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# FULLY MODIFIED ESTIMATION OF SEASONALLY COINTEGRATED PROCESSES

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We extend the framework of the fully modified, ordinary least squares (OLS) estimator introduced by Phillips and Hansen (1990) to the case of seasonally cointegrated processes at a given frequency. First we extend a weak convergence result of sample covariance matrices (Phillips, 1988) to the case of seasonal unit roots. Using a complex number framework, we then show that we can take into account the constraints that exist in a situation of seasonal cointegration as illustrated in Gregoir (1999a) and derive estimates of the cointegration vectors that allow for asymptotic normal inference. This allows us to propose a test whose null hypothesis is the existence of seasonal cointegration. A Monte Carlo exercise investigates the finite sample properties of this test procedure. The paper closes with the analysis of situations in which there exist more than one frequency at which seasonal cointegration can be observed.

## 1. INTRODUCTION

Many macroeconomic intra-annual time series show regular patterns associated to seasonal causes (biological, meteorological, or institutional). In their attempt to model these time series, econometricians are naturally led to raise the question of the parsimonious way to statistically capture these features. Two main approaches are used: on the one hand, a deterministic modeling of the intra-annual movements; on the other hand, the introduction of unit roots at seasonal frequencies related to the observation frequency. Specification tests have been designed to discriminate situations in which one approach is more in concordance with the data than the other one (Hasza and Fuller, 1982; Dickey, Hasza, and Fuller 1984; Hylleberg, Engle, Granger, and Yoo, 1991; Canova and Hansen, 1995; Caner, 1998; Smith and Taylor, 1998; Gregoir, 2006, among others). In practice from empirical studies, it seems that numerous time series can be parsimoniously described as integrated but at a subset of the eligible frequencies. Using tests dealing explicitly with each eligible unit root is therefore recommended (Taylor, 2003).

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When working simultaneously with several seasonally integrated processes in a multivariate setup, the practitioner is naturally confronted by the problem of seasonal cointegration (Engle, Granger, and Hallman, 1989). He must determine the presence of seasonal cointegration at each possible and reasonable frequency and the dimension of each cointegration space to be able to specify and estimate the appropriate vector error correction model (VECM) representation with seasonal error correction terms. This question arises as soon as the practitioner chooses to work with non-seasonally-adjusted processes, but it can also be considered with interest when working with processes that have been seasonally adjusted by the usual software, for we know that some filters used in this statistical treatment can lead to over-differenced processes at the seasonal frequencies and may lead to the nonexistence of finite order VAR approximation (Maravall, 1993). This last situation can be solved by the use of seasonal VECM in the spirit of the approach introduced by Gregoir (1999a), relying on the use of integral operators.

At frequency zero, the set of available statistical procedures to test for the existence and estimate cointegrating relationships can be divided into two main subsets: on the one hand, some estimation and test procedures that need very few hypotheses on the cointegration relationships, mainly the maximum likelihood approach (Johansen, 1988, 1991), the canonical cointegration approach (Park, 1992), and the efficient principal component procedure (Harris, 1997); on the other hand, those that need additional hypotheses on the structure of the cointegration space, mainly the fully modified estimators (Phillips and Hansen, 1990) and the dynamic least squares estimators (Saikkonen, 1991; Stock and Watson, 1993). In these latter frameworks, a subset of variables whose number is equal to the dimension of the cointegration space is supposed to be exclusively present in each cointegration relationship of a set that composes a basis of the cointegration space. This knowledge can be derived from theoretical a prioris. The estimates derived in these frameworks allow the econometrician to use standard asymptotic inference even if some estimators are superconsistent. In case of seasonal cointegration, some results are available. Research has been done to develop maximum likelihood procedures for seasonally cointegrated processes (Lee, 1992; Johansen and Schaumburg, 1998; Cubadda, 2001) or a test procedure based on principal components (Gregoir, 1999b). We propose here to extend the fully-modified approach to seasonal cointegrated systems and in this framework to introduce new tests for seasonal cointegration. We nevertheless may remark that the framework introduced here is also relevant and may help the economist to describe features corresponding to a long-period nonstationary economic cycle (Bierens, 2001).

At last, from a practical point of view, the method introduced here that explicitly uses a complex framework can be easily implemented in software in which complex number computation is a built-in option.[1]

The paper is organized as follows: We first state a general result on the weak convergence of sample covariance matrices that extends to the case of seasonal unit roots, a result established by Phillips (1988). We then give a triangular representation of a set of cointegrated processes at frequency $\omega \in ]0, \pi[$, which extends

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Phillips's (1991) representation and is available once some additional pieces of information are assumed. In a third part, we characterize the asymptotic behavior of the OLS and show that some nuisance parameters that are the complex counterparts of those observed in the case of cointegration at frequency $\omega \in \{0, \pi\}$ are present in their asymptotic distribution. The fourth part is dedicated to the definition of the fully modified estimator in this context and to the consistency properties of the estimates involved in these modifications. Working with the fully modified residuals that are asymptotically free of nuisance parameters, we then introduce a test for cointegration at frequency $\omega \in ]0, \pi[$ that relies on the asymptotic behavior of the complex random walks built upon the estimated residuals and study its small sample properties. The last part briefly considers the situation in which cointegration at various frequencies is present and proposes a estimation strategy. A brief conclusion closes the paper. All the proofs are gathered in the Appendix.

## 2. WEAK CONVERGENCE OF SAMPLE COVARIANCE MATRICES

We first extend a result by Phillips (1988) about weak convergence of sample covariance matrices to the case of seasonal unit roots. This result illustrates the parallel we can draw about the weak convergence of sample moments between processes integrated at frequency 0 (or $\pi$) and those integrated at any frequency $\omega \in ]0, \pi[$. This is already illustrated in the case of univariate time series in Gregoir (2006) and relies on the use of a complex number framework. We consider a general framework to state this result. We first introduce some notations and a multivariate invariance principle. Detailed notations are introduced in the Appendix (for a general framework, see Gregoir, 1999a), we limit our presentation to the minimum. These notations allow us to simplify the formulas and bring to the fore the parallelism that exits between integration and cointegration properties at frequency $\omega \in ]0, \pi[$ and those at frequency $\omega \in \{0, \pi\}$. We start with two definitions. When a real process is integrated at frequency $\omega$, its spectral density has a large peak in the neighborhood of $\omega$. By symmetry with respect to the vertical axis, the same expansion is valid in the neighborhood of $-\omega$. Both roots are necessary present. To obtain a stationary process we must apply to the integrated process the first difference operator at both frequencies.

DEFINITION 2.1. *A univariate process $x_t$, $t \in \mathbb{Z}$, is said to be integrated of order 1 at frequency $\omega \in ]0, \pi[$, which we denote $I_\omega(1)$, if it is such that*

$$\begin{aligned}(1 - 2\cos\omega L + L^2)x_t &= (1 - e^{-i\omega}L)(1 - e^{i\omega}L)x_t \\ &= \delta_\omega(L)\delta_{-\omega}(L)x_t \\ &= \eta_t,\end{aligned}$$

*where $\eta_t, t \in \mathbb{Z}$, is a covariance stationary process such that its spectral density $f_\eta(\varpi), \varpi \in [0, \pi]$, is strictly positive at frequency $\omega$.*

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In the sequel, we denote $\Delta_\omega$ the difference operator at frequency $\omega$ equal to $(1-2\cos\omega L+L^2)$. When we only apply the complex operator $\delta_\omega=(1-e^{-i\omega}L)$ to a real $I_\omega(1)$ process, we get a complex valued process that is still integrated at frequency $-\omega$. We now introduce an integral operator that plays the role of the inverse of the first difference operator at frequency $\omega$, namely $\delta_\omega=(1-e^{-i\omega}L)$.

DEFINITION 2.2. *The integral operator $S_\omega$ associates to any sequence $\varepsilon_t=(\varepsilon_t,\ t=\ldots-1,\ 0,1,\ldots)$ of real numbers a complex valued sequence $S_\omega\varepsilon_t$ defined by*

$$S_\omega\varepsilon_t=\begin{cases}\sum_{\tau=1}^{t}\varepsilon_\tau e^{-i\omega(t-\tau)} & \text{for } t>0\\ 0 & \text{for } t=0\\ -\sum_{\tau=0}^{t+1}\varepsilon_\tau e^{-i\omega(t-\tau)} & \text{for } t<0\end{cases}.$$

Some of its properties are given in the Appendix.

We now turn to the weak convergence properties of $I_\omega(1)$ processes. In this section the innovation processes are denoted $\varepsilon_t, t\in\mathbb{Z}$. We assume that these processes satisfy the following property.

**Assumption 2.1.** Assume $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is an $n$-dimensional martingale difference satisfying $\mathrm{E}(\varepsilon_t|\mathcal{F}_{t-1})=0$, $\mathrm{E}(\varepsilon_t\varepsilon_t'|\mathcal{F}_{t-1})=\Omega$, and $\sup_t \mathrm{E}(\max_i|\varepsilon_{i,t}|^{2+\delta}|\mathcal{F}_{t-1})<+\infty$ a.s. for some $\delta>0$, where $\mathcal{F}_{t-1}$ is the $\sigma$-field generated by $\{\varepsilon_{t-\tau},\tau=1,2,\ldots\}$.

From Chan and Wei (1988) and Tsay and Tiao (1990), we know that when $T$ goes to infinity, the following multivariate convergence in distribution holds, where $[Tt]$ is equal to the integer part of $Tt$:

$$\frac{1}{\sqrt{T}}\sum_{\tau=1}^{[Tt]}e^{i\omega\tau}\varepsilon_\tau=\frac{e^{i\omega[Tt]}}{\sqrt{T}}S_\omega\varepsilon_{[Tt]}\Longrightarrow\frac{1}{\sqrt{2}}W(t),$$

where $W(t)=W_R(t)+iW_I(t)$ and $W_R(t)$ and $W_I(t)$ are two real independent Wiener processes whose variance-covariance matrix is $\Omega$. This presentation with complex number is a convenient way to state the joint convergence of the $\frac{1}{\sqrt{T}}\sum_{\tau=1}^{[Tt]}\varepsilon_\tau\cos\omega\tau$ and $\frac{1}{\sqrt{T}}\sum_{\tau=1}^{[Tt]}\varepsilon_\tau\sin\omega\tau$. Assumption 2.1 is relatively strong. It can be weakened without consequences for the results in this paper in the spirit of Hansen (1992) or in the different approach proposed by Phillips and Solo (1992). Similarly, the assumption that only one pair of complex unit roots is present in the data generating process (DGP) may appear constraining, but a result on asymptotic orthogonality of processes integrated at different frequencies (see Chan and Wei, 1988) ensures that the results obtained in the first parts of the paper remain valid in more general situations. Section 7 explains how to deal with such situations.

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Let us consider an $n$-dimensional $I_\omega(1)$ complex valued process $\{y_t\}_{t\in\mathbb{N}}$ that is generated by

$$y_t = e^{-i\omega} y_{t-1} + u_t \tag{1}$$

for $t = 1, \ldots$, where $\omega \in ]0, \pi[$, $y_0$ is a $O_p(1)$ random vector,[2] and

$$u_t = \sum_{j=-\infty}^{+\infty} C_j \varepsilon_{t-j}, \tag{2}$$

with

$$\sum_{k=1}^{+\infty} \left[ \left\| \sum_{j=k}^{+\infty} C_j \right\| + \left\| \sum_{j=k}^{+\infty} C_{-j} \right\| \right] < +\infty, \tag{3}$$

where $\|.\|$ is the usual matrix norm, i.e., $\|C\|^2 = \mathrm{tr}\,(C\overline{C}')$. When $z$ is a complex number, $\overline{z}$ denotes its conjugate. For the sake of simplicity, in the sequel we limit our study to real matrices $(C_j)$, but a straightforward extension is possible. This linear data generating process is general in the sense that it includes stationary and invertible VARMA processes. Here $\{y_t\}_{t\in\mathbb{N}}$ is a process integrated of order 1 at frequency $\omega$ whose error term is real. We are interested in the weak convergence of

$$\frac{1}{T} \sum_{t=1}^{T} y_{t-1} \overline{u}'_t,$$

and we get the following result.

THEOREM 2.1. *If* $\{y_t\}_{t\in\mathbb{N}}$ *is generated by* (1) *and* $\{u_t\}_{t\in\mathbb{Z}}$ *satisfies* (2) *and* (3) *where* $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ *satisfies Assumption 2.1, then*

$$\frac{1}{T} \sum_{t=1}^{T} y_{t-1} \overline{u}'_t \Longrightarrow e^{i\omega} \left( \frac{1}{2} \int_0^1 B d\overline{B}' + \Lambda \right),$$

*where* $B(t)$ *is an n-dimensional complex Brownian motion whose variance-covariance matrix is* $2\pi f_u(\omega)$, $f_u(\omega)$ *denotes the spectral density matrix of* $\{u_t\}_{t\in\mathbb{Z}}$ *at frequency* $\omega$ *and is such that*

$$2\pi f_u(\omega) = \widetilde{\Omega} + \Lambda + \overline{\Lambda}',$$

*with*

$$\widetilde{\Omega} = \mathrm{E} u_t \overline{u}'_t$$

$$\Lambda = \sum_{k=1}^{+\infty} e^{-i\omega k} \mathrm{E} u_t \overline{u}'_{t+k}.$$

The proof relies on the use of martingale approximations in central limit theory for stationary processes introduced by Hall and Heyde (1980) and used by Phillips (1988) but applied to the demodulated process at frequency $\omega$.

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## 3. REPRESENTATION OF SEASONALLY COINTEGRATED MULTIVARIATE TIME SERIES

We first define more precisely the notion of seasonal integration and cointegration on which our approach relies, then we give a representation theorem in the case of a seasonally cointegrated set of variables of order 1, when cointegration is present at only one frequency.

