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# DYNAMIC HYPERINTENSIONAL BELIEF REVISION

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**Abstract.** We propose a dynamic hyperintensional logic of belief revision for non-omniscient agents, reducing the logical omniscience phenomena affecting standard doxastic/epistemic logic as well as AGM belief revision theory. Our agents don't know all a priori truths; their belief states are not closed under classical logical consequence; and their belief update policies are such that logically or necessarily equivalent contents can lead to different revisions. We model both plain and conditional belief, then focus on dynamic belief revision. The key idea we exploit to achieve non-omniscience focuses on *topic-* or *subject matter-sensitivity*: a feature of belief states which is gaining growing attention in the recent literature.

**§1. Introduction: topicality and non-omniscience.** We can have different attitudes toward necessarily equivalent contents:

1. Bachelors are unmarried.
2. Barium has atomic number 56.
3. $2 + 2 = 4$.
4. No three positive integers $x$, $y$, and $z$ satisfy $x^n + y^n = z^n$ for integer value of $n$ greater than 2.

These sentences are (pairwise) intensionally equivalent: the contents they express are true at the same possible worlds. However, one may believe, or come to believe, the odd items only: one may, for instance, be fluent enough in English to grasp the meaning of 'bachelor' while having little competence in chemistry. One may be on top of enough basic arithmetic while having no clue on Diophantine equations and their solvability in integers. Such phenomena suggest the *topic-sensitivity* of doxastic states, a phenomenon that has recently attracted researchers' attention (see [8, 52]): we may (come to) believe different things given pieces of information true at the same possible worlds, due to differences in what they talk about—their *topic*, or *subject matter*.

It has long been known that our (propositional) mental states—believing, imagining, supposing, hoping, fearing—can be sensitive to hyperintensional distinctions, treating intensionally equivalent contents in different ways. We can think that equilateral triangles are equiangular without thinking that manifolds are topological spaces (we may just have never heard about topological notions). The two contents correspond to

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the same proposition in standard possible worlds semantics. Only the former, however, is *about* equilateral triangles, and made true by how they are.

Hyperintensionality is obviously connected to the issue of logical omniscience, a cluster of closure conditions on knowledge and/or belief. Since Hintikka [22], we've learned to model these as quantifiers over worlds, restricted by an accessibility relation:

(H) $B\varphi$ is true at world $w$ iff $\varphi$ is true at all $w_1$, such that $wRw_1$.

Focusing on belief: agents turn out to believe all logical (a priori) truths, all logical (a priori) consequences of what they believe, and (given that $R$ is serial) never to have inconsistent beliefs. It is generally agreed [13, pp. 34–35], [30, p. 186] that this gives an idealized notion of belief. It is at times suggested that one should read the '$B$' in (H), not as expressing belief, but rather some derivative attitude: 'following logically from what the agent believes', or 'from its total information'. A similar idealization is found in AGM belief revision theory when interpreted as modeling an agent revising beliefs in the light of new information. Among [1]'s postulates, (K$*$1) claims that $K * \varphi$ (belief set $K$ after revision by $\varphi$) is closed under full classical logical consequence; (K$*$5), that, if $\varphi$ is a logical inconsistency, then $K * \varphi = K_{\perp}$, the trivial belief set; and (K$*$6), that $K * \varphi = K * \psi$ for logically equivalent $\varphi$ and $\psi$. Our belief states, instead, needn't be closed under classical (perhaps under any monotonic: see [24]) logical consequence relation; we don't believe all things knowable a priori; nor do we believe everything just by being exposed, as we all occasionally are, to inconsistent information; and it can be the case that we believe $\varphi$ but don't believe $\psi$ for logically equivalent $\varphi$ and $\psi$.

The issue persists in epistemic/doxastic logics more sophisticated than the original Hintikkan approach, which recapture AGM dynamically. Works such as [2, 6, 11, 27, 28, 34, 37, 44, 45] feature operators for conditional belief or (static) belief revision, $B^{\varphi}\psi$ ('If the agent were to learn $\varphi$, they would come to believe that $\psi$ was the case' [6, p. 12]), and for dynamic belief revision, $[\Uparrow\varphi]B\psi$ ('After revision by $\varphi$, the agent believes that $\psi$'), satisfying principles such as:

- From $\varphi \leftrightarrow \psi$ infer $B^{\varphi}\chi \leftrightarrow B^{\psi}\chi$
- From $\varphi \leftrightarrow \psi$ infer $B^{\chi}\varphi \leftrightarrow B^{\chi}\psi$
- From $\varphi \leftrightarrow \psi$ infer $[\Uparrow\varphi]\chi \leftrightarrow [\Uparrow\psi]\chi$

Such operators are, thus, insensitive to hyperintensional differences. The issue persists in approaches resorting to Scott–Montague neighborhood semantics [12, 32, 33]. These allow operators that defy most closure features: agents are not modeled as believing all logical truths and all logical consequences of their beliefs. Such semantics have thus been used to provide (dynamic) epistemic logics for realistic agents, e.g., [3], and also to model allegedly logically anarchic notions such as imagination [50]. But when $\varphi$ and $\psi$ are logically or necessarily equivalent in that they have the same set of worlds as their truth set, they will inevitably have the same neighborhood and as a result, belief in either will automatically entail belief in the other. So even neighborhood-based approaches don't deliver the desired hyperintensionality.

*This* paper aims at improving on the situation. We proceed as follows: in §2, we introduce a basic theory of the topics or subject matters of propositional contents. In §3 we put the theory to work, providing a language and logic for plain, topic-sensitive hyperintensional belief, for which we prove soundness and completeness.

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In §4 we model hyperintensional belief revision in two forms: conditional belief and dynamic belief revision. Both notions turn out to be hyperintensional due to their topic-sensitivity; we provide a sound and complete logic for the former, and a reduction of the latter to the former via reduction principles. In §5, we conclude. Since the technical details are not essential to the main philosophical arguments of the paper, several of the longer proofs are omitted from the main body and collected in appendices.

**§2. Topics.** Arguably, a topic-sensitive account of the content of (de dicto) intentional states should flow from a general theory of propositional content. In various works [7, 8, 9], we have developed a theory of such content in the vicinity of Yablo [51]'s aboutness theory and Kit Fine [16]'s truthmaker semantics. Here's a short recap.

Treating propositional contents as sets of possible worlds gives too coarse-grained an individuation of propositions: '$2 + 2 = 4$' and 'Equilateral triangles are equiangular' are true at the same worlds (all of them), but speak of different things: only one is about equilateral triangles, and made true by how they are. So we need to supplement truth conditions with an account of *aboutness*, 'the relation that meaningful items bear to whatever it is that they are *on* or *of* or that they *address* or *concern*' [51, p. 1]: this is their subject matter or topic. What a sentence is about can be properly included in what another one is about. Contents, thus, can stand in mereological relations [16, sec. 3–5, 51, sec. 2.3]: they can have other contents as their parts and can be fused into wholes which inherit the proper features from the parts. The content of an interpreted sentence is the *thick proposition* it expresses [51, sec. 3.3]. This has two components: (i) intension and (ii) topic or subject matter. $\varphi$ and $\psi$ express the same thick proposition when (i) they are true at the same worlds, and (ii) they have the same topic.

In theories of partial content [15, 16, 51], there is broad agreement that the truth-functional logical connectives must be topic-transparent, i.e., they must add no subject matter of their own. The topic of $\neg\varphi$ must be the same as the topic of $\varphi$. 'Jane is not a lawyer' must be just about what 'Jane is a lawyer' is about: it is hard to come up with a discourse context where 'Jane is a lawyer' is on-topic, but 'Jane is not a lawyer' would be off-topic, or vice versa. What subject matter might 'Jane is not a lawyer' add to 'Jane is a lawyer'? 'Jane is not a lawyer' may speak about Jane, Jane's profession, what Jane does and doesn't do, but doesn't speak about *not*. Similarly, the topic of $\varphi \wedge \psi$ must be the same as that of $\varphi \vee \psi$: a fusion of the topic of $\varphi$ and that of $\psi$. 'Bob is tall and handsome' and 'Bob is tall or handsome' must both be about the same topic, namely the height and looks of Bob; neither can be about *and*, or *or*.

This view grounds patterns of logical validities and invalidities applicable to propositional attitudes. Believing that $\varphi$ requires not only (i) having information or evidence ruling out the non-$\varphi$ worlds (pick your favourite evidence-based theory of belief, see, e.g., [4, 6, 31, 38, 40, 41, 43]), but also (ii) grasping $\varphi$'s topic: what it is about. A first work, [8], presented a topic-sensitive hyperintensional conditional belief operator. This paper will deal not only with conditional belief, but also with plain belief (see, e.g., [6]) and with full-fledged dynamic belief revision operators that work as model-transformers, in the tradition of Dynamic Epistemic Logic [5, 38, 39, 45]. A topic will be assigned to the whole doxastic state of an agent, representing the subject

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matter they have grasped already, and a dynamic of topic-expansions will model how a non-omniscient agent can come to master new subject matter.

Before we venture into formal work, a word on the sense in which the agents we model are *still* logically idealized: they are computationally unbounded. As we will see, they turn out not to believe all a priori truths, and not to treat logical or a priori equivalents equally. Their beliefs are not closed under full, classical logical consequence. Once they receive an amount of information *and* they are on top of its topic, however, nothing stands in the way of their working out all the relevant consequences. Computational time and space limits are one, and perhaps the most compelling, source of non-omniscience for real, finite, and fallible agents. Modeling them is difficult, for these seem to have vague boundaries: they are contingent on time, memory size, psychological resources, attention, efficiency of the available algorithms, etc. We do not rule out the possibility of capturing this source of non-omniscience using Dynamic Epistemic Logic (work in this direction is already being carried out, e.g., [35, 36]). But this lies beyond the scope of the current paper.

**§3. Plain hyperintensional belief.** Learning does not only change the possible worlds space, but—we argue, following Yalcin [52]—also affects the subject matters the agent has grasped. Based on their current information, an agent can believe propositions they are on top of qua subject matter and not propositions about topics they have not grasped yet. We now propose a semantics for plain hyperintensional belief based on this thought. We endow a standard plausibility model for belief with a join semilattice, representing the mereological structure of topics together with the subject matter of the whole doxastic state of a single agent.

***3.1. Syntax and semantics.*** We have a countable set of propositional variables $\mathsf{Prop} = \{p_1, p_2, \dots\}$. The language $\mathcal{L}_{\mathsf{PHB}}$ of *plain hyperintensional belief* is defined by the grammar:

$$\varphi := p_i \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid B\varphi$$

where $p_i \in \mathsf{Prop}$. We often use $p, q, r, \dots$ for propositional variables. We employ the usual abbreviations for propositional connectives $\vee, \rightarrow, \leftrightarrow$ as $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi := \neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$; and for the duals $\Diamond\varphi := \neg\Box\neg\varphi$ and $\hat{B}\varphi := \neg B\neg\varphi$. As for $\bot$, we set $\bot := \neg\top$. We will follow the usual rules for the elimination of the parentheses.

For any $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, $Var(\varphi)$ denotes the set of propositional variables and $\top$ occurring in $\varphi$. In the metalanguage we use variables $x, y, z$ $(x_1, x_2, \dots)$ ranging over elements of $Var(\varphi)$. Another abbreviation will matter in the following: we will use '$\overline{\varphi}$' to denote the tautology $\bigwedge_{x \in Var(\varphi)}(x \vee \neg x)$[1], following a similar idea in [18]. Do not confuse $\overline{\varphi}$ with $\top$: they will turn out to be logically equivalent for all $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, but our belief

[1] In order to have a unique definition of each $\overline{\varphi}$, we set the convention that elements of $Var(\varphi)$ occur in $\bigwedge_{x \in Var(\varphi)}(x \vee \neg x)$ from left-to-right in the order they are enumerated in $\mathsf{Prop} = \{p_1, p_2, \dots\}$. If $\top \in Var(\varphi)$, take $\top \vee \neg\top$ as the first conjunct. For example, for $\varphi := \Box(p_3 \rightarrow p_2) \vee Bp_5$, $\overline{\varphi}$ is $(p_2 \vee \neg p_2) \wedge (p_3 \vee \neg p_3) \wedge (p_5 \vee \neg p_5)$, and not $(p_5 \vee \neg p_5) \wedge (p_3 \vee \neg p_3) \wedge (p_2 \vee \neg p_2)$ or $(p_3 \vee \neg p_3) \wedge (p_5 \vee \neg p_5) \wedge (p_2 \vee \neg p_2)$ etc. For $\psi := B\top \wedge \Box(p_1 \wedge p_3)$, $\overline{\psi}$ abbreviates $(\top \vee \neg\top) \wedge (p_1 \vee \neg p_1) \wedge (p_3 \vee \neg p_3)$. This convention will eventually not matter since our logics cannot differentiate two conjunctions of different order: $\varphi \wedge \psi$ provably and semantically equivalent to $\psi \wedge \varphi$.

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operator will discern them. What $\top$ means will become clearer when our semantics is in, but you may already read it as standing for the total propositional content the agent is (already) on $\top$op of, qua information and subject matter. We read $B\varphi$ as 'The agent believes that $\varphi$', $\Box\varphi$ as a normal epistemic modality (analyticity, or a more generic a priori modality).

The first component of the semantics deals with topicality:

DEFINITION 1 (Doxastic Topic Model for $\mathcal{L}_{\mathsf{PHB}}$). *A doxastic topic model* (*dt-model*) $\mathcal{T}$ *is a tuple* $\langle T, \oplus, \mathfrak{b}, t\rangle$ *where*

1. $T$ *is a nonempty set of possible topics*;
2. $\oplus : T \times T \to T$ *is a binary idempotent, commutative, associative operation*: *topic fusion. We assume unrestricted fusion, that is,* $\oplus$ *is always defined on* $T$: $\forall a, b \in T\ \exists c \in T(c = a \oplus b)$;
3. $\mathfrak{b} \in T$ *is a designated topic, the topic of the agent's belief state representing the totality of subject matter the agent has grasped already*;
4. $t : \mathsf{Prop} \cup \{\top\} \to T$ *is a topic function assigning a topic to each element in* $\mathsf{Prop} \cup \{\top\}$, *such that* $t(\top) = \mathfrak{b}$. $t$ *extends to the whole* $\mathcal{L}_{\mathsf{PHB}}$ *by taking the topic of a sentence* $\varphi$ *as the fusion of the elements in* $Var(\varphi)$:

$$t(\varphi) = \oplus Var(\varphi) = t(x_1) \oplus \cdots \oplus t(x_k)$$

*where* $Var(\varphi) = \{x_1, \ldots, x_k\}$.

In the metalanguage we use variables $a, b, c$ $(a_1, a_2, \ldots)$ ranging over possible topics. We define *topic parthood*, denoted by $\sqsubseteq$, in a standard way as

$$\forall a, b(a \sqsubseteq b \text{ iff } a \oplus b = b).$$

Thus, $(T, \oplus)$ is a *join semilattice* and $(T, \sqsubseteq)$ a *poset*. The *strict topic parthood*, denoted by $\sqsubset$, is defined as usual as $a \sqsubset b$ iff $a \sqsubseteq b$ and $b \not\sqsubseteq a$. The topic of a complex sentence $\varphi$, defined from its primitive components in $Var(\varphi)$ (see Definition 1.4), makes all the logical connectives and modal operators in $\mathcal{L}_{\mathsf{PHB}}$ topic-transparent:

- $t(\Box\varphi) = t(B\varphi) = t(\neg\varphi) = t(\varphi)$;
- $t(\varphi \wedge \psi) = t(\varphi) \oplus t(\psi)$.

Topic fusion and parthood capture the mereological conception of subject matters sketched in §2. Topic-transparency has been advocated there for the truth-functional connectives. It is less straightforward to motivate it for the modal operators $\Box$ and, especially, $B$. The subject matter of $B\varphi$ and that of $\varphi$ don't look quite the same: 'John believes that Jane is a lawyer' is about what John believes, 'Jane is a lawyer' is not. We propose the following story. Call the set $Var(\varphi)$ of propositional variables and $\top$ occurring in a sentence $\varphi$ its *ontic component*. Given $\varphi$, the assigned subject matter $t(\varphi)$ represents its ontic topic: the fusion of the topics of the elements in its ontic component $Var(\varphi)$. Now believing $\varphi$ requires having grasped the topic of its ontic component and having enough available information/evidence supporting it. Once the agent has grasped the topic of $\varphi$, reasoning about $B\varphi$—their own doxastic attitude toward the proposition expressed by $\varphi$—does not require grasping further topics. However, it might require acquiring more information to support the agent's belief in $\varphi$. Our non-omniscient agent can grasp the subject matter of a belief sentence about *their own beliefs* as long as they have mastered the subject matter of its ontic

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component. This idealization we can live with. Still, we have assigned a topic to $\top$ (what the agent is on $\top$op of), as $t(\top) = \mathfrak{b}$, representing the subject matter the agent has already grasped. This will play a role in the semantics of $B$.

The second component in our semantics is familiar:

DEFINITION 2 (Standard Plausibility Frame). *A standard plausibility frame $\mathcal{S}$ is a tuple $\langle W, \geq \rangle$, where*

- *$W$ is a nonempty set of possible worlds*;
- *$\geq$: $W \times W$ is a well-preorder, called the plausibility order. A well-preorder on $W$ is a reflexive and transitive binary relation such that every nonempty subset of $W$ has a minimal element, where the set of minimal elements $Min_{\geq}(P)$ for any $P \subseteq W$ is defined as*

$$Min_{\geq}(P) = \{w \in P : v \geq w \text{ for all } v \in P\}.$$

*$v \geq w$ means 'w is at least as plausible as v'. (Every well-preorder $\geq \subseteq W \times W$ is a total order: either $w \geq v$ or $v \geq w$ for all $w, v \in W$.)*

Such ordering gives an arrangement of worlds, taken as epistemic scenarios, by the degree to which the agent finds them plausible as ways things could be. Thus, $Min_{\geq}(W)$ represents the set of states the agent considers most plausible. Now we merge the two components:

DEFINITION 3 (Topic-sensitive plausibility model for $\mathcal{L}_{\mathsf{PHB}}$). *A topic sensitive plausibility model (tsp-model) $\mathcal{M}$ is a tuple $\langle W, \geq, T, \oplus, \mathfrak{b}, t, v \rangle$ where $\langle W, \geq \rangle$ is a standard plausibility frame, $\langle T, \oplus, \mathfrak{b}, t \rangle$ is a dt-model for $\mathcal{L}_{\mathsf{PHB}}$, and $v : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation function that maps every propositional variable in* Prop *to a set of worlds.*

We define the satisfaction relation $\Vdash$ recursively. The *intension of* $\varphi$ with respect to $\mathcal{M}$ is $|\varphi|_{\mathcal{M}} := \{w \in W : \mathcal{M}, w \Vdash \varphi\}$. We omit the subscript $\mathcal{M}$ when the model is contextually clear.

DEFINITION 4 ($\Vdash$-Semantics for $\mathcal{L}_{\mathsf{PHB}}$). *Given a tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v \rangle$ and a state $w \in W$, the $\Vdash$-semantics for $\mathcal{L}_{\mathsf{PHB}}$ is defined recursively as*:

$$\begin{array}{ll}
\mathcal{M}, w \Vdash \top & \textit{iff always} \\
\mathcal{M}, w \Vdash p & \textit{iff } w \in v(p) \\
\mathcal{M}, w \Vdash \neg\varphi & \textit{iff not } \mathcal{M}, w \Vdash \varphi \\
\mathcal{M}, w \Vdash \varphi \wedge \psi & \textit{iff } \mathcal{M}, w \Vdash \varphi \textit{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \Box\varphi & \textit{iff } \mathcal{M}, u \Vdash \varphi \textit{ for all } u \in W \\
\mathcal{M}, w \Vdash B\varphi & \textit{iff } Min_{\geq}(W) \subseteq |\varphi| \textit{ and } t(\varphi) \sqsubseteq \mathfrak{b}.
\end{array}$$

When it is not the case that $\mathcal{M}, w \Vdash \varphi$, we simply write $\mathcal{M}, w \nVdash \varphi$.

A priori truth, $\Box$, is truth at all worlds. Belief is topic-sensitive: in order for '$B\varphi$' to be true, two things must happen: (i) $\varphi$ is true at all worlds in $Min_{\geq}(W)$ (the worlds considered most plausible); (ii) the topic of $\varphi$ is in $\mathfrak{b}$, i.e., the agent has grasped such subject matter.[2]

[2] Notice that neither of these requirements is state-dependent, therefore, the state of the world has no effect on the agent's beliefs. This is rather standard in single-agent belief logics based on plausibility models as the possible worlds in $w$ are taken to be *epistemically possible* ones which cannot be distinguished with absolute certainty [6]. We could as well interpret $\Box$ as

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The standard plausibility models with the usual semantics for the language $\mathcal{L}_{\mathsf{PHB}}$ are modally equivalent to the tsp-models when the set of possible topics is a singleton. Next comes the definition of *logical consequence* (*with respect to* $\Vdash$): with $\Sigma \subseteq \mathcal{L}_{\mathsf{PHB}}$ and $\varphi, \psi \in \mathcal{L}_{\mathsf{PHB}}$,

- $\Sigma \vDash \varphi$ iff for all models $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ and all $w \in W$: if $\mathcal{M}, w \Vdash \psi$ for all $\psi \in \Sigma$, then $\mathcal{M}, w \Vdash \varphi$.
- For single-premise entailment, we write $\psi \vDash \varphi$ for $\{\psi\} \vDash \varphi$.
- As a special case, *logical validity*, $\vDash \varphi$, truth at all worlds of all models, is $\emptyset \vDash \varphi$, entailment by the empty set of premises. $\varphi$ is called *invalid*, denoted by $\nvDash \varphi$, if it is not a logical validity, that is, if there is a tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ and a possible world $w \in W$ such that $\mathcal{M}, w \nVdash \varphi$.

*Soundness* and *completeness* with respect to the proposed semantics are defined standardly (see, e.g., [10, chap. 4.1]).

The abbreviation $\overline{\varphi} := \bigwedge_{x \in Var(\varphi)}(x \vee \neg x)$ will play a role in formalizing validities and invalidities. Given a tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$, for $B\varphi$ to be true (at $w$) we require (i) $\varphi$ to be true at all worlds in $Min_{\geq}(W)$, and (ii) the topic of $\varphi$ to be in $\mathfrak{b}$. Since $\overline{\varphi}$ is true everywhere and $Var(\overline{\varphi}) = Var(\varphi)$ for any $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, formulas of the form $B\overline{\varphi}$ ($\neg B\overline{\varphi}$) express within the language $\mathcal{L}_{\mathsf{PHB}}$ statements such as 'the agent has (not) grasped the subject matter of $\varphi$':

$$\begin{aligned}\mathcal{M}, w \Vdash B\overline{\varphi} \ &\text{iff } Min_{\geq}(W) \subseteq |\overline{\varphi}| \text{ and } t(\overline{\varphi}) \sqsubseteq \mathfrak{b}\\ &\text{iff } Min_{\geq}(W) \subseteq W \text{ and } t(\varphi) \sqsubseteq \mathfrak{b} \ (t(\overline{\varphi}) = t(\varphi), \text{since} Var(\overline{\varphi}) = Var(\varphi))\\ &\text{iff } t(\varphi) \sqsubseteq \mathfrak{b}.\end{aligned}$$

$B\overline{\varphi}$ is true precisely when the topic of $\varphi$ is included in the topic of the whole doxastic state of the agent. (Of course, $\overline{\varphi}$ as any arbitrary tautology which is a truth-functional compound of $\varphi$, e.g., $\varphi \vee \neg\varphi$, $\varphi \rightarrow \varphi$, $\neg(\varphi \wedge \neg\varphi)$, etc., would do the same job as $\bigwedge_{x \in Var(\varphi)}(x \vee \neg x)$.)

***3.2. Axiomatization, soundness, and completeness.*** Table 1 gives a sound and complete axiomatization PHB of hyperintensional plain belief over $\mathcal{L}_{\mathsf{PHB}}$. We focus on the intuitive readings of the principles presented in Table 1. We then move on to important invalidities concerning the problem of logical omniscience and the role of topicality in achieving non-omniscience. The soundness and completeness proofs are given in Appendices A.2.1 and A.2.2, respectively.

The notion of derivation, denoted by $\vdash_{\mathsf{PHB}}$, in PHB is defined as usual. Thus, $\vdash_{\mathsf{PHB}} \varphi$ means $\varphi$ is a theorem of PHB. We omit the subscript PHB when the logic PHB is contextually clear.

THEOREM 1. *The following are derivable from* PHB:

1. $B\overline{\varphi} \leftrightarrow \bigwedge_{x \in Var(\varphi)} B\overline{x}$
2. $B\overline{\varphi} \rightarrow B\overline{\psi}$, *if* $Var(\psi) \subseteq Var(\varphi)$

a standard S5$\Box$ modality with respect to an equivalence relation and define the plausibility order in a state dependent way with the requirement that the plausibility ordering of a possible world $w$ is total *within* the equivalence class of $w$ (see, e.g., [38] for such a version). This set-up would lead to the same logic and not add much to our conceptual arguments, so we opt for simplicity and work with state-independent plausibility orderings and topic assignments.

