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Bessel Sequences and Its F-Scalability

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Abstract. Frame theory, which contains wavelet analysis and Gabor analysis, has become a powerful tool for many applications of mathematics, engineering and quantum mechanics. The study of extension principles of Bessel sequences to frames is important in frame theory. This paper studies transformations on Bessel sequences to generate frames and Riesz bases in terms of operators and scalability. Some characterizations of operators that mapping Bessel sequences to frames and Riesz bases are given. We introduce the definitions of F-scalable and P-scalable Bessel sequences. F-scalability and P-scalability of Bessel sequences are discussed in this paper, then characterizations of scalings of F-scalable or P-scalable Bessel sequences are established. Finally, a perturbation result on F-scalable Bessel sequences is derived.

AMS subject classifications: 42C15, 46B15

Key words: Frames, Bessel sequences, proper Bessel sequences, F-scalability.

1 Introduction

The concept of frames was first introduced by Duffin and Schaeffer [13]. Then for decades of development both in theory and applications, frame theory are now playing important roles in many applications in mathematics, science and engineering. Frame theory contains wavelet analysis, Gabor analysis and abstract frame theory. Wavelet theory and Gabor analysis are widely used in mathematics, engineering, signal processing and quantum mechanics, see [11,12,16–18]. Finite frame theory is also useful in finite-dimensional quantum mechanics [10,22]. For introductions of fundamental knowledge of frame theory in Hilbert spaces we refer the reader to [4,6], and [8].

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In quantum mechanics, Hilbert spaces (complete inner-product spaces) play a central role in view of the interpretation associated with wave functions [19–21]. Orthonormal basis is a key tool in representations of the state spaces [19]. In the view point of functional analysis, an orthonormal basis gives unique representation of any given object, and therefore would not be able to offer freedoms in representations [11]. Frames serve as a generalized orthonormal basis and offer freedoms on representing the given objects. However, in infinite dimensional Hilbert spaces, it is not easy to verify that a given sequence is a frame or not. To prove that a given sequence is a Bessel sequence or not would be easier [8]. Therefore, the investigation of extension principles of Bessel sequences to frames would be an important topic. In [7], Casazza and Leonhard showed that any Bessel sequence in a finite dimensional space can be extended to a tight frame. Li and Sun extended this result to the infinite dimensional case in [15]. In [9], Christensen et al. showed that in any separable Hilbert space, any pair of Bessel sequences can be extended to a pair of dual frames. Recently, finite extensions of Bessel sequences to frames in infinite dimensional Hilbert spaces was studied by Bakić and Berić [1].

For all the extension principles mentioned above, it is necessary to add new elements, finitely or infinitely, into a given Bessel sequences to generate frames. These extension principles produce better frames, but also lead to additional computational complexity. When one hopes to save computational costs, adding new elements to generate better frames would not be a preferable choice. In this paper we will discuss extension principles of Bessel sequences and frames in terms of operators and scalability in infinite dimensional spaces. Applying operator transformations on Bessel sequences to produce frames can preserve the number of elements in the frames. We find an equivalent condition for operators mapping Bessel sequences to frames in Proposition 3.5. And we prove that, under certain circumstances, operators that mapping Bessel sequences to frames or Riesz bases are unbounded.

Generating tight frames from a given sequence of vectors is also a basic problem in frame theory. The simplest method to generate better frames is by re-scaling the elements in the given complete sequences. Kutyniok et al. introduced scalable frames in [14], a special type of frames which can be modified to generate a Parseval frame by rescaling its frame vectors. Later on, finite scalable frames was investigated in [3] by Cahill and Chen.

We will investigate Bessel sequences which can be scaled to produce frames or Parseval frames. A complete Bessel sequence that can be scaled to generate a frame is called F-scalable in this paper, while it is called P-scalable if it can be scaled to be a Parseval frame. We give some characterizations of F-scalable and P-scalable Bessel sequences, and derive a perturbation result on F-scalable Bessel sequences.

Our paper is organized as follows. In Section 2, we review the needed facts from frame theory and summarize the basic notations in this paper. Section 3 is devoted to the discussions of operators mapping Bessel sequences to frames and Riesz bases. In Section 4, we introduce the definitions of F-scalable and P-scalable Bessel sequences. We investigate the F-scalability and P-scalability of Bessel sequences, then we give charac-

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terizations of the scalings of F-scalable or P-scalable Bessel sequences. We also give a perturbation result about F-scalable Bessel sequences. Finally, in Section 5, we make a conclusion of this paper and summarize the future work.

