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Expansive geodesic flows in manifolds with no conjugate points

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Abstract. Let M be a compact Riemannian manifold with no conjugate points such that its geodesic flow is expansive. We show that there exists a local product structure in the unit tangent bundle of the manifold which is invariant under the geodesic flow. In particular, we have that the set of closed geodesics is dense and that the flow is topologically transitive.

0. *Introduction*

The theory of manifolds with non-positive curvature provides beautiful examples of the interaction between geometry and topology. Among them we have a series of structure theorems concerning the fundamental group of compact manifolds of non-positive curvature and the topological dynamics of the geodesic flow of such a manifold. One of the remarkable properties of the geodesic flow of negatively curved manifolds is their so-called expansiveness.

Definition 1. Let $\psi_t : X \longrightarrow X$ be a continuous flow acting on a metric space (X, d). The flow ψ_t is said to be expansive if there exists a constant $\epsilon > 0$ such that for every $x \in X$ we have the following property: if for a given $y \in X$ there exists a continuous, surjective map $h : R \longrightarrow R$ with $h(0) = 0$ such that

$$d(\psi_t(x), \psi_{h(t)}(y)) \leq \epsilon$$

for every $t \in R$, then there exists t_0 such that $\psi_{t_0}(x) = y$.

In other words, two different orbits of an expansive flow eventually become separated by a distance of at least ϵ. Geodesic flows of compact, negatively curved manifolds have the even stronger property that the distance function between geodesics in the universal covering is bounded below by an exponential function. The study of the divergence of geodesics has played an important role in geometry, dynamical systems and topology.

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It determines, for instance, the exponential growth rate of the fundamental group of compact, negatively curved manifolds and the non-vanishing of the topological entropy.

In this paper we show that several remarkable properties of the topological dynamics of Anosov geodesic flows occur also in the case of expansive geodesic flows. The main result of this article is the following theorem.

THEOREM 1. *Let M be a C^∞ compact Riemannian manifold of dimension n with no conjugate points. Let $\phi_t : T_1M \longrightarrow T_1M$ be the geodesic flow acting in the unit tangent bundle T_1M of M. Then, if ϕ_t is expansive we have that:*

(1) *for every point θ in the unit tangent bundle of M, T_1M, the sets*

$$W^s(\theta) = \{\eta \in T_1M : \lim_{t\to+\infty} d(\phi_t(\theta), \phi_t(\eta)) = 0\}$$
$$W^u(\theta) = \{\eta \in T_1M : \lim_{t\to-\infty} d(\phi_t(\theta), \phi_t(\eta)) = 0\}$$

are C^0 submanifolds of dimension $n-1$;

(2) *the sets W^s, W^u give continuous foliations of T_1M which induce a local product structure.*

Our notion of local product structure is the usual notion appearing, for instance, in the context of hyperbolic dynamical systems, i.e. $\forall\theta \in T_1M$ there is a local transversal section S_θ of θ and a homeomorphism $f : (-1,1)^{2(n-1)} \longrightarrow S_\theta$ such that:

(1) $f((-1,1)^{n-1} \times \{\bar{y_0}\})$ is a subset of the connected component of

$$\bigcup_{t\in R} \phi_t(W^s(f(0,\bar{y_0}))) \cap S_\theta$$

containing $f(0,\bar{y_0})$ for every $\bar{y_0} \in (-1,1)^{n-1}$;

(2) $f(\{\bar{x_0}\} \times (-1,1)^{n-1})$ is a subset of the connected component of

$$\bigcup_{t\in R} \phi_t(W^u(f(\bar{x_0},0))) \cap S_\theta$$

containing $f(\bar{x_0},0)$ for every $\bar{x_0} \in (-1,1)^{n-1}$.