### 3.1. Seasonal Cointegration

When we pile up a set of processes integrated at frequency $\omega$, it may happen as in the case of cointegration at frequency 0 that a linear combination of the components of this vector is covariance stationary. The main difference is that when considering frequency $\omega \in ]0, \pi[$, the current and lagged values of the processes are in general necessary to build up a stationary process. This first was observed by Engle, Granger, Hylleberg, and Lee (1993) and generalized in Gregoir (1999a)

DEFINITION 3.1. *Let $y_t$ be an $n$-dimensional process integrated of order 1 at frequency $\omega$; we say that $y_t$ is cointegrated at frequency $\omega$ if there exist two real $(1 \times n)$ vectors $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ such that $\widetilde{\alpha}_1 y_t + \widetilde{\alpha}_2 y_{t-1}$ is a covariance stationary process.*

The two vectors $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are necessarily constrained. They satisfy the condition that

$$\begin{pmatrix} \widetilde{\alpha}_1 & \widetilde{\alpha}_2 \\ -\widetilde{\alpha}_2 & \widetilde{\alpha}_1 + 2\cos\omega\widetilde{\alpha}_2 \end{pmatrix} \tag{4}$$

is a full rank matrix (its rank here is equal to 2). This is shown in Gregoir (1999a). To get an intuition of the reason why such a property holds, let us consider the lagged value of the cointegration relationship : $\widetilde{\alpha}_1 y_{t-1} + \widetilde{\alpha}_2 y_{t-2}$ is covariance stationary but as $L^2 = \Delta_\omega - 1 + 2\cos\omega L$, $-\widetilde{\alpha}_2 y_t + (\widetilde{\alpha}_1 + 2\cos\omega\widetilde{\alpha}_2) y_{t-1}$ is also a covariance stationary process. We thus have two cointegration relationships. If we now remark that the above matrix (4) can be rewritten as

$$\begin{pmatrix} \widetilde{\alpha}_1 & \widetilde{\alpha}_2 \\ -\widetilde{\alpha}_2 & \widetilde{\alpha}_1 + 2\cos\omega\widetilde{\alpha}_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -e^{i\omega} \\ 1 & -e^{-i\omega} \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{\alpha}_1 + e^{i\omega}\widetilde{\alpha}_2 & 0 \\ 0 & \widetilde{\alpha}_1 + e^{-i\omega}\widetilde{\alpha}_2 \end{pmatrix} \begin{pmatrix} 1 & -e^{i\omega} \\ 1 & -e^{-i\omega} \end{pmatrix} \otimes I_n,$$

we can conclude that, since $\omega \in ]0, \pi[$, the full rankness of (4) corresponds to the full rankness of the complex matrix $\widetilde{\alpha}_1 + e^{i\omega}\widetilde{\alpha}_2$, its nonnullity when we consider only a vector.

We now state a representation theorem that is a particular case of Gregoir (1999a).

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THEOREM 3.1. *Let $y_t$ be an n-dimensional $I_\omega(1)$ process such that (i) the determinant of the polynomial matrix associated to the Wold representation of $\Delta_\omega y_t$ has as only unit roots the pair $\{e^{i\omega}, e^{-i\omega}\}$ with an order of multiplicity equal to r, and (ii) the polynomial matrix associated to the Wold representation evaluated in $e^{i\omega}$ is a complex matrix of rank $n-r$. Then it satisfies the following equation:*

$$\Delta_\omega y_t = \theta_1(\widetilde{\alpha}_1 y_{t-1} + \widetilde{\alpha}_2 y_{t-2}) + \theta_2(\widetilde{\alpha}_1 y_{t-2} + \widetilde{\alpha}_2 y_{t-3}) + \sum_{j=1}^{+\infty} \Phi_j \Delta_\omega y_{t-j} + \varepsilon_t,$$

*where (i) $\varepsilon_t$ is the innovation process of $y_t$, (ii) $\sum_{j=1}^{+\infty} \operatorname{tr}\left(\Phi_j \Phi_j'\right)^{1/2} < +\infty$, (iii) $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are two $(r \times n)$ real matrices such that*

$$\begin{pmatrix} \widetilde{\alpha}_1 & \widetilde{\alpha}_2 \\ -\widetilde{\alpha}_2 & \widetilde{\alpha}_1 + 2\cos\omega\widetilde{\alpha}_2 \end{pmatrix}$$

*is a $(2r \times 2n)$ full rank matrix, and (iv) $\theta_1$ and $\theta_2$ are two $(n \times r)$ matrices such that*

$$\begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 + 2\cos\omega\theta_2 \end{pmatrix}$$

*is a $(2n \times 2r)$ full rank matrix.*

To get an idea of the proof of this theorem, we can draw a parallel with Granger's (1981) approach in the case of usual cointegration at frequency 0. In this case, the polynomial matrix associated to the Wold representation of the first differenced process, $C(L)$, is such that $C(1)$ is not of full rank, it is of rank $(n-r)$. This means that some linear combinations of the process are $I_0(-1)$ (see Gregoir and Laroque, 1993, and the Appendix); i.e., there exists a unit root in their Wold representation. Integrating by summation, these relations do not generate $I_0(1)$ processes but $I_0(0)$ ones, which correspond to the 0 frequency cointegration relationships. This is similar in the case we consider.

The existence of $r$ cointegration relationships of degree 1 corresponds to the fact that the matrix $C(e^{i\omega})$ is a complex matrix of rank $n-r$. Let $\widetilde{\alpha}$ be a complex full ranked $r \times n$ matrix whose conjugate rows span the kernel of $\overline{C(e^{i\omega})}'$, then $\widetilde{\alpha}C(e^{i\omega}) = 0$. The complex processes corresponding to the linear combinations built on $\widetilde{\alpha}$ are $I_\omega(-1)$ and are such that the use of integral operator $S_\omega$ introduced in the previous section does not generate $I_\omega(1)$ processes but stationary ones; in other words, $\widetilde{\alpha}S_\omega\Delta_\omega y_t$ is a complex covariance stationary process.

We can relate the complex matrix $\widetilde{\alpha}$ to the two real matrices $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ we introduced in the above definition of seasonal cointegration by the following simple equation: $\widetilde{\alpha} = \widetilde{\alpha}_1 + e^{i\omega}\widetilde{\alpha}_2$.

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## 4. TRIANGULAR REPRESENTATION

### 4.1. The Necessary Hypotheses

We first emphasize the necessary hypotheses that characterize the situation we consider here. We depart from the VECM representation theorem stated in the above theorem; this requires additional information. These hypotheses are similar to those assumed in the framework of the fully modified estimators introduced by Phillips and Hansen (1990). We need three pieces of information on the set of variables under study. First, we suppose that we know the dimension $r$ of the space of seasonal cointegration. Second, we assume that we know that a subset of $r$ variables, which we set in the first rows of the vector in which we pile up the data and which we denote $y_{1,t}$, is composed of variables that belong to $r$ independent vectors in the cointegration space. Third, polynomial cointegration is not present.

### 4.2. The Triangular Representation

The $n$ variables under study are piled up into a vector $y_t$ we decompose into $\left(y'_{1,t} \;\; y'_{2,t}\right)'$, whose dimensions are, respectively, $(r \times 1)$ and $(n-r \times 1)$. We assume that all the components of $y_t$ are $I_\omega(1)$ and consider the covariance stationary vector $(1-2\cos\omega L+L^2)y_t$. We denote $\varepsilon_t$ its innovation process and $C(L)$ the polynomial matrix associated to the Wold representation of $(\Delta_\omega y_t)_t$.

We restate the above assumptions under the following technical form.

**Assumption 4.1.** Let $\det C(L) = \Delta_\omega^r d(L)$ where all the roots of $d(L)$ have a modulus greater than 1.

**Assumption 4.2.** Let $rank C(e^{i\omega}) = n-r$.

**Assumption 4.3.** Let $\widetilde{\alpha}$ be a $r \times n$ complex matrix whose rows are a basis of the cointegration space, then it can be partitioned into $\left(\widetilde{\alpha}_1 \;\; \widetilde{\alpha}_2\right)$ with $\widetilde{\alpha}_1$ a full-rank $(r \times r)$ complex matrix.

We furthermore assume that the following property is satisfied by the sequence of the Wold representation matrix coefficients.

**Assumption 4.4.** Let $\sum_{j=0}^{+\infty} j \,\mathrm{tr}\,(C_j \overline{C}'_j)^{1/2} < +\infty$.

This last assumption ensures in particular that (3) is satisfied. Applying to the Wold representation matrix polynomial a Beveridge-Nelson type decomposition at frequency $\omega$ (Phillips and Solo, 1992), we get

$$C(L) = C(e^{i\omega}) + \delta_\omega \widetilde{C}(L),$$

where $\widetilde{C}(L)$ is a complex matrix polynomial that, when applied to a $L^2$ process, defines under Assumption 4.4 a $L^2$ process. Simple computations give that $\forall j \in \mathbb{N}$, $\widetilde{C}_j = -\sum_{k=j+1}^{+\infty} e^{i\omega(k-j)} C_k$. In our case, we get that

$$\Delta_\omega y_t = C(e^{i\omega})\varepsilon_t + \delta_\omega \widetilde{C}(L)\varepsilon_t$$

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and thus

$$\Delta_\omega \widetilde{\alpha} y_t = \delta_\omega \widetilde{\alpha} \widetilde{C}(L)\varepsilon_t$$

or

$$\delta_\omega(\widetilde{\alpha}\delta_{-\omega} y_t - \widetilde{\alpha}\widetilde{C}(L)\varepsilon_t) = 0.$$

It follows by Corollary A in the Appendix, there exists a complex vector $\kappa_0$ such that

$$\widetilde{\alpha}\delta_{-\omega} y_t = \widetilde{\alpha}\widetilde{C}(L)\varepsilon_t + e^{-i\omega t}\kappa_0, \tag{5}$$

which conveys that the complex $I_\omega(1)$ process $\delta_{-\omega} y_t$ is cointegrated and that a possible set of independent cointegrating vectors is given by the rows of $\widetilde{\alpha}$. Under Assumption 4.3, $\widetilde{\alpha}$ is an $r \times n$ complex matrix of full rank $r$ such that the square matrix composed of its $r$ first columns $\widetilde{\alpha}_1$ is of full rank:

$$\widetilde{\alpha}_1\delta_{-\omega} y_{1,t} + \widetilde{\alpha}_2\delta_{-\omega} y_{2,t} = \widetilde{\alpha}\widetilde{C}(L)\varepsilon_t + e^{-i\omega t}\kappa_0.$$

Premultiplying by the inverse of this matrix, we get that equation (5) can be rewritten under the following form:

$$\delta_{-\omega} y_{1,t} = \alpha\delta_{-\omega} y_{2,t} + e^{-i\omega t}\kappa + \eta_{1,t},$$

where $\alpha = -\widetilde{\alpha}_1^{-1}\widetilde{\alpha}_2$ is an $r \times (n-r)$ complex matrix, $\kappa = \widetilde{\alpha}_1^{-1}\kappa_0$ is a complex random variable, and $\eta_{1,t}$ is a complex covariance stationary process. We complete this system of $r$ equations with the $(n-r)$ real covariance stationary process $\Delta_\omega y_{2,t}$, between which by assumption there does not exist any seasonal cointegration relationship. We say that $\left(y'_{1,t} \;\; y'_{2,t}\right)'$ satisfies a *complex triangular representation* when it is such that

$$\begin{cases} \delta_{-\omega} y_{1,t} = \alpha\delta_{-\omega} y_{2,t} + e^{-i\omega t}\kappa + \eta_{1,t} \\ \Delta_\omega y_{2,t} = \eta_{2,t} \end{cases},$$

where $\left(\eta'_{1,t} \;\; \eta'_{2,t}\right)'$ is an $n$-dimensional covariance stationary (complex valued for its $r$ first components) process whose determinant of the polynomial matrix associated to its Wold representation has roots of modulus strictly larger than 1.

**Example 4.1.**

Let us consider a bivariate real process integrated of order 1 at frequency $\frac{\pi}{2}$ (and $-\frac{\pi}{2}$) that satisfies the following Wold representation:

$$(1+L^2)\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & -L \\ L & 1 \end{pmatrix}\begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix}.$$

The determinant of the matrix polynomial associated to its Wold representation is equal to $(1+L^2)$, the process is cointegrated at frequency $\frac{\pi}{2}$ (and -$\frac{\pi}{2}$). There

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exist one cointegrating vector at frequency $\frac{\pi}{2}$ and another one (its conjugate) at frequency $-\frac{\pi}{2}$. We have

$$(1+L^2)y_{1,t}+i(1+L^2)y_{2,t}=(1+iL)(\varepsilon_{1,t}+i\varepsilon_{2,t}),$$

whence

$$(1-iL)y_{1,t}+i(1-iL)y_{2,t}=\kappa_0 e^{-i\frac{\pi}{2}t}+\varepsilon_{1,t}+i\varepsilon_{2,t}.$$

We complete with the process $(1+L^2)y_{2,t}$ to get the complex triangular representation

$$\begin{cases}(1-iL)y_{1,t}=-i(1-iL)y_{2,t}+\kappa_0 e^{-i\frac{\pi}{2}t}+\varepsilon_{1,t}+i\varepsilon_{2,t}\\ (1+L^2)y_{2,t}=\varepsilon_{2,t}+\varepsilon_{1,t-1}\end{cases}.$$

**Remarks 4.1.** Any $n$-dimensional process satisfying Assumptions 4.1–4.4 can be represented under a complex triangular representation.

We can rewrite the preceding system with only real quantities:

$$\begin{cases}y_{1,t}-\cos\omega y_{1,t-1}=\Re\alpha y_{2,t}-(\cos\omega\Re\alpha-\sin\omega\Im\alpha)y_{2,t-1}\\ \qquad\qquad +\cos\omega t\Re\kappa+\sin\omega t\Im\kappa+\Re\eta_{1,t},\\ \sin\omega y_{1,t-1}=-\Im\alpha y_{2,t}+(\cos\omega\Im\alpha+\sin\omega\Re\alpha)y_{2,t-1}\\ \qquad\qquad -\cos\omega t\Im\kappa+\sin\omega t\Re\kappa-\Im\eta_{1,t},\\ \Delta_\omega y_{2,t}=\eta_{2,t}.\end{cases}$$

The two first sets of equations correspond to the real and imaginary parts of the complex triangular representation. Up to the multiplication by $\widetilde{\alpha}_1^{-1}$, the first set of equations is equal to the real part of $\widetilde{\alpha}\delta_{-\omega}y_t-e^{-i\omega t}\kappa_0$:

$$\frac{1}{2}\left\{(\widetilde{\alpha}+\overline{\widetilde{\alpha}})y_t-(e^{i\omega}\widetilde{\alpha}+e^{-i\omega}\overline{\widetilde{\alpha}})y_{t-1}-(e^{-i\omega t}\kappa_0+e^{i\omega t}\overline{\kappa}_0)\right\}.$$

From equation (4) we know there exists another set of linear equations that amounts to a rewriting of its lagged value up to a covariance stationary process. Simple algebra gives that it is equal to

$$\frac{1}{2}\left\{(e^{i\omega}\widetilde{\alpha}+e^{-i\omega}\overline{\widetilde{\alpha}})y_t-(e^{2i\omega}\widetilde{\alpha}+e^{-2i\omega}\overline{\widetilde{\alpha}})y_{t-1}-(e^{-i\omega(t-1)}\kappa_0+e^{i\omega(t-1)}\overline{\kappa}_0)\right\},$$

which can be read as the real part of $e^{i\omega}(\widetilde{\alpha}\delta_{-\omega}y_t-e^{-i\omega t}\kappa_0)$. As $i=\frac{e^{i\omega}}{\sin\omega}-\frac{\cos\omega}{\sin\omega}$, we conclude that the imaginary part of $\widetilde{\alpha}\delta_{-\omega}y_t-e^{-i\omega t}\kappa_0$ is a linear combination of the two above sets of equations that correspond to the same set of cointegrating relationships. We thus keep the first set of $r$ equations and get the corresponding

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real triangular representation of $y_t$ composed of a set of $n$ independent equations. Working with such a system is relatively cumbersome, as full rankness conditions exist between the different $r \times n-r$ real matrices involved in this set of linear equations.