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Table 1. *Axiomatization* PHB *of the logic of plain hyperintensional belief (over* $\mathcal{L}_{\mathsf{PHB}}$*)*

| | |
|---|---|
| (CPL) | all classical propositional tautologies and Modus Ponens |
| ($\mathsf{S5}_\Box$) | S5 axioms and rules for $\Box$ |
| | **(I) Axioms for** $B$**:** |
| (Ax$\top$) | $B\top$ |
| ($\mathsf{C}_B$) | $B(\varphi \wedge \psi) \leftrightarrow (B\varphi \wedge B\psi)$ |
| ($\mathsf{D}_B$) | $B\varphi \rightarrow \neg B\neg\varphi$ |
| (Ax1) | $B\varphi \rightarrow B\overline{\varphi}$ |
| | **(II) Axioms connecting** $B$ **and** $\Box$**:** |
| (Ax2) | $(\Box(\varphi \rightarrow \psi) \wedge B\varphi \wedge B\overline{\psi}) \rightarrow B\psi$ |
| (Ax3) | $B\varphi \rightarrow \Box B\varphi$ |

3. $B(\overline{\varphi \wedge \psi}) \leftrightarrow B(\overline{\varphi} \wedge \overline{\psi})$
4. $(B(\varphi \rightarrow \psi) \wedge B\varphi) \rightarrow B\psi$
5. $B\overline{\varphi} \rightarrow \Box B\overline{\varphi}$
6. $\neg B\overline{\varphi} \rightarrow \Box\neg B\overline{\varphi}$
7. $\neg B\varphi \rightarrow \Box\neg B\varphi$
8. $(\Box\varphi \wedge B\overline{\varphi}) \rightarrow B\varphi$
9. $B\varphi \rightarrow BB\varphi$
10. $(\neg B\varphi \wedge B\overline{\varphi}) \rightarrow B\neg B\varphi$

*Proof.* See Appendix A.1. □

Given the semantics of $\Box$, the logic of this modality as $\mathsf{S5}_\Box$ is no surprise. The axioms of Group (I) regulate belief: Ax$\top$ reflects $\top$'s standing for the total propositional content the agent is (already) on top of, qua information and subject matter. $\mathsf{C}_B$ says that belief is fully conjunctive, which imposes some computational idealization on our agents: there may be a syntactic or computational difference between believing $\varphi$ and $\psi$ together and believing them separately (besides, the principle is under discussion in mainstream epistemology due to the Lottery Paradox). But this is no difference in the scenario represented in the mind of an intentional agent: one *cannot* believe that Mary is tall and thin without believing that she is tall, and one cannot believe that Mary is tall and that Mary is thin at the same time without believing that she is tall and thin. $\mathsf{D}_B$ is the so-called axiom of consistency of belief, which is widely taken for granted in recent literature of epistemic logic (see, e.g., [6, 38, 45]). Ax1 expresses in the language the topic-sensitivity of belief: '$B\overline{\varphi}$' is read as 'the agent has grasped the subject matter of $\varphi$'. Thus, the axiom states that to believe $\varphi$, one must have grasped what it's about.

The axioms of Group (II) regulate the interplay between a priori truth $\Box$ and hyperintensional belief $B$. Ax3 (together with Theorem 1.7) has it that $B\varphi$ is state-independent. We will see in §3.2.1 that our agents' beliefs are not closed under a priori consequence. However, Ax2 gives us a more limited topic-sensitive closure principle: one believes those a priori consequences of one's beliefs whose subject matters one has grasped. The principle is a way to phrase the computational idealization of agents flagged in §2: once they are on top of an amount of information and the relevant topics,

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nothing stands in the way of their working out the logical/a priori consequences and believing them.

THEOREM 2. PHB *is a sound and complete axiomatization of* $\mathcal{L}_{\mathsf{PHB}}$ *with respect to the class of tsp-models: for every* $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, $\vdash_{\mathsf{PHB}} \varphi$ *if and only if* $\models \varphi$.

*Proof.* See Appendix A.2 □

*3.2.1. Invalidities.* We now turn to important *in*validities highlighting the hyperintensionality of plain belief. One doesn't believe all logical truths and, in general, all a priori truths, for the following fail[3]:

1. **Omniscience Rule**: from $\varphi$ infer $B\varphi$
2. **Apriori Omniscience**: $\not\models \Box\varphi \rightarrow B\varphi$

One also doesn't believe all logical (a priori) consequences of what one believes, and one can have different attitudes towards logical equivalents—i.e., the following fail:

3. **Closure Under Strict Implication**: $\not\models (\Box(\varphi \rightarrow \psi) \wedge B\varphi) \rightarrow B\psi$
4. **Closure Under Logical Equivalence**: from $\varphi \leftrightarrow \psi$ infer $B\varphi \leftrightarrow B\psi$
   [*Countermodel*: let $W = \{w\}$, $\geq = \{(w, w)\}$, $T = \{a, b\}$, $a \sqsubset b$ such that $\mathfrak{b} = a$, $t(p) = b$, $t(q) = a$, and $v(p) = v(q) = \{w\}$. Therefore, while $p \vee \neg p$ is valid and $w \Vdash \Box(p \vee \neg p)$, we have $w \not\Vdash B(p \vee \neg p)$ since $t(p \vee \neg p) = b \not\sqsubseteq \mathfrak{b}$. Therefore, (1) and (2) fail. Moreover, we have $(q \vee \neg q) \leftrightarrow (p \vee \neg p)$ valid, in turn, $w \Vdash \Box((q \vee \neg q) \rightarrow (p \vee \neg p))$, and $w \Vdash B(q \vee \neg q)$, but $w \not\Vdash B(p \vee \neg p)$. Therefore, (3) and (4) also fail.]

These fail for the right *reason*: the topic-sensitivity of belief. The agent does not believe the proposition in question because they have not grasped its subject matter.

On introspection principles: agents are positively introspective as per Theorems 1.9 and 2, i.e., $B\varphi \rightarrow BB\varphi$ is valid in tsp-models. This is due to (1) the interpretation of the informational content of belief as truth in the set of most plausible states plus invariance of the truth of $B\varphi$ between possible worlds, and (2) the topic transparency of $B$. On the other hand,

5. **Negative Introspection**: $\not\models \neg B\varphi \rightarrow B\neg B\varphi$

fails due to topicality. As a counterexample, we take the one presented above: $\neg Bp \rightarrow B\neg Bp$ is not true at $w$. Given a tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$, $\neg B\varphi$ is true at $w$ (or, equivalently, in every state of the model) iff either $Min_{\geq}(W) \not\subseteq |\varphi|$ or $t(\varphi) \not\sqsubseteq \mathfrak{b}$. Even if the reason for the failure of $B\varphi$ was $Min_{\geq}(W) \not\subseteq |\varphi|$ we would have $Min_{\geq}(W) \subseteq |\neg B\varphi|$, since $|\neg B\varphi| = W$ or $|\neg B\varphi| = \emptyset$. Therefore, in case $\neg B\varphi$ is true at $w$, $B\neg B\varphi$ is false at $w$ iff the agent has not grasped the topic of $\varphi$. In turn, our agent is negatively introspective with respect to the propositions whose subject matter they have grasped already. So we have the following validity (see also Theorems 1.10 and 2):

[3] We say that an inference rule *fails* when it is not validity preserving; and a formula schema *fails* if it has an invalid instance.

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6. **Topic-sensitive Negative Introspection**: $\models (\neg B\varphi \wedge B\overline{\varphi}) \rightarrow B\neg B\varphi$

This approach to logical omniscience has structural similarities to how explicit belief is treated in awareness logics [14, 47, 49]. The closest variant is 'propositionally determined awareness' (see [20, p. 327], which focuses on knowledge): one is aware of $\varphi$ just in case one is aware of all of its atomic constituents taken together. One believes which formulas one is aware of. Awareness has been criticized for mixing syntax and semantics [25]. Our approach doesn't. Whatever topics are (partitions or divisions of the set of worlds as per [51], states or truthmakers as per [16], or whatnot), they are going to be nonlinguistic entities. Our topic function assigns such entities, or constructions thereof, as topics to the formulas in $\mathcal{L}_{\mathsf{PHB}}$ in a recursive way. And the same topic can be assigned to different formulas. This is as syntax-free as one can hope.

Awareness structures are very flexible for they can in principle make as many hyperintensional distinctions as allowed by the syntax of the language itself. It is no surprise, thus, that they can simulate our topic-sensitive account. But they don't seem to be adequate to semantically represent the mereological relations of contents. And while plain belief did not make full use of topicality—we do not yet talk about belief conditional on a piece of explicit input—the importance of the doxastic topic models will become more apparent once we move on to conditional belief and the dynamics of belief.

**§4. Hyperintensional belief revision.** In Dynamic Epistemic Logic (DEL), one makes a distinction between static and dynamic belief revision (see, e.g., [5, 6, 38, 44, 45]). Static belief revision captures the agent's revised beliefs about how the world was before learning new information. Dynamic belief revision captures the agent's revised beliefs about the state of the world after the revision. As standard in the DEL literature, we implement the former by conditional belief modalities $B^{\varphi}\psi$ ('If one were to learn that $\varphi$, one would believe that $\psi$ was the case') and the latter by means of a dynamic operator $[\Uparrow\varphi]\psi$ ('After revision by $\varphi$, $\psi$ holds').

***4.1. Conditional hyperintensional belief.*** We now work with the language $\mathcal{L}_{\mathsf{CHB}}$ of *conditional hyperintensional belief* defined by the grammar:

$$\varphi := p_i \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid [\geq]\varphi \mid B^{\varphi}\varphi$$

where $p_i \in \mathsf{Prop}$. The use of 'to learn' in our reading of '$B^{\varphi}\psi$' above deserves comment: if conditional (as much as plain) belief is topic-sensitive, learning that $\varphi$ doesn't just require that there be information positioning one to rule out the non-$\varphi$ worlds. One must also have grasped what $\varphi$ is about: its subject matter. $[\geq]\varphi$, instead, is a standard Kripke modality corresponding to the plausibility relation. It is standardly used in Dynamic Epistemic Logic to capture a notion of "safe belief" or "indefeasible knowledge".[4] It will help to provide a complete axiomatization of a logic of conditional beliefs.

[4] $[\geq]\varphi$ is read as '$\varphi$ is safely believed' or '$\varphi$ is indefeasibly known' in the sense that no truthful information gain causes the agent to give up their belief/knowledge of $\varphi$ [6]. It is also a commonly used modality in preference logics [29, 42].

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A dt-model for $\mathcal{L}_{\mathsf{CHB}}$ is just as in Definition 1, with $t$ extended to the new language the obvious way. $[\geq]\varphi$ and $B^{\varphi}\psi$ are topic-transparent: $t([\geq]\varphi) = t(\varphi)$ and $t(B^{\varphi}\psi) = t(\varphi) \oplus t(\psi)$. In this section, we only consider doxastic topic models for $\mathcal{L}_{\mathsf{CHB}}$.

DEFINITION 5 ($\Vdash$-Semantics for $\mathcal{L}_{\mathsf{CHB}}$). *Given a tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ and a state $w \in W$, the $\Vdash$-semantics for $\mathcal{L}_{\mathsf{CHB}}$ is as in Definition 4 for the components in $\mathcal{L}_{\mathsf{PHB}}$, plus*:

$$\begin{array}{lll} \mathcal{M}, w \Vdash [\geq]\varphi & \textit{iff} & \mathcal{M}, u \Vdash \varphi \textit{ for all } u \in W \textit{ such that } w \geq u \\ \mathcal{M}, w \Vdash B^{\varphi}\psi & \textit{iff} & \textit{Min}_{\geq}(|\varphi|) \subseteq |\psi| \textit{ and } t(\psi) \sqsubseteq \mathfrak{b} \oplus t(\varphi). \end{array}$$

Conditional belief is topic-sensitive, too. For one to believe $\psi$ conditional on $\varphi$, we require two things to happen: firstly, all the most plausible $\varphi$-worlds must make $\psi$ true. Secondly, the topic of $\psi$ must be in the *fusion* of $\mathfrak{b}$ (the subject matter the agent was already on top of) with the topic of $\varphi$, given that conditionalizing on $\varphi$ requires the agent to have grasped the latter.

Plain belief, $B\varphi$, is now definable in terms of conditional belief, the usual way, as $B\varphi := B^{\top}\varphi$:

$$\begin{array}{lll} \mathcal{M}, w \Vdash B^{\top}\varphi & \text{iff } \textit{Min}_{\geq}(|\top|) \subseteq |\varphi| \text{ and } t(\varphi) \sqsubseteq \mathfrak{b} \oplus t(\top) & \text{(Definition 5)} \\ & \text{iff } \textit{Min}_{\geq}(W) \subseteq |\varphi| \text{ and } t(\varphi) \sqsubseteq \mathfrak{b} \oplus \mathfrak{b} & (\text{since } |\top| = W \text{ and } t(\top) = \mathfrak{b}) \\ & \text{iff } \textit{Min}_{\geq}(W) \subseteq |\varphi| \text{ and } t(\varphi) \sqsubseteq \mathfrak{b} & (\text{since } \oplus \text{ is idempotent}) \end{array}$$

Unlike in [8], plain belief is accommodated in the language $\mathcal{L}_{\mathsf{CHB}}$ of conditional belief via the subject matter the agent is on $\top$op of. (We can also define $\Box\varphi$ as $B^{\neg\varphi}\bot$ in $\mathcal{L}_{\mathsf{CHB}}$; we prefer to take $\Box$ as a primitive operator.) However, we cannot define conditional belief in $\mathcal{L}_{\mathsf{PHB}}$, so $\mathcal{L}_{\mathsf{CHB}}$ is strictly more expressive than $\mathcal{L}_{\mathsf{PHB}}$. As is well known, this is also the case for the usual semantics of conditional and plain belief on the standard plausibility models. All these expressivity results are presented more formally in Lemma 3.

LEMMA 3. *$\mathcal{L}_{\mathsf{CHB}}$ is strictly more expressive than $\mathcal{L}_{\mathsf{PHB}}$ with respect to tsp-models. In fact, the language $\mathcal{L}$ having only conditional belief operators as its modalities (i.e., $\mathcal{L}_{\mathsf{CHB}}$ minus $[\geq]$ and $\Box$) is strictly more expressive than $\mathcal{L}_{\mathsf{PHB}}$ with respect to tsp-models.*

*Proof.* See Appendix B.1 □

Just like in $\mathcal{L}_{\mathsf{PHB}}$, we can express in $\mathcal{L}_{\mathsf{CHB}}$ what subject matters the agent grasps after having grasped further subject matters, via formulas of the form $B^{\overline{\chi}}\overline{\varphi}$:

$$\begin{array}{ll} \mathcal{M}, w \Vdash B^{\overline{\chi}}\overline{\varphi} & \text{iff } \textit{Min}_{\geq}(|\overline{\chi}|) \subseteq |\overline{\varphi}| \text{ and } t(\overline{\varphi}) \sqsubseteq \mathfrak{b} \oplus t(\overline{\chi}) \\ & \text{iff } \textit{Min}_{\geq}(W) \subseteq W \text{ and } t(\varphi) \sqsubseteq \mathfrak{b} \oplus t(\chi) \\ & \quad (t(\overline{\varphi}) = t(\varphi), \text{ since } \textit{Var}(\overline{\varphi}) = \textit{Var}(\varphi)(\text{similarly for} \chi)) \\ & \text{iff } t(\varphi) \sqsubseteq \mathfrak{b} \oplus t(\chi). \end{array}$$

$B^{\overline{\chi}}\overline{\varphi}$ is true precisely when the topic of $\varphi$ is included in the topic of the whole doxastic state of the agent, expanded by the topic of $\chi$. A natural reading of $B^{\overline{\chi}}\overline{\varphi}$, then, is 'The agent would come to grasp the topic of $\varphi$, were they to grasp the topic of $\chi$'. If both

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Fig. 1. Models $\mathcal{M}_1$ and $\mathcal{M}_2$.

$B^{\overline{\chi}}\overline{\varphi}$ and $B^{\overline{\varphi}}\overline{\chi}$ are the case, we say that the topics of $\chi$ and $\varphi$ complement each other with respect to the topic of the agent's belief state.[5]

Notice that, since the topic component of the semantic clause for '$B^{\chi}\varphi$' takes into account the topic of the agent's belief state, $\mathcal{L}_{\mathsf{CHB}}$ is not expressive enough to speak of parthood relations. In $\mathcal{L}_{\mathsf{CHB}}$, we cannot say things like 'The topic of $\varphi$ is included in the topic of $\chi$', or '$\varphi$ and $\chi$ have exactly the same topic'. The opposite is the case in [8]: since the proposal in [8] does not accommodate the whole doxastic state of the agent, $B^{\overline{\chi}}\overline{\varphi}$ *there* states precisely that the topic of $\varphi$ is included in that of $\chi$.

To see that $\mathcal{L}_{\mathsf{CHB}}$ is not expressive enough to state 'The topic of $\varphi$ is included in the topic of $\chi$', consider the models $\mathcal{M}_1 = \langle\{w\}, \geq, \{a_1, b_1, \mathfrak{b}_1\}, \oplus_1, \mathfrak{b}_1, t_1, v\rangle$ and $\mathcal{M}_2 = \langle\{w\}, \geq, \{a_2, b_2, \mathfrak{b}_2\}, \oplus_2, \mathfrak{b}_2, t_2, v\rangle$, where $\geq = \{(w, w)\}$, $v(p) = v(q) = \emptyset$, and $(\{a_1, b_1, \mathfrak{b}_1\}, \oplus_1, t_1)$ and $(\{a_2, b_2, \mathfrak{b}_2\}, \oplus_2, t_2)$ are as given in Figure 1. We have 'The topic of $q$ is included in the topic of $p$' true in $\mathcal{M}_1$ at $w$ (since $t_1(q) = a_1 \sqsubseteq_1 b_1 = t_1(p)$) and false in $\mathcal{M}_2$ at $w$ (since $b_2 = t_2(q) \not\sqsubseteq_2 a_2 = t_2(p)$). However, as shown in Lemma 4, $\mathcal{M}_1, w$ and $\mathcal{M}_2, w$ are modally equivalent with respect to the language $\mathcal{L}_{\mathsf{CHB}}$.[6]

LEMMA 4. *For all* $\varphi \in \mathcal{L}_{\mathsf{CHB}}$, $\mathcal{M}_1, w \Vdash \varphi$ *iff* $\mathcal{M}_2, w \Vdash \varphi$.

*Proof.* The proof follows by induction on the structure of $\varphi$, where cases for the propositional variables, the Boolean connectives, and $\varphi := \Box\psi$ are trivial. The case for $\varphi := [\geq]\psi$ is the same as the one for $\varphi := \Box\psi$ since we have a single possible world in both models. So assume inductively that the result holds for $\psi$ and $\chi$, and show that it holds also for $\varphi := B^{\psi}\chi$. For the direction left-to-right, suppose that $\mathcal{M}_1, w \Vdash B^{\psi}\chi$. This means that $Min_{\geq}|\psi|_{\mathcal{M}_1} \subseteq |\chi|_{\mathcal{M}_1}$ and $t_1(\chi) \sqsubseteq_1 \mathfrak{b}_1 \oplus_1 t_1(\psi)$. Observe that, no matter what the topics of $\chi$ and $\psi$ are, as $\mathfrak{b}_2$ is the top element in $T_2$, we have $t_2(\chi) \sqsubseteq_2 \mathfrak{b}_2 \oplus_2 t_2(\psi)$. Moreover, either $|\psi|_{\mathcal{M}_1} = \{w\}$ or $|\psi|_{\mathcal{M}_1} = \emptyset$. If the former is the case, then $|\chi|_{\mathcal{M}_1} = \{w\}$ as well (since $Min_{\geq}|\psi|_{\mathcal{M}_1} = \{w\} \subseteq |\chi|_{\mathcal{M}_1}$). Then, by induction hypothesis, we have $|\psi|_{\mathcal{M}_2} = \{w\}$ and $|\chi|_{\mathcal{M}_2} = \{w\}$, therefore, $Min_{\geq}|\psi|_{\mathcal{M}_2} \subseteq |\chi|_{\mathcal{M}_2}$. If the latter is the case, then, by induction hypothesis, we

[5] It is easy to see that, for any $\varphi, \chi \in \mathcal{L}_{\mathsf{CHB}}$, sentences $B^{\chi}\overline{\varphi}$ and $B^{\overline{\chi}}\overline{\varphi}$ are logically equivalent with respect to the proposed semantics. We think that the latter is a better fit for the proposed reading as 'The agent would come to grasp the topic of $\varphi$, were they to grasp the topic of $\chi$'.

[6] In figures of tsp-models, circles represent possible worlds, diamonds represent possible topics. Valuation and topic assignment are given by labeling each node with propositional variables. We omit labeling when a node is assigned every element in Prop. Reflexive and transitive arrows on possible worlds are omitted.

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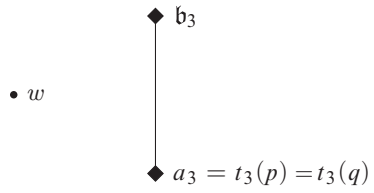



Fig. 2. Model $\mathcal{M}_3$.

Table 2. *Axiomatization* CHB *of the logic of conditional hyperintensional belief (over* $\mathcal{L}_{\mathsf{CHB}}$*)*

| | |
|---|---|
| (CPL) | all classical propositional tautologies and Modus Ponens |
| ($\mathsf{S5}_{\Box}$) | S5 axioms and rules for $\Box$ |
| ($\mathsf{S4}_{[\geq]}$) | S4 axioms and rules for $[\geq]$ |
| | **(I) Axioms for $\Box$ and $[\geq]$:** |
| (Inc) | $\Box\varphi \rightarrow [\geq]\varphi$ |
| (Tot) | $\Box([\geq]\varphi \rightarrow \psi) \vee \Box([\geq]\psi \rightarrow \varphi)$ |
| | **(II) Axioms for $B$:** |
| (Ax$\top$) | $B^{\varphi}\top$ |
| (Ax1) | $B^{\varphi}\overline{\psi}$, if $Var(\psi) \subseteq Var(\varphi)$ |
| (Ax2) | $(B^{\varphi}\overline{\psi} \wedge B^{\psi}\overline{\chi}) \rightarrow B^{\varphi}\overline{\chi}$ |
| (Ax3) | $(B^{\varphi}\psi \wedge B^{\varphi}\chi) \leftrightarrow B^{\varphi}(\psi \wedge \chi)$ |
| | **(III) Axioms connecting $B$, $\Box$, and $[\geq]$:** |
| (Ax4) | $(\Box(\varphi \rightarrow \psi) \wedge B^{\varphi}\overline{\psi}) \rightarrow B^{\varphi}\psi$ |
| (Ax5) | $B^{\varphi}\psi \rightarrow \Box B^{\varphi}\psi$ |
| (Ax6) | $B^{\varphi}\psi \leftrightarrow (\Box(\varphi \rightarrow \langle\geq\rangle(\varphi \wedge [\geq](\varphi \rightarrow \psi))) \wedge B^{\varphi}\overline{\psi})$ |

have $|\psi|_{\mathcal{M}_2} = \emptyset$. Therefore, $Min_{\geq}|\psi|_{\mathcal{M}_2} = \emptyset \subseteq |\chi|_{\mathcal{M}_2}$. We can then conclude that $\mathcal{M}_2, w \Vdash B^{\psi}\chi$. The other direction follows analogously. □

To see that $\mathcal{L}_{\mathsf{CHB}}$ is not expressive enough to state '$\varphi$ and $\chi$ have exactly the same topic', compare the model $\mathcal{M}_1$ given in Figure 1a with $\mathcal{M}_3 = \langle\{w\}, \geq, \{a_3, \mathfrak{b}_3\}, \oplus_3, \mathfrak{b}_3, t_3, v\rangle$, where $(\{a_3, \mathfrak{b}_3\}, \oplus_3, t_3)$ is as given in Figure 2. It is then easy to see that '$p$ and $q$ have exactly the same topic' is true in $\mathcal{M}_3$ at $w$ (since $t_3(p) = a_3 = t_3(q)$), whereas it is false in $\mathcal{M}_1$ at $w$ (since $t_1(q) = a_1 \neq b_1 = t_1(p)$). However, $\mathcal{M}_1, w$ and $\mathcal{M}_3, w$ are modally equivalent with respect to the language $\mathcal{L}_{\mathsf{CHB}}$, that is, for all $\varphi \in \mathcal{L}_{\mathsf{CHB}}$, $\mathcal{M}_1, w \Vdash \varphi$ iff $\mathcal{M}_3, w \Vdash \varphi$ (the proof follows similarly to the proof of Lemma 4).

*4.1.1. Axiomatization, soundness, and completeness.* A sound and complete axiomatization CHB of the logic of *conditional hyperintensional belief* over $\mathcal{L}_{\mathsf{CHB}}$ with respect to tsp-models is presented in Table 2. Following the same structure as in §3.2, we first elaborate on the intuitive readings of the axioms and rules presented in Table 2 together with a few derivable principles given in Theorem 5. We then continue with further (in)validities of interest concerning the hyperintensional nature of static belief revision. The soundness and completeness proofs are given in Appendices B.3.1 and

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B.3.2, respectively, where the latter proof involves a canonical model construction together with a non-trivial topic algebra, as well as a filtration argument.