The sets $W^s(\theta)$ and $W^u(\theta)$ are called the stable and the unstable sets of θ respectively. The sets

$$W^{cs}(\theta) = \bigcup_{t\in R} \phi_t(W^s(\theta)) \quad \text{and} \quad W^{cu}(\theta) = \bigcup_{t\in R} \phi_t(W^u(\theta))$$

are called the center stable and the center unstable sets of θ respectively.

The above definitions tell us that the continuous foliations given by center stable and center unstable sets provide a local product structure in the usual sense of the theory of foliations. The existence of stable and unstable sets for expansive systems is an interesting open question motivated by the well known existence theorems for Anosov systems. The work of Lewowicz [**6**] (see also [**7**], for instance) gives a positive answer to this question in the case of expansive homeomorphisms of surfaces. The suppression of the hyperbolicity hypothesis in an expansive system implies the absence of information concerning the linear part of the dynamics, which is essential to prove the classical existence theorems. Paternain [**7**] has shown that a compact surface with expansive geodesic flow has no conjugate points, so Theorem 1 holds in this case without any hypotheses on the set of conjugate points. From Theorem 1 we obtain the next theorem.

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THEOREM 2. *Let M be a C^∞ compact manifold with no conjugate points such that the geodesic flow is expansive. Then the set of closed orbits is dense and the flow is topologically transitive.*

The idea of the proof of Theorem 1 is to obtain the global expansiveness of the geodesic flow in the universal covering of the manifold, since this will imply a series of regularity properties of the canonical metric of the universal covering. These properties will grant the continuity of the horospheric foliations in the unit tangent bundle, which are not known to be continuous, in general, for any manifold with no conjugate points. Therefore, horospheric foliations become natural candidates to provide the stable and unstable foliations of the flow, and indeed we shall show that expansiveness, together with a degree argument on the horospheres, imply the local product structure of center stable and center unstable foliations.

1. *The continuity of horospheres*

We start by fixing some notations. We shall always consider C^∞ Riemannian manifolds. The universal covering of a Riemannian manifold M endowed with the pull-back of the metric of M by the covering map will be $\tilde{M}$. Given $\theta = (p, v) \in T_1M$ we shall often use the notation $\gamma_\theta(t)$ for a geodesic having initial conditions $\gamma'_\theta(0) = v$, $\gamma_\theta(0) = p$. All the geodesics will be parametrized by arclength. Now, we recall briefly the construction of horospheres and Busemann functions in $\tilde{M}$.

Let γ_θ be a geodesic in $\tilde{M}$. Let $S(t)$ be the sphere of radius $|t|$ centered at $\gamma_\theta(t)$. Notice that if $\theta = (p, v)$, then $p \in S(t)\ \forall t$. Since M is a compact manifold without conjugate points, the sets $S(t)$ approach, as $t \longrightarrow +\infty$ uniformly on compact sets in the C^1 topology, an embedded C^1 submanifold $H^s(\theta)$ of $\tilde{M}$ called the *stable horosphere* of θ. Similarly, letting $t \longrightarrow -\infty$ the spheres $S(t)$ converge in the C^1 topology to the *unstable horosphere* $H^u(\theta)$ which is just $H^s(p, -v)$. For $\theta = (p, v) \in T_1\tilde{M}$ the *Busemann function* $b_\theta : \tilde{M} :\longrightarrow R$ associated to θ is defined by

$$b_\theta(x) = \lim_{t \to +\infty} (d(x, \gamma_\theta(t)) - t).$$

The level sets of b_θ are just the horospheres $H^s(\gamma_\theta(t), \gamma'_\theta(t))$ (notice that $\gamma_\theta(t)$ intersects each level set of b_θ perpendicularly at only one point). Next, we state some basic properties of Busemann functions (see [**13**] for instance).

LEMMA 1.1.