2 Preliminaries

In this section, we review the needed facts from frame theory and summarize the basic notations in this paper.

Throughout this paper, H denotes a separable Hilbert space. We denote the domain, the kernel, the range and the graph of a linear operator T by $dom(T)$, $ker(T)$, $ran(T)$ and $\Gamma(T)$, respectively.

Let X,Y be Banach spaces, we say that the linear operator $A:dom(A)\subseteq X\to Y$ admits a closure if there is a linear operator $B:dom(B)\subseteq X\to Y$ such that $dom(A)\subseteq dom(B)$, $B|_{dom(A)}=A$, and $\Gamma(B)=\overline{\Gamma(A)}$. We call operator B the closure of A, and we can write B as $B=\overline{A}$.

We will give characterizations of scalings of F-scalable Bessel sequences. For that, we have to define the diagonal operator D_c in $l^2(\mathbb{I})$ corresponding to a sequence $c=\{c_i\}\subseteq\mathbb{R}$, which is defined by

$$D_c(d_i):=(c_id_i)_{i\in\mathbb{I}},\quad (d_i)\in l^2(\mathbb{I}),$$

when $dom(D_c):=\{(d_i)\in l^2(\mathbb{I}):(c_id_i)_{i\in\mathbb{I}}\in l^2(\mathbb{I})\}$.

It is easy to see that $dom(D_c)$ is dense in $l^2(\mathbb{I})$. D_c is a self adjoint operator in $l^2(\mathbb{I})$ if and only if $c_i\in\mathbb{R}$, for all $i\in\mathbb{I}$. Note that in this paper, D_c is generally unbounded.

A family $\{f_i\}_{i\in\mathbb{I}}\subseteq H$ is a frame for H if there exist constants $A,B>0$, such that

$$A\|f\|^2\leq\sum_{i\in\mathbb{I}}|\langle f,f_i\rangle|^2\leq B\|f\|^2 \tag{2.1}$$

holds for every $f\in H$, where $\mathbb{I}$ is a countable index set. When $A=B$, we call $\{f_i\}$ an A-tight frame; If $A=B=1$, so that

$$\sum_{i\in\mathbb{I}}|\langle f,f_i\rangle|^2=\|f\|^2,\quad \forall f\in H,$$

we call $\{f_i\}$ a Parseval frame.

The sequence $\{f_i\}_{i\in\mathbb{I}}$ is called a Bessel sequence if at least the upper inequality in (2.1) holds for all $f\in H$; $\{f_i\}$ is called a proper Bessel sequence if only the upper inequality in (2.1) holds for all $f\in H$. A sequence $\{g_i\}$ is called complete if its span is dense in H, i.e., $\overline{span\{g_i\}}=H$; $\{g_i\}$ is called minimal if $g_j\notin\overline{span\{g_i\}}_{i\neq j}$ for all $j\in\mathbb{I}$.

Note that a proper Bessel sequence is also a Bessel sequence. It is easy to show that if $\{f_i\}$ is a proper Bessel sequence, then there exists a sequence of unit norm vectors $\{g_n\}$ such that

$$\sum_i|\langle g_n,f_i\rangle|^2\to 0,\quad \text{when } n\to\infty. \tag{2.2}$$

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Generally, frames are considered as generalized bases, while Parseval frames are generalized orthonormal bases, since every element $f \in H$ can be represented by a Parseval frame $\{f_i\}_{i\in\mathbb{I}}$ of H as:

$$f = \sum_{i\in\mathbb{I}} \langle f, f_i\rangle f_i,$$

which is also a property of orthonormal basis.

For any Bessel sequence $F = \{f_i\}_{i\in\mathbb{I}}$, we can define the associated synthesis operator $T_F : l^2(\mathbb{I}) \to H$ by

$$T_F(\{c_i\}) = \sum_i c_i f_i,$$

and its adjoint operator, the analysis operator $U_F : H \to l^2(\mathbb{K})$ by

$$U_F f = \{\langle f, f_i\rangle\}.$$

The frame operator of a frame $\{f_i\}$ is defined by $S_F = T_F U_F$, i.e.,

$$S_F f = \sum_i \langle f, f_i\rangle f_i, \quad f \in H,$$

which is a bounded, positive, invertible and self-adjoint operator. In particular, $\{f_i\}$ is a Parseval frame is equivalent to $S_F = I$, where I denotes the identity operator on H.