**Example 4.1** (continued)**.**
We obtain the following system of equations with real valued quantities:

$$\begin{cases} y_{1,t} = -y_{2,t-1} + \cos\frac{\pi}{2}t\Re\kappa_0 - \sin\frac{\pi}{2}t\Im\kappa_0 + \varepsilon_{1,t}, \\ y_{1,t-1} = y_{2,t} - \cos\frac{\pi}{2}t\Im\kappa_0 + \sin\frac{\pi}{2}t\Re\kappa_0 - \varepsilon_{2,t}, \\ (1+L^2)y_{2,t} = \varepsilon_{2,t} + \varepsilon_{1,t-1}, \end{cases}$$

where $y_{2,t} = -y_{2,t-2} + \varepsilon_{2,t} + \varepsilon_{1,t-1}$ so that the second equation can be rewritten as $y_{1,t-1} = -y_{2,t-2} - \cos\frac{\pi}{2}t\Im\kappa_0 + \sin\frac{\pi}{2}t\Re\kappa_0 + \varepsilon_{1,t-1}$ and appears redundant. We restrict the triangular representation to the first and third equations.

In the sequel, we limit our attention to the complex triangular representation. The real estimates will be simply derived from the real and imaginary parts of the complex matrices we consider. To justify this approach, we rely on the following lemma, whose proof is given in the Appendix.

LEMMA 4.1. *Let $(u_t)_t$ and $(v_t)_t$ be two sets of complex vectors of respective size $n \times 1$ and $m \times 1$, satisfying the following linear complex model for $t = 1, \ldots, T$:*

$$u_t = \alpha v_t + w_t,$$

*where $\alpha$ is an $(n \times m)$ complex matrix and $w_t$ is an $n \times 1$ random complex vector such that $\mathrm{E}w_t = 0$, $\mathrm{E}w_t\overline{w}'_\tau = 0$ when $t \neq \tau$, $V\Re w_t = \Sigma_R$, $V\Im w_t = \Sigma_I$, $\mathrm{E}\Re w_t \Im w'_t = \Sigma_{RI}$, then the OLS estimate of $(\Re\alpha, \Im\alpha)$ is given by*

$$\widehat{\Re\alpha} + i\widehat{\Im\alpha} = \widehat{\alpha} = u\overline{v}'(v\overline{v}')^{-1},$$

*where $u$ stands for the $n \times T$ matrix whose $t^{th}$ column is equal to $u_t$, and similarly, $v$ stands for the $m \times T$ matrix whose $t^{th}$ column is equal to $v_t$.*

## 4.3. Without Deterministic Term

We first consider a complex triangular representation without deterministic terms. The introduction of such terms is analyzed in Section 4.4. We study the properties of the OLS estimates of the coefficients in the $r$ first equations of the triangular complex representation. As in the case of cointegration at frequency 0, they are superconsistent but include an asymptotic bias and their asymptotic distribution involved nuisance parameters. We determine these parameters to allow for a correction in the spirit of the fully modified approach.

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We thus consider $\begin{pmatrix} y'_{1,t} & y'_{2,t} \end{pmatrix}'$ that satisfies Assumptions 4.1–4.4 and the following *complex triangular representation*:

$$\begin{cases} \delta_{-\omega} y_{1,t} = \alpha \delta_{-\omega} y_{2,t} + \eta_{1,t}, \\ \Delta_{\omega} y_{2,t} = \eta_{2,t}, \end{cases} \tag{6}$$

where $\begin{pmatrix} \eta'_{1,t} & \eta'_{2,t} \end{pmatrix}'$ is an $n$-dimensional complex covariance stationary process. As said above, the OLS estimates of $\alpha$ correspond to the parameter values that minimize the trace of the matrix equal to the sum of the cross-products $\eta_{1,t}\overline{\eta_{1,t}}'$, as $\eta_{1,t}$ is a complex process. Let $\widehat{\alpha}$ be the solution of such a minimization program, and we have the following proposition.

PROPOSITION 4.1. *Let $\widehat{\alpha}$ be the OLS estimate of the first $r$ equations in* (6)*; then when the size of the sample $T$ tends towards $+\infty$, the following weak convergence holds:*

$$T(\widehat{\alpha} - \alpha) \Longrightarrow \left( \frac{1}{2} \int_0^1 dB_1(\tau) \overline{B}'_2(\tau) + \sum_{j=0}^{+\infty} e^{ij\omega} \overline{\Gamma}'_{21}(j) \right) \times \left( \frac{1}{2} \int_0^1 B_2(\tau) \overline{B}'_2(\tau) d\tau \right)^{-1},$$

*where $\begin{pmatrix} B'_1 & B'_2 \end{pmatrix}'$ is an $n$-dimensional complex Brownian motion whose variance-covariance matrix is the complex spectral density matrix of the complex process $\eta'_t = \begin{pmatrix} \eta'_{1,t} & \eta'_{2,t} \end{pmatrix}'$ at frequency $\omega$, multiplied by $2\pi$ (we denote $2\pi F_\eta(\omega)$) and $\overline{\Gamma}'_{21}(j) = \mathrm{E}\eta_{1,t+j}\overline{\eta}'_{2,t}$.*

The covariance function and spectral density of a complex process are introduced for univariate processes in Anderson (1971, Ch. 7) and for multivariate processes in Hannan (1969). They are straightforward extensions to the real case. We detail the properties of the spectral density matrix at the end of this section. Formally the situation is very close to that observed at frequency 0. The asymptotic bias is the one-directional Fourier transform at frequency $\omega$ of a submatrix of the covariance function of the error terms $\eta_t$. The nuisance parameters are related to the spectral density matrix at frequency $\omega$. To get rid of these nuisance parameters and bias, we have to first construct consistent estimates of their value, and second, use these estimates to modify the OLS estimates to derive estimates with asymptotically mixed normal distributions. We are mainly interested in the real estimates of $\Re\alpha$ and $\Im\alpha$. From the above formulas, we obtain:

COROLLARY 4.1. *Let $\widehat{\alpha}$ be the OLS estimate of the first $r$ equations in* (6)*; then when the size of the sample $T$ tends toward $+\infty$, the following joint weak convergence holds:*

$$T\begin{pmatrix} \Re\widehat{\alpha} - \Re\alpha & \Im\widehat{\alpha} - \Im\alpha \end{pmatrix} \Longrightarrow \begin{pmatrix} U_R & U_I \end{pmatrix} V^{-1} \tag{7}$$

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*with*

$$U_R = \frac{1}{2}\int_0^1 dB_{1,R}B'_{2,R} + dB_{1,I}B'_{2,I} + \sum_{j=0}^{+\infty}\cos j\omega\Re\Gamma'_{21}(j) + \sum_{j=0}^{+\infty}\sin j\omega\Im\Gamma'_{21}(j)$$

*and*

$$U_I = \frac{1}{2}\int_0^1 dB_{1,I}B'_{2,R} - dB_{1,R}B'_{2,I} + \sum_{j=0}^{+\infty}\sin j\omega\Re\Gamma'_{21}(j) - \sum_{j=0}^{+\infty}\cos j\omega\Im\Gamma'_{21}(j)$$

*and*

$$V = \frac{1}{2}\begin{pmatrix} \int_0^1 \left(B_{2,R}B'_{2,R} + B_{2,I}B'_{2,I}\right)d\tau & \int_0^1 \left(-B_{2,R}B'_{2,I} + B_{2,I}B'_{2,R}\right)d\tau \\ \int_0^1 \left(B_{2,R}B'_{2,I} - B_{2,I}B'_{2,R}\right)d\tau & \int_0^1 (B_{2,R}B'_{2,R} + B_{2,I}B'_{2,I})\,d\tau \end{pmatrix},$$

*where* $\left(B'_1 \;\; B'_2\right)' = \left(B'_{1,R} \;\; B'_{2,R}\right)' + i\left(B'_{1,I} \;\; B'_{2,I}\right)'$ *and* $\Gamma_{21}(j)$ *are defined in the preceding proposition.*

The above formulas are analytically derived but correspond to the existence of an isomorphism between complex processes and their real representation (see Goodman, 1963) under the following form: To the complex process $W = W_R + iW_I$, we associate the real matrix $\begin{pmatrix} W_R & W_I \\ -W_I & W_R \end{pmatrix}$. We can rewrite the above weak convergence result under the following form:

$$\left(\frac{1}{2}\int_0^1 \left(dB_{1,R} \quad dB_{I,I}\right)B_2^{*\prime} + \sum_{j=0}^{+\infty}\left(\cos j\omega \quad -\sin j\omega\right)\otimes I_r\Gamma_{21}^*(j)\right)$$

$$\times\left(\frac{1}{2}\int_0^1 B_2^*B_2^{*\prime}\,d\tau\right)^{-1},$$

with the following notation:

**Notation 4.1.** For any complex matrix $A = A_R + iA_I$, we denote $A^*$ the real matrix equal to $\begin{pmatrix} A_R & A_I \\ -A_I & A_R \end{pmatrix}$.

We restate a natural result in this framework (Goodman, 1963).

LEMMA 4.2. *If A is an* $(n \times n)$ *complex matrix of full rank, then* $A^*$ *is a* $(2n \times 2n)$ *real matrix of full rank.*

In the same spirit, it may be useful to spare some space to describe the variance-covariance matrix of the $2n \times 1$ real Brownian motion defined by $\widetilde{B} = \left(B'_{1,R} \;\; B'_{2,R}\right.$

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$B'_{1,I} \; B'_{2,I}\big)'$. This variance-covariance matrix is derived from the complex spectral density matrix at frequency $\omega$ of the complex $n$-dimension process $(\eta_t)_t$. It has the form

$$V\widetilde{B}(\tau) = 2\pi\tau \begin{pmatrix} \Re F_\eta(\omega) & -\Im F_\eta(\omega) \\ \Im F_\eta(\omega) & \Re F_\eta(\omega) \end{pmatrix} \tag{8}$$

that is a positive-definite matrix, for $\Im F_\eta(\omega)$ is an antisymmetric real matrix. More precisely, only the $r$ first components of $\eta_t$ equal to $\eta_{1,t}$ are complex: $\eta_{1,t} = \eta_{1,R,t} + i\eta_{1,I,t}$. We thus work with three real processes $(\eta_{1,R,t})_t, (\eta_{1,I,t})_t$, and $(\eta_{2,t})_t$ from which all the elements of $V\widetilde{B}$ can be derived. Straightforward computations give that

$$\Re F_\eta(\omega) = \sum_{j=-\infty}^{+\infty} \cos\omega j \Gamma_R(j) - \sin\omega j \Gamma_I(j) \tag{9}$$

$$\Im F_\eta(\omega) = \sum_{j=-\infty}^{+\infty} \cos\omega j \Gamma_I(j) + \sin\omega j \Gamma_R(j) \tag{10}$$

where

$$\Gamma_R(j) = \begin{pmatrix} \Gamma_{11,R}(j) + \Gamma_{11,I}(j) & \Gamma_{(1,R),2}(j) \\ \Gamma_{2,(1,R)}(j) & \Gamma_{22}(j) \end{pmatrix} \tag{11}$$

$$\Gamma_I(j) = \begin{pmatrix} \Gamma_{(1,I),(1,R)}(j) - \Gamma_{(1,R),(1,I)}(j) & \Gamma_{(1,I),2}(j) \\ \Gamma_{2,(1,I)}(j) & 0 \end{pmatrix} \tag{12}$$

and $a \in \{R, I\}, a' \in \{R, I\}$: $\Gamma_{11,a}(j) = \mathrm{E}\eta_{1,a,t}\eta'_{1,a,t+j}$, $\Gamma_{(1,a),(1,a')}(j) = \mathrm{E}\eta_{1,a,t}$ $\eta'_{1,a',t+j}$, $\Gamma_{(1,a),2}(j) = \mathrm{E}\eta_{1,a,t}\eta'_{2,t+j}$, $\Gamma_{2,(1,a)}(j) = \mathrm{E}\eta_{2,t}\eta'_{1,a,t+j}$, and $\Gamma_{22}(j) = \mathrm{E}\eta_{2,t}\eta'_{2,t+j}$. In practice, it is reasonable to use an estimator of the above quantities that ensures that the variance-covariance matrix we obtain is a positive-definite one. This can be guaranteed by working with the complex processes in order to construct a Hermitian matrix $\widehat{F}_\eta(\omega)$. [3] This will be discussed in Lemma 5.1 in the next section.

### 4.4. With Deterministic Terms

We now consider $\big(y'_{1,t} \; y'_{2,t}\big)'$, which satisfies Assumptions 4.1–4.4 and the following *complex triangular representation*:

$$\begin{cases} \delta_{-\omega} y_{1,t} = \delta_{-\omega}\alpha y_{2,t} + \kappa e^{-i\omega t} + \eta_{1,t}, \\ \Delta_\omega y_{2,t} = \eta_{2,t}, \end{cases} \tag{13}$$

where $\big(\eta'_{1,t} \; \eta'_{2,t}\big)'$ is an $n$-dimensional complex covariance stationary process and $\kappa$ is a constant (random variable). Taking into account the presence of the complex

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deterministic term $(\kappa e^{-i\omega t})_t$ corresponds to the usual introduction of deterministic functions oscillating at the season frequency under consideration in the real counterpart of this system of $r$ (complex) equations. We only consider this deterministic function, as it is the only form that may have an asymptotic influence on the behavior of the OLS estimates. Indeed, all the similar deterministic functions though associated to a different frequency are asymptotically orthogonal to the $I_\omega(1)$ processes. We get the following result:

PROPOSITION 4.2. *Let $\widehat{\alpha}$ be the OLS estimate of the first $r$ equations in* (13)*; then when the size of the sample $T$ tends toward $+\infty$, the following weak convergence holds:*

$$T(\widehat{\alpha}-\alpha) \Longrightarrow \left(\frac{1}{2}\int_0^1 dB_1(\tau)\overline{B^{\mu}}'_2(\tau) + \sum_{j=0}^{+\infty} e^{ij\omega}\overline{\Gamma}'_{21}(j)\right) \times \left(\frac{1}{2}\int_0^1 B_2^{\mu}(\tau)\overline{B^{\mu}}'_2(\tau)\,d\tau\right)^{-1},$$

*where $B_2^{\mu}(\tau) = B_2(\tau) - \int_0^1 B_2(\tau)\,d\tau$ and $\left(B_1' \; B_2'\right)'$ is the $n$-dimensional complex Brownian motion introduced in Proposition 4.1.*

In the presence of the deterministic function $e^{-i\omega t}$, the asymptotic behavior of the OLS estimates are therefore very similar to that obtained in (6); we only have to replace the complex Brownian motion by its demeaned counterpart. This result can be extended to any linear combination of monomial functions $\{t^p e^{-i\omega t}\}_{p\in\mathbb{N}}$.