(1) *b_θ is a C^1 function for every θ.*

(2) *The gradient ∇b_θ has norm equal to one at every point.*

Therefore, the horospheres $H^s(\gamma_\theta(t), \gamma'_\theta(t))$ are C^1 imbedded submanifolds of $\tilde{M}$ of dimension $n - 1$, where n is the dimension of M, which depend continuously on t in the compact open topology, and there is a flow $\psi_t^\theta : \tilde{M} \longrightarrow \tilde{M}$, which we shall call the Busemann flow, whose orbits are geodesics which are everywhere tangential to the vector field $-\nabla b_\theta$. In particular, the geodesic γ_θ is an orbit of this flow and we have that

$$\psi_t^\theta(H^s(\gamma_\theta(t), \gamma'_\theta(t)) = H^s(\gamma_\theta(s + t), \gamma'_\theta(s + t))$$

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for any real numbers t, s. So this flow acts in $\tilde{M}$ perpendicularly to the corresponding horosphere foliation and always preserves it. Although it is clear that $H^s(\gamma_\theta(t), \gamma'_\theta(t))$ depends continuously on t, it is not known whether $H^s(\gamma_\theta(t), \gamma'_\theta(t))$ depends continuously on θ or not. Pesin in [**8**] remarks that there is a large class of manifolds with no conjugate points satisfying this property.

Definition 2. Let M be a Riemannian manifold with no conjugate points. Two geodesics γ, β in $\tilde{M}$ are called asymptotic if there exists $C > 0$ such that

$$d(\gamma(t), \beta(t)) \leq C$$

for every $t \geq 0$.

If γ, β are asymptotic we shall often call β an asymptote of γ. If γ, β are asymptotic and $\bar{\gamma}(t) = \gamma(-t)$, $\bar{\beta}(t) = \beta(-t)$ are also asymptotic then we say that γ, β are *bi-asymptotic*.

Definition 3. Let M be a Riemannian manifold with no conjugate points. We say that $\tilde{M}$ satisfies the *axiom of asymptoticity* if for every $\theta = (p, v)$ in $T_1\tilde{M}$ the following statement holds.

Let $q_n \to q$ be a sequence of points in $\tilde{M}$ and let $\theta_n \to \theta$ be a sequence in $T_1\tilde{M}$. Then for every sequence of real numbers $t_n \to +\infty$ the sequence β_n of geodesic segments joining q_n to $\gamma_{\theta_n}(t_n)$ converges to an asymptote of γ_θ.

Manifolds with no focal points satisfy this axiom, and in particular manifolds of non-positive curvature. Now from [**8**] we have the following lemma.

LEMMA 1.2. *Let M be a Riemannian manifold with no conjugate points. If $\tilde{M}$ satisfies the axiom of asymptoticity, then the vector fields ∇b_θ depend continuously on $\theta \in \tilde{M}$ in the C^0 topology, and the horospheres $H^s(\theta)$ depend continuously on $\theta \in T_1\tilde{M}$ in the compact open topology of $\tilde{M}$.*

A straightforward consequence of this lemma is the following.

COROLLARY 1.1. *Let M have no conjugate points and suppose that $\tilde{M}$ satisfies the axiom of asymptoticity. Then for every q in $\tilde{M}$ such that*

$$q \in H^s(\gamma_\theta(t), \gamma'_\theta(t)) \bigcap H^u(\gamma_\theta(t), \gamma'_\theta(t))$$

for some $t \in R$, there exists a geodesic β in $\tilde{M}$, with $\beta(0) = q$, which is bi-asymptotic to γ_θ.

This comes from the fact that $H^s(\theta)$ and $H^u(\theta)$ always intersect at points of tangency.