Definition 2.1. Let H be a Hilbert space. A countable complete set $\{f_i\}_{i\in\mathbb{I}} \subset H$ is called a Riesz basis for H if there exist constants $A, B > 0$ such that

$$A\sum_i |c_i|^2 \le \left\|\sum_i c_i f_i\right\|^2 \le B\sum_i |c_i|^2, \quad \forall \{c_i\} \in l^2(\mathbb{I}). \tag{2.3}$$

The numbers A, B in (2.3) are called upper and lower Riesz basis bounds.

Note that the bounds we use in this paper are optimal bounds. Here optimal bounds are the largest possible value for A and the smallest possible value for B. It is well known that a Riesz basis is also a frame, with the frame bounds equal to the Riesz bounds. And obviously, for any element in a Riesz basis $\{g_i\}$, we have $A \le \|g_i\| \le B$ holds for all $i \in \mathbb{I}$, where A, B are the Riesz bounds.

Lemma 2.1 and Lemma 2.2 are cited from [8] Theorem 3.2.3 and Theorem 6.1.1.

Lemma 2.1. *Let $\{f_i\}$ be a sequence in H. Then $\{f_i\}$ is a Bessel sequence with Bessel bound B if and only if*

$$T\{c_i\} \to \sum_i c_i f_i$$

is a well defined and bounded operator from $l^2(\mathbb{I})$ into H and $\|T\| \le \sqrt{B}$.

Lemma 2.2. *Let $\{f_i\}$ be a frame for H. Then $\{f_i\}$ is a Riesz basis if and only if it is minimal.*

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By [8] Lemma 5.2.1, we have

Lemma 2.3. *Let $F=\{f_i\}$ be a Bessel sequence in H, and $T_F: l^2(\mathbb{I})\to H$ its synthesis operator, which is bounded. Then we have,*
(a) F is a frame for H if and only if T_F is surjective.
(b) F is a Riesz basis for H if and only if T_F is invertible.

3 Operators mapping Bessel sequences to frames

In this section, we discuss operators that mapping Bessel sequences to frames and Riesz bases. We will prove that an linear operator T mapping frames to frames if and only if it is bounded and surjective. But operators mapping Bessel sequences to frames are probably unbounded. We give examples for bounded and unbounded operators that mapping proper Bessel sequences to frames. We also find an equivalent condition for operators that mapping Bessel sequence to frames in Proposition 3.5.

Note that in finite dimension spaces, we have:

Proposition 3.1. Suppose that H is an n-dimensional Hilbert space. Let $F=\{f_i\}_{i\in\mathbb{I}}$ be a Bessel sequence, A be an operator, then $G=\{Af_i\}_{i\in\mathbb{I}}$ is a frame if and only if F is a frame and A is invertible.

Proof. Consider F, G as matrices in $\mathbb{C}^{n\times|\mathbb{I}|}$, whose columns are the frame elements, where $|\mathbb{I}|$ denotes the cardinal of the index set $\mathbb{I}$. And A as an operator on n-dimensional spaces can be presented as a matrix in $\mathbb{C}^{n\times n}$. If $G=\{Af_i\}$ is a frame, we have $G=AF$ is row full rank. And since

$$rank(G)=\min\{rank(A),rank(F)\},$$

therefore, we have $rank(G)=rank(F)=rank(A)=n$. That is F is a frame and A is invertible. □

In infinite dimensional spaces, $G=\{Af_i\}$ is a frame does not imply that $F=\{f_i\}$ is a frame or A is invertible.

Example 3.1. Let $\{e_1,e_2,\cdots,e_n,\cdots\}$ be an orthonormal basis of a Hilbert space H, and

$$F=\left\{e_1,\frac{1}{3}e_3,\frac{1}{5}e_5,\cdots,\frac{1}{2n-1}e_{2n-1},\cdots\right\},\quad G=\{e_1,e_2,\cdots,e_n,\cdots\}.$$

Define an operator $L:H\to H$ by

$$L(e_{2n-1})=(2n-1)e_n,\quad L(e_{2n})=0,\quad n\in\mathbb{N}.$$

Then we have $LF=G$. Obviously, L is not bounded and not invertible. Also, it is easy to see that F is only a non-complete proper Bessel sequence and G is a frame since it is an orthonormal basis for H.