## 5. FULLY MODIFIED OLS ESTIMATORS

By computing fully modified OLS estimators, we want to construct estimates that have no nuisance parameter in their asymptotic distribution and allow the practitioners to use standard normal asymptotic inference when investigating the properties of the cointegration space. We follow Phillips and Hansen (1990) and partition the spectral density matrix $F_\eta(\omega)$ of the complex covariance stationary process $\eta_t$ at frequency $\omega$ conformably to the dimensions of $y_{1,t}$ and $y_{2,t}$:

$$F_\eta(\omega) = \begin{pmatrix} F_{\eta,11}(\omega) & F_{\eta,12}(\omega) \\ F_{\eta,21}(\omega) & F_{\eta,22}(\omega) \end{pmatrix}$$

and the one-directional Fourier transform of the *complex* variance-covariance matrix function $\Gamma(h) = \mathrm{E}\begin{pmatrix}\eta_{1,t}\\ \eta_{2,t}\end{pmatrix}\left(\overline{\eta}'_{1,t+h}\; \overline{\eta}'_{2,t+h}\right)$:

$$G_\eta(\omega) = \begin{pmatrix} G_{\eta,11}(\omega) & G_{\eta,12}(\omega) \\ G_{\eta,21}(\omega) & G_{\eta,22}(\omega) \end{pmatrix} = \sum_{h=0}^{+\infty} e^{-ih\omega}\Gamma(h).$$

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We then set

$$\delta_{-\omega} y_{1,t}^{+} = \delta_{-\omega} y_{1,t} - F_{\eta,12}(\omega) F_{\eta,22}(\omega)^{-1} \Delta_{\omega} y_{2,t},$$

$$\overline{G}_{\eta,21}^{\prime +}(\omega) = \overline{G}_{\eta,21}^{\prime}(\omega) - F_{\eta,12}(\omega) F_{\eta,22}(\omega)^{-1} \overline{G}_{\eta,22}^{\prime}(\omega),$$

and define the fully modified OLS estimators by the following equation:

$$\widehat{\alpha}^{+} = \left( \delta_{-\omega} y_1^{+} \overline{\delta_{-\omega} y_2}^{\prime} - T \overline{G}_{\eta,21}^{\prime +} \right) \left( \delta_{-\omega} y_2 \overline{\delta_{-\omega} y_2}^{\prime} \right)^{-1}, \tag{14}$$

where for sake of notational simplicity $\delta_{-\omega} y_j$ stands for the $(\dim y_j \times T)$ matrix whose $t^{\text{th}}$ column is equal to $\delta_{-\omega} y_{j,t}$, $j \in \{1, 2\}$.

PROPOSITION 5.1. *Let $\widehat{\alpha}^{+}$ be the FM-OLS estimate of the coefficients in the first $r$ equations as defined above; then when the size of the sample $T$ tends toward $+\infty$, the following weak convergence holds:*

$$T\left(\widehat{\alpha}^{+} - \alpha\right) \Longrightarrow \left( \int_0^1 dB_{1.2}(\tau) \overline{B}_2^{\prime}(\tau) \right) \left( \int_0^1 B_2(\tau) \overline{B}_2^{\prime}(\tau)\, d\tau \right)^{-1},$$

*where $B_{1.2}$ and $B_2$ are two complex Brownian motions of, respectively, $r$ dimensions and $n - r$ dimensions, which are not correlated.*

We now state the same result for the real parameters involved in the above set of equations.

COROLLARY 5.1. *Let $\widehat{\alpha}^{+}$ be the OLS estimate of the first $r$ equations in* (6)*; then when the size of the sample $T$ tends toward $+\infty$, the following joint weak convergence holds:*

$$T\left( \Re\widehat{\alpha}^{+} - \Re\alpha \quad \Im\widehat{\alpha}^{+} - \Im\alpha \right) \Longrightarrow \left( U_R \quad U_I \right) V^{-1}, \tag{15}$$

*with*

$$U_R = \int_0^1 dB_{1.2,R} B_{2,R}^{\prime} + dB_{1.2,I} B_{2,I}^{\prime},$$

$$U_I = \int_0^1 dB_{1.2,I} B_{2,R}^{\prime} - dB_{1.2,R} B_{2,I}^{\prime},$$

*and*

$$V = \begin{pmatrix} \int_0^1 \left( B_{2,R} B_{2,R}^{\prime} + B_{2,I} B_{2,I}^{\prime} \right) d\tau & \int_0^1 \left( -B_{2,R} B_{2,I}^{\prime} + B_{2,I} B_{2,R}^{\prime} \right) d\tau \\ \int_0^1 \left( B_{2,R} B_{2,I}^{\prime} - B_{2,I} B_{2,R}^{\prime} \right) d\tau & \int_0^1 \left( B_{2,R} B_{2,R}^{\prime} + B_{2,I} B_{2,I}^{\prime} \right) d\tau \end{pmatrix},$$

*where $\left( B_{1.2}^{\prime} \; B_2^{\prime} \right)^{\prime} = \left( B_{1.2,R}^{\prime} \; B_{2,R}^{\prime} \right)^{\prime} + i \left( B_{1.2,I}^{\prime} \; B_{2,I}^{\prime} \right)^{\prime}$ is defined in the preceding proposition.*

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The variance-covariance matrix of the complex Brownian motion $B_{1.2}(\tau)$ is equal to

$$\begin{aligned}\mathrm{E}B_{1.2}(\tau)\overline{B_{1.2}(\tau)}' &= 2\pi\,\tau\,(F_{\eta,11}(\omega) - F_{\eta,12}(\omega)F_{\eta,22}^{-1}(\omega)F_{\eta,21}(\omega)) \\ &= 2\pi\,\tau\,F_{\eta,1.2}(\omega).\end{aligned}$$

The variance-covariance matrix of the $(2r \times 1)$ associated real process $\left(B'_{1.2,R}(\tau)\right.$ $\left.B'_{1.2,I}(\tau)\right)'$ is more cumbersome to derive. It has the form

$$2\pi\,\overline{F_{\eta,1.2}(\omega)}^{*} = 2\pi \begin{pmatrix} \Re F_{\eta,1.2}(\omega) & -\Im F_{\eta,1.2}(\omega) \\ \Im F_{\eta,1.2}(\omega) & \Re F_{\eta,1.2}(\omega) \end{pmatrix},$$

where

$$\begin{aligned}\Re F_{\eta,1.2}(\omega) = {} & \Re F_{\eta,11}(\omega) + \Im F_{\eta,12}(\omega)\Im F_{\eta}^{22}(\omega)\Re F'_{\eta,12}(\omega) \\ & - \Re F_{\eta,12}(\omega)\Re F_{\eta}^{22}(\omega)\Re F'_{\eta,12}(\omega) \\ & - \Im F_{\eta,12}(\omega)\Re F_{\eta}^{22}(\omega)\Im F'_{\eta,12}(\omega) \\ & - \Re F_{\eta,12}(\omega)\Im F_{\eta}^{22}(\omega)\Im F'_{\eta,12}(\omega),\end{aligned}$$

$$\begin{aligned}\Im F_{\eta,1.2}(\omega) = {} & \Im F_{\eta,11}(\omega) + \Re F_{\eta,12}(\omega)\Re F_{\eta}^{22}(\omega)\Im F'_{\eta,12}(\omega) \\ & - \Im F_{\eta,12}(\omega)\Re F_{\eta}^{22}(\omega)\Re F'_{\eta,12}(\omega) \\ & - \Re F_{\eta,12}(\omega)\Im F_{\eta}^{22}(\omega)\Re F'_{\eta,12}(\omega) \\ & - \Im F_{\eta,12}(\omega)\Im F_{\eta}^{22}(\omega)\Im F'_{\eta,12}(\omega),\end{aligned}$$

$$\Re F_{\eta}^{22}(\omega) = (\Re F_{22}(\omega) + \Im F_{22}(\omega)(\Re F_{22}(\omega))^{-1}\Im F_{22}(\omega))^{-1},$$

and

$$\begin{aligned}\Im F_{\eta}^{22}(\omega) = {} & -(\Re F_{22}(\omega))^{-1}\Im F_{22}(\omega) \\ & \times \left(\Re F_{22}(\omega) + \Im F_{22}(\omega)(\Re F_{22}(\omega))^{-1}\Im F_{22}(\omega)\right)^{-1}.\end{aligned}$$

The above result holds also if we use consistent estimates of $F_{\eta,12}(\omega)$, $F_{\eta,22}(\omega)$, $G_{\eta,12}(\omega)$, and $G_{\eta,22}(\omega)$ whose rate of convergence is fast enough. $F_{\eta,12}(\omega)$ and $G_{\eta,12}(\omega)$ are computed with the complex $r$-dimensional covariance stationary process $(\eta_{1,t})_t$ and the real $(n-r)$-dimensional covariance stationary process $(\eta_{2,t})_t$, whereas $F_{\eta,22}(\omega)$ and $G_{\eta,22}(\omega)$ are computed with only the real $(n-r)$-dimensional covariance stationary process $(\eta_{2,t})_t$. If we want to construct an estimator that ensures that the spectral density matrix estimate is Hermitian, we have

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to work with the complex process $(\eta_t)_t$ in an approach very similar to that used when working with real processes.

We start from the estimated residuals of a first-step OLS regression $(\widehat{\eta}_{1,t})_t$ with $\widehat{\eta}_{1,t} = \delta_{-\omega} y_{1,t} - \widehat{\alpha}\delta_{-\omega} y_{2,t}$. Direct computations give $\widehat{\eta}_{1,t} = \eta_{1,t} + O_p(T^{-1/2})$. The empirical estimates $\widehat{\Gamma_\eta(j)}$ of $\Gamma_\eta(j)$ are consistent when its real and imaginary parts are consistent. According to the preceding section, these two sets of matrices can be decomposed into submatrices that are (auto)covariance matrices of three covariance stationary real processes $(\eta_{1,R,t})_t$, $(\eta_{1,I,t})_t$, and $(\eta_{2,t})_t$. Their empirical estimates are consistent when the polynomial matrix applied to $\varepsilon_t$ in defining these three processes derived from $\eta_t$ satisfies Assumption 4.4, as this assumption implies absolute convergence of the matrix coefficients. We assumed this is satisfied. We in fact look for an estimate of the complex spectral matrix at frequency $\omega$ that is Hermitian. It is natural to use a kernel estimator with a Bartlett window that has the following form (see Hannan, 1970):

$$2\pi \widehat{F}_\eta(\omega) = \sum_{j=-m}^{+m} \left(1 - \frac{|j|}{m+1}\right) e^{i\omega j} \widehat{\Gamma_\eta(j)}, \tag{16}$$

where $m = O_p(T^{1/3})$, for we have the following result that extends the Bartlett window property to produce definite positive matrix estimator to frequencies different from 0.

LEMMA 5.1. *The estimate* $2\pi \widehat{F}_\eta(\omega)$ *given by* (16) *is a Hermitian matrix.*

Working with such a fully modified estimate, it is then possible to construct Wald test statistics that are asymptotically distributed as $\chi^2$ distributions. We must nevertheless take into account that we work with complex numbers. The matrix of constrains on the real and imaginary parts of the complex estimates $\widehat{\alpha}^+$ may take a complicated form. We indeed can rewrite a set of constraints on the complex representation as a set of constraints on the real and imaginary parts of the estimates, but in this case we implicitly assume to keep the full-rank property of the matrix of constraints that we consider an even number of constraints. The proof given in the Appendix deals with this situation. Nevertheless, the test statistic can also be derived from the corresponding real expressions, and we state a result in this setup. Let $R$ be a $k \times 2r(n-r)$ matrix of full rank and $q$ a $k \times 1$ vector; we consider the following set of constraints:

$$H_0 : R \operatorname{vec} \begin{pmatrix} \Re\alpha \\ \Im\alpha \end{pmatrix} = q.$$

From the above result we conclude the following.

PROPOSITION 5.2. *Let* $\widehat{\alpha}^+$ *be the FM-OLS estimate defined in* (14) *under the null that* $H_0$ *is satisfied, and let* $V\widehat{\alpha}^+$ *be the empirical estimate of the variance-covariance of this estimate; then the following convergence holds when* $T$

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*goes to* $+\infty$

$$\left(R\,\mathrm{vec}\begin{pmatrix}\Re\widehat{\alpha}^{+}\\ \Im\widehat{\alpha}^{+}\end{pmatrix}-q\right)'\left(R'V\widehat{\alpha}^{+}R\right)^{-1}\left(R\,\mathrm{vec}\begin{pmatrix}\Re\widehat{\alpha}^{+}\\ \Im\widehat{\alpha}^{+}\end{pmatrix}-q\right)\Longrightarrow\chi_2(k),$$

*where* $V\widehat{\alpha}^{+}=V_{emp}y_2\otimes 2\pi\,\overline{F_{\eta,1.2}(\omega)}^{*}$,

$$V_{emp}y_2=\begin{pmatrix}\Re\delta_{-\omega}y_2\Re\delta_{-\omega}y_2'+\Im\delta_{-\omega}y_2\Im\delta_{-\omega}y_2' & \Im\delta_{-\omega}y_2\Re\delta_{-\omega}y_2'-\Re\delta_{-\omega}y_2\Im\delta_{-\omega}y_2'\\ -\Im\delta_{-\omega}y_2\Re\delta_{-\omega}y_2'+\Re\delta_{-\omega}y_2\Im\delta_{-\omega}y_2 & \Re\delta_{-\omega}y_2\Re\delta_{-\omega}y_2'+\Im\delta_{-\omega}y_2\Im\delta_{-\omega}y_2'\end{pmatrix},$$

*with* $\Re\delta_{-\omega}y_2=y_2-\cos\omega y_{2,-1}$, $\Im\delta_{-\omega}y_2=-\sin\omega y_{2,-1}$, *and* $2\pi\,\overline{F_{\eta,1.2}(\omega)}^{*}$ *is defined above.*

## 6. A TEST FOR SEASONAL COINTEGRATION

### 6.1. Design of the Test

In this part we introduce a test whose null hypothesis corresponds to the existence of $r$ relations of cointegration between the $n$ variables considered above with the associated set of constraints. It relies on the same idea as that in Kwiatkowski, Phillips, Schmidt, and Shin (1992) to test for the stationarity of a univariate process. It was used in the case of seasonal integration by Canova and Hansen (1995) and in the test for cointegration at frequency 0 by Shin (1994). Under the null, the error terms $\eta_{1,t}$ of equation

$$\delta_{-\omega}y_{1,t}=\alpha\delta_{-\omega}y_{2,t}+\eta_{1,t}$$

are complex covariance stationary processes. We know that if we construct from these processes the complex random walks at frequency $\omega$ : $S_\omega\eta_{1,t}$, these $I_\omega(1)$ processes are not cointegrated, otherwise this would mean that we are in case of polynomial cointegration and would be in contradiction with Assumptions 4.1 and 4.2. This implies that their spectral density matrix or equivalently their long-term variance-covariance matrix at frequency $\omega$ is a full rank matrix. Unfortunately, we cannot work with the simple OLS residuals of the above set of equations that are consistent estimators but involve nuisance parameters. We propose to use the fully modified estimates of the residuals,

$$\widehat{\eta}_{1,t}^{+}=\delta_{-\omega}y_{1,t}^{+}-\widehat{\alpha}^{+}\delta_{-\omega}y_{2,t},$$

in a test procedure whose asymptotic test statistic distribution is nuisance-parameter-free. We propose to consider the following test statistic:

$$\zeta_T=\frac{1}{2\pi T^2}\mathrm{tr}\left(\left(\widehat{F}_{\widehat{\eta},11}(\omega)-\widehat{F}_{\widehat{\eta},12}(\omega)\widehat{F}_{\widehat{\eta},22}^{-1}(\omega)\widehat{F}_{\widehat{\eta},21}(\omega)\right)^{-1}\sum_{t=1}^{T}S_\omega\widehat{\eta}_{1,t}^{+}\overline{S_\omega\widehat{\eta}_{1,t}^{+}}'\right),$$

which has the property stated in the following proposition.