THEOREM 5. *The following are derivable from* CHB:

1. $\Diamond\varphi \to \neg B^{\varphi}\bot$
2. $B^{\varphi}\varphi$
3. *from* $\varphi \leftrightarrow \chi$, $B^{\overline{\varphi}}\overline{\chi}$, *and* $B^{\overline{\chi}}\overline{\varphi}$ *infer* $B^{\varphi}\psi \leftrightarrow B^{\chi}\psi$
4. *from* $\varphi \leftrightarrow \chi$, $B^{\overline{\varphi}}\overline{\chi}$, *and* $B^{\overline{\chi}}\overline{\varphi}$ *infer* $B^{\psi}\varphi \leftrightarrow B^{\psi}\chi$

*Proof.* See Appendix B.2. □

The fragment of CHB having only $\Box$ and $[\geq]$ as modal operators is the well-known normal modal logic of total preorders extended with the global modality (here denoted by $\Box$). Therefore, the axiomatization of $\Box$ and $[\geq]$ as $\mathsf{S5}_{\Box}$ and $\mathsf{S4}_{[\geq]}$, respectively, together with the so-called inclusion axiom Inc is standard [19]. Axiom Tot guarantees that the plausibility order is total. Axioms of Group (II) states the properties of the conditional belief operators. Ax$\top$ is a generalization of the axiom of the same name for plain belief given in Table 1. Ax1 states that learning $\varphi$ involves having grasped every subject matter that is included in the subject matter of $\varphi$. This axiom is keyed to the assumption that the subject matter of a complex sentence is the fusion of the subject matters of its primitive components. Ax2 simply says that topic parthood is a transitive relation. Ax3 is the conditional belief counterpart of $\mathsf{C}_B$ for plain belief given in Table 1: conditional belief as well is fully conjunctive. The axioms in Group (III) regulate the relationship between conditional beliefs, $\Box$, and $[\geq]$. Ax4 and Ax5 are generalizations of Ax2 and Ax3 of Table 1, respectively. The last axiom Ax6 on the other hand explicates the link between conditional beliefs, plausibility ordering, and topic sensitivity of static belief revision: if one were to learn that $\varphi$, one would believe that $\psi$ was the case (i.e., $B^{\varphi}\psi$ is true) iff $\psi$ is true in the most plausible $\varphi$-worlds (the first conjunct of the right-hand-side) and the topic of $\psi$ is included in the topic of $\varphi$ (the second conjunct of the right-hand-side, also see footnote 5).

Finally, we focus on the derivable principles given in Theorem 5. The principle in Theorem 5.1 is called *Consistency of Revision*: one would not believe a falsehood was the case as long as one were to receive consistent information. Moreover, conditional belief satisfies a counterpart of the AGM Success postulate $B^{\varphi}\varphi$ (Theorem 5.2): one who learns that $\varphi$ comes to believe that it was the case (as is well known, Success is not problematic for static belief revision, whereas it needs to be handled carefully for dynamic belief revision due to the Moore sentences: see [23]). The last two derivable inference rules (Theorems 5.3 and 5.4) are topic-sensitive versions of replacement of provable equivalents rules and are elaborated further in §4.1.2.

THEOREM 6. CHB *is a sound and complete axiomatization of* $\mathcal{L}_{\mathsf{CHB}}$ *with respect to the class of tsp-models*: *for every* $\varphi \in \mathcal{L}_{\mathsf{CHB}}$, $\vdash_{\mathsf{CHB}} \varphi$ *if and only if* $\vDash \varphi$.

*Proof.* See Appendix B.3 □

*4.1.2. Further validities and invalidities.* The validities and invalidities we get for conditional belief are the same as the ones in [8], to which we refer for a fuller discussion, and where one also finds the various semantic proofs/countermodels. Here we limit ourselves to a few remarks.

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The next two valid principles are often called, respectively, Cut or Limited Transitivity, and Cautious Monotonicity, in the literature on nonmonotonic logics and operators [26]:[7]

1. $\models (B^{\varphi}\psi \wedge B^{\varphi\wedge\psi}\chi) \rightarrow B^{\varphi}\chi$
2. $\models (B^{\varphi}\psi \wedge B^{\varphi}\chi) \rightarrow B^{\varphi\wedge\psi}\chi$

They are advocated in [11] as 'principles of informational economy' and [17] takes them as minimal conditions a nonmonotonic entailment ought to obey. It may therefore be taken as a good feature of our conditional belief operator that, in spite of its being weaker than its non-hyperintensional counterpart, it still satisfies them.

Conditional belief is, as usual, nonmonotonic:

3. $\not\models B^{\varphi}\psi \rightarrow B^{\varphi\wedge\chi}\psi$
   [*Countermodel*: let $W = \{w, u\}$, $\geq = \{(w, w), (u, u), (u, w)\}$, $T = \{\mathfrak{b}\}$, $t(p) = \{\mathfrak{b}\}$ for all $p \in \mathsf{Prop}$, and $v(p) = \{w, u\}$, $v(q) = \{w\}$, and $v(r) = \{u\}$. Since the model has only one possible topic, the topic component in this particular case does not play any essential role. We then have that $w \Vdash B^{p}q$ since $Min_{\geq}|p| = \{w\} = v(q)$. However, $w \not\Vdash B^{p\wedge r}q$ since $Min_{\geq}|p \wedge r| = \{u\} \not\subseteq v(q) = \{w\}$.]

Having learned that Franz has been lecturing all day $(p)$, you come to believe that he must have been at the University of Amsterdam $(q)$. But if you learned that Franz has been lecturing all day and that he has recently been hired by the University of St. Andrews $(p \wedge r)$, you would not come to believe that he must have been at the University of Amsterdam.

The next principle is a notable failure due to topic-sensitivity, and not found in more standard settings for conditional belief:

4. $\not\models B^{\varphi}\chi \rightarrow B^{\varphi}(\psi \vee \chi)$

If you learned that Sonja is in Amsterdam, you'd come to believe that she was in the Netherlands. You wouldn't thereby automatically come to believe that Sonja was either in the Netherlands or on planet Kepler-442b (you may never have heard of Kepler-442b to begin with!), though the former logically entails the latter. Disjunction Introduction can fail in this setting because it can take you off-topic.

Further invalidities illustrate conditional belief's lack of closure with respect to a priori implications:

5. $\not\models \Box(\varphi \rightarrow \psi) \rightarrow B^{\varphi}\psi$
6. $\not\models (\Box(\varphi \rightarrow \psi) \wedge B^{\chi}\varphi) \rightarrow B^{\chi}\psi$

Non-omniscience is further modeled by our agent's failing to believe everything conditional on inconsistent information, and by their failing to conditionally believe all logical truths:

7. $\not\models B^{\varphi\wedge\neg\varphi}\psi$

[7] As per Theorem 6, both of these principles are derivable in CHB. Their derivations use Ax6 (together with other axioms and inference rules) and require tedious syntactic manipulations involving the operators $\Box$ and $[\geq]$. As their derivations are quite long and, for our purposes in this paper, not instructive, we prefer to state them as validities.

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8. $\nvDash B^{\varphi}(\neg\psi \vee \psi)$
[*Countermodel*: Let $\mathcal{M}_6 = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$, where $W = \{w\}$, $\geq = \{(w,w)\}$, $T = \{a, \mathfrak{b}\}$, $\mathfrak{b} \sqsubset a$ such that $\mathfrak{b} = t(q)$, $t(p) = a$, and $v(p) = v(q) = \{w\}$. For invalidities (4), (7), and (8): take $\varphi := q$, $\chi := q$, and $\psi := p$. We then have $w \Vdash B^q q$ but $w \nVdash B^q(p \vee q)$ since $t(p \vee q) = a \not\sqsubseteq \mathfrak{b} \oplus t(q) = \mathfrak{b}$, therefore, (4) is invalid. Similarly, $t(p) = t(p \vee \neg p) = a \not\sqsubseteq \mathfrak{b} \oplus t(q) = \mathfrak{b} \oplus t(q \wedge \neg q) = \mathfrak{b}$, therefore, $w \nVdash B^{q \wedge \neg q} p$ and $w \nVdash B^q(p \vee \neg p)$. Hence, (7) and (8) are also invalid. For invalidities (5) and (6): take $\varphi := q$, $\chi := q$, and $\psi := p \vee \neg p$ and the argument follows similarly.]

The remaining (in)validities concerning introspection principles and Replacement of (provable) Equivalents rules (RE) are not of focus in [8]. In particular, the ones involving plain belief cannot be formalised in the logic of [8], as it studies only conditional beliefs.

The reasoning behind the following valid and invalid introspection principles is similar to the one behind the analogous principles we have for plain belief:

9. **Positive Introspection:** $\vDash B^{\varphi}\psi \to B^{\varphi}B^{\varphi}\psi$
*Proof*: Let $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ be a tsp-model and $w \in W$ such that $\mathcal{M}, w \Vdash B^{\varphi}\psi$. This means that $Min_{\geq}(|\varphi|) \subseteq |\psi|$ and $t(\psi) \sqsubseteq \mathfrak{b} \oplus t(\varphi)$. By the latter, we have that $t(B^{\varphi}\psi) = t(\psi) \oplus t(\varphi) \sqsubseteq \mathfrak{b} \oplus t(\varphi)$. Moreover, since the truth of $B^{\varphi}\psi$ is state-independent, we have $|B^{\varphi}\psi| = W$, therefore, $Min_{\geq}(|\varphi|) \subseteq |B^{\varphi}\psi|$. We then conclude that $\mathcal{M}, w \Vdash B^{\varphi}B^{\varphi}\psi$.]

10. **Negative Introspection:** $\nvDash \neg B^{\varphi}\psi \to B^{\varphi}\neg B^{\varphi}\psi$
[*Countermodel*: Consider the tsp-model $\mathcal{M}_6$ given above. We then have $w \Vdash \neg B^q p$, i.e., $w \nVdash B^q p$, since $t(p) = a \not\sqsubseteq \mathfrak{b} \oplus t(q) = \mathfrak{b}$. Similarly, $t(\neg B^q p) = t(p) \oplus t(q) = a \not\sqsubseteq \mathfrak{b} \oplus t(q) = \mathfrak{b}$, thus $w \nVdash B^q \neg B^q p$.]

11. **Topic-sensitive Negative Introspection:** $\vDash (\neg B^{\varphi}\psi \wedge B\overline{\psi}) \to B^{\varphi}\neg B^{\varphi}\psi$
[*Proof*: Let $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ be a tsp-model and $w \in W$ such that $\mathcal{M}, w \Vdash \neg B^{\varphi}\psi$ and $\mathcal{M}, w \Vdash B\overline{\psi}$. While the former means that $|\neg B^{\varphi}\psi| = W$, the latter means $t(\psi) \sqsubseteq \mathfrak{b}$. Therefore, $Min_{\geq}(|\varphi|) \subseteq |\neg B^{\varphi}\psi|$ and $t(\neg B^{\varphi}\psi) = t(\psi) \oplus t(\varphi) \sqsubseteq \mathfrak{b} \oplus t(\varphi)$. We then conclude that $\mathcal{M}, w \Vdash B^{\varphi}\neg B^{\varphi}\psi$.]

However, one notable failure of positive introspection involving both plain and conditional belief is the following, and it fails due to topicality:

12. $\nvDash B^{\varphi}\psi \to BB^{\varphi}\psi$
[*Countermodel*: let $W = \{w\}$, $\geq = \{(w,w)\}$, $T = \{a, \mathfrak{b}\}$, $\mathfrak{b} \sqsubset a$ such that $t(q) = (p) = a$, and $v(p) = v(q) = \{w\}$. Obviously, $w \Vdash B^q p$. However, since $t(B^q p) = t(p) \oplus t(q) = a \not\sqsubseteq \mathfrak{b}$, $w \nVdash BB^q p$.]

Having learned that Achilles tendon rupture causes walking difficulties, you'd come to believe that Tom cannot run 10km with his ruptured tendon. Nevertheless, you wouldn't plainly believe that you'd come to believe that Tom cannot run 10 km with a broken Achilles tendon if you were to learn that such tendon rupture causes walking difficulties: you may have never heard of Achilles tendon to begin with.

Hyperintensionality is further displayed by failure of RE both in antecedent and in consequent position for conditional belief:

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13. **RE**$_B$1: from $\varphi \leftrightarrow \psi$, infer $B^{\varphi}\chi \leftrightarrow B^{\psi}\chi$
14. **RE**$_B$2: from $\varphi \leftrightarrow \psi$, infer $B^{\chi}\varphi \leftrightarrow B^{\chi}\psi$
[*Countermodel*: See the tsp-model $\mathcal{M}_6$ given above, and take $\varphi := p \vee \neg p$, $\psi := q \vee \neg q$, and $\chi := p \vee \neg p$ for RE$_B$1, and $\chi := q \vee \neg q$ for RE$_B$2.]

Yet we have the following weaker, topic-sensitive versions of RE rules valid (see Theorems 5.3, 5.4, and 6):

15. **Topic-sensitive RE**$_B$1: from $\varphi \leftrightarrow \chi$, $B^{\overline{\varphi}}\overline{\chi}$, and $B^{\overline{\chi}}\overline{\varphi}$ infer $B^{\varphi}\psi \leftrightarrow B^{\chi}\psi$
16. **Topic-sensitive RE**$_B$2: from $\varphi \leftrightarrow \chi$, $B^{\overline{\varphi}}\overline{\chi}$, and $B^{\overline{\chi}}\overline{\varphi}$ infer $B^{\psi}\varphi \leftrightarrow B^{\psi}\chi$

If $\varphi$ and $\chi$ are logically equivalent *and* their topics complement each other with respect to the topic of the agent's belief state, they are interchangeable both in antecedent and in consequent position of conditional belief. The validity of this twofold rule allows us to prove a completeness result for the dynamic extension, by using reduction axioms (see [45, sec. 7.4] for a detailed presentation of completeness by reduction).

***4.2. Dynamic hyperintensional belief revision.*** We now extend $\mathcal{L}_{\mathsf{CHB}}$ with a dynamic *topic-sensitive lexicographic upgrade* operator, $[\Uparrow\varphi]$. We work with the language $\mathcal{L}_{\mathsf{DHB}}$ of *dynamic hyperintensional belief* defined by the grammar:

$$\varphi := p_i \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid [\geq]\varphi \mid B^{\varphi}\varphi \mid [\Uparrow\varphi]\varphi$$

One can read $[\Uparrow\varphi]\psi$ as 'After revision by $\varphi$, $\psi$ holds' but a less terse reading makes clear that dynamic belief revision is topic-sensitive, too: 'After the agent has received information $\varphi$ *and* has come to grasp the topic of $\varphi$, $\psi$ holds'. The semantics for this will require an innovative twofold model-transforming technique: the traditional dynamic of lexicographic upgrade will be paired with a dynamic of topics.

A dt-model for $\mathcal{L}_{\mathsf{DHB}}$ is defined as for $\mathcal{L}_{\mathsf{CHB}}$, where $t$ extends to $\mathcal{L}_{\mathsf{DHB}}$ in a similar way. $[\Uparrow]$ is also taken to be topic transparent: $t([\Uparrow\varphi]\psi) = t(\varphi) \oplus t(\psi)$. From now on we only consider doxastic topic models for $\mathcal{L}_{\mathsf{DHB}}$. The semantics for '$[\Uparrow\varphi]\psi$' needs auxiliary definitions:

Definition 6 (Updated dt-Model). *Given a dt-model $\mathcal{T} = \langle T, \oplus, \mathfrak{b}, t\rangle$ and $a \in T$, the update of $\mathcal{T}$ by $a$ is the tuple $\mathcal{T}^a = \langle T, \oplus, \mathfrak{b}^a, t^a\rangle$ where*

1. $\mathfrak{b}^a = \mathfrak{b} \oplus a$,
2. $t^a(\varphi) = \begin{cases} t(\varphi) \oplus a, & \text{if } \top \in Var(\varphi) \\ t(\varphi) & \text{otherwise.} \end{cases}$

Lemma 7. *Given a dt-model $\langle T, \oplus, \mathfrak{b}, t\rangle$ and $a \in T$, the update $\mathcal{T}^a = \langle T, \oplus, \mathfrak{b}^a, t^a\rangle$ of $\mathcal{T}$ by $a$ is a dt-model.*

*Proof.* Observe that $\mathcal{T}$ and $\mathcal{T}^a$ share exactly the same join semilattice $(T, \oplus)$. Therefore, as $\mathcal{T}$ is a topic model, item 2 of Definition 1 for $\mathcal{T}^a$ is already satisfied. Given unrestricted fusion, $\mathfrak{b} \oplus a$ always exists in $T$, thus, $\mathfrak{b}^a = \mathfrak{b} \oplus a \in T$. Now, let $\varphi \in \mathcal{L}_{\mathsf{DHB}}$ such that $Var(\varphi) = \{x_1, \ldots, x_n\}$. Observe that, by Definition 6.2, $t^a(x) = t(x)$ for all $p \in \mathsf{Prop}$ and $t^a(\top) = \mathfrak{b} \oplus a = \mathfrak{b}^a$. If $\top \notin Var(\varphi)$, we have $t^a(\varphi) = t(\varphi) = t(x_1) \oplus \cdots \oplus t(x_n) = t^a(x_1) \oplus \cdots \oplus t^a(x_n)$. If $\top \in Var(\varphi)$, i.e., (w.l.o.g.) $\top = x_n$, we obtain that $t^a(\varphi) = t(\varphi) \oplus a = t(x_1) \oplus \cdots \oplus t(x_n) \oplus a = t(x_1) \oplus \cdots \oplus t(\top) \oplus a = t(x_1) \oplus \cdots \oplus (\mathfrak{b} \oplus a) = t(x_1) \oplus \cdots \oplus \mathfrak{b}^a = t^a(x_1) \oplus \cdots \oplus t^a(\top) = t^a(x_1) \oplus \cdots \oplus t^a(x_n)$. We therefore conclude that $\mathcal{T}^a$ satisfies Definition 1.4 as well. □

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Such model-transformation represents the agent's coming to grasp new subject matter. The dynamic operation is purely doxastic: it models what further subject matter the agent grasps after an informative event.

DEFINITION 7 (Upgraded plausibility frame). *Given a plausibility frame* $\langle W, \geq \rangle$ *and* $P \subseteq W$, *the upgraded frame by* $P$ *is the tuple* $\mathcal{S}^{\Uparrow P} = \langle W, \geq^{\Uparrow P} \rangle$, *where* $\geq^{\Uparrow P}$ *is the new ordering such that* $v \geq^{\Uparrow P} w$ *iff* (1) $v \geq w$ *and* $w \in P$, *or* (2) $v \geq w$ *and* $v \in W \backslash P$, *or* (3) ($v \geq w$ *or* $w \geq v$) *and* $w \in P$ *and* $v \in W \backslash P$.

This is the well-known lexicographic upgrade operator making all $P$-worlds more plausible than all $W \backslash P$-worlds, and keeping the ordering the same within those two zones.

DEFINITION 8 ($\Vdash$-Semantics for $\mathcal{L}_{\mathsf{DHB}}$). *Given a tsp-model* $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v \rangle$ *and a state* $w \in W$, *the* $\Vdash$*-semantics for* $\mathcal{L}_{\mathsf{DHB}}$ *is as before for the components in* $\mathcal{L}_{\mathsf{CHB}}$, *plus*:

$$\mathcal{M}, w \Vdash [\Uparrow \varphi]\psi \quad \textit{iff} \quad \mathcal{M}^{\Uparrow \varphi}, w \Vdash \psi$$

*where* $\mathcal{M}^{\Uparrow \varphi} = \langle W, \geq^{\varphi}, T, \oplus, \mathfrak{b}^{\varphi}, t^{\varphi}, v \rangle$ *such that* $\geq^{\varphi} = \geq^{\Uparrow |\varphi|_{\mathcal{M}}}$, $\mathfrak{b}^{\varphi} = \mathfrak{b}^{t(\varphi)}$, *and* $t^{\varphi} = t^{t(\varphi)}$ *as described in Definition* 6.

For $[\Uparrow \varphi]\psi$ to hold, $\psi$ must hold in the model transformed across two dimensions: firstly, all the $\varphi$-worlds must have become more plausible than all the $\neg\varphi$-worlds. Secondly, the agent must have grasped the topic of $\varphi$, merging it with $\mathfrak{b}$ – the overall subject matter they were already on $\top$op of before.

PROPOSITION 8. *Given a tsp-model* $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v \rangle$ *and* $\varphi \in \mathcal{L}_{\mathsf{DHB}}$, $\mathcal{M}^{\Uparrow \varphi} = \langle W, \geq^{\varphi}, T, \oplus, \mathfrak{b}^{\varphi}, t^{\varphi}, v \rangle$ *is a tsp-model.*

*Proof.* It is easy to verify that $\geq^{\varphi}$ is a well-order. Moreover, by Lemma 7, we know that $\langle T, \oplus, \mathfrak{b}^{\varphi}, t^{\varphi} \rangle$ is a dt-model. □

We now turn to the important invalidities and axiomatization of our dynamic logic for belief revision. As intended, our dynamic operator is sensitive to hyperintensional differences, that is, the following dynamic RE rule is invalid:

**Dynamic RE:** from $\varphi \leftrightarrow \chi$ infer $[\Uparrow \varphi]\psi \leftrightarrow [\Uparrow \chi]\psi$

Note that this is the case only when $\psi$ is a doxastic sentence. Therefore, in particular, the principle

**Dynamic RE$_B$**: from $\varphi \leftrightarrow \chi$ infer $[\Uparrow \varphi]B\psi \leftrightarrow [\Uparrow \chi]B\psi$

is not valid (see the counterexample for RE$_B$1 in §4.1.2). One can come to believe different things after revising one's beliefs with equivalent pieces of information which differ in topic. As expected though, the topic-sensitive version of Dynamic RE is valid:

**Topic-sensitive Dynamic RE:** from $\varphi \leftrightarrow \chi$, $B^{\overline{\varphi}}\overline{\chi}$, and $B^{\overline{\chi}}\overline{\varphi}$ infer $[\Uparrow \varphi]\psi \leftrightarrow [\Uparrow \chi]\psi$

Moreover, our topic-sensitive dynamic operator complies with the standard reduction axioms of lexicographic upgrade (presented, e.g., in [38, 42, 49]). We therefore obtain a completeness result for the logic of dynamic hyperintensional belief revision, DHB:

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Table 3. *Reduction Axioms and Inference Rules for* $\Uparrow$ *(over* $\mathcal{L}_{\mathsf{DHB}}$*)*

| | |
|---|---|
| $(\mathrm{Nec}_{\Uparrow})$ | from $\psi$ infer $[\Uparrow\varphi]\psi$ |
| $(\mathrm{R}_{\top})$ | $[\Uparrow\varphi]\top \leftrightarrow (\top \wedge \overline{\varphi})$ |
| $(\mathrm{R}_{p})$ | $[\Uparrow\varphi]p \leftrightarrow (p \wedge \overline{\varphi})$ |
| $(\mathrm{R}_{\neg})$ | $[\Uparrow\varphi]\neg\psi \leftrightarrow \neg[\Uparrow\varphi]\psi$ |
| $(\mathrm{R}_{\wedge})$ | $[\Uparrow\varphi](\psi \wedge \chi) \leftrightarrow ([\Uparrow\varphi]\psi \wedge [\Uparrow\varphi]\chi)$ |
| $(\mathrm{R}_{\Box})$ | $[\Uparrow\varphi]\Box\psi \leftrightarrow \Box[\Uparrow\varphi]\psi$ |
| $(\mathrm{R}_{[\geq]})$ | $[\Uparrow\varphi][\geq]\psi \leftrightarrow ((\neg\varphi \rightarrow [\geq][\Uparrow\varphi]\psi) \wedge (\neg\varphi \rightarrow \Box(\varphi \rightarrow [\Uparrow\varphi]\psi)) \wedge [\geq](\varphi \rightarrow [\Uparrow\varphi]\psi))$ |
| $(\mathrm{R}_{B})$ | $[\Uparrow\varphi]B^{\psi}\chi \leftrightarrow ((\Diamond(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{\varphi\wedge[\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi) \vee (\neg\Diamond(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{[\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi))$ |

THEOREM 9. DHB *is soundly and completely axiomatized by the static base logic of conditional belief* CHB *in Table 2 plus the set of axioms and rules for* $[\Uparrow]$ *given in Table 3*:

*Proof.* See Appendix B.4. □

The reduction axioms formalize the effect of the dynamic operator $[\Uparrow]$ on each component of the language $\mathcal{L}_{\mathsf{DHB}}$ and give us a recursive rewriting algorithm to step-by-step translate every formula containing the topic-sensitive lexicographic upgrade operator to a provably equivalent formula in the language $\mathcal{L}_{\mathsf{CHB}}$. Most of the axioms and rules in Table 3 are standard for lexicographic upgrade, so we refer to [38, p. 143] for their intuitive readings. However, notice the nonstandard formulations of $\mathrm{R}_{\top}$ and $\mathrm{R}_{p}$. For the reduction procedure to go through, we need both of them to be in the shape given in Table 3 rather than their standard forms $[\Uparrow\varphi]\top \leftrightarrow \top$ and $[\Uparrow\varphi]p \leftrightarrow p$, respectively. This is due to the fact that the replacement rules $\mathrm{RE}_{B}1$ and $\mathrm{RE}_{B}2$ are not valid on tsp-models, however, their topic-sensitive versions are (see Theorems 5.3 and 5.4). $\mathrm{R}_{\top}$ and $\mathrm{R}_{p}$ in their current form guarantee that the sentences on both sides of the equivalences have the same topic (see Lemma 46 for the use of topic-sensitive RE rules). Reduction axiom $R_{[\geq]}$ encodes precisely how upgrades change the plausibility relation: after an upgrade with $\varphi$ all $\geq$-accessible worlds make $\psi$ true iff (1) if the actual world is a $\neg\varphi$-world then every $\geq$-accessible world will become a $\psi$-world after the upgrade, (2) if the actual world is a $\neg\varphi$-world then every $\varphi$-world will become a $\psi$-world after the upgrade, and (3) all $\geq$-accesible $\varphi$-worlds become $\psi$-worlds after the upgrade [49].