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Proposition 3.2. Suppose that $F=\{f_i\}$ is a Bessel sequence in H, and L is a linear, closed operator on H such that $\{Lf_i\}$ is a frame for H, then L is surjective.

Proof. Since F is a Bessel sequence, by Lemma 2.1, its synthesis operator T_F is well-defined and bounded. Since L is closed and $\{Lf_i\}$ is a frame, then the synthesis operator for LF is

$$T_{LF}=LT_F.$$

And therefore the frame operator of LF,

$$S_{LF}=LT_FT_F^*L^*$$

is invertible, which implies that L is surjective. □

Proposition 3.3. Suppose that $\{f_i\}$ is a proper Bessel sequence in H. Let L be a linear, operator on H. If $\{Lf_i\}$ and $\{L^*f_i\}$ are frames for H, then L is not bounded.

Proof. Since $\{f_i\}$ is a proper Bessel sequence, then there exists a sequence of unit norm vectors $\{g_n\}$ such that

$$\sum_i|\langle g_n,f_i\rangle|^2\to 0,\quad \text{when } n\to\infty. \tag{3.1}$$

If $\{Lf_i\}$ and $\{L^*f_i\}$ are frames for H, let the frame bounds of $\{L^*f_i\}$ be A, B. If L is bounded, let $\|L\|\le M$, then by Proposition 3.2, L and L^* are surjective, which implies that L is bijective. Therefore, there exists h_n such that $g_n=Lh_n$, for any $n\in\mathbb{N}$. And since $1=\|g_n\|=\|Lh_n\|\le\|L\|\|h_n\|$, then we have

$$\|h_n\|\ge\frac{1}{\|L\|}\ge\frac{1}{\|M\|}.$$

Therefore we have

$$\sum_i|\langle g_n,f_i\rangle|^2=\sum_i|\langle Lh_n,f_i\rangle|^2=\sum_i|\langle h_n,L^*f_i\rangle|^2\ge A\|h_n\|^2\ge AM^{-2},$$

which is contradict with (3.1). Therefore, L is not bounded. □

For operators mapping frames to frames, we have

Proposition 3.4. Suppose that $\{f_i\}$ is a frame for H with bounds A, B. Let $L:H\to H$ be a linear operator, then $\{Lf_i\}$ is a frame for H if and only if L is surjective and bounded.

Proof. If L is surjective and bounded, by [8] Corollary 5.3.2, $\{Lf_i\}$ is a frame for H with frame bounds $A\|L^+\|^{-2}$ and $B\|L\|^2$.

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Suppose that $\{Lf_i\}$ is a frame for H with frame bounds A', B'. If L is not bounded, then so is its adjoint operator L^*. Thus for any $N\in\mathbb{N}$, there exists g_N such that $\|g_N\|=1$ and $\|L^*g_N\|\geq N$. Since $\{f_i\}$ is a frame for H with bounds A, B, we have

$$AN\leq A\|L^*g_N\|\leq\sum_i|\langle L^*g_N,f_i\rangle|^2=\sum_i|\langle g_N,Lf_i\rangle|^2\leq B',$$

which is a contradiction.

Therefore, L is bounded. Then by Proposition 3.2, L is surjective. □

Proposition 3.4 is not true for operators from proper Bessel sequences to frames, even for operators that mapping complete proper Bessel sequences to frames.

Example 3.2. Let $F=\{e_1,e_2/2,\cdots,e_n/n,\cdots\}$, $G=\{e_1,e_2,\cdots,e_n,\cdots\}$. Define an operator $L:H\to H$ by

$$L(e_n)=ne_n \quad \text{for all } n\in\mathbb{N}.$$

Then we have $LF=G$. Then it is easy to see that F is a complete proper Bessel sequence and G is a frame, but L is not bounded.

Meanwhile, we can define another operator $Q:H\to H$ by:

$$Qe_n=\frac{1}{n}e_n \quad \text{for all } n\in\mathbb{N}.$$

Then Q is bounded but not surjective, since for $\sum_{k=1}^{\infty}\frac{1}{k}e_k\in H$, we can not find a vector $f\in H$, such that $Q(f)=\sum_{k=1}^{\infty}\frac{1}{k}e_k$. In fact, we have $Q(\sum_{k=1}^{n}e_k)=\sum_{k=1}^{n}\frac{1}{k}e_k$ for any $n\in\mathbb{N}$, but $\sum_{k=1}^{\infty}e_k\notin H$ since $\|\sum_{k=1}^{\infty}e_k\|=\infty$. Therefore, $QG=F$ is not a frame.