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PROPOSITION 6.1. *Under Assumptions 4.1, 4.2, and 4.4, if* $\widehat{F}_{\widehat{\eta}}(\omega)$ *is a consistent estimator of the spectral matrix of the process* $\eta_t$*, then when* $T$ *tends toward* $+\infty$*, we have*

$$\zeta_T \Longrightarrow \operatorname{tr}\left(\frac{1}{2}\int_0^1 RW(\tau)\overline{RW}'(\tau)\,d\tau\right),$$

*where* $RW(\tau) = W_1(\tau) - \int_0^1 dW_1(s)\overline{W_2}'(s)\left(\int_0^1 W_2(s)\overline{W_2}'(s)\,ds\right)^{-1}\int_0^\tau W_2(s)\,ds$ *and* $W_1$ *and* $W_2$ *are two independent standard complex Brownian motions of respective dimension* $r$ *and* $n-r$*.*

The distribution of the test statistic depends on the dimensions of the cointegration space and the number of variables involved in these long-term relations. It also depends on the presence of a deterministic term in the cointegration equation. In presence of a deterministic function $e^{-i\omega t}$ in the regression, the above asymptotic distribution must be modified as follows: We have to replace the fully modified residuals by

$$\widehat{\eta}^+_{1,t} = \delta_{-\omega}y^+_{1,t} - \widehat{\alpha}^+\delta_{-\omega}y_{2,t} - \widehat{\kappa}e^{-\omega t}$$

and $W_2(\tau)$ by $\begin{pmatrix} 1 \\ W_2(\tau) \end{pmatrix}$ in the above formula in Proposition 6.1, which gives a test statistic $\zeta_T^\mu$. In practice, it is this more general situation that seems to be the most relevant.

Under the alternative that there exists at least one relationship that is not cointegrating in the set of fully modified regressions considered above, the test statistic is $O_p(\frac{T}{m})$, where $m$ is the truncation parameter introduced in the estimation of the Hermitian spectral density matrix (see equation 16). This ensures the consistency of the test procedure. Critical values for $\zeta_T$ statistics are given in Table 1 for $n = 1, \ldots, 6$ and $r = 1, \ldots, n-1$. They are computed by simulation (50,000 simulations of multivariate seasonal random walk). Various sample sizes have been considered (from $T = 100$ to $T = 2,500$) but this does not significantly affect the distribution.

## 6.2. Small Sample Size and Power Properties

We proceed in this section to a simulation study of the properties of the test procedures introduced just above. We restrict our study to the situation of one cointegration relationship between two univariate processes. We particularly want to investigate how well the asymptotic theory describes the small sample properties of our test procedure.

We proceed as follows: We consider the following DGP, which extends the bivariate process considered in Example 4.1:

$$\begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & \xi L \\ -\xi L & 1 \end{pmatrix}\begin{pmatrix} v_{R,t} \\ v_{I,t} \end{pmatrix}, \tag{17}$$

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**TABLE 1.** Critical values of $\zeta_T^{\mu}$

| $n$ | $r$ | 0.50 | 0.80 | 0.90 | 0.95 | 0.972 | 0.999 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 0.091 | 0.148 | 0.193 | 0.240 | 0.293 | 0.364 |
| 3 | 1 | 0.067 | 0.104 | 0.133 | 0.164 | 0.197 | 0.245 |
| 3 | 2 | 0.196 | 0.285 | 0.351 | 0.417 | 0.481 | 0.573 |
| 4 | 1 | 0.053 | 0.079 | 0.098 | 0.119 | 0.142 | 0.174 |
| 4 | 2 | 0.142 | 0.199 | 0.241 | 0.287 | 0.332 | 0.394 |
| 4 | 3 | 0.301 | 0.416 | 0.500 | 0.577 | 0.656 | 0.757 |
| 5 | 1 | 0.044 | 0.063 | 0.077 | 0.091 | 0.107 | 0.132 |
| 5 | 2 | 0.111 | 0.150 | 0.179 | 0.209 | 0.240 | 0.285 |
| 5 | 3 | 0.217 | 0.291 | 0.346 | 0.401 | 0.458 | 0.541 |
| 5 | 4 | 0.404 | 0.546 | 0.644 | 0.739 | 0.827 | 0.948 |
| 6 | 1 | 0.037 | 0.052 | 0.063 | 0.073 | 0.085 | 0.101 |
| 6 | 2 | 0.091 | 0.120 | 0.140 | 0.161 | 0.182 | 0.212 |
| 6 | 3 | 0.170 | 0.220 | 0.256 | 0.293 | 0.332 | 0.385 |
| 6 | 4 | 0.294 | 0.382 | 0.446 | 0.511 | 0.579 | 0.664 |
| 6 | 5 | 0.512 | 0.677 | 0.786 | 0.894 | 0.995 | 1.113 |

such that when $|\xi| < 1$, the bivariate process is covariance stationary without any unit root in the determinant of the polynomial matrix associated to its Wold representation, but when $\xi = 1$, $i$ and $-i$ are the roots of the determinant and there exist two conjugate cointegration relationships between the $I_{\frac{\pi}{2}}(1)$ complex processes $S_{\frac{\pi}{2}}\eta_{1,t}$ and $S_{\frac{\pi}{2}}\eta_{2,t}$ on the one hand and the conjugate processes on the other hand. This gives

$$S_{\frac{\pi}{2}}\eta_{1,t} = \alpha S_{\frac{\pi}{2}}\eta_{2,t} + \kappa e^{-i\frac{\pi}{2}t} + \upsilon_{R,t} - i\upsilon_{I,t} \tag{18}$$

and its conjugate, where $\alpha = i$ (in real terms $(0\ 1)$).

The two $I_{\omega}(1)$ cointegrated processes in the above regression equation (18) are generated from (17) with various data generating processes of $(\upsilon_{R,t})_t$ and $(\upsilon_{I,t})_t$ that have been considered elsewhere in the literature (Phillips and Perron, 1988; Schwert, 1989; DeJong, Nankervis, Savin, and Whiteman, 1992; Elliott, Rothenberg, and Stock 1996, among others), namely:

1. MA(1): $\upsilon_{R \text{ or } I,t} = \eta_t - \theta\eta_{t-1} \qquad (\theta = .8, .5, 0, -.5, -.8)$,
2. AR(1): $\upsilon_{R \text{ or } I,t} = \phi\upsilon_{R \text{ or } I,t-1} + \eta_t \qquad (\phi = .5, -.5)$,
3. GARCH-MA(1):
$$\begin{cases} \upsilon_{R \text{ or } I,t} = \zeta_t - \theta\zeta_{t-1}, & (\theta = .5, 0, -.5) \\ \zeta_t = h_t^{1/2}\eta_t & \\ h_t = 1 + .65h_{t-1} + .25\zeta_{t-1}^2, & h_0 = 0 \end{cases}.$$

For each type of model (the same one for the real and imaginary parts), we generate a large number (10,000) of processes $(\eta_{1,t}\ \eta_{2,t})'$, construct the

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complex processes $(S_\omega \eta_{1,t}, S_\omega \eta_{2,t})_{t=1,\ldots,T}$, and run the OLS and FM-OLS of equation (18).

In each simulation of these models, the initial conditions are set to 0. The autocorrelation structure of bivariate process is assumed to be unknown to the econometrician, so we have to estimate the nuisance parameters it involves.

The estimation of $F_\eta(\omega)$ and $G_\eta(\omega)$ relies on the properties of kernel estimator of spectral density. These nonparametric estimators entail the choice of a "bandwidth" number ($m$) whose influence on the quality of the estimate can be drastic. We detail now the way we propose to proceed in the choice of this parameter. At frequency 0, Newey and West (1994) proposed optimal (in a certain sense) choice procedure of this parameter. They select parameters that minimize the asymptotic mean square errors of the estimate of the spectral density of a linear combination of the components of the vector under study. This results from the fact that there is a trade-off between the effect of the bandwidth parameter on the variance of the estimate and its bias. This approach was suggested by Hannan (1970), who derived the formula of the optimal bandwidth number. We implement in this Monte Carlo study a rule similar to the procedure proposed by Newey and West (1994) but at the frequency under study. A proof is provided in Gregoir (2006) in the univariate case, but extension to a multivariate setup is straightforward, as it relies on the estimate of the spectral matrix of a linear combination of the components. Let $l$ denote the complex vector whose components are the weights associated to each component of the process under study. Here is the adapted procedure when using a Bartlett window, the truncation parameter $\widehat{m} = [\widehat{g}_B T^{1/3}]$ with

$$n = \left[4\left(\frac{T}{100}\right)^{2/9}\right],$$

$$\widehat{g}_B = 1.1447\left(\frac{s_1^2}{s_0^2}\right)^{1/3},$$

where

$$s_0 = \sum_{j=-n}^{n} e^{i\omega j} l' \hat{\Gamma}(j) \bar{l},$$

$$s_1 = \sum_{j=-n}^{n} j e^{i\omega j} l' \hat{\Gamma}(j) \bar{l},$$

and

$$\hat{\Gamma}(j) = \frac{1}{T}\sum_{t=1}^{T-j} \eta_t \overline{\eta}'_{t+j}.$$

In the simulations below, the complex vector $l$ is such that the weights of the real and imaginary parts for the complex components of the spectral matrix are equal.

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We also want to study the power properties of the test procedure. In similar circumstances, some authors focus on the case of near-integrated regressors in the cointegration equation (see Elliott, 1998). As shown in Gregoir (2006), this leads to asymptotic distribution that can be described with complex Orstein-Uhlenbeck processes. We do not follow this way. We consider processes $S_{\frac{\pi}{2}}\eta_{1,t}$ and $S_{\frac{\pi}{2}}\eta_{2,t}$ constructed from $\eta_{1,t}$ and $\eta_{2,t}$ generated by (17) when $\xi \neq 1$. In this case, these series are not cointegrated. We let the parameter $\xi$ vary between 0.80 and 0.99.

We compute the empirical size and the size-adjusted power of the cointegration test. In the computation of the nuisance parameters, mainly the spectral density in $\omega$, we use the above selection procedure but also consider cases in which $\widehat{m}$ is a priori fixed to the value 5. The results are given in Table 2 and Figures 3–5. This brings about some comments.

We first comment on the empirical distribution of the OLS and FM-OLS estimates of $\alpha = i$. We observe in Figure 1 that the distributions of the real part of the OLS and FM-OLS estimators are quite close (in Figures 1 and 2, "auto" means that the truncation parameter is estimated from the data with the above selection procedure). This is consistent with the fact that the DGP (17) is such that there is no bias for this coefficient, as the corresponding quantity $\Re(\sum_{j=0}^{+\infty} e^{ij\omega}\overline{\Gamma}'_{21})$ is equal to 0. This is not the case for the imaginary part. The distribution

**TABLE 2.** Empirical sizes for 5% level test

| DGP | Truncation | $T = 100$ | $T = 200$ | $T = 400$ |
|---|---|---|---|---|
| MA(1):$\theta = 0.8$ | automatic | 0.0404 | 0.0450 | 0.0498 |
| | 5 | 0.0329 | 0.0439 | 0.0533 |
| MA(1):$\theta = 0.5$ | automatic | 0.0448 | 0.0471 | 0.0511 |
| | 5 | 0.0322 | 0.0423 | 0.0531 |
| MA(1):$\theta = 0$ | automatic | 0.0702 | 0.0704 | 0.0687 |
| | 5 | 0.0320 | 0.0418 | 0.0528 |
| MA(1):$\theta = -0.5$ | automatic | 0.0728 | 0.0692 | 0.0729 |
| | 5 | 0.0321 | 0.0419 | 0.0537 |
| MA(1):$\theta = -0.8$ | automatic | 0.0638 | 0.0603 | 0.069 |
| | 5 | 0.0328 | 0.0418 | 0.0538 |
| AR(1):$\phi = 0.5$ | automatic | 0.0293 | 0.0336 | 0.0331 |
| | 5 | 0.0154 | 0.0218 | 0.0275 |
| AR(1):$\phi = -0.5$ | automatic | 0.0218 | 0.0272 | 0.0273 |
| | 5 | 0.0158 | 0.0224 | 0.0284 |
| GARCH-MA(1):$\theta = 0.5$ | automatic | 0.0494 | 0.0497 | 0.0502 |
| | 5 | 0.0334 | 0.0429 | 0.0483 |
| GARCH-MA(1):$\theta = 0$ | automatic | 0.0706 | 0.0744 | 0.0637 |
| | 5 | 0.0334 | 0.0417 | 0.0478 |
| GARCH-MA(1):$\theta = -0.5$ | automatic | 0.072 | 0.0724 | 0.0683 |
| | 5 | 0.0319 | 0.0417 | 0.0491 |

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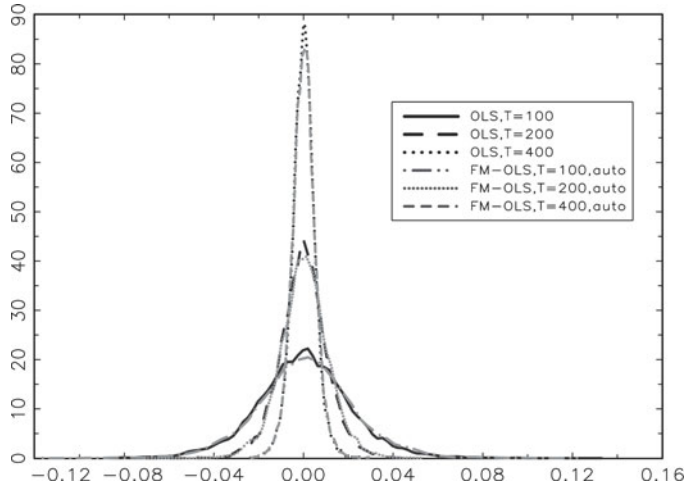



**FIGURE 1.** Empirical distributions of $\Re\hat{\alpha}_{OLS}$ and $\Re\hat{\alpha}_{FM-OLS}$, $\alpha = 0 + 1.i$, $T = 100, 200, 400$.