Definition 4. Let $\theta = (p, v) \in T_1\tilde{M}$. The canonical lifting $\mathcal{F}^s(\theta)$ of $H^s(\theta) \in T_1\tilde{M}$ is defined by

$$\mathcal{F}^s(\theta) = \{(q, w) \in T_1\tilde{M} : q \in H^s(\theta), w = -\nabla b_\theta\}.$$

The canonical lifting $\mathcal{F}^u(\theta)$ of $H^u(\theta)$ is defined by

$$\mathcal{F}^u(\theta) = \{(q, w) \in T_1\tilde{M} : q \in H^u(\theta), w = \nabla b_{(p,-v)}\}.$$

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It is clear that $\phi_t(\mathcal{F}^s(\theta)) = \mathcal{F}^s(\phi_t(\theta))$, and $\phi_t(\mathcal{F}^u(\theta)) = \mathcal{F}^u(\phi_t(\theta))$. Since the horospheres are, in general, C^1 submanifolds of the universal covering of M then $\mathcal{F}^s(\theta)$ and $\mathcal{F}^s(\theta)$ are only C^0 submanifolds of the unit tangent bundle of dimension $n-1$. It is well known that the differentiability of horospheres is closely related to the dynamics of the geodesic flow and curvature restrictions. For instance, if the curvature of $\tilde{M}$ is bounded above by a negative constant then horospheres are C^∞ submanifolds, but if the curvature is just non-positive then horospheres are C^2 and this result is sharp [**1**]. For more detailed references and interesting examples see [**1**]. Now we state the main result of this section.

PROPOSITION 1.1. *Let M be a compact Riemannian manifold with no conjugate points. If the geodesic flow of M is expansive, then the universal covering satisfies the axiom of asymptoticity.*

The proof of Proposition 1.1 will follow from a series of lemmas.

LEMMA 1.3. *Let M be a compact manifold having no conjugate points and let $\phi_t : T_1M \longrightarrow T_1M$ be expansive. Then there exists $n_0 > 0$ such that for every $n \geq n_0$ there exists $\delta = \delta(n) > 0$ with $\lim_{n\to+\infty} \delta(n) = 0$ such that for every pair of geodesics γ, β in $\tilde{M}$ satisfying*

$$\sup\{d(\gamma(t), \beta(t)), d(\gamma(s), \beta(s))\} \leq \frac{1}{n}$$

for some $t < s$, then we have that

$$\sup_{r\in[t,s]} d(\gamma(r), \beta(r)) \leq \delta.$$

The proof of the above lemma is shown in [**11**], and it is inspired by an analogous result for expansive homeomorphisms proved by Walters [**14**].

Let us denote by $[a, b]$ the geodesic segment joining $a, b \in \tilde{M}$.

Definition 5. We say that $\tilde{M}$ is quasi-convex if there exist constants $K, C > 0$ such that

$$d([a, b], [p, q]) \leq K \sup\{d(a, p), d(b, q)\} + C,$$

where $d([a, b], [p, q])$ denotes the Hausdorff distance between the geodesic segments.

We introduce this notion at this point for the following lemma.

LEMMA 1.4. *Let M be a Riemannian manifold with no conjugate points. If $\tilde{M}$ is quasi-convex, then it satisfies the axiom of asymptoticity.*

The proof of this lemma is a straightforward consequence of the definitions. Roughly speaking, the quasi-convexity condition allows us to control the convergence of sequences of geodesics of the kind stated in the definition of the axiom of asymptoticity. Instead of proving Lemma 1.4 we shall prove the following lemma that will be useful for us in the forthcoming arguments, and has the same line of reasoning as the proof of Lemma 1.4.

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LEMMA 1.5. *Let M have no conjugate points and suppose that $\tilde{M}$ is a K, C quasi-convex space. If γ, β are asymptotic, then*

$$\sup_{t\geq 0} d(\gamma(t), \beta(t)) \leq K d(\gamma(0), \beta(0)) + C.$$

Proof. From the definition of horospheres it is easy to conclude that β is obtained as the limit of the sequence $[\beta(0), \gamma(s)]$—the geodesic segments joining $\beta(0)$ to $\gamma(s)$—letting s go to ∞. But from the quasi-convexity we get

$$d([\beta(0), \gamma(s)],\ [\gamma(0), \gamma(s)]) \leq K d(\gamma(0), \beta(0)) + C$$

for every s, where $d(\ ,\ )$ in the left-hand side of the inequality means Hausdorff distance. Since this inequality is preserved by limits we conclude the proof of the statement. □

So we would get Proposition 1.1 if we could show that in our hypotheses, the metric in $\tilde{M}$ is quasi-convex. We have the following lemma.