But for Bessel sequences, we have the following result.

Proposition 3.5. Suppose that $F=\{f_i\}$ is a Bessel sequence in H. Let $L:V:=ran(T_F)\to H$ be a linear, closed operator, then $\{Lf_i\}$ is a frame for H if and only if LT_F is surjective and bounded, where T_F is the synthesis operator of F.

Proof. Since L is a closed operator, and the synthesis operator of $G=\{Lf_i\}$ is LT_F, by Lemma 2.1 and Lemma 2.3, the conclusion above is true. □

Note that an operator L maps a Bessel sequence to a frame is possibly bounded. For example, let $F=\{e_n+e_{n+1}\}_{n=1}^{\infty}$, where $\{e_n\}$ is an orthonormal basis of H, then F is a Bessel sequence. Define an operator by:

$$L(e_{2n-1})=e_n, L(e_{2n})=0 \quad \text{for all } n\in\mathbb{N}.$$

Then $LF=\{e_1\}\bigcup\{e_n,e_n\}_{n=2}^{\infty}$ is a frame with frame bounds 1 and 2. Obviously, L is bounded.

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Proposition 3.6. Suppose that $\{f_i\}$ is a Bessel sequence which is not norm-bounded below, i.e., there does not exist a positive constant $\delta>0$ such that $\|f_i\|\geq\delta$ for all $i\in\mathbb{I}$. Let $L:H\to H$ be a linear operator, such that $\{Lf_i\}$ is a Riesz basis, then L is not bounded.

Proof. Suppose that $\{Lf_i\}$ is a Riesz basis with bounds A, B. Then we have $A\leq\|Lf_i\|\leq B$. If L is bounded, we have $\|Lf_i\|\leq\|L\|\|f_i\|$ tends to 0, since $\{f_i\}$ is not lower bounded. It contradicts with $0<A\leq\|Lf_i\|$. Therefore, L is not bounded. □

4 F-scalable Bessel sequences

The simplest method to generate a frame from a given Bessel sequence is by scaling its elements. In this section, in infinite dimensional spaces, we will investigate complete Bessel sequences that can be scaled to generate frames, which will be called F-scalable Bessel sequences. A Bessel sequence will be called P-scalable if it can be scaled to produce a Parseval frame. We give some characterizations of F-scalable and P-scalable Bessel sequences in this section. Furthermore, we will establish a perturbation result about F-scalable Bessel sequences.

Definition 4.1. A complete Bessel sequence $F=\{f_i\}$ in H is called F-scalable if there exist scalars $c_i\geq 0$, $i\in\mathbb{I}$, such that $\{c_if_i\}$ is a frame for H. If $\{c_if_i\}$ is a Parseval frame for H, then F is called P-scalable. In addition, if $c_i>0$ for all $i\in\mathbb{I}$, then F is called positively F-scalable or P-scalable. If there exists $\delta>0$, such that $c_i\geq\delta$, for all $i\in\mathbb{I}$, then F is called strictly F-scalable or P-scalable.

Note that, F is P-scalable implies that it is F-scalable. But F-scalability do not implies P-scalability, since not every frame is scalable [14]. In [2], the authors proved that if $\{f_i\}$ is a frame, and there exist $M_1,M_2>0$, such that $M_1\leq c_i\leq M_2$ holds for all $i\in\mathbb{I}$, then $\{c_if_i\}$ is still a frame. But such a sequence of scalars can not be used as the scaling to generate a frame from a proper Bessel sequence.

Proposition 4.1. Suppose that $F=\{f_i\}$ is a complete proper Bessel sequence in H. If there exists a sequence $\{c_i\}$ such that $\{c_if_i\}$ is a frame for H, then $\{c_i\}$ is not bounded.

Proof. If $F=\{f_i\}$ is a complete proper Bessel sequence in H, then there exists a sequence of unit norm vectors $\{g_n\}$, i.e., $\|g_n\|=1$ for all $n\in\mathbb{N}$, such that $\sum_i|\langle g_n,f_i\rangle|^2\to 0$, when $n\to\infty$.