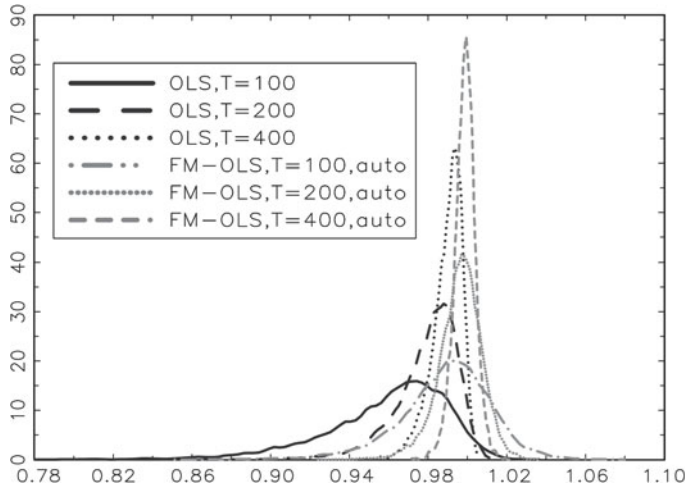



**FIGURE 2.** Empirical distributions of $\Im\hat{\alpha}_{OLS}$ and $\Im\hat{\alpha}_{FM-OLS}$, $\alpha = 0 + 1.i$, $T = 100, 200, 400$.

of the corresponding OLS estimator is shifted to the left with respect to the true value (1), whereas the FM-OLS one is correctly centered on this true value.

We turn to the empirical sizes we get in our experiment. Empirical sizes are close to the true level of the test except when the DGP of the error terms are AR(1)-type models. In this case the sizes are smaller than the nominal level. The use of an automatic selection procedure of the truncation parameter does not improve the test size properties; the empirical sizes are not necessarily closer

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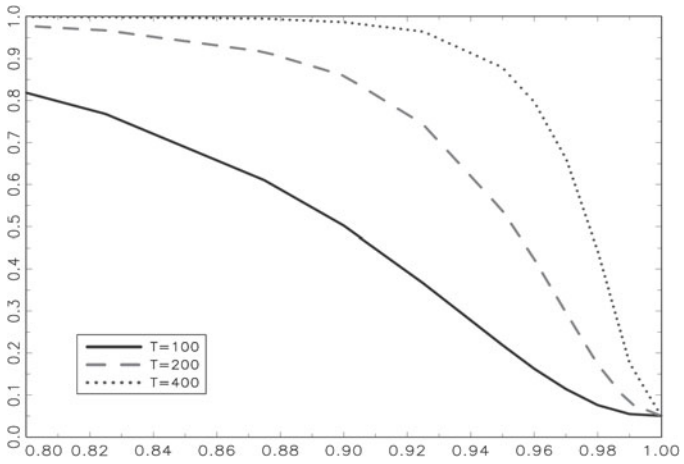



**FIGURE 3.** Power functions for various sample sizes, $\xi \in [0.8, 1]$, DGP:MA(1)-($\theta = 0$).

to the nominal level. They are larger when the DGP involves an MA model with a negative parameter value, whereas the empirical sizes obtained with a fixed truncation parameter are smaller than but closer to the nominal value.

We now turn to the size-adjusted power. We summarize the results with several plots. Figure 3 shows that power properties improve with the size of the sample; this is observed in all the situations. We can nevertheless notice that the power properties are not good for sample sizes of about 100 observations. Regarding the various DGPs of the error terms, these simulation exercises do not bring to the fore contrasted properties. Power properties seem better for MA-type DGPs when parameter values are negative and GARCH features absent.

## 7. MULTIPLE SEASONAL UNIT ROOT CASE

We at last address the general case in which several seasonal cointegration relationships at various frequencies exist simultaneously. We know that there exists a kind of asymptotic orthogonality between processes that are integrated at different frequencies (Chan and Wei, 1988). We propose to use this property. Two points of view can be adopted. On the one hand, we may be interested in estimating and testing for the existence of cointegration relationships between processes that appear to be integrated at the same frequency in the initial set of data under study. This is the natural approach. On the other hand, as cointegration corresponds from an algebraic point of view to the existence of a unit root in the determinant of the polynomial matrix associated to the Wold representation of the "stationarized" processes, we may be interested in estimating and testing for the existence of cointegration relationships between all the integrated processes at a given frequency built upon the covariance stationary variables. This approach

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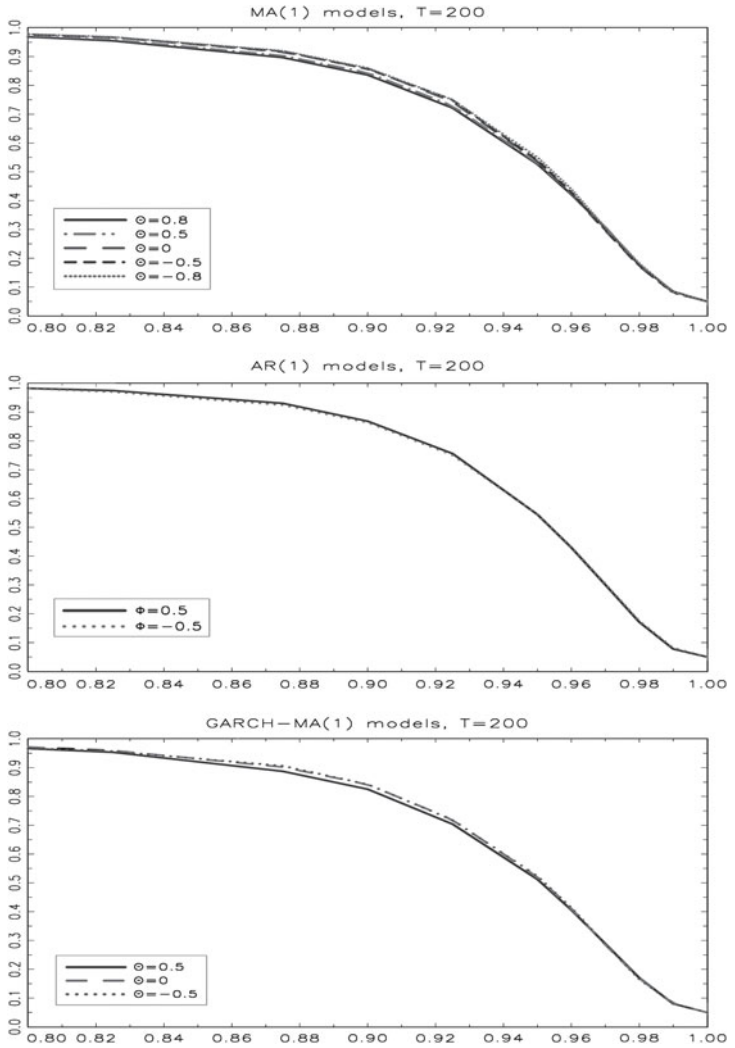



**FIGURE 4.** Power functions for various DGPs, $T = 200$, $\xi \in [0.8, 1]$.

was used by Gregoir and Laroque (1994) for testing for the presence of polynomial cointegration. In both approaches, it is important to stress that as long as we disregard polynomial cointegration, the above triangular representation always exists at each eligible frequency for the exactly integrated of order one processes that are cointegrated. More precisely, let $x_t$ be the vector composed of the covariance stationary processes derived from a set of integrated processes $y_t$. The above reasoning can be applied without modification on each set of exactly integrated of order one processes at each different frequency $(S_\omega x_t)$. These processes can

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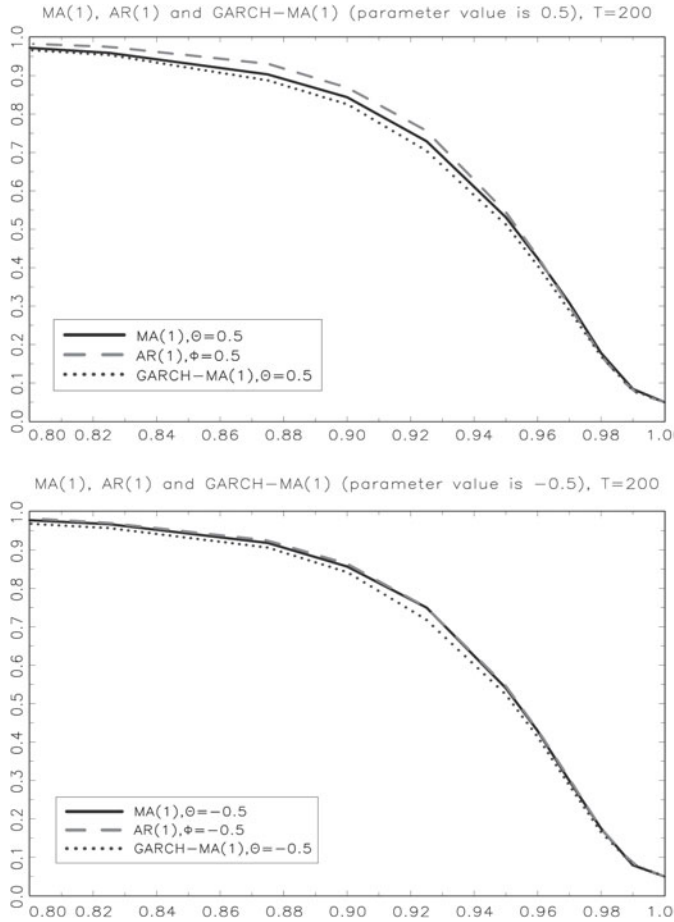



**FIGURE 5.** Power functions for various DGPs, $T = 200$, $\xi \in [0.8, 1]$.

be constructed by applying the integral operator at frequency $\omega$ on the processes $x_t$ or by applying the appropriate difference operators at all the frequencies but $\omega$ on the available processes $y_t$. For instance, when working with a set of processes that are integrated of order 1 at frequencies 0, $\pi$, and $\frac{\pi}{2}$, such that $x_t(1-L^4)y_t$ is $I(0)$, we can separately study the cointegration properties of $(1+L+L^2+L^3)y_t$ that is $I_0(1)$, $(1-L+L^2-L^3)y_t$ that is $I_\pi(1)$, and $(1+iL-L^2-iL^3)y_t$ that is $I_{-\pi/2}(1)$ (or $(1-iL-L^2+iL^3)y_t$ that is $I_{\pi/2}(1)$ and its conjugate).

From this discussion, we may conclude that it is important to correctly define the set of simultaneously covariance stationary processes, in other words, the appropriate order of integration of each time series at each eligible frequency. A powerful test must be looked for (see Gregoir, 2006).

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## 8. CONCLUSION

In this paper we develop a simple approach that extends the FM-OLS approach proposed by Phillips and Hansen (1990) and the cointegration test procedure proposed by Shin (1994) and Harris (1997) to the case of seasonal cointegration. This is done in a framework that explicitly uses complex numbers and in doing so, makes the derivations and the formulas easy to read and formally close to those obtained in the case of cointegration at frequency 0. Implementing these formulas is easy with software with built-in complex number computations.

### NOTES

1. Gauss (Aptec System Inc.) programs related to the procedure introduced in this paper can be provided by the author on request.

2. Influence of the initial conditions has been considered by Elliott (1999) and Müller and Elliott (2003). Similar considerations apply in this framework.

3. A Hermitian matrix (or self-adjoint matrix) is equal to its own conjugate transpose. When deterministic relationship exists between the components of a multivariate process, the spectral density matrix is Hermitian positive definite.

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# APPENDIX: Proofs

**Notations.** When a process is overdifferenced of order one at frequency $\omega$, we say that we deal with an $I_\omega(-1)$ process. More precisely, we have the following.

DEFINITION A. *A univariate process $x_t$, $t \in \mathbb{Z}$, is said to be integrated of order –1 at frequency $\omega \in ]0, \pi[$, if it is such that*

$$x_t = (1-e^{-i\omega}L)(1-e^{i\omega}L)\eta_t = (1-2\cos\omega L + L^2)\eta_t,$$

*where $\eta_t, t \in \mathbb{Z}$, is a covariance stationary process such that its spectral density $f_\eta(\varpi)$, $\varpi \in [0, \pi]$, is strictly positive at frequency $\omega$.*

In the sequel, we denote $\Delta_\omega$ the difference operator at frequency $\omega$ equal to $(1 - 2\cos\omega L + L^2)$. When we only apply the complex operator $\delta_\omega = (1 - e^{-i\omega}L)$ to a real $I_\omega(1)$ process, we get a complex process that is still integrated at frequency $-\omega$.

Some algebraic properties of the integral operator $S_\omega$ are summed up in the following statement.

COROLLARY A. *If $Id$ denotes the identity operator,*

$$(i)\ \delta_\omega S_\omega = Id, \tag{A.1}$$

$$(ii)\ \forall (y_t)_t, S_\omega \delta_\omega y_t = y_t - y_0 e^{-i\omega t}. \tag{A.2}$$

We stress that the operators $\delta_\omega$ and $S_\omega$ do not commute. The constant term that appears in (A.2) is due to our definition of the integral operator that constructs integrated of order 1 process that are always equal to 0 at $t = 0$. Alternative definitions with different conventions are possible.

The proofs follow.