LEMMA 1.6. *Let M be a compact Riemannian manifold with no conjugate points. If the geodesic flow of M is expansive then the universal covering $\tilde{M}$ is a quasi-convex metric space.*

Proof. Although the proof of this lemma is performed in [**11**], we shall give it here for the sake of completeness.

Let $[a, b]$, $[p, q]$ be two geodesic segments in $\tilde{M}$ and assume without loss of generality that $d(a, p) \geq d(b, q)$. Let n_0 be as in Lemma 1.3 and let $N_0 = [\![d(a, p)n_0]\!]$, where $[\![r]\!]$ is the integer part of the number r. Let us make a partition of the geodesic segments

$$[a, p] = \bigcup_{i=0}^{N_0} [a_i, a_{i+1}] \quad [b, q] = \bigcup_{i=0}^{N_0} [b_i, b_{i+1}]$$

with $a_0 = a$, $b_0 = b$, $a_{N_0+1} = p$, $b_{N_0+1} = q$ such that:

(1) length $[a_i, a_{i+1}] = 1/n_0$ for every $0 \leq i \leq N_0$ and length $[a_{N_0}, a_{N_0+1}] \leq 1/n_0$ (notice that a_{N_0} may be equal to a_{N_0+1});

(2) length $[b_i, b_{i+1}] = d(b, q)/N_0 + 1 \leq 1/n_0$ for every i.

From Lemma 1.3 and properties (1) and (2) above we deduce that

$$\begin{aligned} d([a, b], [p, q]) &\leq \sum_{i=0}^{N_0} d([a_i, b_i], [a_{i+1}, b_{i+1}]) \\ &\leq (N_0 + 1)\delta(n_0) \\ &= \delta(n_0)([\![n_0 d(a, p)]\!] + 1) \\ &\leq \delta(n_0) n_0 d(a, p) + \delta(n_0) \\ &= K \sup(d(a, p), d(b, q)) + C, \end{aligned}$$

where $K = \delta(n_0)n_0$ and $C = \delta(n_0)$. This finishes the proof of the lemma. □

2. *Global divergence of geodesics and the existence of invariant sets*

Let $\pi : \tilde{M} \longrightarrow M$ be the covering map and let $\bar{\pi} : T_1\tilde{M} \longrightarrow T_1M$ be defined by $\bar{\pi}(p, v) = (\pi(p), d_p\pi(v))$. The main result of this section is the following proposition.

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PROPOSITION 2.1. *Let M be a compact Riemannian manifold with no conjugate points having an expansive geodesic flow. Then $W^s(\theta) = \bar{\pi}(\mathcal{F}^s(\bar{\pi}^{-1}(\theta)))$ and $W^u(\theta) = \bar{\pi}(\mathcal{F}^u(\bar{\pi}^{-1}(\theta)))$, where $\bar{\pi}^{-1}(\theta)$ is any preimage of θ by $\bar{\pi}$.*

We shall subdivide the proof of Proposition 2.1 into several lemmas. First we show the following lemma (compare with [**11**]).

LEMMA 2.1. *Let M satisfy the hypotheses of Proposition 2.1. If γ, β are two geodesics in $\tilde{M}$ satisfying*

$$d(\gamma, \beta) \leq C,$$

where $d(\gamma, \beta)$ is the Hausdorff distance, then there exists t_0 such that $\gamma(t) = \beta(t + t_0)$.