Suppose that $F=\{c_if_i\}$ is a frame for H with bounds A, B. If $\{c_i\}$ is bounded, i.e., there exists a positive constant M such that $|c_i|\leq M$ for all $i\in\mathbb{I}$. Thus, we have,

$$A\leq\sum_i|c_i|^2|\langle g_n,f_i\rangle|^2\leq M^2\sum_i|\langle g_n,f_i\rangle|^2\to 0,$$

which is a contradiction. Therefore, $\{c_i\}$ is not bounded. □

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Proposition 4.2. Let L be a bounded invertible operator on H. Then the Bessel sequence $F=\{f_i\}$ is F-scalable if and only if $LF=\{Lf_i\}$ is F-scalable. In this case, $\{f_i\}$ and $\{Lf_i\}$ share the same scalings.

Proof. Suppose that F is F-scalable with scaling $c=\{c_i\}$, by which we can define the diagonal operator D_c. Then by Lemma 2.3, the synthesis operator of the scaled frame $G=\{c_if_i\}$, T_G is surjective. Then for any bounded invertible operator L on H, the synthesis operator of LG, $T_{LG}=LT_G$ is also surjective, which implies that $LG=\{c_iLf_i\}$ is a frame. Therefore, LF is also F-scalable and shares the same scaling c with F. The converse can be proved similarly. □

Note that not every Bessel sequence can be scaled to generate a frame.

Example 4.1. Let $F=\{e_n+e_{n+1}\}\subseteq H$, where $\{e_n\}$ is an orthonormal basis of H. It is well known that F is minimal, complete, but is only a proper Bessel sequence in H. We can prove that F is not F-scalable.

Proof. Suppose that there exists scalars $\{c_n\}$, such that $G=\{c_n(e_n+e_{n+1})\}$ is a frame for H with frame bounds A, B. Then by Lemma 2.2, G is also a Riesz basis for H with Riesz bounds A, B. Thus we have

$$A\leq\|c_n(e_n+e_{n+1})\|=\sqrt{2}|c_n|\leq B\quad\text{for all }\ n\in\mathbb{N}.$$

The inequality above leads to a contradiction, since $\{|c_n|\}$ is not bounded according to Proposition 4.1. Therefore, F is not F-scalable. □

In fact, we have an immediate corollary of Proposition 4.1.

Corollary 4.1. *For any complete proper Bessel sequence, if it is norm-bounded below, then it is not F-scalable, and therefore, in particular, not P-scalable.*

Proof. Suppose that $F=\{f_i\}$ is a F-scalable proper Bessel sequence with scaling $c=\{c_i\}$. Let the frame bounds of $G=\{c_if_i\}$ be A, B. Then we have $\|c_if_i\|=|c_i|\|f_i\|\leq B$.

Suppose that $\|f_i\|$ is bounded below with bound δ, i.e., $\|f_i\|\geq\delta$. Then we have $\delta|c_i|\leq B$. According to Proposition 4.1, $|c_i|$ is unbounded, which leads to a contradiction. Therefore, F is not F-scalable. □

However, we can prove that every frame can be scaled to be a complete proper Bessel sequence.

Proposition 4.3. Every frame can be scaled to generate a complete proper Bessel sequence.

Proof. Suppose that $\{f_i\}_{i\in\mathbb{N}}$ is a frame with frame bounds A, B. Let c_i be a positive decreasing scalar sequence such that $c_i\to 0$ when $i\to\infty$. Then $\{c_if_i\}$ is a complete proper

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Bessel sequence. In fact, let $V_N=\overline{span\{f_i\}_{i=1}^N}$, take $g_N\in V_N^\perp$, the orthogonal complement of V_N, such that $\|g_N\|=1$ for all $N\in\mathbb{N}$. Thus we have,

$$\sum_i|\langle g_N,c_if_i\rangle|^2=\sum_{i=N}|\langle g_N,c_if_i\rangle|^2\le|c_N|^2|\sum_{i=N}\langle g_N,f_i\rangle|^2\le|c_N|^2B.$$

Since $c_N\to 0$, $|c_N|^2B\to 0$ when $N\to\infty$. Therefore, $\{c_if_i\}$ is a complete proper Bessel sequence. □

Proposition 4.4. Let $F=\{f_i\}$ be a Bessel sequence in H with analysis operator U_F, such that F is F-scalable with scaling $\{c_i\}$, then we have, $ran(U_F)\subset dom(D_c)$, and there exists $\delta>0$ such that $\|D_cd\|\ge\delta\|d\|$ for all $d\in ran(U_F)$. In particular, in this case, the frame operator of the frame $G=\{c_if_i\}$ is given by:

$$S_G=(D_cU_F)^*(D_cU_F)=\overline{U_F^*D_c}D_cU_F,$$

where $\overline{U_F^*D_c}$ denotes the closure of the operator $U_F^*D_c$.

