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The departure process from a queueing system

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Introduction. Consider a single-server queueing system with a Poisson arrival process at rate λ and positive service requirements independently distributed with common distribution function $B(z)$ and finite expectation

$$\beta = \int_0^\infty z \, \mathrm{d}B(z),$$

where $\beta\lambda < 1$, i.e. an $M/G/1$ system. When the queue discipline is first come first served, or last come first served without pre-emption, the stationary departure process is Poisson if and only if $G = M$ (i.e. $B(z) = 1 - \exp(-z/\beta)$); see (8), (4) and (2). In this paper it is shown that when the queue discipline is last come first served with pre-emption the stationary departure process is Poisson whatever the form of $B(z)$. The method used is adapted from the approach of Takács(10) and Shanbhag and Tambouratzis(9).

Results. Let $\mathscr{A}(\mathscr{D})$ be the set of arrival (departure) epochs after time $t = 0$. In this paper the system will be studied at epochs $e \in \mathscr{E} = \mathscr{A} \cup \mathscr{D}$. At epoch $e \in \mathscr{E}$ let $\rho_e = +1$ or -1 according as $e \in \mathscr{A}$ or $e \in \mathscr{D}$. If $e \in \mathscr{A}(\mathscr{D})$ let ν_e be the number of customers in the queue immediately before (after) epoch e. If a customer arrives at epoch $a \in \mathscr{A}$ he takes up position $\nu_a + 1$ in the queue. The server devotes his attention to the last arrival and so if a customer leaves the queue at epoch $d \in \mathscr{D}$ he leaves from position $\nu_d + 1$. Let χ_{je} be the remaining service requirement at epoch e of the customer occupying position j in the queue, for $j = 1, 2, \ldots, \nu_e$. If a customer's service is interrupted owing to the arrival of another customer his remaining service requirement remains constant until the server can attend to him again. Let η_e be the time elapsed since the last departure epoch before epoch e. Where its omission can cause no confusion the subscript e will be dropped. It is clear that the process $(\rho, \nu, \chi_1, \chi_2, \ldots, \chi_\nu, \eta)$ observed at successive epochs $e \in \mathscr{E}$ is a Markov process. Define

$$P_k = (1 - \lambda\beta)(\lambda\beta)^k \quad (k = 0, 1, 2, \ldots)$$

and

$$B^*(x) = \beta^{-1} \int_0^x (1 - B(z))\, dz \quad (x \geqslant 0).$$

THEOREM. *A stationary distribution for* $(\rho, \nu, \chi_1, \chi_2, \ldots, \chi_\nu, \eta)$ *observed at successive epochs* $e \in \mathscr{E}$ *is*

$$P\{\rho = r, \nu = k, \chi_1 \leqslant x_1, \chi_2 \leqslant x_2, \ldots, \chi_k \leqslant x_k, \eta \leqslant y\} = \tfrac{1}{2} P_k (1 - e^{-\lambda y}) \prod_{j=1}^{k} B^*(x_j). \quad (1)$$

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Proof. If at the first epoch in $\mathscr{E}(\rho, \nu, \chi_1, \chi_2, \ldots, \chi_\nu, \eta)$ has distribution (1) then at every subsequent epoch in $\mathscr{E}$ it will have the same distribution if the following equations hold:

$$\tfrac{1}{2}P_0(1-e^{-\lambda y}) = \tfrac{1}{2}P_0 \operatorname{Prob}\{\tilde{w} \leqslant y\}, \tag{2}$$

$$\tfrac{1}{2}P_k(1-e^{-\lambda y}) \prod_{j=1}^{k} B^*(x_j) = \tfrac{1}{2}P_{k-1} \prod_{j=1}^{k-1} B^*(x_j) \operatorname{Prob}\{\tilde{w} \leqslant \tilde{z} \leqslant \tilde{w}+x_k, \tilde{y}+\tilde{w} \leqslant y\}$$
$$+\tfrac{1}{2}P_k \prod_{j=1}^{k-1} B^*(x_j) \operatorname{Prob}\{\tilde{w} \leqslant \tilde{x} \leqslant \tilde{w}+x_k, \tilde{w} \leqslant y\} \tag{3}$$

for $k = 1, 2, \ldots$, and

$$\tfrac{1}{2}P_k(1-e^{-\lambda y}) \prod_{j=1}^{k} B^*(x_j) = \tfrac{1}{2}P_k \prod_{j=1}^{k} B^*(x_j) \operatorname{Prob}\{\tilde{z} \leqslant \tilde{w}, \tilde{y}+\tilde{z} \leqslant y\}$$
$$+\tfrac{1}{2}P_{k+1} \prod_{j=1}^{k} B^*(x_j) \operatorname{Prob}\{\tilde{x} \leqslant \tilde{w}, \tilde{x} \leqslant y\} \tag{4}$$