**Proof of Theorem 2.1.** Let $\mathcal{F}_t = \sigma\{\varepsilon_s, s \le t\}$ be the $\sigma$-fields generated by $\{\varepsilon_s, s \le t\}$; we introduce two processes,

$$\xi_t = \sum_{j=-\infty}^{+\infty} \left[ \mathrm{E}(e^{i\omega j} u_j \,|\mathcal{F}_t) - \mathrm{E}(e^{i\omega j} u_j \,|\mathcal{F}_{t-1}) \right] \tag{A.3}$$

$$= \left( \sum_{k=-\infty}^{+\infty} e^{i\omega k} C_k \right) e^{i\omega t} \varepsilon_t \tag{A.4}$$

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and

$$\zeta_t = \sum_{j=t}^{+\infty} \mathrm{E}\left(e^{i\omega j} u_j \,|\mathcal{F}_{t-1}\right) - \sum_{j=-\infty}^{t-1} \left[e^{i\omega j} u_j - \mathrm{E}\left(e^{i\omega j} u_j \,|\mathcal{F}_{t-1}\right)\right], \tag{A.5}$$

and (3) implies that

$$\sum_{k=-\infty}^{+\infty} \left\| e^{i\omega k} C_k \right\| = \sum_{k=-\infty}^{+\infty} \|C_k\| < +\infty.$$

From (A.4) we observe that $\{\xi_t, \mathcal{F}_t\}$ is a martingale difference sequence whose properties are derived from those of $\{e^{i\omega t}\varepsilon_t\}$, which are similar to those $\{\varepsilon_t\}$. Its differences are stationary, square integrable, and satisfy SLLN, CLT, and FCLT with $\tau \in [0, 1]$,

$$\frac{1}{\sqrt{T}} \sum_{j=1}^{[T\tau]} \xi_j \Longrightarrow \frac{1}{\sqrt{2}} \left( \sum_{k=-\infty}^{+\infty} e^{i\omega k} C_k \right) W(\tau).$$

The covariance matrix of $\xi_t$ is

$$V\xi_t = \left( \sum_{k=-\infty}^{+\infty} e^{i\omega k} C_k \right) \Omega \left( \sum_{k=-\infty}^{+\infty} e^{-i\omega k} C_k' \right).$$

$$= 2\pi f_u(\omega).$$

The same kinds of properties are satisfied by $\{\zeta_t\}$: It is stationary, square integrable, and satisfies SLLN and CLT.

With this construction, we have

$$e^{i\omega t} u_t = \xi_t - (\zeta_{t+1} - \zeta_t).$$

Note that

$$y_t = e^{-i\omega} y_{t-1} + u_t$$

so that

$$e^{i\omega t} y_t = e^{i\omega(t-1)} y_{t-1} + e^{i\omega t} u_t,$$

$$y_t = e^{-i\omega t} \left( \sum_{j=1}^{t} e^{i\omega j} u_j + y_0 \right),$$

or

$$y_t = S_\omega u_t + y_0 e^{-i\omega t},$$

whence

$$\frac{1}{T} \sum_{t=1}^{T} y_{t-1} \overline{u}_t' = \frac{1}{T} e^{i\omega} \sum_{t=1}^{T} (e^{i\omega(t-1)} y_{t-1}) \left(\overline{e^{i\omega t} u_t}\right)'$$

$$= \frac{1}{T} e^{i\omega} \sum_{t=1}^{T} \left( \sum_{j=1}^{t-1} \xi_j - \zeta_t + \zeta_1 + y_0 \right) \left( \overline{\xi}_t' - (\overline{\zeta}_{t+1}' - \overline{\zeta}_t') \right)$$

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which we decompose into five quantities. First, by construction of martingale difference $\{\xi_t, \mathcal{F}_t\}$, we have

$$\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{t-1}\xi_j\right)\overline{\xi}'_t \Longrightarrow \frac{1}{2}\int_0^1 B d\overline{B}',$$

where $B$ is a complex Brownian motion whose variance-covariance matrix is $2\pi f_u(\omega)$ (Tsay and Tiao, 1990; Gregoir, 1999b). Second,

$$-\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{t-1}\xi_j\right)\left(\overline{\zeta}'_{t+1}-\overline{\zeta}'_t\right) = -\frac{1}{T}\sum_{j=1}^{T-1}\xi_j\left(\overline{\zeta}'_{T+1}-\overline{\zeta}'_{j+1}\right)$$

that converges by the SLLN to $-\mathrm{E}\xi_t\overline{\zeta}'_{t+1}$. Third,

$$\frac{1}{T}\sum_{t=1}^{T}(-\zeta_t+\zeta_1)\overline{\xi}'_t = -\frac{1}{T}\sum_{t=1}^{T}\zeta_t\overline{\xi}'_t + \zeta_1\left(\frac{1}{T}\sum_{t=1}^{T}\overline{\xi}'_t\right)$$

that converges to $-\mathrm{E}\zeta_t\overline{\xi}'_t$. Fourth,

$$\frac{1}{T}\sum_{t=1}^{T}(\zeta_t-\zeta_1)\left(\overline{\zeta}'_{t+1}-\overline{\zeta}'_t\right) = \frac{1}{T}\sum_{t=1}^{T}\zeta_t\left(\overline{\zeta}'_{t+1}-\overline{\zeta}'_t\right) - \zeta_1\left(\frac{1}{T}\sum_{t=1}^{T}(\overline{\zeta}'_{t+1}-\overline{\zeta}'_t)\right)$$

that converges to $\mathrm{E}\zeta_t\left(\overline{\zeta}'_{t+1}-\overline{\zeta}'_t\right)$. At last, under our assumption about $y_0$,

$$y_0\frac{1}{T}\sum_{t=1}^{T}\left(\overline{\xi}'_t-\left(\overline{\zeta}'_{t+1}-\overline{\zeta}'_t\right)\right)$$

converges to 0. We therefore get

$$\frac{1}{T}\sum_{t=1}^{T}y_{t-1}\overline{u}'_t \Longrightarrow e^{i\omega}\left(\frac{1}{2}\int_0^1 B d\overline{B}' + \mathrm{E}\xi_t\overline{\zeta}'_{t+1} - \mathrm{E}\zeta_t\left(e^{-i\omega t}\overline{u}'_t\right)\right).$$

One the one hand,

$$\mathrm{E}\zeta_t\left(e^{-i\omega t}\overline{u}'_t\right) = \mathrm{E}\left[\sum_{j=-\infty}^{+\infty}\mathrm{E}\left(e^{i\omega j}u_j\,|\mathcal{F}_{t-1}\right)e^{-i\omega t}\overline{u}'_t\right]$$

$$-\mathrm{E}\left[\left(\sum_{j=-\infty}^{t-1}e^{i\omega j}u_j\right)e^{-i\omega t}\overline{u}'_t\right],$$

where the second term is equal to $-\sum_{j=1}^{+\infty}e^{-i\omega j}\mathrm{E}u_t\overline{u}'_{t+j}$. On the other hand,

$$\mathrm{E}\xi_t\overline{\zeta}'_{t+1} = \mathrm{E}\left[\sum_{j=-\infty}^{+\infty}\xi_t\mathrm{E}\left(e^{-i\omega j}\overline{u}'_j\,|\mathcal{F}_t\right)\right] - \mathrm{E}\sum_{j=-\infty}^{t}\xi_t e^{-i\omega j}\overline{u}'_j$$

$$= \mathrm{E}\sum_{j=t+1}^{+\infty}\xi_t e^{-i\omega j}\overline{u}'_j,$$

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but as $\xi_t = \sum_{j=-\infty}^{+\infty}[\mathrm{E}(e^{i\omega j}u_j|\mathcal{F}_t) - \mathrm{E}(e^{i\omega j}u_j|\mathcal{F}_{t-1})]$, we can rewrite this last equation as follows :

$$\begin{aligned}
\mathrm{E}\xi_t\overline{\zeta}'_{t+1} &= \mathrm{E}\sum_{j=1}^{+\infty}\xi_{t-j}e^{-i\omega t}\overline{u}'_t \\
&= \sum_{j=1}^{+\infty}\mathrm{E}\left[\sum_{k=-\infty}^{+\infty}\mathrm{E}\left(e^{i\omega k}u_k|\mathcal{F}_{t-j}\right)e^{-i\omega t}\overline{u}'_t\right. \\
&\qquad \left. - \sum_{k=-\infty}^{+\infty}\mathrm{E}\left(e^{i\omega k}u_k|\mathcal{F}_{t-j-1}\right)e^{-i\omega t}\overline{u}'_t\right] \\
&= \sum_{k=-\infty}^{+\infty}\left[\sum_{j=1}^{+\infty}\mathrm{E}\left(\mathrm{E}\left(e^{i\omega k}u_k\,|\mathcal{F}_{t-j}\right)e^{-i\omega t}\overline{u}'_t\right)\right. \\
&\qquad \left. - \sum_{j=1}^{+\infty}\mathrm{E}\left(\mathrm{E}\left(e^{i\omega k}u_k\,|\mathcal{F}_{t-j-1}\right)e^{-i\omega t}\overline{u}'_t\right)\right] \\
&= \sum_{j=-\infty}^{+\infty}\mathrm{E}\left(\mathrm{E}\left(e^{i\omega j}u_j\,|\mathcal{F}_{t-1}\right)e^{-i\omega t}\overline{u}'_t\right),
\end{aligned}$$

since $\mathrm{E}(e^{i\omega t}u_t|\mathcal{F}_{-\infty}) = 0$ almost surely, and we conclude that $\mathrm{E}\xi_t\overline{\zeta}'_{t+1} - \mathrm{E}\zeta_t(e^{-i\omega t}\overline{u}'_t) = \sum_{j=1}^{+\infty}e^{-i\omega j}\mathrm{E}u_t\overline{u}'_{t+j}$. ■

**Proof of Lemma 4.1.** We start from the complex multivariate equation

$$u_t = \alpha v_t + w_t, \tag{A.6}$$

and we rewrite with the real quantities (real parts are indexed by $R$ and imaginary ones by I)

$$\begin{pmatrix} u_{R,t} \\ u_{I,t} \end{pmatrix} = \begin{pmatrix} \alpha_R v_{R,t} - \alpha_I v_{I,t} \\ \alpha_R v_{I,t} + \alpha_I v_{R,t} \end{pmatrix} + \begin{pmatrix} w_{R,t} \\ w_{I,t} \end{pmatrix} \tag{A.7}$$

or

$$\begin{pmatrix} u_{R,t} \\ u_{I,t} \end{pmatrix} = \begin{pmatrix} v'_{R,t}\otimes I_n & -v'_{I,t}\otimes I_n \\ v'_{I,t}\otimes I_n & v'_{R,t}\otimes I_n \end{pmatrix}\begin{pmatrix} \mathrm{vec}\,\alpha_R \\ \mathrm{vec}\,\alpha_I \end{pmatrix} + \begin{pmatrix} w_{R,t} \\ w_{I,t} \end{pmatrix}, \tag{A.8}$$

$$\begin{pmatrix} u_{R,t} \\ u_{I,t} \end{pmatrix} = \left(\left(I_2\otimes v'_{R,t} + J_2\otimes v'_{I,t}\right)\otimes I_n\right)\begin{pmatrix} \mathrm{vec}\,\alpha_R \\ \mathrm{vec}\,\alpha_I \end{pmatrix} + \begin{pmatrix} w_{R,t} \\ w_{I,t} \end{pmatrix} \tag{A.9}$$

where $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and such that $J_2^2 = -I_2$ but $J_2J_2' = I_2$, so that piling up first all the real components and second the imaginary ones into a $2nT \times 1$ vector, we get

$$\begin{pmatrix} u_R \\ u_I \end{pmatrix} = \left(\left(I_2\otimes v'_R + J_2\otimes v'_I\right)\otimes I_n\right)\begin{pmatrix} \mathrm{vec}\,\alpha_R \\ \mathrm{vec}\,\alpha_I \end{pmatrix} + \begin{pmatrix} w_R \\ w_I \end{pmatrix},$$

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with

$$V\begin{pmatrix} w_R \\ w_I \end{pmatrix} = \begin{pmatrix} I_T \otimes \Sigma_R & I_T \otimes \Sigma_{RI} \\ I_T \otimes \Sigma_{RI} & I_T \otimes \Sigma_I \end{pmatrix}.$$

The OLS estimates of $\begin{pmatrix} \operatorname{vec} \alpha_R \\ \operatorname{vec} \alpha_I \end{pmatrix}$ are given by

$$\begin{pmatrix} \operatorname{vec} \widehat{\alpha_R} \\ \operatorname{vec} \widehat{\alpha_I} \end{pmatrix} = A^{-1} B,$$

where

$$\left(\left(I_2 \otimes v_R' + J_2 \otimes v_I'\right) \otimes I_n\right)' \left(\left(I_2 \otimes v_R' + J_2 \otimes v_I'\right) \otimes I_n\right) = A,$$

$$\left(\left(I_2 \otimes v_R' + J_2 \otimes v_I'\right) \otimes I_n\right)' \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes u_R + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes u_I\right) = B,$$

that are such that

$$\left(I_2 \otimes \left(v_R v_R' + v_I v_I'\right) + J_2 \otimes \left(v_R v_I' - v_I v_R'\right)\right) \otimes I_n = A,$$

$$\begin{pmatrix} \operatorname{vec}\left(u_R v_R' - u_I v_I'\right) \\ \operatorname{vec}\left(u_R v_I' + u_I v_R'\right) \end{pmatrix} = B.$$

If we now compute

$$\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \otimes I_{nm} \begin{pmatrix} \operatorname{vec} \widehat{\alpha_R} \\ \operatorname{vec} \widehat{\alpha_I} \end{pmatrix} = \begin{pmatrix} \operatorname{vec} \widehat{\alpha} \\ \operatorname{vec} \widehat{\overline{\alpha}} \end{pmatrix},$$

we get

$$\begin{pmatrix} \operatorname{vec} \widehat{\alpha} \\ \operatorname{vec} \widehat{\overline{\alpha}} \end{pmatrix} = (C \otimes I_n) B,$$

where

$$\left(\left(I_2 \otimes \left(v_R v_R' + v_I v_I'\right) + J_2 \otimes \left(v_R v_I' - v_I v_R'\right)\right) \left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix} \otimes I_m\right)\right)^{-1} = C,$$

$$\begin{pmatrix} v\overline{v}' & 0 \\ 0 & \overline{v} v' \end{pmatrix}^{-1} \left(\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \otimes I_m\right) = C,$$

whence

$$\operatorname{vec} \widehat{\alpha} = \operatorname{vec}\left(u\overline{v}' \left(v\overline{v}'\right)^{-1}\right). \qquad \blacksquare$$