Proof. Since F is F-scalable, then let the scaled frame be $G=\{c_if_i\}$, and denote its analysis operator by U_G. Then for any $f\in H$, the j-th component of U_Gf is given by

$$(U_Gf)_j=\langle f,c_jf_j\rangle=c_j\langle f,f_j\rangle=(D_cU_Ff)_j.$$

Thus, $U_G=D_cU_F$. Since $dom(U_G)=H$, then we have $ran(U_F)\subseteq dom(D_c)$.

And since G is a frame, there exist $A,B>0$, such that $A\|f\|^2\le\|D_cU_Ff\|^2\le B\|f\|^2$ for all $f\in H$. Therefore, in particular, for any $d=U_Ff\in ran(U_F)$, we have:

$$\|D_cd\|^2=\|D_cU_Ff\|^2\ge A\|f\|\ge A\|U_F\|^{-2}\|d\|^2.$$

Then let $\delta=A\|U_F\|^{-2}$, we have $\|D_cd\|\ge\delta\|d\|$ for all $d\in ran(U_F)$.

Finally, we prove that $(D_cU_F)^*=\overline{U_F^*D_c}$. In fact, since D_cU_F is bounded, so is its adjoint $(D_cU_F)^*$. By a standard calculation, we can see that $U_F^*D_cd=(D_cU_F)^*d$ for all d in the dense subspace $dom(D_c)$. Hence $U_F^*D_c$ is densely defined, and therefore admits a closure $(D_cU_F)^*$. That is, $(D_cU_F)^*=\overline{U_F^*D_c}$. □

Note that the converse of Proposition 4.4 is generally not holds. In fact, for any scalar sequence $\{c_i\}$ which is upper and lower bounded, i.e., there exist constants δ_1, δ_2, such that $\delta_1<|c_i|<\delta_2$, by which, we can define a diagonal operator D_c. Then $\|D_cd\|\ge\delta_1\|d\|$ holds for all $d\in ran(U_F)$. But $\{c_if_i\}$ is not a frame, since the scaling of a proper Bessel sequence is unbounded.

Proposition 4.5. Let $F=\{f_i\}$ be a Bessel sequence in H. Then the following conditions are equivalent.
(a) F is (positively, strictly) P-scalable.
(b) There exists a non-negative (positive, strictly, respectively) diagonal operator D_c on $l^2(\mathbb{I})$ corresponding to the scalar sequence $c=(c_i)$ such that $\overline{U_F^*D_c}D_cU_F=I$.

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Proof. Suppose that F is (positively, strictly) P-scalable with the scaling $c=(c_i)$. Then $G=\{c_if_i\}$ is a Parseval frame. By Proposition 4.4, the operator of G is given by $S_G=\overline{U_F^*D_c}D_cU_F$, where D_c is the diagonal operator corresponding to c. Since the frame operator of a Parseval frame coincides with the identity operator, therefore, we have

$$\overline{U_F^*D_c}D_cU_F=I.$$

Conversely, suppose that there exists a non-negative (positive, strictly, respectively) diagonal operator D_c on $l^2(\mathbb{I})$ corresponding to the scalar sequence $c=(c_i)$ such that

$$\overline{U_F^*D_c}D_cU_F=I.$$

Then D_cU_F is well defined on H. Similar to the proof of Proposition 4.4,

$$\overline{U_F^*D_c}D_cU_F=(D_cU_F)^*D_cU_F=I,$$

which implies that, for any $f\in H$, we have $\|D_cU_Ff\|^2=\|f\|^2$, i.e.,

$$\sum_i|\langle f,c_if_i\rangle|^2=\|f\|^2.$$

Therefore, $G=\{c_if_i\}$ is a Parseval frame. That is, F is (positively, strictly) P-scalable with the scaling c.

