A NOTE ON POSITIVE HARMONIC FUNCTIONS

By S. VERBLUNSKY

Received 15 March 1947

1. If $H(\xi, \eta)$ is a harmonic function which is defined and positive in $\eta > 0$, then there is a non-negative number D and a bounded non-decreasing function $G(x)$ such that

$$H(\xi, \eta) = D\eta + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta(1+x^2)}{\eta^2 + (x-\xi)^2} dG(x). \tag{1}$$

(For a proof, see Loomis and Widder, *Duke Math. J.* 9 (1942), 643–5.) If we write

$$\rho e^{i\phi} = u + iv = (\xi + i\eta)^\lambda = (re^{i\theta})^\lambda, \tag{2}$$

where $\lambda > 1$, then the equation

$$h(u, v) = H(\xi, \eta) \quad (0 < \theta < \pi/\lambda)$$

defines a harmonic function h which is positive in $v > 0$. Hence there is a non-negative number d and a bounded non-decreasing function $g(x)$ such that

$$h(u, v) = dv + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(1+x^2)}{v^2 + (x-u)^2} dg(x). \tag{3}$$

The problem of finding the connexion between the functions $G(x)$ and $g(x)$ has been mentioned by Loomis (*Trans. American Math. Soc.* 53 (1943), 239–50, 244). In this note, the connexion is deduced from the following theorem:

Theorem I. *If $0 \leqslant \delta < 2$ and $a > 0$, then either*

$$\int_0^a \frac{H}{r^\delta} dr \tag{4}$$

is infinite for $0 < \theta < \pi$, or finite for all such θ. In the latter case, if $x = a$ is a point of continuity of $G(x)$, then

$$\lim_{\theta \to 0} \int_0^a \frac{H}{r^\delta} dr = \int_0^a \frac{1+x^2}{r^\delta} dG,$$

both sides being finite.

The simplest case in which (4) is finite is when $G(x) = O(|x|^\alpha)$ for small x, where $2 > \alpha > \delta$. In that case, a straightforward argument shows that when $\theta = \frac{1}{2}\pi$, i.e. $\xi = 0$, $\eta = r$, we have $H = O(r^{\alpha-1})$.

The connexion between g and G can be expressed as follows.

Theorem II. *If $G(x) = O(|x|^\alpha)$ for small x, where $2 > \alpha > 1 - \lambda(1-\delta)$, and $0 \leqslant \delta < 2$, then*

$$\lambda \int_0^{a^{1/\lambda}} \frac{1+x^2}{x^{1-\lambda(1-\delta)}} dG = \int_0^a \frac{1+x^2}{x^\delta} dg,$$

provided that $g(x)$ is continuous at $x = a$ and $G(x)$ is continuous at $x = a^{1/\lambda}$.

2. The deduction of Theorem II from Theorem I is simple. By Theorem I,

$$\lim_{\phi \to 0} \int_0^a \frac{h}{\rho^\delta} d\rho = \int_0^a \frac{1+x^2}{x^\delta} dg, \tag{5}$$

provided that

$$\int_0^a \frac{h}{\rho^\delta} d\rho < \infty. \tag{6}$$

https://doi.org/10.1017/S0305004100024257 Published online by Cambridge University Press

The change of variable (2) shows that

$$\int_0^a \frac{h}{\rho^\delta} d\rho = \lambda \int_0^{a^{1/\lambda}} \frac{H}{r^{1-\lambda(1-\delta)}} dr, \tag{7}$$

and since $\phi = \lambda\theta$,

$$\lim_{\theta \to 0} \lambda \int_0^{a^{1/\lambda}} \frac{H}{r^{1-\lambda(1-\delta)}} dr = \int_0^a \frac{1+x^2}{x^\delta} dg \tag{8}$$

if (6) holds, i.e. if the second member of (7) is finite. By the remark following Theorem I, this is the case since $H = O(r^{\alpha-1})$. By Theorem I, the first member of (8) has the required value.

3. It is easy to see that Theorem I follows from the following lemmas:

LEMMA 1. *If $x = a$ is a point of continuity of $G(x)$, and $0 < \delta < 2$, then*

$$\int_0^a \frac{H}{r^\delta} dr = \frac{\sin(1-\delta)\theta}{\sin \pi\delta} \int_{-1}^0 \frac{1+x^2}{|x|^\delta} dG + \frac{\sin(1-\delta)(\pi-\theta)}{\sin \pi\delta} \int_0^a \frac{1+x^2}{x^\delta} dG + R,$$

where R is bounded for $0 < \theta < \pi$, and tends to 0 with θ.

LEMMA 2. *If $x = a$ is a point of continuity of $G(x)$ then*

$$\int_0^a H dr = \frac{\sin\theta}{\pi} \int_{-1}^a (1+x^2) \log \frac{1}{|x|} dG + \int_0^a (1+x^2) dG + R,$$

where R satisfies the conditions of Lemma 1.