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**Proof of Proposition 4.1.** Under the null, we have

$$\widehat{\alpha}-\alpha=\left(\sum_{t=1}^{T}\eta_{1,t}\overline{\delta_{-\omega}y_{2,t}}'\right)\left(\sum_{t=1}^{T}\delta_{-\omega}y_{2,t}\overline{\delta_{-\omega}y_{2,t}}'\right)^{-1}, \tag{A.10}$$

as by construction $\delta_{-\omega}y_{2,t}=S_\omega\eta_{2,t}$, we have to study the joint weak convergence of $\sum_{t=1}^{T}\eta_{1,t}\overline{S_\omega\eta_{2,t}}'$ and $\sum_{t=1}^{T}S_\omega\eta_{2,t}\overline{S_\omega\eta_{2,t}}'$, we rely on results in Appendix A in Gregoir (1999b) and Theorem 2.1. We start from the representation of $\eta_t$ as a polynomial matrix in the lag operator applied to $\varepsilon_t$ :

$$\begin{pmatrix}\eta_{1t}\\ \eta_{2t}\end{pmatrix}=D(L)\varepsilon_t, \tag{A.11}$$

where $D(L)$ is a complex polynomial matrix whose determinant has roots of modulus strictly larger than 1 (it is a linear combination of the polynomial matrix of the Wold representation of $y_t$). Using Lemma A.1 in Gregoir (1999b) and the framework developed in Gregoir (2006), we get that

$$\frac{1}{T^2}\sum_{t=1}^{T}S_\omega\eta_t\overline{S_\omega\eta_t}'=\frac{1}{T^2}D(e^{i\omega})\Omega^{1/2}\sum_{t=1}^{T}\Omega^{-1/2}S_\omega\varepsilon_t\overline{S_\omega\varepsilon_t}'\Omega^{-1/2}\Omega^{1/2}\overline{D(e^{i\omega})}'+o_p(1), \tag{A.12}$$

and, as under Assumption 2.1, $\frac{1}{\sqrt{T}}\Omega^{-1/2}S_\omega\varepsilon_{[Tt]}$ satisfies a FCLT, by the CMT, we get

$$\frac{1}{T^2}\sum_{t=1}^{T}S_\omega\eta_t\overline{S_\omega\eta_t}'\Longrightarrow\frac{1}{2}\int_0^1 B\overline{B}'d\tau, \tag{A.13}$$

where $B$ is an $n$-dimensional Brownian whose variance-covariance matrix is $D(e^{i\omega})\Omega\overline{D(e^{i\omega})}'=2\pi F_\eta(\omega)$. The submatrix $\frac{1}{T^2}\sum_{t=1}^{T}S_\omega\eta_{2,t}\overline{S_\omega\eta_{2,t}}'$ therefore weakly converges toward the corresponding submatrix $\frac{1}{2}\int_0^1 B_2\overline{B}_2'd\tau$. We now turn to the second term. We have

$$\sum_{t=1}^{T}\eta_t\overline{S_\omega\eta_t}'=\sum_{t=1}^{T}\eta_t\left(\overline{\eta}_t'+e^{i\omega}\overline{S_\omega\eta_{t-1}}'\right). \tag{A.14}$$

From Theorem 2.1, using conjugate quantities, we get

$$\frac{e^{i\omega}}{T}\sum_{t=1}^{T}\eta_t\overline{S_\omega\eta_{t-1}}'\Longrightarrow\left(\frac{1}{2}\int dB\overline{B}'+\sum_{j=1}^{+\infty}e^{ij\omega}\mathrm{E}\eta_t\overline{\eta}_{t+j}'\right),$$

to which we add $\mathrm{E}(\eta_t\overline{\eta}_t')$, whence the result in selecting the appropriate submatrix. ■

**Proof of Corollary 4.1.** This proof relies on standard arguments and is left to the reader.

**Proof of Proposition 5.1.** This is an extension to the proof of Proposition 4.1. We have

$$\widehat{\alpha}^+=\alpha+\left(\left(\eta_1-F_{12}(\omega)F_{22}(\omega)^{-1}\eta_2\right)\overline{S_\omega\eta_2}'-TG_{\eta,12}^+\right)\left(S_\omega\eta_2\overline{S_\omega\eta_2}'\right)^{-1}, \tag{A.15}$$

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where $\eta_1$, $\eta_2$, and $S_\omega\eta_2$ stand for, respectively, the $(n \times T)$, $(m \times T)$, and $(m \times T)$ matrix whose $t^{\text{th}}$ column is equal to $\eta_{1,t}$, $\eta_{2,t}$, and $S_\omega\eta_{2,t}$. The main convergence results in the preceding proposition are still valid, and if we define $B_{1.2}(\tau) = B_1(\tau) - F_{12}(\omega)F_{22}(\omega)^{-1}B_2(\tau)$, that is such that $\mathrm{E}B_{1.2}(\tau)\overline{B}_2'(\tau) = 0$, we get that

$$\frac{1}{T}(\eta_1 - F_{12}(\omega)F_{22}(\omega)^{-1}\eta_2)\overline{S_\omega\eta_2}' \Longrightarrow \frac{1}{2}\int_0^1 dB_{1.2}(\tau)\overline{B}_2'(\tau)\,d\tau + G^+_{\eta,12}, \tag{A.16}$$

whence the result. ■

**Proof of Lemma 5.1.** Let $2\pi\,\widehat{F}_\eta(\omega) = \sum_{j=-m}^{+m}\left(1 - \frac{|j|}{m+1}\right)e^{i\omega j}\widehat{\Gamma_\eta(j)}$, *where* $\widehat{\Gamma_\eta(j)}$ *are the empirical estimates, be a Hermitian matrix :* Let $v$ be an $n \times 1$ complex-valued vector; we get that $2\pi\,T\overline{v}'\widehat{F}_\eta(\omega)v$ is equal to

$$\sum_{j=-m}^{-1}\sum_{t=-j+1}^{T}\left(1 + \frac{j}{m+1}\right)\zeta_t\overline{\zeta}'_{t+j} + \sum_{t=+1}^{T}\zeta_t\overline{\zeta}'_t + \sum_{j=1}^{m}\sum_{t=1}^{T-j}\left(1 - \frac{j}{m+1}\right)\zeta_t\overline{\zeta}'_{t+j}, \tag{A.17}$$

where $\zeta_t = e^{-i\omega t}\overline{v}'\eta_t$, so that using similar arguments as those used by Newey and West (1988), we get the result. ■

**Proof of Proposition 5.2.** For the sake of notational simplicity, we propose here to consider the case of constraints that are expressed on complex quantities. We consider the following set of constraints: $Rvec\alpha = q$, where $R$ and $q$ are two complex-valued matrices of respective dimensions $k \times r(n-r)$ and $k \times 1$. Under the null, we have $R\,\mathrm{vec}\,\widehat{\alpha}^+ - q = R\,\mathrm{vec}\,(\widehat{\alpha}^+ - \alpha)$. The convergence result on the FM estimator is such that

$$T(\widehat{\alpha}^+ - \alpha) \Longrightarrow \int_0^1 dB_{1.2}(\tau)\overline{B}_2'(\tau)\left(\int_0^1 B_2(\tau)\overline{B}_2'(\tau)\,d\tau\right)^{-1}, \tag{A.18}$$

and therefore

$$T\,R\,\mathrm{vec}\,(\widehat{\alpha}^+ - \alpha) \Longrightarrow R\int_0^1\left(\int_0^1 \overline{B}_2(\tau)B_2'(\tau)\,d\tau\right)^{-1}\overline{B}_2(\tau)\otimes dB_{1.2}(\tau). \tag{A.19}$$

From a distributional point of view,

$$T\,\mathrm{vec}\left(\int_0^1 dB_{1.2}(\tau)\overline{B}_2'(\tau)\left(\int_0^1 B_2(\tau)\overline{B}_2'(\tau)\,d\tau\right)^{-1/2}\right) \equiv \mathcal{N}_c(0, I_{n-r}\otimes 2\pi\,F_{1.2}(\omega)), \tag{A.20}$$

where $\mathcal{N}_c(0, I_{n-r}\otimes 2\pi\,F_{1.2}(\omega))$ is defined in Goodman (1963) or Brockwell and Davies (1991, p. 444). On the other hand, using arguments similar to those used in the proof of Proposition 5.1, we have the following weak convergence:

$$R\left(\left(\frac{1}{T^2}\sum_{t=1}^{T}\overline{S_\omega\eta_{2,t}}S_\omega\eta'_{2,t}\right)^{-1}\otimes 2\pi\,F_{1.2}(\omega)\right)\overline{R}'$$
$$\Longrightarrow\quad R\left(\left(\int_0^1\overline{B}_2(\tau)B_2'(\tau)\,d\tau\right)^{-1}\otimes 2\pi\,F_{1.2}(\omega)\right)\overline{R}'. \tag{A.21}$$

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We deduce that the test statistic

$$T\overline{R\,\mathrm{vec}(\widehat{\alpha}^{+}-\alpha)}'\left(R\left(\left(\frac{1}{T^2}\sum_{t=1}^{T}\overline{S_\omega\eta_{2,t}}S_\omega\eta_{2,t}'\right)^{-1}\otimes 2\pi F_{1.2}(\omega)\right)\overline{R}'\right)^{-1}T\,R\,\mathrm{vec}(\widehat{\alpha}^{+}-\alpha) \tag{A.22}$$

weakly converges to

$$\left(\int_0^1 B_2'(\tau)\left(\int_0^1 \overline{B}_2(\tau)B_2'(\tau)\,d\tau\right)^{-1}\otimes d\overline{B}_{1.2}'(\tau)\right)\overline{R}'$$

$$\left(R\left(\left(\int_0^1 \overline{B}_2(\tau)B_2'(\tau)\,d\tau\right)^{-1}\otimes 2\pi F_{1.2}(\omega)\right)\overline{R}'\right)^{-1}$$

$$R\int_0^1\left(\int_0^1 \overline{B}_2(\tau)\,B_2'(\tau)\,d\tau\right)^{-1}\overline{B}_2(\tau)\otimes dB_{1.2}(\tau).$$

We now observe that $B_2(\tau)$ is independent of $B_{1.2}(\tau)$ and thus, conditional on $\mathcal{F}_2=\sigma$-fields $\{B_2(\tau),0\le\tau\le 1\}$, we have

$$R\int_0^1\left(\int_0^1 \overline{B}_2(\tau)B_2'(\tau)\,d\tau\right)^{-1}\overline{B}_2(\tau)\otimes dB_{1.2}(\tau)|\mathcal{F}_2$$

$$\equiv\mathcal{N}_c\left(0,R\left(\left(\int_0^1 \overline{B}_2(\tau)B_2'(\tau)\,d\tau\right)^{-1}\otimes 2\pi F_{1.2}(\omega)\right)\overline{R}'\right),$$

so that conditional on $\mathcal{F}_2$, (A.22) is $\chi_2(2k)$. Since this distribution is independent of $\mathcal{F}_2$, the result holds unconditionally, whence the result.

At last, we notice that a set of constraints on a complex valued matrix can be rewritten as a set of constraints on associated real matrices. Simple computations give that $R\,\mathrm{vec}\,\alpha=q$ is equivalent to

$$\overline{R}^{*}\begin{pmatrix}\mathrm{vec}\,\Re\alpha\\ \mathrm{vec}\,\Im\alpha\end{pmatrix}=\begin{pmatrix}\Re q\\ \Im q\end{pmatrix}. \tag{A.23}$$

This nonetheless does not cover all the possible situations, as the number of constraints on the real quantities is necessarily even. The proof of Proposition 5.2 on real quantities is very similar to what has been done here and is left to the reader. ■

**Proof of Proposition 6.1.** We start from

$$\widehat{\alpha}^{+}-\alpha=\frac{1}{T}\left(\frac{1}{T}\eta_{12}\overline{S_\omega\eta_2}'-G_{\eta,12}^{+}\right)\left(\frac{1}{T^2}S_\omega\eta_2\overline{S_\omega\eta_2}'\right)^{-1}$$

$$=\frac{1}{T}L_T,$$

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where $\eta_{12} = \eta_1 - F_{12}(\omega)F_{22}(\omega)^{-1}\eta_2$. We have

$$\begin{aligned}\widehat{\eta}^+_{1,t} &= \delta_{-\omega} y^+_{1,t} - \widehat{\alpha}^+ \delta_{-\omega} y_{2,t} \\ &= \delta_{-\omega} y_{1,t} - F_{\eta,12}(\omega)F_{\eta,22}(\omega)^{-1}\Delta_\omega y_{2,t} - \left(\alpha + \frac{1}{T}L_T\right)\delta_{-\omega} y_{2,t} \\ &= \delta_{-\omega} y_{1,t} - \alpha\delta_{-\omega} y_{2,t} - F_{\eta,12}(\omega)F_{\eta,22}(\omega)^{-1}\Delta_\omega y_{2,t} - \frac{1}{T}L_T\delta_{-\omega} y_{2,t} \\ &= \eta_{12,t} - \frac{1}{T}L_T S_\omega \eta_{2,t},\end{aligned}$$

whence

$$\frac{e^{i\omega[T\tau]}}{\sqrt{T}} S_\omega \widehat{\eta}^+_{1,[T\tau]} = \frac{e^{i\omega[T\tau]}}{\sqrt{T}} S_\omega \eta_{12,t} - L_T \sum_{j=1}^{[T\tau]} \frac{e^{i\omega j}}{\sqrt{T}} S_\omega \eta_{2,j}\left(\frac{j}{T} - \frac{j-1}{T}\right),$$

and by Property A.1 in Gregoir (1999b), we get the weak convergence

$$\frac{e^{i\omega[T\tau]}}{\sqrt{T}} S_\omega \widehat{\eta}^+_{1,[T\tau]} \Longrightarrow \frac{1}{\sqrt{2}}\left(B_{12}(\tau) - \int_0^1 dB_{12}\overline{B}_2'\left(\int_0^1 B_2\overline{B}_2'\right)^{-1}\int_0^\tau B_2(s)\,ds\right),$$

so that if we set $W_2 = (2\pi F_{\eta,22}(\omega))^{-1/2}B_2$, a standard $n-r$-dimensional complex Wiener process, and $W_1 = (2\pi(F_{\eta,11}(\omega) - F_{\eta,12}(\omega)F^{-1}_{\eta,22}(\omega)F_{\eta,21}(\omega)))^{-1/2}B_{12}$, a standard $r$-dimensional complex Wiener process independent from $W_2$, and we have at our disposal an estimate of $F_{\eta,11}(\omega) - F_{\eta,12}(\omega)F^{-1}_{\eta,22}(\omega)F_{\eta,21}(\omega)$ that converges in probability, we get by the FCLT that

$$\zeta_T \Longrightarrow \operatorname{tr}\left(\frac{1}{2}\int_0^1 RW(\tau)\overline{RW}'(\tau)\,d\tau\right)$$

with $RW(\tau) = W_1(\tau) - \int_0^1 dW_1(s)\overline{W_2}'(s)\left(\int_0^1 W_2(s)\overline{W_2}'(s)\,ds\right)^{-1}\int_0^\tau W_2(s)\,ds$. When a deterministic function $e^{-i\omega t}$ is present in the regression, the same reasoning and computation apply. Remark that by applying a Frisch-Waugh decomposition in this case, the test distribution is expressed in terms of a complex Brownian bridge in place of $W_1$ and $W_2^\mu$ in place of $W_2$. ■

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