The next interesting reduction axiom is $\mathrm{R}_{B}$. While it seems complicated to parse, it simply gives us the case distinction determining the resulting upgraded order and indicates the behavior of topic fusion. Regarding the former, we have the reading given in [38]: after the $\varphi$-upgrade all most plausible $\psi$-worlds satisfy $\chi$ iff if there is a $\varphi$-state which makes $[\Uparrow\varphi]\psi$ true, then the most plausible $[\Uparrow\varphi]\psi$-worlds with respect to $\geq^{\varphi}$ are the same as the most plausible $\varphi \wedge [\Uparrow\varphi]\psi$-worlds with respect to $\geq$, else the $\geq^{\varphi}$-order

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among the $[\Uparrow\varphi]\psi$-worlds is the same as the $\geq$-order. Regarding topicality: on the left-hand-side we have that after the agent grasps the topic of $\varphi$, the agent would come to grasp the topic of $\chi$, were they to grasp the topic of $\psi$. In other words, the topic of $\chi$ is included in the topic of the whole doxastic state of the agent, expanded by the topics of $\varphi$ and $\psi$. On the right-hand-side, given the topic transparency of $\wedge$ and $[\Uparrow]$, we obtain the same reading via the conditional belief operators of both disjuncts (see Appendix B.4 for the soundness proof).

Some features of our topic-sensitive lexicographic upgrade operator might remind the reader of the explicit upgrade/observation operators of Velázquez–Quesada [46, 48, 49]. In both approaches, well-known DEL methods of modeling information change, such as lexicographic upgrade and world elimination, are equipped with additional tools to render the modeled agents non-omniscient. Our approach to dynamic hyperintensional belief revision departs from that of [49] in various ways. First, while we use topic-sensitive models, [49] appeals to awareness structures to account for non-omniscient agents. Next, the dynamic logic presented in [49] does not contain hyperintensional belief or knowledge operators as primitives. They are, rather, defined in terms of awareness and normal modal operators. What we are interested in, on the other hand, is the effect of learning on the static *hyperintensional* belief operators taken as primitives.

**§5. Conclusion and future work.** A first area for further work is philosophical. Topicality is a general semantic feature of propositional content, whose exploration is still in its infancy (see [21] for an overview and discussion). If intentional states are generally hyperintensional because of their having topic-sensitive propositional content, one can expect frameworks broadly similar to the one explored above to apply to intentional states ranging from knowledge to supposition and imagination (see [7, 9] for some initial work in this area). One may conjecture that a general topic-sensitive semantics for the intentional states of non-omniscient agents may resort to neighborhood structures [32], given that the topicality filter fixes precisely the main shortcoming of neighborhoods with respect to hyperintensional phenomena: its forcing agents to have the same attitudes towards all logical equivalents.

A more technical area of further investigation is in the ballpark of axiomatization and completeness results. We have completeness for our hyperintensional plain and conditional belief logics, and we have a reduction of our dynamics of belief revision to our static conditional belief. However, completeness for the latter is obtained via the help of the normal modal operator $[\geq]\varphi$. Whereas this is pretty standard and well-motivated in the literature on Dynamic Epistemic Logic, one may wish for a complete axiomatization of the hyperintensional conditional belief logic without this modality. Moreover, the appropriate topic-sensitive versions of various dynamic attitudes, such as conservative upgrade and public announcements, are still to be investigated.

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## §A. Proofs of §3.

***A.1. Proof of Theorem 1.*** Axiom labels refer to the ones in Table 1 and $\vdash$ abbreviates $\vdash_{\mathsf{PHB}}$.

1. $\vdash B\overline{\varphi} \leftrightarrow \bigwedge_{x\in Var(\varphi)} B\overline{x}$:
   Note that $B\overline{\varphi} := B(\bigwedge_{x\in Var(\varphi)}(x \vee \neg x)) := B(\bigwedge_{x\in Var(\varphi)} \overline{x})$. Then, by $\mathsf{C}_B$, we have $\vdash B(\bigwedge_{x\in Var(\varphi)} \overline{x}) \leftrightarrow (\bigwedge_{x\in Var(\varphi)} B\overline{x})$, i.e., $\vdash B\overline{\varphi} \leftrightarrow \bigwedge_{x\in Var(\varphi)} B\overline{x}$.

2. $\vdash B\overline{\varphi} \rightarrow B\overline{\psi}$, if $Var(\psi) \subseteq Var(\varphi)$:
   Follows from Theorem 1.1 and the fact that $(\varphi \wedge \psi) \rightarrow \varphi$ is a theorem of CPL.

3. $\vdash B(\overline{\varphi \wedge \psi}) \leftrightarrow B(\overline{\varphi} \wedge \overline{\psi})$:
   From left-to-right follows from Theorem 1.2 and $\mathsf{C}_B$. From right-to-left follows from Ax1 and Theorem 1.2.

4. $\vdash (B(\varphi \rightarrow \psi) \wedge B\varphi) \rightarrow B\psi$:
   1. $\vdash (B(\varphi \rightarrow \psi) \wedge B\varphi) \rightarrow (\Box(((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \psi) \wedge B((\varphi \rightarrow \psi) \wedge \varphi) \wedge B\overline{\psi})$ — $\mathsf{S5}_\Box$, $\mathsf{C}_B$, Thm 1.2
   2. $\vdash (\Box(((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \psi) \wedge B((\varphi \rightarrow \psi) \wedge \varphi) \wedge B\overline{\psi}) \rightarrow B\psi$ — Ax2
   3. $\vdash (B(\varphi \rightarrow \psi) \wedge B\varphi) \rightarrow B\psi$ — 2, 1, CPL.

5. $\vdash B\overline{\varphi} \rightarrow \Box B\overline{\varphi}$:
   An easy instance of Ax3.

6. $\vdash \neg B\overline{\varphi} \rightarrow \Box\neg B\overline{\varphi}$:
   1. $\vdash \neg B\overline{\varphi} \rightarrow \Box\neg\Box B\overline{\varphi}$ — $\mathsf{S5}_\Box$ ($\vdash \neg\psi \rightarrow \Box\neg\Box\psi$)
   2. $\vdash B\overline{\varphi} \leftrightarrow \Box B\overline{\varphi}$ — $\mathsf{T}_\Box$, Thm 1.5
   3. $\vdash \Box\neg\Box B\overline{\varphi} \leftrightarrow \Box\neg B\overline{\varphi}$ — 2, $\mathsf{S5}_\Box$, CPL
   4. $\vdash \neg B\overline{\varphi} \rightarrow \Box\neg B\overline{\varphi}$ — 1, 3, CPL

7. $\vdash \neg B\varphi \rightarrow \Box\neg B\varphi$:
   Similar to the above case: replace $B\overline{\varphi}$ by $B\varphi$, and Thm 1.5 by Ax3.

8. $\vdash (\Box\varphi \wedge B\overline{\varphi}) \rightarrow B\varphi$:
   1. $\vdash (\Box(\top \rightarrow \varphi) \wedge B\top \wedge B\overline{\varphi}) \rightarrow B\varphi$ — Ax2
   2. $\vdash (\Box\varphi \wedge B\overline{\varphi}) \leftrightarrow (\Box(\top \rightarrow \varphi) \wedge B\top \wedge B\overline{\varphi})$ — Ax$\top$, CPL, $\mathsf{S5}_\Box$
   3. $\vdash (\Box\varphi \wedge B\overline{\varphi}) \rightarrow B\varphi$ — 1, 2, CPL

9. $\vdash B\varphi \rightarrow BB\varphi$:
   1. $\vdash B\varphi \rightarrow (\Box B\varphi \wedge B\overline{B\varphi})$ — Ax3, Ax1, Thm 1.2
   2. $\vdash (\Box B\varphi \wedge B\overline{B\varphi}) \rightarrow BB\varphi$ — Thm 1.8
   3. $\vdash B\varphi \rightarrow BB\varphi$ — 1, 2, CPL

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10. $\vdash (\neg B\varphi \wedge B\overline{\varphi}) \rightarrow B\neg B\varphi$

| | | |
|---|---|---|
| 1. | $\vdash (\neg B\varphi \wedge B\overline{\varphi}) \rightarrow (\Box\neg B\varphi \wedge B\overline{\neg B\varphi})$ | Thm 1.7, Thm 1.2 |
| 2. | $\vdash (\Box\neg B\varphi \wedge B\overline{\neg B\varphi}) \rightarrow B\neg B\varphi$ | Thm 1.8 |
| 3. | $\vdash (\neg B\varphi \wedge B\overline{\varphi}) \rightarrow B\neg B\varphi$ | 1, 2, CPL |

### *A.2. Proof of Theorem 2: soundness and completeness of* PHB.

*A.2.1. Soundness of* PHB. Soundness is a matter of routine validity check, so we spell out only the relatively tricky cases.

*Proof.* Let $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ be a tsp-model and $w \in W$. Checking the soundness of the system S5 for $\Box$ is standard: recall that $\Box$ is interpreted as the global modality on tsp-models. Validity of $B\top$ is keyed to the stipulation $t(\top) = \mathfrak{b}$. Validity of $\mathsf{D}_B$ is guaranteed since the plausibility relation $\geq$ is well-founded. Validity of Ax1 is an immediate consequence of the semantic clause for $B$ and the definition of $\overline{\varphi}$. Ax3 is valid since truth of a belief sentence $B\varphi$ is state independent: it is easy to see that either $|B\varphi| = W$ or $|B\varphi| = \emptyset$, for any $\varphi \in \mathcal{L}_{\mathsf{PHB}}$. Here we spell out the details only for $\mathsf{C}_B$ and Ax2.

$\mathsf{C}_B$:

$$\begin{aligned}
\mathcal{M}, w \Vdash B(\varphi \wedge \psi) \text{ iff } & Min_{\geq}(W) \subseteq |\varphi \wedge \psi| \text{ and } t(\varphi \wedge \psi) \sqsubseteq \mathfrak{b} \\
\text{iff } & Min_{\geq}(W) \subseteq |\varphi| \cap |\psi| \text{ and } t(\varphi) \oplus t(\psi) \sqsubseteq \mathfrak{b} \\
\text{iff } & (Min_{\geq}(W) \subseteq |\varphi| \text{ and } Min_{\geq}(W) \subseteq |\psi|) \text{ and } (t(\varphi) \sqsubseteq \mathfrak{b} \text{ and } t(\psi) \sqsubseteq \mathfrak{b}) \\
\text{iff } & (Min_{\geq}(W) \subseteq |\varphi| \text{ and } t(\varphi) \sqsubseteq \mathfrak{b}) \text{ and } (Min_{\geq}(W) \subseteq |\psi| \text{ and } t(\psi) \sqsubseteq \mathfrak{b}) \\
\text{iff } & \mathcal{M}, w \Vdash B\varphi \wedge B\psi
\end{aligned}$$

Ax2:

Suppose that $\mathcal{M}, w \Vdash \Box(\varphi \rightarrow \psi) \wedge B\varphi \wedge B\overline{\psi}$, i.e., (1) $\mathcal{M}, w \Vdash \Box(\varphi \rightarrow \psi)$, (2) $\mathcal{M}, w \Vdash B\varphi$, and (3) $\mathcal{M}, w \Vdash B\overline{\psi}$. (1) means that $|\varphi| \subseteq |\psi|$, (2) implies that $Min_{\geq}(W) \subseteq |\varphi|$. Therefore, (1) and (2) together implies that $Min_{\geq}(W) \subseteq |\psi|$. Moreover, (3) is the case if and only if $t(\psi) \sqsubseteq \mathfrak{b}$. We therefore conclude that $\mathcal{M}, w \Vdash B\psi$. □

*A.2.2. Completeness of* PHB. The completeness proof is presented in full detail. Since the intensional component of the belief operator $B$ is interpreted as truth in the most plausible states—rather than as a standard Kripke operator—completeness is proven via a detour into an alternative semantics for $\mathcal{L}_{\mathsf{PHB}}$ based on, what we call, *topic-sensitive relational models* (or, in short, *tsr-models*). This semantics is closer in style to the standard relational semantics for modal logic, where $\Box$ is again interpreted as the global modality and $B$ as the standard KD45 modality with a topic component as before. These models will be proven to be equivalent to our tsp-models with respect to the language $\mathcal{L}_{\mathsf{PHB}}$. Therefore, completeness for our intended tsp-models follows from the completeness for the tsr-models. We then establish the completeness result via a *canonical topic-sensitive relational model* construction. Our canonical model construction is heavily inspired by the one presented in [18].

*From topic-sensitive relational to topic-sensitive plausibility models.*

DEFINITION 9 (Topic-sensitive relational model for $\mathcal{L}_{\mathsf{PHB}}$). *A topic-sensitive relational model (tsr-model) is a tuple* $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v\rangle$ *where* $W$, $T$, $\oplus$, $\mathfrak{b}$, $t$, *and* $v$ *are as*

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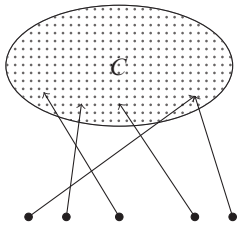



Fig. 3. $R_B = W \times C$, where the top ellipse illustrates the final clusters $C$ and an arrow relates the state it started from to every element in the cluster via $R_B$.

*before, and* $R_B \subseteq W \times W$ *is a serial relation such that*

$$\textit{for all } w, w' \in W,\ R_B(w) = R_B(w') \quad (\mathsf{Const} - R_B)$$

*where* $R_B(w) = \{v \in W : wR_Bv\}$.[8]

We recursively define the satisfaction relation $|\!\models$ with respect to tsr-models as follows. The reader should note the notational difference between $|\!\models$ and $\Vdash$, and recall that the latter denotes the semantics with respect to topic-sensitive *plausibility* models.

Definition 10 ($|\!\models$-Semantics for $\mathcal{L}_{\mathsf{PHB}}$ on tsr-models). *Given a tsr-model* $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v\rangle$ *and a state* $w \in W$, *the* $|\!\models$*-semantics for* $\mathcal{L}_{\mathsf{PHB}}$ *is as in Definition* 4 *for the propositional variables, Booleans, and* $\Box\varphi$, *plus*:

$$\mathfrak{M}, w \ |\!\models\ B\varphi \quad \textit{iff} \quad \forall v \in W (\textit{if } wR_Bv \textit{ then } \mathfrak{M}, v \ |\!\models\ \varphi) \textit{ and } t(\varphi) \sqsubseteq \mathfrak{b}.$$

We define the *intension of* $\varphi$ with respect to tsr-models $\mathfrak{M}$ as $[\![\varphi]\!]_{\mathfrak{M}} := \{w \in W : \mathfrak{M}, w \ |\!\models\ \varphi\}$, omit the subscript $\mathfrak{M}$ when the model is contextually clear.

Toward establishing the connection between tsr- and tsp-models, consider the tsr-model $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v\rangle$. Due to condition $(\mathsf{Const} - R_B)$, we have $R_B = W \times C$ for some nonempty subset $C$ of $W$ (see Figure 3). In fact, $C = R_B(w)$ for any arbitrary $w \in W$. Therefore, (1) since $R_B$ is serial, it is guaranteed that $C$ is nonempty, and (2) since $R_B$ satisfies $(\mathsf{Const} - R_B)$, every tsr-model has a unique such $C$ and we call it the *final cluster*.

Given a tsr-model $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v\rangle$, we can define a well-preorder $\geq_{R_B}$ on $W$ as

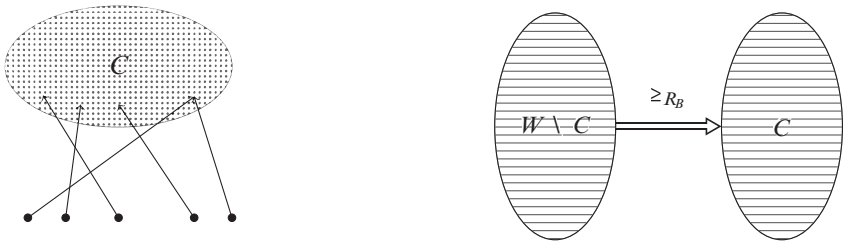
$$\geq_{R_B} = \underbrace{(W \times C)}_{1} \cup \underbrace{((W \backslash C) \times (W \backslash C))}_{2},$$

where $C$ is the final cluster of $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v\rangle$. In other words, $\geq_{R_B} = R_B \cup ((W \backslash C) \times (W \backslash C))$. Figure 4 illustrates this construction.

Lemma 10. *For every tsr-model* $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v\rangle$, *the relation* $\geq_{R_B}$ *is a well-preorder on* $W$. *Moreover,* $Min_{\geq_{R_B}}(W) = C$.

[8] It is easy to prove that $R_B$ is also transitive and Euclidean. $R_B$ is transitive: let $v \in R_B(w)$ and $u \in R_B(v)$. Since $R_B$ satisfies $(\mathsf{Const} - R_B)$, we have $R_B(w) = R_B(v)$. Therefore, $u \in R_B(v)$ implies that $u \in R_B(w)$. $R_B$ is Euclidean: let $v \in R_B(w)$ and $u \in R_B(w)$. Again, by the property $(\mathsf{Const} - R_B)$, we have $R_B(w) = R_B(v)$. Therefore, $u \in R_B(w)$ implies that $u \in R_B(v)$.

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(a) $R_B = (W \times C)$ (b) $(W, \geq_{R_B})$

Fig. 4. Construction of $(W, \geq_{R_B})$, given a tsr-model $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v \rangle$.

*Proof.* Let $v_1, v_2, v_3 \in W$

reflexivity: if $v_1 \in C$, the $v_1 \geq_{R_B} v_1$ by (1) in the definition of $\geq_{R_B}$. If $v_1 \in W \setminus C$, the $v_1 \geq_{R_B} v_1$ by (2) in the definition of $\geq_{R_B}$.

transitivity: suppose that $v_1 \geq_{R_B} v_2$ and $v_2 \geq_{R_B} v_3$. We then have two cases:
Case 1: $v_3 \in C$.
Then, by (1) in the definition of $\geq_{R_B}$, we obtain that $v_1 \geq_{R_B} v_3$.
Case 2: $v_3 \in W \setminus C$.
Then, $v_2 \geq_{R_B} v_3$ means that $v_2 \in W \setminus C$. Similarly, $v_1 \geq_{R_B} v_2$ means that $v_1 \in W \setminus C$. Therefore, by (2) in the definition of $\geq_{R_B}$, we obtain that $v_1 \geq_{R_B} v_3$.

well-foundedness: let $P \subseteq W$ be nonempty and show that $Min_{\geq_{R_B}}(P) \neq \emptyset$. It is not difficult to see, by the definition of $\geq_{R_B}$, that

$$Min_{\geq_{R_B}}(P) = \begin{cases} P \cap C, \text{ if } P \cap C \neq \emptyset \\ P \cap (W \setminus C), \text{ otherwise.} \end{cases}$$

In both cases $Min_{\geq_{R_B}}(P) \neq \emptyset$. Therefore, $\geq_{R_B}$ is a well-preorder on $W$. Finally, as $W \cap C = C \neq \emptyset$, we obtain that $Min_{\geq_{R_B}}(W) = W \cap C = C$. □

We are now ready to show the correspondence between tsr-models and tsp-models.

THEOREM 11. *Given a tsr-model* $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v \rangle$, *for all* $w \in W$ *and* $\varphi \in \mathcal{L}_{\mathsf{PHB}}$,

$$\mathfrak{M}, w \models\!\!\models \varphi \text{ iff } \mathcal{M}_{\geq_{R_B}}, w \Vdash \varphi,$$

*where* $\mathcal{M}_{\geq_{R_B}} = \langle W, \geq_{R_B}, T, \oplus, \mathfrak{b}, t, v \rangle$ *is the corresponding tsp-model.*

*Proof.* The proof follows by induction on the structure of $\varphi$, where cases for the propositional variables, the Boolean connectives, and $\Box$ are elementary. So assume inductively that the result holds for $\psi$ and show that it holds also for $B\psi$. Note that the induction hypothesis implies that $[\![\psi]\!]_{\mathfrak{M}} = |\psi|_{\mathcal{M}_{\geq_{R_B}}}$.

Case $\varphi := B\psi$

$$\begin{aligned}
\mathfrak{M}, w \models\!\!\models B\psi \quad & \text{iff } R_B(w) \subseteq [\![\psi]\!]_{\mathfrak{M}} \text{ and } t(\psi) \sqsubseteq \mathfrak{b} && \text{(Definition 10)} \\
& \text{iff } C \subseteq [\![\psi]\!]_{\mathfrak{M}} \text{ and } t(\psi) \sqsubseteq \mathfrak{b} && \text{(where } C = R_B(w), \text{the final cluster)} \\
& \text{iff } Min_{\geq_{R_B}}(W) \subseteq [\![\psi]\!]_{\mathfrak{M}} \text{ and } t(\psi) \sqsubseteq \mathfrak{b} && \text{(Lemma 10)} \\
& \text{iff } Min_{\geq_{R_B}}(W) \subseteq |\psi|_{\mathcal{M}_{\geq_{R_B}}} \text{ and } t(\psi) \sqsubseteq \mathfrak{b} && \text{(induction hypothesis)} \\
& \text{iff } \mathcal{M}_{\geq_{R_B}}, w \Vdash B\psi && \text{(Definition 4)}
\end{aligned}$$

□

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For any set of formulas $\Gamma \subseteq \mathcal{L}_{\mathsf{PHB}}$ and any $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, we write $\Gamma \vdash_{\mathsf{PHB}} \varphi$ if there exists a finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_{\mathsf{PHB}} (\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \varphi$. We say that $\Gamma$ is PHB-*consistent* if $\Gamma \nvdash_{\mathsf{PHB}} \bot$, and PHB-*inconsistent* otherwise. A sentence $\varphi$ is PHB-consistent with $\Gamma$ if $\Gamma \cup \{\varphi\}$ is PHB-consistent (or, equivalently, if $\Gamma \nvdash_{\mathsf{PHB}} \neg\varphi$). Finally, a set of formulas $\Gamma$ is a *maximally* PHB-*consistent set* (or, in short, *mcs*) if it is PHB-consistent and any set of formulas properly containing $\Gamma$ is PHB-inconsistent [10].[9] We drop mention of the logic PHB when it is clear from the context.

LEMMA 12. *For every mcs $w$ of* PHB *and $\varphi, \psi \in \mathcal{L}_{\mathsf{PHB}}$, the following hold*:

1. $w \vdash_{\mathsf{PHB}} \varphi$ *iff* $\varphi \in w$,
2. *if* $\varphi \in w$ *and* $\varphi \rightarrow \psi \in w$, *then* $\psi \in w$,
3. *if* $\vdash_{\mathsf{PHB}} \varphi$ *then* $\varphi \in w$,
4. $\varphi \in w$ *and* $\psi \in w$ *iff* $\varphi \wedge \psi \in w$,
5. $\varphi \in w$ *iff* $\neg\varphi \notin w$.

*Proof.* Standard. □

In the following proofs, we make repeated use of Lemma 12 in a standard way and often omit mention of it.

LEMMA 13 (Lindenbaum's Lemma). *Every* PHB-*consistent set can be extended to a maximally* PHB-*consistent one.*

*Proof.* Standard. □

Let $\mathcal{W}^c$ be the set of all maximally consistent sets of PHB. Define $\sim_{\Box}$ and $\rightarrow_B$ on $\mathcal{W}^c$ as

$$\begin{array}{ll} w \sim_{\Box} v & \text{iff } \{\varphi \in \mathcal{L}_{\mathsf{PHB}} : \Box\varphi \in w\} \subseteq v, \\ w \rightarrow_B v & \text{iff } \{\varphi \in \mathcal{L}_{\mathsf{PHB}} : B\psi \wedge \Box(\psi \rightarrow \varphi) \in w \text{ for some } \psi \in \mathcal{L}_{\mathsf{PHB}}\} \subseteq v. \end{array}$$

Since $\Box$ is an S5 modality, $\sim_{\Box}$ is an equivalence relation. Moreover, due to Ax$\top$, we also have $\rightarrow_B \subseteq \sim_{\Box}$ (see the proof Lemma 15, item (1)).

