac{1}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A}{B}\,C(\xi)^2\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d\partial\psi_k\partial\psi_\ell} = -2\,\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\cos(k-\ell)\xi}{B}\,C(\xi)\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d\partial\psi_k\partial\phi_t} = \frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\alpha_k\beta_t}{B^2}\,C(\xi)\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d\partial\psi_k\partial\omega} = \frac{-1}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\alpha_k}{B}\,C(\xi)\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d\partial\phi_t\partial\phi_u} = -2\,\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}AC(\xi)\cos\xi s[-B^{-2}\cos(t-u)\xi + B^{-3}\beta_t\beta_u]\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d\partial\phi_t\partial\omega} = \frac{1}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{A\beta_t}{B^2}\,C(\xi)\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial d\partial\omega^2} = 0,$$

$$\frac{\partial^3\gamma(s)}{\partial\psi_k\partial\psi_\ell\partial\psi_s} = 0,$$

$$\frac{\partial^3\gamma(s)}{\partial\psi_k\partial\psi_\ell\partial\phi_t} = -2\,\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\beta_t\cos(k-\ell)\xi}{B^2}\cos\xi s\,d\xi,$$

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$$\frac{\partial^3\gamma(s)}{\partial\psi_k\partial\psi_\ell\partial\omega} = \frac{2}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\cos(k-\ell)\xi}{B}\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial\psi_k\partial\phi_t\partial\phi_u} = 2\,\frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}\alpha_k\cos\xi s[-B^{-2}\cos(t-u)\xi + B^{-3}\beta_t\beta_u]\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial\psi_k\partial\phi_t\partial\omega} = -\frac{1}{\pi}\int_0^\pi e^{-dC(\xi)}\,\frac{\alpha_k\beta_t}{B^2}\cos\xi s\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial\phi_t\partial\phi_u\partial\phi_v} = \frac{\omega}{\pi}\int_0^\pi e^{-dC(\xi)}A\cos\xi s\{-6B^{-4}\beta_t\beta_u\beta_v + 4B^{-3}[\beta_t\cos(u-v)\xi + \beta_u\cos(t-v)\xi + \beta_v\cos(t-u)\xi]\}\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial\phi_t\partial\phi_u\partial\omega} = \frac{2}{\pi}\int_0^\pi e^{-dC(\xi)}A\cos\xi s[-B^{-2}\cos(t-u)\xi + B^{-3}\beta_t\beta_u]\,d\xi,$$

$$\frac{\partial^3\gamma(s)}{\partial\phi_t\partial\omega^2} = \frac{\partial^3\gamma(s)}{\partial\omega^3} = 0.$$

The derivatives are evaluated by numerical integration over the interval $[0,\pi]$ of the relevant integrand derivatives.

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