for $k = 0, 1, 2, \ldots$, where $\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}$ are independent random variables with distribution functions $1-\exp(-\lambda w)$, $B^*(x)$, $1-\exp(-\lambda y)$, $B(z)$ respectively. To obtain equation (2) note that the event $\{\rho = +1, \nu = 0, \eta \leqslant y\}$ will occur at an epoch if and only if at the preceding epoch the event $\{\rho = -1, \nu = 0\}$ occurred and the time between these two epochs is $\leqslant y$. Equation (3) is obtained from the consideration that the event $\{\rho = +1, \nu = k\}$ can occur at an epoch y only if at the preceding epoch either of the events $\{\rho = +1, \nu = k-1\}$ or $\{\rho = -1, \nu = k\}$ occurred. Similarly equation (4) reflects the fact that the event $\{\rho = -1, \nu = k\}$ must be preceded by one of the events $\{\rho = +1, \nu = k\}$ or $\{\rho = -1, \nu = k+1\}$. The validity of equations (2), (3) and (4) is easily checked.

The process $(\rho, \nu, \chi_1, \chi_2, \ldots, \chi_\nu, \eta)$ observed at successive epochs $e \in \mathscr{E}$ is a periodic process and hence although expression (1) is a stationary distribution it is not in general a limiting distribution.

COROLLARY 1. *The unique stationary distribution for* $(\nu, \chi_1, \chi_2, \ldots, \chi_\nu)$ *observed at successive epochs* $a \in \mathscr{A}$ *is*

$$P\{\nu = k, \chi_1 \leqslant x_1, \chi_2 \leqslant x_2, \ldots, \chi_k \leqslant x_k\} = P_k \prod_{j=1}^{k} B^*(x_j). \tag{5}$$

Proof. Observed at successive epochs $a \in \mathscr{A}$ the event that an arriving customer finds the queue empty (the event $\{\nu = 0\}$) is, for the process $(\nu, \chi_1, \chi_2, \ldots, \chi_\nu)$, an aperiodic recurrent event with finite mean recurrence time. Hence this process has a unique limiting distribution ((3), p. 319, Theorem 2; although the statement of Feller's theorem assumes the state space of the process is countable, the proof readily extends to the uncountable case). This limiting distribution must be a stationary distribution ((1), pp. 133–4) and so the unique stationary distribution.

Now suppose that the process $\xi_e = (\rho_e, \nu_e, \chi_{1e}, \chi_{2e}, \ldots, \chi_{\nu_e e}, \eta_e)$ observed at successive epochs $e \in \mathscr{E}$ is stationary with distribution (1). Then the process

$$\zeta_a = (\rho_a, \nu_a, \chi_{1a}, \chi_{2a}, \ldots, \chi_{\nu_a a})$$

observed at successive epochs $a \in \mathscr{A}$ will *not* be stationary. However, for both processes the number of visits to the set $\{\rho = +1, \nu = k, \chi_1 \leqslant x_1, \chi_2 \leqslant x_2, \ldots, \chi_k \leqslant x_k\}$ in the

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interval $[O, T]$ expressed as a proportion of the number of visits to the set $\{\rho = +1\}$ in the interval $[O, T]$ will be the same. Since ξ_e is stationary with distribution (1) this proportion must converge as $T \to \infty$ to the right-hand side of equation (5) (the ergodic theorem for stationary processes, (1), p. 118, Theorem 6.28); but this proportion must also converge to the limiting distribution for ζ_a (the ergodic theorem for Markov chains, (7), theorem 8, (5), p. 74). Hence the result.

COROLLARY 2. *The unique stationary distribution for* $(\nu, \chi_1, \chi_2, \ldots, \chi_\nu, \eta)$ *observed at successive epochs* $d \in \mathscr{D}$ *is*

$$P\{\nu = k, \chi_1 \leqslant x_1, \chi_2 \leqslant x_2, \ldots, \chi_k \leqslant x_k, \eta \leqslant y\} = P_k(1 - e^{-\lambda y}) \prod_{j=1}^{k} B^*(x_j). \tag{6}$$

The stationary departure process is Poisson.

Proof. Observed at successive epochs $d \in \mathscr{D}$ the event that a departing customer leaves the queue empty is, for the process $(\nu, \chi_1, \chi_2, \ldots, \chi_\nu, \eta)$, an aperiodic recurrent event with finite mean recurrence time. The first part of the Corollary can hence be established in the same manner as Corollary 1.

From equation (6) it follows that at a departure epoch d the time elapsed since the last departure epoch, η_d, is exponentially distributed and independent of the number of customers left in the queue and of their remaining service requirements. Thus η_d is independent of future inter-departure times, and so the departure process is Poisson.

Further remarks. The method of this paper can also be used to show that the stationary departure process is Poisson when the server divides his effort equally between all the customers present (so that a customer's remaining service requirement decreases at rate k^{-1} while there are k customers in the queue). When the service requirement distribution can be expressed as a finite mixture of gamma distributions a more direct technique can be used to establish the results of this paper (6).

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