To simplify the notation in the following proofs, let $w[B] := \{\varphi \in \mathcal{L}_{\mathsf{PHB}} : B\psi \wedge \Box(\psi \rightarrow \varphi) \in w \text{ for some } \psi \in \mathcal{L}_{\mathsf{PHB}}\}$, where $w \in \mathcal{W}^c$. Therefore, we can equivalently write

$$w \rightarrow_B v \text{ iff } w[B] \subseteq v.$$

DEFINITION 11 (Canonical tsr-model for $w_0$). *Let $w_0$ be a mcs of* PHB. *The canonical tsr-model for $w_0$ is a tuple $\mathfrak{M}^c = \langle W^c, R^c_B, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c \rangle$ where*

- $W^c = \{w \in \mathcal{W}^c : w_0 \sim_{\Box} w\}$,
- $R^c_B = \rightarrow_B \cap (W^c \times W^c)$,
- $T^c = \{a, b\}$, *where* $a = \{x \in \mathsf{Prop} \cup \{\top\} : \neg B\overline{x} \in w_0\}$ *and* $b = \{x \in \mathsf{Prop} \cup \{\top\} : B\overline{x} \in w_0\}$,
- $\oplus^c : T^c \times T^c \rightarrow T^c$ *such that* $a \oplus^c a = a$, $b \oplus^c b = b$, $a \oplus^c b = b \oplus^c a = a$,
- $\mathfrak{b}^c = b$, *and*

[9] Notions of derivation, (in)consistent, and maximally consistent sets for the systems studied in §4 are defined similarly.

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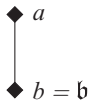



Fig. 5. The canonical topic model $\langle T^c, \oplus^c, \mathfrak{b}^c, t^c\rangle$ for $w_0$, where $b \sqsubset^c a$.

- $t^c : \mathcal{L}_{\mathsf{PHB}} \to T^c$ *such that, for every* $a \in T^c$ *and* $x \in \mathsf{Prop} \cup \{\top\}$,
$$t^c(x) = a \text{ iff } x \in a,$$
*and* $t^c$ *extends to* $\mathcal{L}_{\mathsf{PHB}}$ *by* $t^c(\varphi) = \oplus^c Var(\varphi)$ (*see Figure* 5).
- $w \in v^c(p)$ *iff* $p \in w$, *for all* $p \in \mathsf{Prop}$.

*The canonical topic parthood on* $T^c$, *denoted by* $\sqsubseteq^c$, *is defined in a standard way as in Definition* 1.

LEMMA 14. *Given a mcs* $w$, $\bigwedge_{i\leq n}\varphi_i \in w[B]$ *for all finite* $\{\varphi_1, \dots, \varphi_n\} \subseteq w[B]$.

*Proof.* Let $\{\varphi_1, \dots, \varphi_n\} \subseteq w[B]$. This means that, for each $\varphi_i$ with $i \leq n$, there is a $\psi_i \in \mathcal{L}_{\mathsf{PHB}}$ such that $B\psi_i \wedge \Box(\psi_i \to \varphi_i) \in w$. Thus, $\bigwedge_{i\leq n} B\psi_i \wedge \bigwedge_{i\leq n}\Box(\psi_i \to \varphi_i) \in w$. Then, by $\mathsf{C}_B$, we obtain that $B(\bigwedge_{i\leq n}\psi_i) \in w$. By $\mathsf{S5}_\Box$, we also have $\Box(\bigwedge_{i\leq n}\psi_i \to \bigwedge_{i\leq n}\varphi_i)$. Therefore, $\bigwedge_{i\leq n}\varphi_i \in w[B]$. □

LEMMA 15. $\mathfrak{M}^c = \langle W^c, R^c_B, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c\rangle$ *is a tsr-model.*

*Proof.*

1. $R^c_B$ is serial, i.e., that for all $w \in W^c$ there is a mcs $v \in W^c$ such that $wR^c_Bv$: to show this, we need to show that (a) $\to_B \subseteq \sim_\Box$, and (b) $w[B]$ is a consistent set.
to prove (a): let $w, v \in \mathcal{W}^c$ such that $w \to_B v$, i.e., that $w[B] \subseteq v$. Now, let $\chi \in \mathcal{L}_{\mathsf{PHB}}$ such that $\Box\chi \in w$. Then, we have that $\Box(\top \to \chi) \in w$ (by $\mathsf{S5}_\Box$). By $\mathsf{Ax}\top$, we also have that $B\top \in w$. Hence, we obtain $\chi \in w[B] \subseteq v$, implying that $\chi \in v$. Therefore, $w \sim_\Box v$.
to prove (b): let $w \in W^c$ and suppose, toward contradiction, that $w[B]$ is not consistent, i.e., $w[B] \vdash \bot$. This means that there is a finite subset $A = \{\varphi_1, \dots, \varphi_n\} \subseteq w[B]$ such that $\vdash \bigwedge A \to \neg\varphi_i$ for some $i \leq n$. By Lemma 14, we have that $\bigwedge A \in w[B]$, thus, there is a $\psi$ such that $B\psi \in w$ and $\Box(\psi \to \bigwedge A) \in w$. Since $\vdash \bigwedge A \to \neg\varphi_i$, by $\mathsf{S5}_\Box$, we also have $\Box(\psi \to \neg\varphi_i) \in w$. Hence, $\neg\varphi_i \in w[B]$ too. As $\varphi_i \in w[B]$, we also have a $\psi'$ with $B\psi' \in w$ and $\Box(\psi' \to \varphi_i) \in w$. From $\Box(\psi \to \neg\varphi_i) \in w$ and $\Box(\psi' \to \varphi_i) \in w$, by $\mathsf{S5}_\Box$, we obtain that $\Box(\psi \to \neg\psi') \in w$. As $B\psi' \in w$, by Ax1 and Theorem 1.2, $B\overline{\neg\psi'} \in w$. Therefore, $B\overline{\neg\psi'} \in w$, $\Box(\psi \to \neg\psi') \in w$, $B\psi \in w$, by Ax2, implies that $B\neg\psi' \in w$, contradicting the consistency of $w$: $B\psi' \in w$ implies $\neg B\neg\psi' \in w$, by $\mathsf{D}_B$. Therefore, $w[B]$ is consistent. By Lindenbaum's Lemma, we can then extend it to a mcs $v$. As $w[B] \subseteq v$, we obtain that $w \to_B v$. Then, by (a), we know that $w \sim_\Box v$. Therefore, as $w \in W^c$, we also have $v \in W^c$ (by the definition of $W^c$ and since $\sim_\Box$ is transitive). Hence, we conclude that $wR^c_Bv$, that is, $R^c_B$ is serial.
2. $R^c_B(v) = R^c_B(v')$ for all $v, v' \in W^c$: let $u \in R^c_B(v)$, i.e., that $v[B] \subseteq u$. Suppose $\chi \in v'[B]$. This implies that there is a $\psi \in \mathcal{L}_{\mathsf{PHB}}$ such that $B\psi \wedge \Box(\psi \to \chi) \in v'$. By Ax3 and $\mathsf{S5}_\Box$, we also have that $\Box B\psi \in v'$ and $\Box\Box(\psi \to \chi) \in v'$. Since $v' \sim_\Box v$ (as $v, v' \in W^c$ and $\sim_\Box$ is an equivalence relation), we obtain that $B\psi \in v$ and

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$\Box(\psi \to \chi) \in v$. Hence, $\chi \in v[B]$. Therefore, by the first assumption, we have $\chi \in u$, implying that $v'[B] \subseteq u$, i.e., $u \in R^c_B(v')$. The opposite direction is analogous.

3. $\oplus^c$ is idempotent, commutative, and associative: easy to see from the definition of $\oplus^c$.
4. $t^c$ is well-defined: Observe that, since $w_0$ is consistent and by Ax$\top$, we have $a \neq b$. Let $\varphi_1, \varphi_2 \in \mathcal{L}_{\mathsf{PHB}}$ such that $t^c(\varphi_1) \neq t^c(\varphi_2)$. This means, wlog, that $t^c(\varphi_1) = a$ and $t^c(\varphi_2) = b$. I.e., $\oplus^c Var(\varphi_1) = a$ and $\oplus^c Var(\varphi_2) = b$. While the former means that there is a $x \in Var(\varphi_1)$ such that $\neg B\overline{x} \in w_0$, the latter implies that for all $y \in Var(\varphi_2)$, we have $B\overline{y} \in w_0$. Since $w_0$ is consistent, $x \notin Var(\varphi_2)$. Therefore, $\varphi_1 \neq \varphi_2$.
5. $t^c(\top) = \mathfrak{b}^c$: By Ax$\top$ and Ax1, we have $B\overline{\top} \in w_0$. And, $Var(\top) = \{\top\}$. Hence, $Var(\top) \subseteq b = \mathfrak{b}^c$. Then, by the definition of $t^c$, we have that $t^c(\top) = \mathfrak{b}^c$. And, obviously, $\forall a, b \in T^c\ \exists b' \in T^c (b' = a \oplus^c b)$. □

LEMMA 16. *Given the canonical tsr-model* $\mathfrak{M}^c = \langle W^c, R^c_B, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c\rangle$, *for any* $w \in W^c$ *and* $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, $B\overline{\varphi} \in w$ *iff* $B\overline{x} \in w$ *for all* $x \in Var(\varphi)$.

*Proof.* The direction from left-to-right follows from Theorem 1.2. For the opposite direction, let $Var(\varphi) = \{x_1, \dots, x_n\}$ and observe that $\overline{\varphi} := \overline{x_1} \wedge \dots \wedge \overline{x_n}$. If $B\overline{x_i} \in w$ for all $x_i \in \{x_1, \dots, x_n\}$, then $\bigwedge_{i \leq n} B\overline{x_i} \in w$ (by Lemma 12.4). Then, by $\mathsf{C}_B$, we obtain that $B(\bigwedge_{i \leq n} \overline{x_i}) \in w$, i.e., $B\overline{\varphi} \in w$. □

COROLLARY 17. *Given the canonical tsr-model* $\mathfrak{M}^c = \langle W^c, R^c_B, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c\rangle$, *for any* $w \in W^c$, *and* $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, $B\overline{\varphi} \in w$ *iff* $t^c(\varphi) \sqsubseteq \mathfrak{b}^c$.

*Proof.*

| | | |
|---|---|---|
| $B\overline{\varphi} \in w$ | iff $B\overline{x} \in w$ for all $x \in Var(\varphi)$ | (Lemma 16) |
| | iff $B\overline{x} \in w_0$ for all $x \in Var(\varphi)$ | (Ax3 and the definition of $W^c$) |
| | iff $t^c(x) = b$ for all $x \in Var(\varphi)$ | (by the definitions of $b$ and $t^c$) |
| | iff $t^c(\varphi) = b$ | (by the definition of $(T^c, \oplus^c)$ and $t^c(\varphi)$) |
| | iff $t^c(\varphi) \sqsubseteq \mathfrak{b}^c$ | (since $b = \mathfrak{b}^c$ and $b \sqsubseteq a$ for all $a \in T^c$) |

□

LEMMA 18. *For every mcs* $w$ *and* $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, *if* $w[B] \vdash \varphi$ *and* $B\overline{\varphi} \in w$, *then* $B\varphi \in w$.

*Proof.* Suppose $w[B] \vdash \varphi$ and $B\overline{\varphi} \in w$. Then, there is a finite set $A \subseteq w[B]$ such that $\vdash \bigwedge A \to \varphi$. By Lemma 14, we know that $\bigwedge A \in w[B]$. This means that there is a $\psi$ such that $B\psi \wedge \Box(\psi \to \bigwedge A) \in w$. Then, by $\mathsf{S5}_\Box$, we obtain that $\Box(\psi \to \varphi) \in w$. We also have $B\overline{\varphi} \in w$ and $B\psi \in w$. Therefore, by Ax2, we conclude that $B\varphi \in w$. □

LEMMA 19 (Truth Lemma). *Let* $w_0$ *be a mcs of* PHB *and* $\mathfrak{M}^c = \langle W^c, R^c_B, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c\rangle$ *the canonical tsr-model for* $w_0$. *Then, for all* $\varphi \in \mathcal{L}_{\mathsf{PHB}}$ *and* $w \in W^c$, *we have* $\mathfrak{M}^c, w \models \varphi$ *iff* $\varphi \in w$.

*Proof.* The proof follows by induction on the structure of $\varphi$. The cases for the propositional variables, Booleans, and $\varphi := \Box\psi$ are standard. We here prove the case $\varphi := B\psi$.

($\Leftarrow$) Suppose $B\psi \in w$. Since $B\psi \in w$, by Ax1, $B\overline{\psi} \in w$. Thus, by Corollary 17, $t^c(\psi) \sqsubseteq \mathfrak{b}^c$. Now let $v \in R^c_B(w)$, i.e., $w[B] \subseteq v$. As $B\psi \in w$ and $\Box(\psi \to \psi) \in w$ (the latter is by $\mathsf{S5}_\Box$), we have that $\psi \in w[B]$. Therefore, since $w[B] \subseteq v$, we have $\psi \in v$. Then, by the induction hypothesis, we have $\mathfrak{M}^c, v \models \psi$. As $v$ has been chosen arbitrarily, we obtain that $\mathfrak{M}^c, w \models B\psi$.

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($\Rightarrow$) Suppose $\mathfrak{M}^c, w \mid\!\models B\psi$, i.e., for all $v \in R_B^c(w)$, $\mathfrak{M}^c, v \mid\!\models \psi$ and $t^c(\psi) \sqsubseteq \mathfrak{b}^c$. By Corollary 17, the latter means that $B\overline{\psi} \in w$. Moreover, the former, by the induction hypothesis, implies that $\psi \in v$ for all $v \in R_B^c(w)$. In other words, for all $v \in W^c$ with $w[B] \subseteq v$, we have that $\psi \in v$. This implies that $w[B] \vdash \psi$. Otherwise, $w[B] \cup \{\neg\psi\}$ would be consistent, thus, by Lemma 13, there exists a mcs $v'$ such that $w[B] \cup \{\neg\psi\} \subseteq v'$. As $w[B] \subseteq v'$, we have $w \to_B v'$. Since $\to_B \subseteq \sim_\Box$ and $w \in W^c$, we obtain that $v' \in W^c$, therefore, $v' \in R_B^c(w)$. Therefore, that $\neg\psi \in v'$ contradicts with the assumption that $\psi \in v$ for all $v \in R_B^c(w)$. Since $B\overline{\psi} \in w$, by Lemma 18, we obtain that $B\psi \in w$. □

COROLLARY 20. PHB *is complete with respect to the class of tsr-models.*

*Proof.* Let $\varphi \in \mathcal{L}_{\mathsf{PHB}}$ such that $\nvdash \varphi$. This mean that $\{\neg\varphi\}$ is consistent. Then, by Lindenbaum's Lemma (Lemma 13), there exists a mcs $w$ such that $\varphi \notin w$. Therefore, by Lemma 19, we conclude that $\mathfrak{M}^c, w \not\mid\!\models \varphi$, where $\mathfrak{M}^c$ is the canonical tsr-model for $w$. □

COROLLARY 21. PHB *is complete with respect to the class of tsp-models.*

*Proof.* Let $\varphi \in \mathcal{L}_{\mathsf{PHB}}$ such that $\nvdash \varphi$. Then, by Corollary 20, there is a tsr-model $\mathfrak{M} = \langle W, R_B, T, \oplus, \mathfrak{b}, t, v\rangle$ and $w \in W$ such that $\mathfrak{M}, w \not\mid\!\models \varphi$. Therefore, by Theorem 11, we conclude that $\mathcal{M}_{\geq_{R_B}}, w \nVdash \varphi$ where $\mathcal{M}_{\geq_{R_B}} = \langle W, \geq_{R_B}, T, \oplus, \mathfrak{b}, t, v\rangle$ is the corresponding tsp-model. □

## §B. Proofs of §4.

***B.1. Proof of Lemma 3.*** Let $\mathcal{L}$ be the language defined in Lemma 3. More precisely, $\mathcal{L}$ is the language defined by the grammar

$$\varphi := p_i \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid B^{\varphi}\varphi$$

where $p_i \in \mathsf{Prop}$.

DEFINITION 12 (Translation from $\mathcal{L}_{\mathsf{PHB}}$ to $\mathcal{L}$). *Let $e : \mathcal{L}_{\mathsf{PHB}} \to \mathcal{L}$ be the map such that*

$$\begin{aligned} e(p) &= p \\ e(\top) &= \top \\ e(\neg\varphi) &= \neg e(\varphi) \\ e(\varphi \wedge \psi) &= e(\varphi) \wedge e(\psi) \\ e(\Box\varphi) &= B^{\neg e(\varphi)}\bot \\ e(B\varphi) &= B^{\top} e(\varphi) \end{aligned}$$

LEMMA 22. *Given a tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ and $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, $t(\varphi) \sqsubseteq \mathfrak{b}$ iff $t(e(\varphi)) \sqsubseteq \mathfrak{b}$.*

*Proof.* The proof is by induction on the structure of $\varphi$. The cases for the propositional variables and Booleans straightforwardly follow from the definition of $e$ and the fact that Boolean connectives are topic transparent. Cases for $\varphi := \Box\psi$ and $\varphi := B\psi$ follow similarly to each other so we give the details of only the former case. Suppose inductively that the statement holds for $\psi$.

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Case $\varphi := \Box\psi$:

$$
\begin{array}{lll}
t(\Box\psi) \sqsubseteq \mathfrak{b} & \text{iff } t(\psi) \sqsubseteq \mathfrak{b} & (t(\psi) = t(\Box\psi)) \\
& \text{iff } t(e(\psi)) \sqsubseteq \mathfrak{b} & \text{(induction hypothesis)} \\
& \text{iff } t(e(\psi)) \oplus \mathfrak{b} \sqsubseteq \mathfrak{b} & \text{(by the definition of } \oplus) \\
& \text{iff } t(e(\psi)) \oplus t(\bot) \sqsubseteq \mathfrak{b} & (t(\bot) = t(\neg\top) = \mathfrak{b}) \\
& \text{iff } t(\neg e(\psi)) \oplus t(\bot) \sqsubseteq \mathfrak{b} & (t(\neg e(\psi)) = t(e(\psi))) \\
& \text{iff } t(B^{\neg e(\psi)}\bot) \sqsubseteq \mathfrak{b} & (t(B^{\neg e(\psi)}\bot) = t(\neg e(\psi)) \oplus t(\bot)) \\
& \text{iff } t(e(\Box\psi)) \sqsubseteq \mathfrak{b} & \text{(by the definition of } e) \quad \Box
\end{array}
$$

LEMMA 23. *For all* $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, *we have* $\vDash \varphi \leftrightarrow e(\varphi)$.

*Proof.* The proof is by induction on the structure of $\varphi$. The cases for the propositional variables and Booleans straightforwardly follow from the definition of $e$. Towards showing the cases for $\varphi := \Box\psi$ and $\varphi := B\psi$, suppose inductively that the statement holds for $\psi$. Let $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ be a tsp-model and $w \in W$.

Case $\varphi := \Box\psi$:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash e(\Box\psi) & \text{iff } \mathcal{M}, w \Vdash B^{\neg e(\psi)}\bot & \text{(by the definition of } e) \\
& \text{iff } Min_{\geq}|\neg e(\psi)| \subseteq |\bot| \text{ and } t(\bot) \sqsubseteq \mathfrak{b} \oplus t(e(\psi)) & \text{(Definition 5)} \\
& \text{iff } Min_{\geq}|\neg e(\psi)| \subseteq |\bot| & (\text{since } t(\bot) = \mathfrak{b}) \\
& \text{iff } |\neg e(\psi)| = \emptyset & (\text{since } |\bot| = \emptyset \text{ and } \geq \text{ is well-founded}) \\
& \text{iff } |e(\psi)| = W & \text{(by Definition 5)} \\
& \text{iff } |\psi| = W & \text{(induction hypothesis)} \\
& \text{iff } \mathcal{M}, w \Vdash \Box\psi &
\end{array}
$$

Case $\varphi := B\psi$:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash e(B\psi) & \text{iff } \mathcal{M}, w \Vdash B^{\top} e(\psi) & \text{(by the definition of } e) \\
& \text{iff } Min_{\geq}|\top| \subseteq |e(\psi)| \text{ and } t(e(\psi)) \sqsubseteq \mathfrak{b} \oplus t(\top) & \text{(by Definition 5)} \\
& \text{iff } Min_{\geq}(W) \subseteq |e(\psi)| \text{ and } t(e(\psi)) \sqsubseteq \mathfrak{b} & (\text{since } t(\top) = \mathfrak{b} \text{ and } \oplus \text{ is idempotent}) \\
& \text{iff } Min_{\geq}(W) \subseteq |\psi| \text{ and } t(\psi) \sqsubseteq \mathfrak{b} & \text{(induction hypothesis, Lemma 22)} \\
& \text{iff } \mathcal{M}, w \Vdash B\psi & \text{(Definition 4)}
\end{array}
$$

COROLLARY 24. *For every* $\varphi \in \mathcal{L}_{\mathsf{PHB}}$, *there exists* $\psi \in \mathcal{L}$ *such that* $\vDash \varphi \leftrightarrow \psi$. *In other words,* $\mathcal{L}$ *is at least as expressive as* $\mathcal{L}_{\mathsf{PHB}}$ *with respect to tsp-models.*

**Proof of Lemma 3:** Corollary 24 shows that $\mathcal{L}$ is at least as expressive as $\mathcal{L}_{\mathsf{PHB}}$ with respect to tsp-models. To show that it is indeed strictly more expressive, consider the models $\mathcal{M}_4 = \langle\{w, u, v\}, \geq_4, \{\mathfrak{b}\}, \oplus, t, v\rangle$ and $\mathcal{M}_5 = \langle\{w, u, v\}, \geq_5, \{\mathfrak{b}\}, \oplus, t, v\rangle$ such that $t(p) = \{\mathfrak{b}\}$ for all $p \in \mathsf{Prop}$, posets $\langle\{w, u, v\}, \geq_4\rangle$ and $\langle\{w, u, v\}, \geq_5\rangle$ are as given in Figure 6, $v(p) = \{v, u\}$, $v(q) = \{w, v\}$. Since the models have only one possible topic, the topic component in this particular case does not play any essential role. The two models differ only in their plausibility ordering, while having exactly the same most plausible world, namely $w$. It is then easy to see that $\mathcal{M}_4, w \Vdash B^p q$ (since $Min_{\geq}|p|_{\mathcal{M}_4} = \{v\} \subseteq \{w, v\} = |q|_{\mathcal{M}_4}$), whereas $\mathcal{M}_5, w \nVdash B^p q$ (since $Min_{\geq}|p|_{\mathcal{M}_5} = \{u\} \not\subseteq \{w, v\} = |q|_{\mathcal{M}_5}$). So, $\mathcal{L}$ can distinguish $\mathcal{M}_4, w$ from $\mathcal{M}_5, w$. However, for all $\varphi \in \mathcal{L}_{\mathsf{PHB}}$ and $w' \in W$, $\mathcal{M}_4, w' \Vdash \varphi$ iff $\mathcal{M}_5, w' \Vdash \varphi$, i.e., $|\varphi|_{\mathcal{M}_4} = |\varphi|_{\mathcal{M}_5}$. This follows easily by an inductive proof on the structure of $\varphi$. Therefore, $\mathcal{L}$ is strictly more expressive than $\mathcal{L}_{\mathsf{PHB}}$. Since $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{CHB}}$, it also follows that $\mathcal{L}_{\mathsf{CHB}}$ is strictly more expressive than $\mathcal{L}_{\mathsf{PHB}}$.

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(a) $\mathcal{M}_4$ (b) $\mathcal{M}_5$

Fig. 6. Models $\mathcal{M}_4$ and $\mathcal{M}_5$. Circles represent possible worlds, diamonds represent possible topics. Valuation and topic assignment are given by labelling each node with propositional variables. We omit labelling when a node is assigned every element in Prop. Arrows represent the plausibility relation $\geq$ and point to more plausible worlds. Reflexive and transitive arrows are omitted.