Positive and strict scalability of F can be proved similarly. □

At the end of this section, we give a perturbation result about F-scalable Bessel sequences. For that, we need the following result which was given by Casazza and Christensen in [5]:

Theorem 4.1. *Let $\{f_i\}$ be a frame for H with bounds A, B. Let $\{g_i\}\subseteq H$ and assume that there exist constants $\lambda_1,\lambda_2,\mu\geq 0$ such that $\max(\lambda_1+\mu/\sqrt{A})<1$ and*

$$\Big\|\sum_{i=1}^n d_i(f_i-g_i)\Big\|\leq\lambda_1\Big\|\sum_{i=1}^n d_if_i\Big\|+\lambda_2\Big\|\sum_{i=1}^n d_ig_i\Big\|+\mu\Big(\sum_{i=1}^n|d_i|^2\Big)^{\frac{1}{2}} \tag{4.1}$$

holds for all finite scalar sequences $\{d_1,d_2,\cdots,d_n\}$ $(n\in\mathbb{N})$. Then $\{g_i\}$ is a frame for H with bounds

$$A\Big(1-\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{A}}}{1+\lambda_2}\Big)^2,\quad B\Big(1+\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}\Big)^2.$$

Theorem 4.2. *Suppose that $\{f_i\}$ is a positive F-scalable Bessel sequence in H with scaling $c=\{c_i\}$ and the frame bounds for $\{c_if_i\}$ are A, B. Let $\{g_i\}\subseteq H$ and assume that there exist constants $\lambda_1,\lambda_2,\mu\geq 0$ such that $\max(\lambda_1+\mu/\sqrt{A})<1$ and*

$$\Big\|\sum_{i=1}^n d_i(f_i-g_i)\Big\|\leq\lambda_1\Big\|\sum_{i=1}^n d_if_i\Big\|+\lambda_2\Big\|\sum_{i=1}^n d_ig_i\Big\|+\mu\Big(\sum_{i=1}^n\Big|\frac{d_i}{c_i}\Big|^2\Big)^{\frac{1}{2}} \tag{4.2}$$

holds for all finite scalar sequences $\{d_1,d_2,\cdots,d_n\}$ $(n\in\mathbb{N})$. Then $\{g_i\}$ is also F-scalable. In this case, $\{g_i\}$ shares the same scalings with $\{f_i\}$.

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Proof. By a transform associated with the scaling c in (4.2), we have:

$$\Big\|\sum_{i=1}^{n}\frac{d_i}{c_i}c_i(f_i-g_i)\Big\|\leq\lambda_1\Big\|\sum_{i=1}^{n}\frac{d_i}{c_i}c_if_i\Big\|+\lambda_2\Big\|\sum_{i=1}^{n}\frac{d_i}{c_i}c_ig_i\Big\|+\mu\Big(\sum_{i=1}^{n}\Big|\frac{d_i}{c_i}\Big|^2\Big)^{\frac{1}{2}}. \tag{4.3}$$

Let $F_i=c_if_i$, $G_i=c_ig_i$ and $b_i=d_i/c_i$, then we have

$$\Big\|\sum_{i=1}^{n}b_i(F_i-G_i)\Big\|\leq\lambda_1\Big\|\sum_{i=1}^{n}b_iF_i\Big\|+\lambda_2\Big\|\sum_{i=1}^{n}b_iG_i\Big\|+\mu\Big(\sum_{i=1}^{n}|b_i|^2\Big)^{\frac{1}{2}} \tag{4.4}$$

holds for all finite scalar sequences $\{b_1,b_2,\cdots,b_n\}$ $(n\in\mathbb{N})$ when $\max(\lambda_1+\mu/\sqrt{A})<1$. Since $\{F_i\}$ is a frame, by Theorem 4.1, $\{G_i\}=\{c_ig_i\}$ is also a frame. That is, $\{g_i\}$ is F-scalable and share the same scalings with $\{f_i\}$. □

5 Conclusions and future work

In this paper, the extension principles of Bessel sequences to frames in terms of operators and scalability were investigated. We proved that operators, that mapping Bessel sequences to frames or Riesz bases, are unbounded under certain circumstances, and established the equivalent condition for operators that mapping Bessel sequences to frames. By introducing the definitions of F-scalable and P-scalable Bessel sequences, we also gave some characterizations of F-scalable and P-scalable Bessel sequences, and derived a perturbation result on F-scalable Bessel sequences.

As the fact that Bessel sequences are generally not complete, extension principles of non-complete Bessel sequences to scalable frames or scalable (F-scalable or P-scalable) Bessel sequences will be an interesting and important topic associated with this paper, and the study of this topic will be given in the future. Also, the study of scalability of wavelet and Gabor sequences would be useful in the applications of quantum mechanics.

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