To prove these results, write $\xi = r\cos\theta$, $\eta = r\sin\theta$ in (1). Multiply (1) by $r^{-\delta}$, $0 \leqslant \delta < 2$, and integrate. Then

$$\int_0^a \frac{H}{r^\delta} dr = D \sin\theta \frac{a^{2-\delta}}{2-\delta} + \int_{-\infty}^{\infty} K(1+x^2) dG, \tag{9}$$

where

$$K = \frac{1}{\pi} \int_0^a \frac{r^{1-\delta} \sin\theta}{r^2 + x^2 - 2rx\cos\theta} dr. \tag{10}$$

The first term on the right of (9) can be included in R. We have to examine the second term. This term is certainly infinite if G is discontinuous at $x = 0$. In that case, the lemmas are true since both sides of the two equations are infinite. We may therefore suppose that G is continuous when $x = 0$.

4. We first consider the case $0 < \delta < 2$. Then $K = O(\sin\theta x^{-2})$ if $x \leqslant -1$, and also if $x \geqslant ak$, where k is a number (independent of θ) which exceeds 1. It can be shown (e.g. by contour integration) that

$$\frac{1}{\pi} \int_0^\infty \frac{t^{1-\delta} \sin\theta}{1+t^2-2t\cos\theta} dt = \frac{\sin(1-\delta)(\pi-\theta)}{\sin \pi\delta} \quad (0 < \delta < 2). \tag{11}$$

For $x > 0$,

$$K = \frac{1}{\pi x^\delta} \int_0^{a/x} \frac{t^{1-\delta} \sin\theta}{1+t^2-2t\cos\theta} dt. \tag{12}$$

Hence for $x \geqslant a/k$, $K = O(1)$ and

$$\int_{a/k}^{ak} K(1+x^2) dG = O[G(ak) - G(a/k)]. \tag{13}$$

Since G is continuous at $x = a$, we may suppose that k has been chosen so that the preceding expression is less than an assigned positive ϵ.

https://doi.org/10.1017/S0305004100024257 Published online by Cambridge University Press

For $0 < x \leqslant a/k$, we write the integral in (12) as

$$\int_0^\infty - \int_{a/x}^\infty,$$

and in the last integral we observe that

$$1 + t^2 - 2t\cos\theta > (t-1)^2 > t^2\left(\frac{k-1}{k}\right)^2,$$

since $t \geqslant a/x > k$. Thus, the last integral is less than

$$\left(\frac{k}{k-1}\right)^2 \sin\theta \int_{a/x}^\infty \frac{dt}{t^{1+\delta}} = O(x^\delta \sin\theta).$$

By (11) and (12),
$$K = \frac{1}{x^\delta}\frac{\sin(1-\delta)(\pi-\theta)}{\sin\pi\delta} + O(\sin\theta).$$

Finally, for $-1 \leqslant x < 0$, we write $x = -p$. By (10),

$$K = \frac{1}{\pi p^\delta}\int_0^{a/p} \frac{t^{1-\delta}\sin\theta}{1+t^2+2t\cos\theta}\,dt.$$

We express the integral as
$$\int_0^\infty - \int_{a/p}^\infty,$$

and to evaluate the first integral, we replace θ by $\pi - \theta$ in (11). The last integral is $O(p^\delta \sin\theta)$. Thus

$$K = \frac{1}{|x|^\delta}\frac{\sin(1-\delta)\theta}{\sin\pi\delta} + O(\sin\theta).$$

Using these results, we obtain Lemma 1 with a/k instead of a on the right-hand side, and with R bounded for $0 < \theta < \pi$ and $\overline{\lim}\,|R| \leqslant \epsilon$ as $\theta \to 0$. But we can let $\epsilon \to 0$, $k \to 1$, and the result follows.

5. To prove Lemma 2, we use (9) and (10) with $\delta = 0$. We have to consider the last term in (9) where

$$K = \frac{1}{\pi}\int_0^a \frac{r\sin\theta}{r^2+x^2-2rx\cos\theta}\,dr.$$

For $x \leqslant -1$, and for $x \geqslant a/k$, the argument is the same as in § 4. For $0 < x \leqslant a/k$, we observe that

$$K = \frac{1}{2\pi}\sin\theta\log\frac{a^2+x^2-2ax\cos\theta}{x^2} + \frac{\cos\theta}{\pi}\left\{\tan^{-1}\left(\frac{a-x\cos\theta}{x\sin\theta}\right) + \frac{\pi}{2} - \theta\right\}.$$

Here, the first term is
$$\frac{1}{\pi}\sin\theta\log\frac{1}{x} + O(\sin\theta),$$

while the second term is between -1 and 1 and tends to 1 as $\theta \to 0$.

If $-1 \leqslant x < 0$, then

$$K = \frac{1}{2\pi}\sin\theta\log\frac{a^2+x^2-2ax\cos\theta}{x^2} - \frac{\cos\theta}{\pi}\left\{\tan^{-1}\frac{a-x\cos\theta}{|x|\sin\theta} - \frac{\pi}{2} + \theta\right\}.$$

The first term is
$$\frac{1}{\pi}\sin\theta\log\frac{1}{|x|} + O(\sin\theta),$$

and the second term is between -1 and 1 and tends to 0 as $\theta \to 0$.

Using these results, we complete the proof as in § 4.

QUEEN'S UNIVERSITY
BELFAST

https://doi.org/10.1017/S0305004100024257 Published online by Cambridge University Press