***B.2. Proof of Theorem 5.*** Axiom labels refer to the ones in Table 2.

1. $\vdash \Diamond\varphi \to \neg B^{\varphi}\bot$
   1. $\vdash B^{\varphi}\bot \to (\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \bot))))$ — Ax6
   2. $\vdash (\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \bot)))) \leftrightarrow (\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq]\neg\varphi)))$ — $(\vdash (\varphi \to \bot) \leftrightarrow \neg\varphi)$
   3. $\vdash (\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq]\neg\varphi))) \leftrightarrow (\Box(\varphi \to \langle\geq\rangle\bot))$ — $(\vdash (\varphi \wedge [\geq]\neg\varphi) \leftrightarrow \bot)$
   4. $\vdash \Box(\varphi \to \langle\geq\rangle\bot) \leftrightarrow \Box\neg\varphi$ — $(\vdash \langle\geq\rangle\bot \leftrightarrow \bot)$
   5. $\vdash B^{\varphi}\bot \to \Box\neg\varphi$ — 1-4, CPL
   6. $\vdash \Diamond\varphi \to \neg B^{\varphi}\bot$ — 5, CPL
2. $\vdash B^{\varphi}\varphi$
   An easy consequence of Ax1 and Ax6, together with $\mathsf{S5}_{\Box}$ and $\mathsf{S4}_{[\geq]}$.
3. from $\vdash \varphi \leftrightarrow \chi$, $\vdash B^{\overline{\varphi}}\overline{\chi}$, and $\vdash B^{\overline{\chi}}\overline{\varphi}$ infer $\vdash B^{\varphi}\psi \leftrightarrow B^{\chi}\psi$
   First observe that $\vdash B^{\psi}\overline{\varphi} \leftrightarrow B^{\overline{\psi}}\overline{\varphi}$. This is an easy consequence of Ax1 and Ax2 (use $\vdash B^{\overline{\psi}}\psi$ and $\vdash B^{\psi}\overline{\psi}$). Let us denote this theorem by $\star$. Then:
   1. $\vdash \varphi \leftrightarrow \chi$ — ass. 1
   2. $\vdash B^{\overline{\varphi}}\overline{\chi}$ — ass. 2
   3. $\vdash B^{\overline{\chi}}\overline{\varphi}$ — ass. 3
   4. $\vdash B^{\varphi}\overline{\psi} \leftrightarrow B^{\chi}\overline{\psi}$ — ass. 2-3, $\star$, Ax2
   5. $\vdash B^{\varphi}\psi \leftrightarrow (\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))) \wedge B^{\varphi}\overline{\psi})$ — Ax6
   6. $\vdash (\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))) \wedge B^{\varphi}\overline{\psi}) \leftrightarrow (\Box(\chi \to \langle\geq\rangle(\chi \wedge [\geq](\chi \to \psi))) \wedge B^{\varphi}\overline{\psi})$ — $\mathsf{S5}_{\Box}$, $\mathsf{S4}_{[\geq]}$, ass. 1
   7. $\vdash (\Box(\chi \to \langle\geq\rangle(\chi \wedge [\geq](\chi \to \psi))) \wedge B^{\varphi}\overline{\psi}) \leftrightarrow (\Box(\chi \to \langle\geq\rangle(\chi \wedge [\geq](\chi \to \psi))) \wedge B^{\chi}\overline{\psi})$ — 4, CPL
   8. $(\Box(\chi \to \langle\geq\rangle(\chi \wedge [\geq](\chi \to \psi))) \wedge B^{\chi}\overline{\psi}) \leftrightarrow B^{\chi}\psi$ — Ax6
   9. $\vdash B^{\varphi}\psi \leftrightarrow B^{\chi}\psi$ — 5–8, CPL
4. from $\vdash \varphi \leftrightarrow \chi$, $\vdash B^{\overline{\varphi}}\overline{\chi}$, and $\vdash B^{\overline{\chi}}\overline{\varphi}$ infer $\vdash B^{\psi}\varphi \leftrightarrow B^{\psi}\chi$
   Similar to the proof of Theorem 5.3.

***B.3. Proof of Theorem 6: soundness and completeness of*** CHB**.**

*B.3.1. Soundness of* CHB. Soundness is a matter of routine validity check, so we spell out only the relatively tricky cases.

*Proof.* Let $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ be a tsp-model and $w \in W$. The validity of Inc is due to the fact that $\Box$ is the global modality. Validity of $B^{\varphi}\top$ follows from the stipulation $t(\top) = \mathfrak{b}$. Validity of Ax1 follows immediately from the semantic clause for $B^{\varphi}\psi$ and the definition of $t$. Ax2 is valid since $\sqsubseteq$ is a transitive relation. Validity

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of Ax3 and Ax4 follow similarly to those of $\mathsf{C}_B$ and Ax2 in Table 1, respectively (see the proof of Theorem 2 in Appendix A.2.1). Ax5 is valid since truth of a conditional belief sentence $B^{\varphi}\psi$ is state independent: it is easy to see that either $|B^{\varphi}\psi| = W$ or $|B^{\varphi}\psi| = \emptyset$, for any $\varphi, \psi \in \mathcal{L}_{\mathsf{CHB}}$. The validity proofs of Tot and Ax6 are spelled out.

Tot: Suppose that $\mathcal{M}, w \not\Vdash \Box([\geq]\varphi \to \psi)$ and let $v \in W$ such that $\mathcal{M}, v \Vdash [\geq]\psi$. The former means that there is $v_0 \in W$ such that $v_0 \Vdash [\geq]\varphi \wedge \neg\psi$. As $\mathcal{M}, v \Vdash [\geq]\psi$ and $\mathcal{M}, v_0 \Vdash \neg\psi$, we have $v \not\geq v_0$. Thus, as $\geq$ is a total order, we obtain that $v_0 \geq v$. Therefore, since $v_0 \Vdash [\geq]\varphi$, we conclude that $\mathcal{M}, v \Vdash \varphi$.

Ax6: We first show that

$$\mathcal{M}, w \Vdash \Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))) \text{ iff } \mathit{Min}_{\geq}(|\varphi|) \subseteq |\psi| \tag{1}$$

($\Rightarrow$) Suppose that $\mathcal{M}, w \Vdash \Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi)))$ and let $v \in \mathit{Min}_{\geq}(|\varphi|)$. As $\mathit{Min}_{\geq}(|\varphi|) \subseteq |\varphi|$, we obtain by the first assumption that $\mathcal{M}, v \Vdash \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))$. This means that there is $u \in W$ such that $v \geq u$ and $\mathcal{M}, u \Vdash \varphi \wedge [\geq](\varphi \to \psi)$. As $v \in \mathit{Min}_{\geq}|\varphi|$, $u \in |\varphi|$, and $v \geq u$, we have $u \in \mathit{Min}_{\geq}|\varphi|$. Therefore, $u \geq v$. Hence, as $\mathcal{M}, u \Vdash [\geq](\varphi \to \psi)$ and $\mathcal{M}, v \Vdash \varphi$, we conclude that $\mathcal{M}, v \Vdash \psi$.
($\Leftarrow$) Suppose that $\mathit{Min}_{\geq}(|\varphi|) \subseteq |\psi|$ and let $v \in W$ such that $\mathcal{M}, v \Vdash \varphi$. Since $\geq$ is well-founded and $|\varphi| \neq \emptyset$, we have $\mathit{Min}_{\geq}(|\varphi|) \neq \emptyset$, i.e., there is a $u \in \mathit{Min}_{\geq}(|\varphi|)$. Since $\geq$ is a total order, we obtain that $v \geq u$. Moreover, as $u \in \mathit{Min}_{\geq}(|\varphi|)$, if $u \geq u'$ and $\mathcal{M}, u' \Vdash \varphi$, we have $u' \in \mathit{Min}_{\geq}(|\varphi|)$. Therefore, since $\mathit{Min}_{\geq}(|\varphi|) \subseteq |\psi|$, we obtain that $\mathcal{M}, u \Vdash [\geq](\varphi \to \psi)$. As $u \in \mathit{Min}_{\geq}(|\varphi|)$, we moreover have that $\mathcal{M}, u \Vdash \varphi \wedge [\geq](\varphi \to \psi)$. Since $v \geq u$, we obtain that $\mathcal{M}, v \Vdash \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))$. As $v$ has been chosen arbitrarily from $W$, we conclude that $\mathcal{M}, w \Vdash \Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi)))$. To conclude,

$$\begin{aligned}
\mathcal{M}, w \Vdash B^{\varphi}\psi \text{ iff } & \mathit{Min}_{\geq}(|\varphi|) \subseteq |\psi| \text{ and } t(\psi) \sqsubseteq \mathfrak{b} \oplus t(\varphi) \\
\text{iff } & \mathcal{M}, w \Vdash \Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))) \text{ and } \mathcal{M}, w \Vdash B^{\varphi}\overline{\psi} \\
& \qquad\qquad \text{(By (1) above and the semantics of } B^{\varphi}\overline{\psi}) \\
\text{iff } & \mathcal{M}, w \Vdash \Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))) \wedge B^{\varphi}\overline{\psi}
\end{aligned}$$

□

*B.3.2. Completeness of* CHB. We first show completeness with respect to *quasi* tsp-models via a canonical model construction. Building the canonical quasi tsp-model involves a non-trivial topic algebra construction. A quasi tsp-model is like a tsp-model except that its plausibility ordering is not guaranteed to be a well-order. We then continue proving the finite quasi tsp-model property for CHB via a filtration argument. As every finite quasi tsp-model is a tsp-model, we establish the completeness of CHB with respect to the class of tsp-models.

DEFINITION 13 (Quasi tsp-model for $\mathcal{L}_{\mathsf{CHB}}$). *A quasi tsp-model is a tuple* $\langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ *where* $\langle W, \geq\rangle$ *is a total preorder,* $\langle T, \oplus, \mathfrak{b}, t\rangle$ *is a dt-model for* $\mathcal{L}_{\mathsf{CHB}}$, *and* $v : \mathsf{Prop} \to \mathcal{P}(W)$ *is a valuation function that maps every propositional variable in* Prop *to a set of worlds.*

So, a quasi tsp-model for $\mathcal{L}_{\mathsf{CHB}}$ is just like a tsp-model except that the order $\geq$ is not guaranteed to be a well-order, that is, it is not guaranteed that $\mathit{Min}_{\geq}(P) \neq \emptyset$ for all nonempty $P \subseteq W$.

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LEMMA 25. *For every mcs $w$ of* CHB *and $\varphi, \psi \in \mathcal{L}_{\mathsf{CHB}}$, the following hold*:

1. $w \vdash_{\mathsf{CHB}} \varphi$ *iff* $\varphi \in w$,
2. *if* $\varphi \in w$ *and* $\varphi \to \psi \in w$, *then* $\psi \in w$,
3. *if* $\vdash_{\mathsf{CHB}} \varphi$ *then* $\varphi \in w$,
4. $\varphi \in w$ *and* $\psi \in w$ *iff* $\varphi \wedge \psi \in w$,
5. $\varphi \in w$ *iff* $\neg\varphi \notin w$.

*Proof.* Standard. □

LEMMA 26 (Lindenbaum's Lemma). *Every* CHB*-consistent set can be extended to a maximally* CHB*-consistent one.*

*Proof.* Standard. □

Let $\mathcal{X}^c$ be the set of all maximally consistent sets of CHB. Define $\sim_\Box$ and $\geq$ on $\mathcal{X}^c$ as

$$w \sim_\Box v \text{ iff } \{\varphi \in \mathcal{L}_{\mathsf{CHB}} : \Box\varphi \in w\} \subseteq v,$$
$$w \geq v \text{ iff } \{\varphi \in \mathcal{L}_{\mathsf{CHB}} : [\geq]\varphi \in w\} \subseteq v.$$

Since $\Box$ is an S5 modality, $\sim_\Box$ is an equivalence relation. Similarly, since $[\geq]$ is an S4 modality, $\geq$ is a preorder. Moreover, Inc guarantees that $\geq$ is a subset of $\sim_\Box$, and axiom Tot that $\geq$ is a total order within each equivalence class induced by $\sim_\Box$ (see Lemma 28, item (1)).

To define the canonical quasi tsp-model, we need some auxiliary definitions and lemmas. For $w \in \mathcal{X}^c$, let $\approx_w \subseteq \mathcal{L}_{\mathsf{CHB}} \times \mathcal{L}_{\mathsf{CHB}}$ such that

$$\varphi \approx_w \psi \text{ iff } B^\varphi\overline{\psi}, B^\psi\overline{\varphi} \in w.$$

In the following proofs, we make repeated use of Lemma 25 in a standard way and omit mention of it.

LEMMA 27. *For all $w \in \mathcal{X}^c$, $\approx_w$ is an equivalence relation. Moreover, for all $w, v \in \mathcal{X}^c$ such that $w \sim_\Box v$, we have $\approx_w = \approx_v$.*

*Proof.* Let $w \in \mathcal{X}^c$ and $\varphi, \psi, \chi \in \mathcal{L}_{\mathsf{CHB}}$.

- reflexivity: By Ax1, we have $\vdash_{\mathsf{CHB}} B^\varphi\overline{\varphi}$, thus, $\varphi \approx_w \varphi$.
- symmetry: Suppose that $\varphi \approx_w \psi$. This means, by the definition of $\approx_w$, that $B^\varphi\overline{\psi}, B^\psi\overline{\varphi} \in w$. Therefore, $B^\psi\overline{\varphi}, B^\varphi\overline{\psi} \in w$, i.e., $\psi \approx_w \varphi$.
- transitivity: Suppose that $\varphi \approx_w \psi$ and $\psi \approx_w \chi$. This means that (a) $B^\psi\overline{\varphi} \in w$, (b) $B^\varphi\overline{\psi} \in w$, (c) $B^\psi\overline{\chi} \in w$, and (d) $B^\chi\overline{\psi} \in w$. Then, by Ax2, (b), and (c), $B^\varphi\overline{\chi} \in w$. Similarly, by Ax2, (a), and (d), $B^\chi\overline{\varphi} \in w$. Therefore, $\varphi \approx_w \chi$.

For the last part, let $w, v \in \mathcal{X}^c$ such that $w \sim_\Box v$ and suppose that $\varphi \approx_w \psi$. The latter means that $B^\varphi\overline{\psi}, B^\psi\overline{\varphi} \in w$. Then, by Ax5, we obtain that $\Box B^\varphi\overline{\psi}, \Box B^\psi\overline{\varphi} \in w$. As $w \sim_\Box v$, we conclude that $B^\varphi\overline{\psi}, B^\psi\overline{\varphi} \in v$, i.e., $\varphi \approx_v \psi$. For the other direction, use the fact that $\sim_\Box$ is symmetric. □

Let $[\varphi]_w = \{\psi \in \mathcal{L}_{\mathsf{CHB}} : \varphi \approx_w \psi\}$, i.e., $[\varphi]_w$ is the equivalence class of $\varphi$ with respect to $\approx_w$.

DEFINITION 14 (Canonical quasi tsp-model for $w_0$). *Let $w_0$ be a mcs of* CHB. *The canonical model for $w_0$ is the tuple $\mathcal{M}^c = \langle W^c, \geq^c, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c \rangle$, where*

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- $W^c = \{w \in \mathcal{X}^c : w_0 \sim_\Box w\}$,
- $\geq^c = \geq \cap (W^c \times W^c)$,
- $T^c = \{[\varphi]_{w_0} : \varphi \in \mathcal{L}_{\mathsf{CHB}}\}$ (*we omit the subscript* $w_0$ *when it is clear from the context*),
- $\oplus^c : T^c \times T^c \to T^c$ *such that* $[\varphi] \oplus^c [\psi] = [\varphi \wedge \psi]$,
- $\mathfrak{b}^c = [\top]$,
- $t^c : \mathcal{L}_{\mathsf{CHB}} \to T^c$ *such that, for all* $x \in \mathsf{Prop} \cup \{\top\}$, $t^c(x) = [x]$ *and* $t^c(\varphi) = \oplus^c Var(\varphi)$,
- $v^c : \mathsf{Prop} \to \mathcal{P}(W^c)$ *such that* $v^c(p) = \{w \in W^c : p \in w\}$.

*The canonical topic parthood on* $T^c$, *denoted by* $\sqsubseteq^c$, *is defined in a standard way as in Definition* 1.

LEMMA 28. *For any mcs* $w_0$ *of* CHB, *the canonical model* $\mathcal{M}^c = \langle W^c, \geq^c, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c\rangle$ *for* $w_0$ *constructed as in Definition* 14 *is a quasi tsp-model for* $\mathcal{L}_{\mathsf{CHB}}$.

*Proof.*

1. $\geq^c$ is a total preorder on $W^c$:
That $\geq^c$ is reflexive and transitive follows from the fact that $[\geq]$ is an S4 modality. To prove that $\geq^c$ is total in $W^c$, let $w, v \in W^c$ and assume, toward contradiction, that $w \not\geq^c v$ and $v \not\geq^c w$. Then, by the definition of $\geq^c$, there exist $\psi, \chi \in \mathcal{L}_{\mathsf{CHB}}$ such that $[\geq]\psi \in w$ but $\psi \notin v$; and $[\geq]\chi \in v$ but $\chi \notin w$. Therefore, $[\geq]\psi \wedge \neg\chi \in w$ and $[\geq]\chi \wedge \neg\psi \in v$. By Tot, we have that $\Box([\geq]\psi \to \chi) \vee \Box([\geq]\chi \to \psi) \in w$, i.e., that $\Box([\geq]\psi \to \chi) \in w$ or $\Box([\geq]\chi \to \psi) \in w$. If $\Box([\geq]\psi \to \chi) \in w$, then (by $\mathsf{T}_\Box$) $[\geq]\psi \to \chi \in w$, contradicting consistency of $w$. If $\Box([\geq]\chi \to \psi) \in w$, then (since $w \sim_\Box v$), $[\geq]\chi \to \psi \in v$, contradicting consistency of $v$. Therefore, for all $w, v \in W^c$, we obtain that either $w \geq^c v$ or $v \geq^c w$.
2. $\oplus^c$ is idempotent, commutative, and associative: Follows easily from Ax1 and the fact that $[\varphi]_{w_0}$ is an equivalence class for each $\varphi \in \mathcal{L}_{\mathsf{CHB}}$.
3. $\oplus^c$ is always defined on $T^c$, that is, $\forall a, b \in T^c\, \exists c \in T^c (c = a \oplus^c b)$ : Let $a, b \in T^c$. By the definition of $T^c$, we have that $a = [\varphi]_{w_0}$ and $b = [\psi]_{w_0}$ for some $\varphi, \psi \in \mathcal{L}_{\mathsf{CHB}}$. As $\varphi \wedge \psi \in \mathcal{L}_{\mathsf{CHB}}$ and $a \oplus^c b = [\varphi \wedge \psi] \in T^c$, we obtain the result.
4. $t^c(\top) = \mathfrak{b}^c$: Easy to see by the definitions of $t^c$ and $\mathfrak{b}^c$. □

LEMMA 29. *The following are derivable in* CHB:

1. $B^\varphi(\varphi \wedge \top)$
2. $B^\varphi\psi \to B^\varphi\overline{\psi}$.

*Proof.*

1. $\vdash B^\varphi(\varphi \wedge \top)$

| | |
|---|---|
| 1. $\vdash B^\varphi\varphi \wedge B^\varphi\top$ | Theorem 5.2, Ax$\top$ |
| 2. $\vdash (B^\varphi\varphi \wedge B^\varphi\top) \to B^\varphi(\varphi \wedge \top)$ | Ax3 |
| 3. $\vdash B^\varphi(\varphi \wedge \top)$ | 1, 2, CPL |

2. $\vdash B^\varphi\psi \to B^\varphi\overline{\psi}$: An easy consequence of Ax6. □

LEMMA 30. *For all* $\varphi \in \mathcal{L}_{\mathsf{CHB}}$, *we have*

1. $t^c(\varphi) = [\varphi]$, *and*
2. $[\varphi \wedge \top] = [\varphi]$.

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*Proof.* Let $\varphi \in \mathcal{L}_{\mathsf{CHB}}$ such that $Var(\varphi) = \{x_1, \ldots, x_n\}$.

1. By the definitions of $t^c$ and $\oplus^c$, we have

$$t^c(\varphi) = \oplus^c Var(\varphi) = t^c(x_1) \oplus^c \cdots \oplus^c t^c(x_n) = [x_1] \oplus^c \cdots \oplus^c [x_n] = [x_1 \wedge \cdots \wedge x_n].$$

Therefore, as $[x_1 \wedge \cdots \wedge x_n]$ is an equivalence class, it suffices to show that $\varphi \in [x_1 \wedge \cdots \wedge x_n]$. It follows by Ax1 that $B^{\varphi}(\overline{x_1 \wedge \cdots \wedge x_n}), B^{x_1 \wedge \cdots \wedge x_n}\overline{\varphi} \in w_0$. Therefore, $[\varphi] = [x_1 \wedge \cdots \wedge x_n] = t^c(\varphi)$.
2. By Ax1, we have $B^{\varphi \wedge \top}\overline{\varphi} \in w_0$. Moreover, by Lemmas 29.1 and 29.2, we obtain that $B^{\varphi}(\overline{\varphi \wedge \top}) \in w_0$. Hence, $\varphi \approx_{w_0} (\varphi \wedge \top)$, i..e., $[\varphi] = [\varphi \wedge \top]$. □

LEMMA 31. *For all $w \in W^c$ and $\varphi, \psi \in \mathcal{L}_{\mathsf{CHB}}$, $t^c(\psi) \sqsubseteq^c t^c(\varphi) \oplus^c \mathfrak{b}^c$ iff $B^{\varphi}\overline{\psi} \in w$.*

*Proof.* Observe that

$$\begin{aligned}
t^c(\psi) \sqsubseteq^c t^c(\varphi) \oplus^c \mathfrak{b}^c \text{ iff } & t^c(\psi) \oplus^c (t^c(\varphi) \oplus^c \mathfrak{b}^c) = t^c(\varphi) \oplus^c \mathfrak{b}^c \\
& \qquad \text{(by the definition of } \sqsubseteq^c) \\
\text{iff } & [\psi] \oplus^c ([\varphi] \oplus^c \mathfrak{b}^c) = [\varphi] \oplus^c \mathfrak{b}^c \qquad \text{(Lemma 30.1)} \\
\text{iff } & [\psi] \oplus^c ([\varphi] \oplus^c [\top]) = [\varphi] \oplus^c [\top] \qquad \text{(by the definition of } \mathfrak{b}^c) \\
\text{iff } & [(\psi \wedge \varphi) \wedge \top] = [\varphi \wedge \top] \qquad \text{(by the definition of } \oplus^c) \\
\text{iff } & [\psi \wedge \varphi] = [\varphi] \qquad \text{(Lemma 30.2)} \\
\text{iff } & B^{\varphi}(\overline{\psi \wedge \varphi}), B^{\psi \wedge \varphi}\overline{\varphi} \in w_0 \\
\text{iff } & B^{\varphi}(\overline{\psi \wedge \varphi}), B^{\psi \wedge \varphi}\overline{\varphi} \in w \text{ for all } w \in W^c \\
& \qquad \text{(Ax5 and the definition of } W^c)
\end{aligned}$$

Let $w \in W^c$:
($\Rightarrow$) Assume that $t^c(\psi) \sqsubseteq^c t^c(\varphi) \oplus^c \mathfrak{b}^c$. Then, by the above reasoning, we have $B^{\varphi}(\overline{\psi \wedge \varphi}) \in w$. Moreover, by Ax1, we also have $B^{\psi \wedge \varphi}\overline{\psi} \in w$. Then, by Ax2, we obtain that $B^{\varphi}\overline{\psi} \in w$.
($\Leftarrow$) Assume that $B^{\varphi}\overline{\psi} \in w$. By Ax1, we also have $B^{\varphi}\overline{\varphi} \in w$. Then, by Ax3, we obtain that $B^{\varphi}(\overline{\psi} \wedge \overline{\varphi}) \in w$. By Ax1, we also that $B^{\overline{\psi} \wedge \overline{\varphi}}(\overline{\psi \wedge \varphi}) \in w$. Thus, by Ax2, we obtain $B^{\varphi}(\overline{\psi \wedge \varphi}) \in w$. Moreover, by Ax1, $B^{\psi \wedge \varphi}\overline{\varphi} \in w$. Therefore, by the above reasoning, we conclude that $t^c(\psi) \sqsubseteq^c t^c(\varphi) \oplus^c \mathfrak{b}^c$. □

LEMMA 32 (Existence Lemma for $\Box$). *Let $w$ be a mcs and $\varphi \in \mathcal{L}_{\mathsf{CHB}}$. Then, $\Box\varphi \notin w$ iff there is $v \in \mathcal{X}^c$ such that $w \sim_{\Box} v$ and $\varphi \notin v$.*

*Proof.* Suppose that $\Box\varphi \notin w$. This implies that $\{\psi \in \mathcal{L}_{\mathsf{CHB}} : \Box\psi \in w\} \cup \{\neg\varphi\}$ is a consistent set. Otherwise, as the standard argument goes (see, e.g., [10, Lemma 4.20]), we could prove that $\Box\varphi \in w$, contradicting the first assumption. Therefore, by Lemma 26, there is a mcs $v$ such that $\{\psi \in \mathcal{L}_{\mathsf{CHB}} : \Box\psi \in w\} \cup \{\neg\varphi\} \subseteq v$. Thus $\varphi \notin v$ and, since $\{\psi \in \mathcal{L}_{\mathsf{CHB}} : \Box\psi \in w\} \subseteq v$, we have $w \sim_{\Box} v$. The other direction follows from the definition of $\sim_{\Box}$. □

LEMMA 33 (Existence Lemma for $[\geq]$). *Let $w$ be a mcs and $\varphi \in \mathcal{L}_{\mathsf{CHB}}$. Then, $[\geq]\varphi \notin w$ iff there is $v \in \mathcal{X}^c$ such that $w \geq v$ and $\varphi \notin v$.*

*Proof.* Similar to the proof of Lemma 32. □

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COROLLARY 34. *Let $w_0$ be a mcs and $\mathcal{M}^c = \langle W^c, \geq^c, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c\rangle$ be the canonical quasi tsp-model for $w_0$. Then, for all $\varphi \in \mathcal{L}_{\mathsf{CHB}}$ and $w \in W^c$, $[\geq]\varphi \notin w$ iff there is $v \in W^c$ such that $w \geq^c v$ and $\varphi \notin v$.*

*Proof.* From left-to-right follows from Lemma 33 and the fact that $\geq \subseteq \sim_{\Box}$. The other direction follows from the definition of $\geq^c$ and the fact that $\geq^c \subseteq \geq$. □

LEMMA 35 (Truth Lemma). *Let $w_0$ be a mcs of* CHB *and $\mathcal{M}^c = \langle W^c, \geq^c, T^c, \oplus^c, \mathfrak{b}^c, t^c, v^c\rangle$ be the canonical quasi tsp-model for $w_0$. Then, for all $w \in W^c$ and $\varphi \in \mathcal{L}_{\mathsf{CHB}}$,*

$$\mathcal{M}^c, w \Vdash \varphi \text{ iff } \varphi \in w.$$

*Proof.* The proof is by induction on the structure of $\varphi$. The cases for the propositional variables, Booleans, $\varphi := \Box\psi$, and $\varphi := [\geq]\psi$ are standard, where the latter two cases use Lemma 32 and Corollary 34, respectively. Toward showing the case for $\varphi := B^{\psi}\chi$, suppose inductively that the statement holds for $\psi$ and $\chi$.

Case $\varphi := B^{\psi}\chi$:

$$\begin{aligned}
\mathcal{M}^c, w \Vdash B^{\psi}\chi \text{ iff } & Min_{\geq^c}|\psi| \subseteq |\chi| \text{and } t^c(\chi) \sqsubseteq^c t^c(\psi) \oplus^c \mathfrak{b}^c && \text{(by the semantics)}\\
\text{iff } & Min_{\geq^c}|\psi| \subseteq |\chi| \text{ and } B^{\psi}\overline{\chi} \in w && \text{(Lemma 31)}\\
\text{iff } & \mathcal{M}^c, w \Vdash \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi))) \text{ and } B^{\psi}\overline{\chi} \in w\\
& && \text{(by (1) in the soundness proof, Appendix B.3.1)}\\
\text{iff } & \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi))) \in w \text{ and } B^{\psi}\overline{\chi} \in w\\
& && \text{(induction hypothesis and Lemmas 25, 32, and Corollary 34)}\\
\text{iff } & \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi))) \wedge B^{\psi}\overline{\chi} \in w\\
\text{iff } & B^{\psi}\chi \in w && \text{(Ax6)}
\end{aligned}$$

□

COROLLARY 36. CHB *is complete with respect to the class of quasi tsp-models.*

*Proof.* Let $\varphi \in \mathcal{L}_{\mathsf{CHB}}$ such that $\nvdash_{\mathsf{CHB}} \varphi$. This means that $\{\neg\varphi\}$ is CHB-consistent. Then, by Lindenbaum's Lemma (Lemma 26), there exists a mcs $w$ such that $\varphi \notin w$. Then, by Truth Lemma (Lemma 35), we conclude that $\mathcal{M}^c, w \nVdash \varphi$, where $\mathcal{M}^c$ is the canonical quasi tsp-model for $w$. □

*B.3.3. Finite quasi-model property for* CHB. Corollary 36 does not yet entail that CHB is complete with respect to the class of tsp-models since the plausibility order of a quasi tsp-models is not necessarily a well-preorder. However, as shown below, every quasi tsp-model is modally equivalent to a finite quasi tsp-model with respect to the language $\mathcal{L}_{\mathsf{CHB}}$. We establish this result via a filtration argument. Since every finite total preorder is a well-order, modal equivalence between quasi tsp-models and finite quasi tsp-models yields that CHB is also complete with respect to the class of tsp-models.

By a *finite model* $\langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$, we mean a model in which both the set of possible worlds $w$ and the set of possible topics $T$ are finite. Although we only need $W$ to be finite to show the completeness of CHB with respect to the class of tsp-models via Corollary 36, it is nevertheless not too complicated to construct a model whose set of possible topics is also finite.

For the filtration argument, we need a few auxiliary definitions and lemmas.

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DEFINITION 15 (Subformula closed set). *A set of formulas* $\Sigma \subseteq \mathcal{L}_{\mathsf{CHB}}$ *is called subformula closed if for all* $\varphi, \psi \in \mathcal{L}_{\mathsf{CHB}}$ *we have*

- *if* $\neg\varphi \in \Sigma$, *then* $\varphi \in \Sigma$ (*and similarly for* $\Box\varphi$ *and* $[\geq]\varphi$),
- *if* $\varphi \wedge \psi \in \Sigma$ *then* $\varphi \in \Sigma$ *and* $\psi \in \Sigma$ (*and similarly for* $B^{\varphi}\psi$).

*For any* $\varphi \in \mathcal{L}_{\mathsf{CHB}}$, $Sub(\varphi)$ *denotes the subformula closure of* $\varphi$. *Any formula* $\psi \in Sub(\varphi)\backslash\{\varphi\}$ *is called a proper subformula of* $\varphi$.

Let $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ be a quasi tsp-model and $\Sigma$ a *finite* subset of $\mathcal{L}_{\mathsf{CHB}}$ that satisfies the following closure conditions:

C1 $\top \in \Sigma$,
C2 $\Sigma$ is subformula closed, and
C3 if $B^{\varphi}\psi \in \Sigma$ then $\Box(\varphi \rightarrow \langle\geq\rangle(\varphi \wedge [\geq](\varphi \rightarrow \psi))) \in \Sigma$.

For $w, v \in W$, put

$$w \leftrightsquigarrow_{\Sigma} v \text{ iff } \forall\varphi \in \Sigma\ (\mathcal{M}, w \Vdash \varphi \text{ iff } \mathcal{M}, v \Vdash \varphi),$$

and denote by $[w]_{\Sigma}$ the equivalence class of $w$ modulo $\leftrightsquigarrow_{\Sigma}$. We omit the subscript $\Sigma$ and write $\leftrightsquigarrow$ and $[w]$ when the corresponding set of formulas $\Sigma$ is contextually clear. We define the *filtration* $\mathcal{M}^f = \langle W^f, \geq^f, T^f, \oplus^f, \mathfrak{b}^f, t^f, v^f\rangle$ of $\mathcal{M}$ through $\Sigma$ as follows:

- $W^f = \{[w] : w \in W\}$,
- For any $[w], [v] \in W^f$,

$[w] \geq^f [v]$ iff for all $\varphi \in \Sigma$(if $[\geq]\varphi \in \Sigma$ and $\mathcal{M}, w \Vdash [\geq]\varphi$ then $\mathcal{M}, v \Vdash \varphi \wedge [\geq]\varphi$),

- $T^f = \{a \in T : a = \oplus\{t(\varphi) : \varphi \in \Sigma'\}$ for some nonempty $\Sigma' \subseteq \Sigma\}$,
- $\oplus^f : T^f \times T^f \rightarrow T^f$ such that $a \oplus^f b = a \oplus b$,
- $\mathfrak{b}^f = \mathfrak{b}$,
- $t^f : \mathcal{L}_{\mathsf{CHB}} \rightarrow T^f$ such that $t^f(x) = \begin{cases} t(x) \text{ if } x \in \Sigma \\ \mathfrak{b}^f, \text{ otherwise,} \end{cases}$ and $t^f(\varphi) = \oplus^f Var(\varphi)$,
- $v^f(p) = \begin{cases} \{[w] \in W^f : w \in v(p)\} \text{ if } p \in \Sigma \\ \emptyset, \text{ otherwise.} \end{cases}$

LEMMA 37. *Let* $\mathcal{M}^f = \langle W^f, \geq^f, T^f, \oplus^f, \mathfrak{b}^f, t^f, v^f\rangle$ *be the filtration of a quasi tsp-model* $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ *through a finite set of formulas* $\Sigma$ *that satisfies the closure conditions* $(C1)$-$(C3)$. *Then, for all* $w, v \in W$ *and* $\varphi \in \mathcal{L}_{\mathsf{CHB}}$,

1. *if* $w \geq v$ *then* $[w] \geq^f [v]$,
2. *if* $[w] \geq^c [v]$ *then for all* $[\geq]\varphi \in \Sigma$, *if* $\mathcal{M}, w \Vdash [\geq]\varphi$ *then* $\mathcal{M}, v \Vdash \varphi$,
3. *if* $\varphi \in \Sigma$ *then* $t(\varphi) \in T^f$ *and* $t(\varphi) = t^f(\varphi)$,
4. *for all* $a, b \in T^f$, $a \sqsubseteq b$ *iff* $a \sqsubseteq^f b$.

*Proof.*

1. Suppose that $w \geq v$ and let $\varphi \in \Sigma$ such that $[\geq]\varphi \in \Sigma$ and $\mathcal{M}, w \Vdash [\geq]\varphi$. The latter, by the semantics of $[\geq]$, means that $\mathcal{M}, v \Vdash \varphi$. Moreover, $\mathcal{M}, w \Vdash [\geq]\varphi$ and transitivity of $\geq$ entails that $\mathcal{M}, v \Vdash [\geq]\varphi$. Therefore, $\mathcal{M}, v \Vdash \varphi \wedge [\geq]\varphi$. Hence, $[w] \geq^f [v]$.

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2. Suppose that $[w] \geq^f [v]$ and let $[\geq]\varphi \in \Sigma$ such that $\mathcal{M}, w \Vdash [\geq]\varphi$. Since $\Sigma$ is subformula closed, we also have $\varphi \in \Sigma$. Therefore, by the definition of $\geq^f$, we obtain that $\mathcal{M}, v \Vdash \varphi \wedge [\geq]\varphi$. Hence, $\mathcal{M}, v \Vdash \varphi$.
3. For any $\varphi \in \Sigma$, that $t(\varphi) \in T^f$ follows from the definition of $T^f$. To show $t(\varphi) = t^f(\varphi)$, suppose $Var(\varphi) = \{x_1, \ldots, x_n\}$. Then,

$$\begin{aligned} t(\varphi) &= t(x_1) \oplus \cdots \oplus t(x_n) && \text{(by the definition of } t(\varphi)) \\ &= t(x_1) \oplus^f \cdots \oplus^f t(x_n) \\ &\quad \text{(by the definition of } \oplus^f \text{ and } t(x_i) \in T^f \text{ (since } \Sigma \text{ is subformula closed))} \\ &= t^f(x_1) \oplus^f \cdots \oplus^f t^f(x_n) && \text{(by the definition of } t^f \text{ and each } x_i \in \Sigma) \\ &= t^f(\varphi) && \text{(by the definition of } t^f) \end{aligned}$$

4. Let $a, b \in T^f$, i.e., $a = \oplus\{t(\varphi) : \varphi \in \Sigma_1\}$ and $b = \oplus\{t(\varphi) : \varphi \in \Sigma_2\}$ for some nonempty $\Sigma_1, \Sigma_2 \subseteq \Sigma$. Then,

$$a \sqsubseteq b \text{ iff } a \oplus b = b \text{ iff } a \oplus^f b = b \text{ iff } a \sqsubseteq^f b. \qquad \square$$

LEMMA 38. *Given a quasi tsp-model* $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ *and a finite set* $\Sigma \subseteq \mathcal{L}_{\mathsf{CHB}}$ *satisfying the closure conditions* (C1)-(C3), *the filtration* $\mathcal{M}^f = \langle W^f, \geq^f, T^f, \oplus^f, \mathfrak{b}^f, t^f, v^f\rangle$ *of* $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ *through* $\Sigma$ *is a quasi tsp-model. Moreover,* $W^f$ *and* $T^f$ *are both finite. Therefore,* $\mathcal{M}^f$ *is a tsp-model.*

*Proof.* We first show that $\mathcal{M}^f = \langle W^f, \geq^f, T^f, \oplus^f, \mathfrak{b}^f, t^f, v^f\rangle$ is a quasi tsp-model:

- $\geq^f$ is a total preorder: Let $\varphi \in \Sigma$ such that $[\geq]\varphi \in \Sigma$ and $[w], [v], [u] \in W^f$.
  For reflexivity, suppose that $\mathcal{M}, w \Vdash [\geq]\varphi$. Since $\geq$ is reflexive, we also have that $\mathcal{M}, w \Vdash \varphi$. Therefore, $\mathcal{M}, w \Vdash \varphi \wedge [\geq]\varphi$, i.e., $[w] \geq^f [w]$.
  For transitivity, suppose that $[w] \geq^f [v] \geq^f [u]$ and $\mathcal{M}, [w] \Vdash [\geq]\varphi$. Then, since $[w] \geq^f [v]$, we have $\mathcal{M}, v \Vdash \varphi \wedge [\geq]\varphi$. This implies that $\mathcal{M}, v \Vdash [\geq]\varphi$. Similarly, since $[v] \geq^f [u]$, we conclude that $\mathcal{M}, u \Vdash \varphi \wedge [\geq]\varphi$. Therefore, $[w] \geq^f [u]$.
  Finally, by the totality of $\geq$ and Lemma 37.1, we obtain that $[w] \geq^f [v]$ or $[v] \geq^f [w]$. Therefore, $\geq^f$ is a total order.
- For all $a, b \in T^f$, $a \oplus b \in T^f$ and $\oplus^f$ is well-defined: Let $a, b \in T^f$. Then, by the definition of $T^f$, it is easy to see that $a \oplus b = (\oplus\{t(\varphi) : \varphi \in \Sigma_1\}) \oplus (\oplus\{t(\varphi) : \varphi \in \Sigma_2\}) = \oplus\{t(\varphi) : \varphi \in \Sigma_1 \cup \Sigma_2\}$ for some nonempty $\Sigma_1, \Sigma_2 \subseteq \Sigma$. Since $\Sigma_1 \cup \Sigma_2 \subseteq \Sigma$, we obtain that $a \oplus b \in T^f$. To prove that $\oplus^f$ is well-defined, let $(a, b), (c, d) \in T^f \times T^f$ such that $(a, b) = (c, d)$. This means that $a = c$ and $b = d$. Then,

$$\begin{aligned} a \oplus^f b &= a \oplus b && \text{(by the definition of } \oplus^f) \\ &= c \oplus d && \text{(since } a = c, b = d, \text{ and } \oplus \text{ is well-defined)} \\ &= c \oplus^f d && \text{(by the definition of } \oplus^f) \end{aligned}$$

  Therefore, $\oplus^f$ is well-defined.
- $\forall a, b \in T^f \exists c \in T^f (a \oplus^f b = c)$: Let $a, b \in T^f$. Then, $a \oplus^f b = a \oplus b \in T^f$ (by the above clause).
- $\oplus^f$ is idempotent, commutative, and associative: Let $a, b, c \in T^f$, then:
  idempotance: $a \oplus^f a = a \oplus a = a$, by the definition of $\oplus^f$ and idempotance of $\oplus$.

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commutativity: $a \oplus^f b = a \oplus b = b \oplus a = b \oplus^f a$, by the definition of $\oplus^f$ and commutativity of $\oplus$.
associativity: $(a \oplus^f b) \oplus^f c = (a \oplus b) \oplus c = a \oplus (b \oplus c) = a \oplus^f (b \oplus^f c)$, by the definition of $\oplus^f$ and associativity of $\oplus$.

- $t^f$ is well-defined: follows easily since $t$ and $\oplus^f$ are well-defined.
- $t^f(\top) = \mathfrak{b}^f \in T^f$ : First observe that $t(\top) \in T^f$, since $\top \in \Sigma$ (by the closure condition C1). We then have,

$$\begin{aligned} t^f(\top) &= t(\top) && \text{(by the definition of } t^f, \text{ since } \top \in \Sigma) \\ &= \mathfrak{b} && \text{(by the definition of quasi tsp-models )} \\ &= \mathfrak{b}^f && \text{(by the definition of } \mathfrak{b}^f) \end{aligned}$$

This completes the proof that $\mathcal{M}^f = \langle W^f, \geq^f, T^f, \oplus^f, \mathfrak{b}^f, t^f, v^f \rangle$ is a quasi tsp-model. Moreover, since $\Sigma$ is finite, there are only finitely many equivalence classes $[w]$ modulo $\leftrightsquigarrow_\Sigma$. Therefore, $W^f$ is finite. Similarly, it is easy to observe that $T^f$ can have at most as many elements as $\mathcal{P}(\Sigma)$. Therefore, since $\Sigma$ is finite, $T^f$ is finite as well. Finally, since $W^f$ is finite, $\geq^f$ is a total preorder on $W^f$. Therefore, as every finite total preorder is a well-preorder, we conclude that $\mathcal{M}^f$ is a tsp-model. □

The Filtration Theorem (Theorem 40) is proved by induction on a complexity measure defined by means of a measure that counts the $B$-depth of formulas in $\mathcal{L}_{\mathsf{CHB}}$.

DEFINITION 16 ($B$-depth of formulas in $\mathcal{L}_{\mathsf{CHB}}$). *The $B$-depth $d(\varphi)$ of a formula $\varphi \in \mathcal{L}_{\mathsf{CHB}}$ is a natural number recursively defined as*:

$$\begin{aligned} d(\top) = d(p) &= 0, \\ d(\neg\varphi) = d(\Box\varphi) = d([\geq]\varphi) &= d(\varphi), \\ d(\varphi \wedge \psi) &= max\{d(\varphi), d(\psi)\}, \\ d(B^\varphi \psi) &= 1 + max\{d(\varphi), d(\psi)\}. \end{aligned}$$

LEMMA 39. *There is a well-founded strict partial order $\prec$ on $\mathcal{L}_{\mathsf{CHB}}$ such that, for all $\varphi, \psi \in \mathcal{L}_{\mathsf{CHB}}$,*

1. *if $\psi$ is a proper subformula of $\varphi$ then $\psi \prec \varphi$, and*
2. $\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))) \prec B^\varphi \psi$.

*Proof.* For any $\varphi, \psi \in \mathcal{L}_{\mathsf{CHB}}$, define

$$\psi \prec \varphi \quad \text{iff either } d(\psi) < d(\varphi), \text{ or } d(\psi) = d(\varphi) \text{ and } \psi \in Sub(\varphi)\backslash\{\varphi\}, \quad (\prec\text{-order})$$

where $<$ represents the standard order on natural numbers. It is a routine exercise to check that $\prec$ is a well-founded strict partial order on $\mathcal{L}_{\mathsf{CHB}}$. To prove the lemma, let $\varphi, \psi \in \mathcal{L}_{\mathsf{CHB}}$.

1. Suppose that $\psi$ is a proper subformula of $\varphi$, i.e., that $\psi \in Sub(\varphi)\backslash\{\varphi\}$. Then, by the definition of $d$ (Definition 16), either $d(\psi) < d(\varphi)$ or $d(\psi) = d(\varphi)$. Therefore, by the definition of $\prec$, we obtain that $\psi \prec \varphi$.
2. It is easy to see, by the definition of $d$, that $d(\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi)))) = max\{d(\varphi), d(\psi)\}$ and $d(B^\varphi\psi) = 1 + max\{d(\varphi), d(\psi)\}$. Therefore, $d(\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi)))) < d(B^\varphi\psi)$. We then immediately conclude that $\Box(\varphi \to \langle\geq\rangle(\varphi \wedge [\geq](\varphi \to \psi))) \prec B^\varphi\psi$. □

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THEOREM 40 (Filtration Theorem for $\mathcal{L}_{\mathsf{CHB}}$). *Let $\mathcal{M}^f = \langle W^f, \geq^f, T^f, \oplus^f, \mathfrak{b}^f, t^f, v^f \rangle$ be the filtration of a quasi tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v \rangle$ through a finite set $\Sigma \subseteq \mathcal{L}_{\mathsf{CHB}}$ which satisfies the closure conditions (C1)–(C3). Then for all $w \in W$ and $\varphi \in \Sigma$, we have*

$$\mathcal{M}, w \Vdash \varphi \text{ iff } \mathcal{M}^f, [w] \Vdash \varphi.$$

*Proof.* The proof is by $\prec$-induction on the structure of $\varphi$ (see the proof of Lemma 39 for the definition of $\prec$). The cases for the propositional variables, Booleans, and $\varphi := \Box\psi$ are standard. Toward showing the cases for $\varphi := [\geq]\psi$ and $\varphi := B^{\psi}\chi$, suppose inductively that the statement holds for all $\psi$ such that $\psi \prec \varphi$.

Case $\varphi := [\geq]\psi$: Observe that $\psi \in Sub(\varphi)\backslash\{\varphi\}$. Thus, by Lemma 39.1, we have $\psi \prec \varphi$. Moreover, $\psi \in \Sigma$ since $\Sigma$ is subformula closed. Therefore, we can apply the induction hypothesis on $\psi$.

($\Rightarrow$) Suppose that $\mathcal{M}, w \Vdash [\geq]\psi$ and let $[v] \in W^f$ such that $[w] \geq^f [v]$. Then, by Lemma 37.2 and the fact that $[\geq]\psi \in \Sigma$, we have $\mathcal{M}, v \Vdash \psi$. Therefore, by the induction hypothesis, we obtain that $\mathcal{M}^f, [v] \Vdash \psi$. Since $[v]$ has been chosen arbitrarily from $W^f$ with $[w] \geq^f [v]$, we conclude that $\mathcal{M}^f, [w] \Vdash [\geq]\psi$.

($\Leftarrow$) Suppose that $\mathcal{M}^f, [w] \Vdash [\geq]\psi$ and let $v \in W$ such that $w \geq v$. The latter, by Lemma 37.1, implies that $[w] \geq^f [v]$. Then, by the first assumption, we obtain that $\mathcal{M}^f, [v] \Vdash \psi$. Thus, by the induction hypothesis, we have $\mathcal{M}, v \Vdash \psi$. Since $v$ has been chosen arbitrarily from $W$ with $w \geq v$, we conclude that $\mathcal{M}, w \Vdash [\geq]\psi$.

Case $\varphi := B^{\psi}\chi$: Observe that $d(\varphi) = d(B^{\psi}\chi) = 1 + max\{d(\psi), d(\chi)\}$ and $d(\Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi)))) = max\{d(\psi), d(\chi)\}$. Therefore, $\Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi))) \prec \varphi$. Moreover, $\Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi))) \in \Sigma$ (by the closure condition C3, since $B^{\psi}\chi \in \Sigma$). Thus, we can apply the induction hypothesis on $\Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi)))$.

$\mathcal{M}, w \Vdash B^{\psi}\chi$ iff $\mathcal{M}, w \Vdash \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi))) \wedge B^{\psi}\overline{\chi}$ (validity of Ax6)

iff $\mathcal{M}, w \Vdash \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi)))$ and $\mathcal{M}, w \Vdash B^{\psi}\overline{\chi}$ (by the semantics)

iff $\mathcal{M}, w \Vdash \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi)))$ and $t(\chi) \sqsubseteq t(\psi) \oplus \mathfrak{b}$ (by the semantics)

iff $\mathcal{M}, w \Vdash \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi)))$ and $t^f(\chi) \sqsubseteq^f t^f(\psi) \oplus^f \mathfrak{b}^f$ (Lemmas 37.3 and 37.4)

iff $\mathcal{M}^f, [w] \Vdash \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi)))$ and $t^f(\chi) \sqsubseteq^f t^f(\psi) \oplus^f \mathfrak{b}^f$ (induction hypothesis)

iff $\mathcal{M}^f, [w] \Vdash \Box(\psi \to \langle\geq\rangle(\psi \wedge [\geq](\psi \to \chi)))$ and $\mathcal{M}^f, [w] \Vdash B^{\psi}\overline{\chi}$ (by the semantics)

iff $\mathcal{M}^f, [w] \Vdash B^{\psi}\chi$ (validity of Ax6)

□

COROLLARY 41. CHB *is complete with respect to the class of finite tsp-models.*

*Proof.* Let $\varphi \in \mathcal{L}_{\mathsf{CHB}}$ such that $\nvdash_{\mathsf{CHB}} \varphi$. Then, by Corollary 36, there is a quasi tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v \rangle$ and $w \in W$ such that $\mathcal{M}, w \nVdash \varphi$. Let $\Sigma$ be the set of formulas obtained by closing $\{\varphi\}$ under the closure conditions (C1)–(C3) and $\mathcal{M}^f = \langle W^f, \geq^f, T^f, \oplus^f, \mathfrak{b}^f, t^f, v^f \rangle$ be the filtration of $\mathcal{M}$ through $\Sigma$. Then, by Theorem 40,

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we obtain that $\mathcal{M}^f, [w] \not\Vdash \varphi$. By Lemma 38, we know that $\mathcal{M}^f$ is a tsp-model, in fact, it is a finite tsp-model. We therefore conclude that CHB is complete with respect to the class of finite tsp-models. □

### *B.4. Proof of Theorem 9: soundness and completeness of* DHB.

#### *B.4.1. Soundness of* DHB.

LEMMA 42. *Given a join semi-lattice* $(T, \oplus)$ *and* $a, b, c \in T$,

$$c \sqsubseteq a \oplus b \text{ iff } a \oplus c \sqsubseteq a \oplus b.$$

*Proof.*

$$\begin{aligned}(a \oplus b) \oplus (a \oplus c) &= ((a \oplus b) \oplus a) \oplus c && \text{(associativity of } \oplus)\\ &= (a \oplus (a \oplus b)) \oplus c && \text{(commutativity of } \oplus)\\ &= ((a \oplus a) \oplus b) \oplus c && \text{(associativity of } \oplus)\\ &= (a \oplus b) \oplus c && \text{(idempotence of } \oplus)\end{aligned}$$

Therefore, $(a \oplus b) \oplus c = a \oplus b$ iff $(a \oplus b) \oplus (a \oplus c) = a \oplus b$, i.e., $c \sqsubseteq a \oplus b$ iff $a \oplus c \sqsubseteq a \oplus b$. □

The following observation follows directly from Definition 8.

OBSERVATION 43. *For all* $\varphi \in \mathcal{L}_{\mathsf{DHB}}$ *and tsp-model* $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$, *we have*

$$|[\Uparrow\varphi]\psi|_{\mathcal{M}} = |\psi|_{\mathcal{M}^{\Uparrow\varphi}}.$$

For the soundness of DHB, we need to check the validity of the axioms and rules in Table 3. All the cases except for $R_{[\geq]}$ and $R_B$ are straightforward. We thus only spell out the details for $R_{[\geq]}$ and $R_B$. Let $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ be a tsp-model and $w \in W$.

$R_{[\geq]}$:

($\Rightarrow$) Suppose that $\mathcal{M}, w \Vdash [\Uparrow\varphi][\geq]\psi$. This means that $\mathcal{M}^{\Uparrow\varphi}, w \Vdash [\geq]\psi$, i.e., for all $v \in W$ such that $w \geq^{\varphi} v$, we have $\mathcal{M}^{\Uparrow\varphi}, v \Vdash \psi$. To prove $\mathcal{M}, w \Vdash \neg\varphi \to [\geq][\Uparrow\varphi]\psi$, suppose that $\mathcal{M}, w \Vdash \neg\varphi$. Moreover, let $v' \in W$ such that $w \geq v'$. As $\mathcal{M}, w \not\Vdash \varphi$, by the definition of $\geq^{\varphi}$, we have that $w \geq^{\varphi} v'$ as well. Thus, by the first assumption, we obtain that $\mathcal{M}^{\Uparrow\varphi}, v' \Vdash \psi$. Therefore, by the semantics, $\mathcal{M}, v' \Vdash [\Uparrow\varphi]\psi$. As $v'$ has been chosen arbitrarily from $W$ with $w \geq v'$, we conclude that $\mathcal{M}, w \Vdash [\geq][\Uparrow\varphi]\psi$. Hence, $\mathcal{M}, w \Vdash \neg\varphi \to [\geq][\Uparrow\varphi]\psi$. To show that $\mathcal{M}, w \Vdash \neg\varphi \to \Box(\varphi \to [\Uparrow\varphi]\psi)$, suppose that $\mathcal{M}, w \Vdash \neg\varphi$ and let $v \in W$ such that $\mathcal{M}, v \Vdash \varphi$. Then, by the definition of $\geq^{\varphi}$, we have $w \geq^{\varphi} v$. Therefore, by the first assumption, we obtain that $\mathcal{M}^{\Uparrow\varphi}, v \Vdash \psi$, i.e., that $\mathcal{M}, v \Vdash [\Uparrow\varphi]\psi$. Hence, $\mathcal{M}, v \Vdash \varphi \to [\Uparrow\varphi]\psi$. As $v$ has been chosen arbitrarily from $W$, we obtain that $\mathcal{M}, w \Vdash \Box(\varphi \to [\Uparrow\varphi]\psi)$, thus, $\mathcal{M}, w \Vdash \neg\varphi \to \Box(\varphi \to [\Uparrow\varphi]\psi)$. To prove $\mathcal{M}, w \Vdash [\geq](\varphi \to [\Uparrow\varphi]\psi)$, let $v \in W$ such that $w \geq v$ and $\mathcal{M}, v \Vdash \varphi$. Then, by the definition of $\geq^{\varphi}$, we have that $w \geq^{\varphi} v$. Hence, by the first assumption, we obtain that $\mathcal{M}^{\Uparrow\varphi}, v \Vdash \psi$. I.e., $\mathcal{M}, v \Vdash [\Uparrow\varphi]\psi$. Thus, $\mathcal{M}, v \Vdash \varphi \to [\Uparrow\varphi]\psi$. As $v$ has been chosen arbitrarily from $W$ with $w \geq v$, we conclude that $\mathcal{M}, w \Vdash [\geq](\varphi \to [\Uparrow\varphi]\psi)$. Therefore, $\mathcal{M}, w \Vdash (\neg\varphi \to [\geq][\Uparrow\varphi]\psi) \wedge (\neg\varphi \to \Box(\varphi \to [\Uparrow\varphi]\psi)) \wedge [\geq](\varphi \to [\Uparrow\varphi]\psi)$.

($\Leftarrow$) Suppose that $\mathcal{M}, w \Vdash (\neg\varphi \to [\geq][\Uparrow\varphi]\psi) \wedge (\neg\varphi \to \Box(\varphi \to [\Uparrow\varphi]\psi)) \wedge [\geq](\varphi \to [\Uparrow\varphi]\psi)$. So, (1) $\mathcal{M}, w \Vdash \neg\varphi \to [\geq][\Uparrow\varphi]\psi$, (2) $\mathcal{M}, w \Vdash \neg\varphi \to \Box(\varphi \to [\Uparrow\varphi]\psi)$, and (3) $\mathcal{M}, w \Vdash [\geq](\varphi \to [\Uparrow\varphi]\psi)$. Now let $v' \in W$ such that $w \geq^{\varphi} v'$ and show that $\mathcal{M}^{\Uparrow\varphi}, v' \vDash \psi$. Since $\geq$ is a total order, we have two case:

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Case $w \geq v'$: Then, since $w \geq^{\varphi} v'$, we obtain by the definition of $\geq^{\varphi}$ that either $\mathcal{M}, w \not\Vdash \varphi$ or both $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, v' \Vdash \varphi$. If $\mathcal{M}, w \not\Vdash \varphi$, by assumption (1), we have $\mathcal{M}, w \Vdash [\geq][\Uparrow\varphi]\psi$. Therefore, as $w \geq v'$, we obtain that $\mathcal{M}^{\Uparrow\varphi}, v' \Vdash \psi$. If both $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, v' \Vdash \varphi$, since $w \geq v'$ and $\mathcal{M}, w \Vdash [\geq](\varphi \to [\Uparrow\varphi]\psi)$ (assumption (3)), we have $\mathcal{M}, v' \vDash [\Uparrow\varphi]\psi$, i.e., $\mathcal{M}^{\Uparrow\varphi}, v' \vDash \psi$.

Case $v' \geq w$: Then, since $w \geq^{\varphi} v'$, we have $\mathcal{M}, w \not\Vdash \varphi$ and $\mathcal{M}, v' \Vdash \varphi$. Since $\mathcal{M}, w \not\Vdash \varphi$, by assumption (2), we have $\mathcal{M}, w \Vdash \Box(\varphi \to [\Uparrow\varphi]\psi)$. Therefore, since $\mathcal{M}, v' \Vdash \varphi$, we obtain that $\mathcal{M}, v' \Vdash [\Uparrow\varphi]\psi$, i.e., that $\mathcal{M}^{\Uparrow\varphi}, v' \Vdash \psi$.

Since $v'$ has been chosen arbitrarily from $W$ with $w \geq^{\varphi} v'$, we conclude that $\mathcal{M}^{\Uparrow\varphi}, w \Vdash [\geq]\psi$, i.e., that $\mathcal{M}, w \Vdash [\Uparrow\varphi][\geq]\psi$.

$\mathrm{R}_B$:

($\Rightarrow$) Suppose $\mathcal{M}, w \Vdash [\Uparrow\varphi]B^{\psi}\chi$. We then have

$$\begin{aligned}
\mathcal{M}, w \Vdash [\Uparrow\varphi]B^{\psi}\chi \text{ iff } & \mathcal{M}^{\Uparrow\varphi}, w \Vdash B^{\psi}\chi && \text{(by the } \Vdash\text{-semantics)}\\
\text{iff } & Min_{\geq^{\varphi}}|\psi|_{\mathcal{M}^{\Uparrow\varphi}} \subseteq |\chi|_{\mathcal{M}^{\Uparrow\varphi}} \text{ and } t^{\varphi}(\chi) \sqsubseteq \mathfrak{b}^{\varphi} \oplus t^{\varphi}(\psi) && \text{(by the } \Vdash\text{-semantics)}\\
\text{iff } & Min_{\geq^{\varphi}}|[\Uparrow\varphi]\psi|_{\mathcal{M}} \subseteq |[\Uparrow\varphi]\chi|_{\mathcal{M}} \text{ and } t(\chi) \sqsubseteq (\mathfrak{b} \oplus t(\varphi)) \oplus t(\psi) && \text{(Observation 43, the definitions of } \mathfrak{b}^{\varphi} \text{ and } t^{\varphi})
\end{aligned}$$

If $\top \in Var(\chi)$, use Lemma 42 in the last step. We then have two cases due the definition of $\geq^{\varphi}$:

If $\mathcal{M}, w \Vdash \Diamond(\varphi \wedge [\Uparrow\varphi]\psi)$, then $Min_{\geq^{\varphi}}|[\Uparrow\varphi]\psi|_{\mathcal{M}} = Min_{\geq}|\varphi \wedge [\Uparrow\varphi]\psi|_{\mathcal{M}}$: if there is a world in $W$ which makes $\varphi \wedge [\Uparrow\varphi]\psi$ true in $\mathcal{M}$, then the most plausible $[\Uparrow\varphi]\psi$-worlds with respect to $\geq^{\varphi}$ are the same as the most plausible $\varphi \wedge [\Uparrow\varphi]\psi$-worlds with respect to $\geq$, since the lexicographic upgrade with $\varphi$ makes all $\varphi$-worlds more plausible than $\neg\varphi$-worlds. For topicality, we have

$$\begin{aligned}
t([\Uparrow\varphi]\chi) &= t(\varphi) \oplus t(\chi)\\
&\sqsubseteq t(\varphi) \oplus ((\mathfrak{b} \oplus t(\varphi)) \oplus t(\psi)) && \text{(by the assumption)}\\
&= (\mathfrak{b} \oplus (t(\varphi) \oplus t(\psi)) && \text{(the properties of } \oplus)\\
&= \mathfrak{b} \oplus t(\varphi \wedge [\Uparrow\varphi]\psi) && \text{(since } t(\varphi \wedge [\Uparrow\varphi]\psi) = t(\varphi) \oplus t(\psi))
\end{aligned}$$

Therefore, $\mathcal{M}, w \Vdash B^{\varphi \wedge [\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi$.

If $\mathcal{M}, w \not\Vdash \Diamond(\varphi \wedge [\Uparrow\varphi]\psi)$, then $Min_{\geq^{\varphi}}|[\Uparrow\varphi]\psi|_{\mathcal{M}} = Min_{\geq}|[\Uparrow\varphi]\psi|_{\mathcal{M}}$: if there is no state that makes $\varphi \wedge [\Uparrow\varphi]\psi$ true in $\mathcal{M}$, then by the definition of lexicographic upgrade, the $\geq^{\varphi}$-order among the $[\Uparrow\varphi]\psi$-worlds is the same as the $\geq$-order. For topicality, we have $t([\Uparrow\varphi]\chi) \sqsubseteq (\mathfrak{b} \oplus t(\varphi)) \oplus t(\psi) = \mathfrak{b} \oplus t([\Uparrow\varphi]\psi)$ (similar to the above case). Therefore, $\mathcal{M}, w \Vdash B^{[\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi$.

($\Leftarrow$) Suppose $\mathcal{M}, w \Vdash (\Diamond(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{\varphi \wedge [\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi) \vee (\neg\Diamond(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{[\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi)$. Following a similar reasoning as above, we obtain $Min_{\geq^{\varphi}}|[\Uparrow\varphi]\psi|_{\mathcal{M}} \subseteq |[\Uparrow\varphi]\chi|_{\mathcal{M}}$. For topicality, we need Lemma 42. By spelling out the topicality component of the assumption, we obtain that $t(\varphi) \oplus t(\chi) \sqsubseteq (\mathfrak{b} \oplus t(\varphi)) \oplus t(\psi)$. Then, by Lemma 42, we obtain that $t(\chi) \sqsubseteq (\mathfrak{b} \oplus t(\varphi)) \oplus t(\psi)$.

### *B.4.2. Completeness of* DHB.

DEFINITION 17 (Complexity measure for $\mathcal{L}_{\mathsf{DHB}}$). *The complexity $c(\varphi)$ of a formula $\varphi \in \mathcal{L}_{\mathsf{DHB}}$ is a natural number recursively defined as*

$$c(\top) = c(p) = 1$$

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$$\begin{aligned} c(\neg\varphi) = c(\Box\varphi) = c([\geq]\varphi) &= c(\varphi) + 1 \\ c(\varphi \wedge \psi) &= 1 + max\{c(\varphi), c(\psi)\} \\ c(B^{\varphi}\psi) &= 1 + c(\varphi) + c(\psi) \\ c([\Uparrow\varphi]\psi) &= (6 + |Var(\varphi)| + c(\varphi)) \cdot c(\psi), \end{aligned}$$

*where* $|Var(\varphi)|$ *is the number of elements in* $Var(\varphi)$.

LEMMA 44. *For all* $\varphi, \psi, \chi \in \mathcal{L}_{\mathsf{DHB}}$:

1. $c(\varphi) > c(\psi)$ *if* $\psi$ *is a proper subformula of* $\varphi$,
2. $c([\Uparrow\varphi]\top) > c(\top \wedge \overline{\varphi})$,
3. $c([\Uparrow\varphi]p) > c(p \wedge \overline{\varphi})$,
4. $c([\Uparrow\varphi]\neg\psi) > c(\neg[\Uparrow\varphi]\psi)$,
5. $c([\Uparrow\varphi](\psi \wedge \chi)) > c([\Uparrow\varphi]\psi \wedge [\Uparrow\varphi]\chi)$,
6. $c([\Uparrow\varphi]\Box\psi) > c(\Box[\Uparrow\varphi]\psi)$,
7. $c([\Uparrow\varphi][\geq]\psi) > c((\neg\varphi \rightarrow [\geq][\Uparrow\varphi]\psi) \wedge (\neg\varphi \rightarrow \Box(\varphi \rightarrow [\Uparrow\varphi]\psi)) \wedge [\geq](\varphi \rightarrow [\Uparrow\varphi]\psi))$,
8. $c([\Uparrow\varphi]B^{\psi}\chi) > c((\Diamond(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{\varphi \wedge [\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi) \vee (\neg\Diamond(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{[\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi))$.

*Proof.* Follows by easy calculations using Definition 17. □

DEFINITION 18 (Translation $f : \mathcal{L}_{\mathsf{DHB}} \rightarrow \mathcal{L}_{\mathsf{CHB}}$). *The translation* $f : \mathcal{L}_{\mathsf{DHB}} \rightarrow \mathcal{L}_{\mathsf{CHB}}$ *is defines as follows*:

$$\begin{aligned} f(\top) &= \top \\ f(p) &= p \\ f(\neg\varphi) &= \neg f(\varphi) \\ f(\Box\varphi) &= \Box f(\varphi) \\ f(\varphi \wedge \psi) &= f(\varphi) \wedge f(\psi) \\ f(B^{\varphi}\psi) &= B^{f(\varphi)} f(\psi) \\ f([\Uparrow\varphi]\top) &= f(\top \wedge \overline{\varphi}) \\ f([\Uparrow\varphi]p) &= f(p \wedge \overline{\varphi}) \\ f([\Uparrow\varphi]\neg\varphi) &= f(\neg[\Uparrow\varphi]\varphi) \\ f([\Uparrow\varphi](\psi \wedge \chi)) &= f([\Uparrow\varphi]\psi \wedge [\Uparrow\varphi]\chi) \\ f([\Uparrow\varphi]\Box\varphi) &= f(\Box[\Uparrow\varphi]\varphi) \\ f([\Uparrow\varphi][\geq]\psi) &= f((\neg\varphi \rightarrow [\geq][\Uparrow\varphi]\psi) \wedge (\neg\varphi \rightarrow \Box(\varphi \rightarrow [\Uparrow\varphi]\psi)) \\ &\quad \wedge [\geq](\varphi \rightarrow [\Uparrow\varphi]\psi)) \\ f([\Uparrow\varphi]B^{\varphi}\psi) &= f((\Diamond(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{\varphi \wedge [\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi) \vee (\neg\Diamond(\varphi \wedge [\Uparrow\varphi]\psi) \\ &\quad \wedge B^{[\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi)) \\ f([\Uparrow\varphi][\Uparrow\psi]\chi) &= f([\Uparrow\varphi] f([\Uparrow\psi]\chi)). \end{aligned}$$

We need the following lemma in order to be able to use the topic-sensitive RE rules (Theorems 5.3 and 5.4) in the completeness proof of DHB. For this lemma to go through, it is crucial that the reduction axioms $\mathrm{R}_{\top}$ and $\mathrm{R}_{p}$ have occurrences of each element in $Var(\varphi)$ on the right-hand-side of the equation, where $\varphi$ is the sentence inside the dynamic operator.

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LEMMA 45. *For all* $\varphi \in \mathcal{L}_{\mathsf{DHB}}$, $Var(\varphi) = Var(f(\varphi))$.

*Proof.* The proof follows by an easy $c$-induction on the structure of $\varphi$ and uses Lemma 44. Note that the case for $\varphi := [\Uparrow\psi]\chi$ requires subinduction on $\chi$. □

LEMMA 46. *For all* $\varphi \in \mathcal{L}_{\mathsf{DHB}}$, $\vdash_{\mathsf{DHB}} \varphi \leftrightarrow f(\varphi)$.

*Proof.* The proof follows by $c$-induction on the structure of $\varphi$ and uses Lemma 44 and the reduction axioms given in Table 3. Cases for the propositional variables, the Boolean connectives, $\varphi := \Box\psi$, and $\varphi := [\geq]\psi$ are elementary. Here we only show the cases for $\varphi := B^{\psi}\chi$ and $\varphi := [\Uparrow\psi]\chi$, where the latter requires subinduction on $\chi$. Suppose inductively that $\vdash_{\mathsf{DHB}} \psi \leftrightarrow f(\psi)$, for all $\psi$ with $c(\psi) < c(\varphi)$.

Case $\varphi := B^{\psi}\chi$

By Lemma 44.1 and the induction hypothesis (IH), we have $\vdash_{\mathsf{DHB}} \psi \leftrightarrow f(\psi)$. Moreover, by Lemma 45, we have $Var(\psi) = Var(f(\psi))$. It is therefore easy to see, by Ax1 in Table 2, that $\vdash_{\mathsf{DHB}} B^{\overline{\psi}}\overline{f(\psi)}$ and $\vdash_{\mathsf{DHB}} B^{\overline{f(\psi)}}\overline{\psi}$. Then, by Theorem 5.3, we obtain $\vdash_{\mathsf{DHB}} B^{\psi}\chi \leftrightarrow B^{f(\psi)}\chi$. Similarly, we also have $\vdash_{\mathsf{DHB}} \chi \leftrightarrow f(\chi)$ and $Var(\chi) = Var(f(\chi))$, thus, $\vdash_{\mathsf{DHB}} B^{\overline{\chi}}\overline{f(\chi)}$ and $\vdash_{\mathsf{DHB}} B^{\overline{f(\chi)}}\overline{\chi}$. Then, by Theorem 5.4, we obtain $\vdash_{\mathsf{DHB}} B^{f(\psi)}\chi \leftrightarrow B^{f(\psi)}f(\chi)$. Therefore, by CPL, we conclude that $\vdash_{\mathsf{DHB}} B^{\psi}\chi \leftrightarrow B^{f(\psi)}f(\chi)$, with $B^{f(\psi)}f(\chi) = f(B^{\psi}\chi)$ by Definition 18.

Case $\varphi := [\Uparrow\psi]\chi$: we prove only the cases $\chi := \top$ and $\chi := [\Uparrow\theta]\alpha$. All the other cases follow similarly by using the corresponding reduction axiom, Lemma 44, and Definition 18.

Subcase $\chi := \top$

| | |
|---|---|
| 1. $\vdash_{\mathsf{DHB}} [\Uparrow\psi]\top \leftrightarrow (\top \wedge \overline{\psi})$ | $\mathrm{R}_{\top}$ |
| 2. $\vdash_{\mathsf{DHB}} (\top \wedge \overline{\psi}) \leftrightarrow f(\top \wedge \overline{\psi})$ | Lemma 44.2, induction hypothesis |
| 3. $\vdash_{\mathsf{DHB}} [\Uparrow\psi]\top \leftrightarrow f(\top \wedge \overline{\psi})$ | 1, 2, CPL |

And, $f(\top \wedge \overline{\psi}) = f([\Uparrow\psi]\top)$ by Definition 18.

Subcase $\chi := [\Uparrow\theta]\alpha$

By Lemma 44.1 and induction hypothesis, we know that $\vdash_{\mathsf{DHB}} [\Uparrow\theta]\alpha \leftrightarrow f([\Uparrow\theta]\alpha)$

| | |
|---|---|
| 1. $\vdash_{\mathsf{DHB}} [\Uparrow\theta]\alpha \leftrightarrow f([\Uparrow\theta]\alpha)$ | Lemma 44.1, ind. hyp. |
| 2. $\vdash_{\mathsf{DHB}} [\Uparrow\psi]([\Uparrow\theta]\alpha \leftrightarrow f([\Uparrow\theta]\alpha))$ | $\mathrm{Nec}_{\Uparrow}$ |
| 3. $\vdash_{\mathsf{DHB}} [\Uparrow\psi][\Uparrow\theta]\alpha \leftrightarrow [\Uparrow\psi]f([\Uparrow\theta]\alpha)$ | $\mathrm{R}_{\neg}$, $\mathrm{R}_{\wedge}$ |
| 4. $\vdash_{\mathsf{DHB}} [\Uparrow\psi]f([\Uparrow\theta]\alpha) \leftrightarrow f([\Uparrow\psi]f([\Uparrow\theta]\alpha))$ | induction hypothesis |
| 5. $\vdash_{\mathsf{DHB}} [\Uparrow\psi][\Uparrow\theta]\alpha \leftrightarrow f([\Uparrow\psi]f([\Uparrow\theta]\alpha))$ | 3, 4, CPL |

And, $f([\Uparrow\psi]f([\Uparrow\theta]\alpha)) = f([\Uparrow\psi][\Uparrow\theta]\alpha)$ by Definition 18. □

COROLLARY 47. *For all* $\varphi \in \mathcal{L}_{\mathsf{DHB}}$ *there is a* $\psi \in \mathcal{L}_{\mathsf{CHB}}$ *such that* $\vdash_{\mathsf{DHB}} \varphi \leftrightarrow \psi$.

*Proof.* Follows from Lemma 46, since $f(\varphi) \in \mathcal{L}_{\mathsf{CHB}}$. □

The following lemma is straightforward.

LEMMA 48. *For any tsp-model* $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ *for* $\mathcal{L}_{\mathsf{CHB}}$, *there is a tsp-model* $\mathcal{M}' = \langle W, \geq, T, \oplus, \mathfrak{b}, t', v\rangle$ *for* $\mathcal{L}_{\mathsf{DHB}}$ *such that for all* $w \in W$ *and* $\varphi \in \mathcal{L}_{\mathsf{CHB}}$, *we have* $\mathcal{M}, w \vDash \varphi$ *iff* $\mathcal{M}', w \vDash \varphi$.

*Proof.* Recall that a tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ for $\mathcal{L}_{\mathsf{CHB}}$ ($\mathcal{L}_{\mathsf{DHB}}$) is a structure as described in Definition 3, where $t$ is defined for the whole language $\mathcal{L}_{\mathsf{CHB}}$ ($\mathcal{L}_{\mathsf{DHB}}$). Now, given $t$, define $t'$ as an extension of $t$ such that for $\varphi \in \mathcal{L}_{\mathsf{DHB}}$,

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$t'(\varphi) = t(x_1) \oplus \cdots \oplus t(x_k)$ (where $Var(\varphi) = \{x_1, \ldots, x_k\}$). It is easy to see that $t'$ satisfies Definition 1.4 for elements of $\mathcal{L}_{\mathsf{DHB}}$, so $\mathcal{M}' = \langle W, \geq, T, \oplus, \mathfrak{b}, t', v\rangle$ is a tsp-model for $\mathcal{L}_{\mathsf{DHB}}$. Notice that the only difference between the two tsp-models $\mathcal{M}$ and $\mathcal{M}'$ is that topics of sentences of the form $[\Uparrow\psi]\chi$ are defined in $\mathcal{M}'$, but not in $\mathcal{M}$. Therefore, we obtain the result by an easy induction on the structure of $\varphi \in \mathcal{L}_{\mathsf{CHB}}$. □

**Proof of Completeness:** Let $\varphi \in \mathcal{L}_{\mathsf{DHB}}$ such that $\nvdash_{\mathsf{DHB}} \varphi$. Then, by Corollary 47, there is a $\psi \in \mathcal{L}_{\mathsf{CHB}}$ such that $\vdash_{\mathsf{DHB}} \varphi \leftrightarrow \psi$. This implies, since $\mathsf{CHB} \subseteq \mathsf{DHB}$ and $\nvdash_{\mathsf{DHB}} \varphi$, that $\nvdash_{\mathsf{CHB}} \psi$. Then, by Theorem 6, there is a tsp-model $\mathcal{M} = \langle W, \geq, T, \oplus, \mathfrak{b}, t, v\rangle$ for $\mathcal{L}_{\mathsf{CHB}}$ and $w \in W$ such that $\mathcal{M}, w \nVdash \psi$. By Lemma 48, we then have that $\mathcal{M}', w \nVdash \psi$, where $\mathcal{M}'$ is a tsp-model for $\mathcal{L}_{\mathsf{DHB}}$ as described in the proof of Lemma 48. Then, by the soundness of DHB and $\vdash_{\mathsf{DHB}} \varphi \leftrightarrow \psi$, we conclude that $\mathcal{M}', w \nVdash \varphi$.

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