Proof. Let ϵ be the the expansiveness constant of the geodesic flow. It is clear that there exists $\bar{\epsilon}$ such that two geodesic segments of length greater than one in M have Hausdorff distance less than $\bar{\epsilon}$ if and only if their Hausdorff distance in $T_1\tilde{M}$ as pieces of orbits of the geodesic flow is less than ϵ, so we shall assume without loss of generality that $\bar{\epsilon} = \epsilon$. Now, let us assume that the Hausdorff distance between γ and β is bounded above by C. Then these geodesics are bi-asymptotic and we have

$$d(\gamma(t), \beta(t)) \leq C'$$

for every $t \in R$, where C' depends on C. Keeping the notation of the previous section, consider the geodesic segments $\sigma_m = [\gamma(m), \beta(m)]$. All their lengths are less than or equal to C'. According to Lemma 1.3 there exists n_0 such that the corresponding $\delta(n_0)$ is less than ϵ. Chose an integer $A \geq \epsilon/n_0$. Then we can find $t_0 < t_1 < \cdots < t_A$ and parametrizations of the σ_m such that for all $m \in Z$ we have $\sigma_m(t_0) = \gamma(m)$, $\sigma_m(t_A) = \beta(m)$ and

$$\sup_i \{d(\sigma_m(t_i), \sigma_m(t_{i+1}))\} \leq \frac{1}{n_0}.$$

Now consider the geodesic segments

$$\alpha_m^i = [\sigma_{-m}(t_i), \sigma_m(t_i)]$$

for $0 \leq i \leq A$ and $m \geq 1$. From the construction we get that

$$d(\alpha_m^i, \alpha_m^{i+1}) \leq \delta(n_0) \leq \epsilon$$

for every $0 \leq i \leq A$ and $m \geq 1$.

From the quasi-convexity of $\tilde{M}$ we get convergent subsequences of the segments α_m^i as m goes to ∞ for each fixed i. In this way we obtain a collection of $A + 1$ geodesics α^i, where $\alpha^0 = \gamma$, $\alpha^A = \beta$, and such that

$$d(\alpha^i, \alpha^{i+1}) \leq \epsilon$$

$\forall 0 \leq i \leq A$. However, from the definition of expansiveness we get that $\gamma = \alpha^i = \beta$ for every i, thus proving the lemma. □

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COROLLARY 2.1. *Let M be as in Proposition 2.1. Then if $\gamma \neq \beta$ are geodesics in $\tilde{M}$ we have that*

$$\lim_{t\to+\infty} d(\gamma(t), \beta(t)) = +\infty$$

or

$$\lim_{t\to-\infty} d(\gamma(t), \beta(t)) = +\infty.$$

Proof. From Lemma 2.1 we deduce that if γ and β are two different geodesics in $\tilde{M}$, then their Hausdorff distance is not finite, or equivalently

$$\limsup_{t\in R} d(\gamma(t), \beta(t)) = +\infty.$$

But again from the quasi-convexity of $\tilde{M}$ we get that

$$\lim_{t\to+\infty} d(\gamma(t), \beta(t)) = +\infty$$

or

$$\lim_{t\to-\infty} d(\gamma(t), \beta(t)) = +\infty.$$

Otherwise there would exist $D > 0$ and sequences $s_n \to +\infty$ and $r_n \to -\infty$ such that

$$d(\gamma(s_n), \beta(s_n)) \leq D \quad \text{and} \quad d(\gamma(r_n), \beta(r_n)) \leq D,$$

which means by the quasi-convexity that there are constants K, C such that

$$d(\gamma[r_n, s_n], \beta[r_n, s_n]) \leq KD + C,$$

where $d(\ ,\ )$ in the above formula means Hausdorff distance. Letting the sequences go to ∞ we get that the Hausdorff distance between γ and β is finite, contradicting Lemma 2.1. This ends the proof of the corollary. □

As a consequence of Corollary 2.1 we obtain the following result.

COROLLARY 2.2. *Let M be a compact Riemannian manifold with no conjugate points whose geodesic flow is expansive. Then given $0 < r < D$ there exists $T > 0$ such that for every $\theta = (p, v) \in T_1\tilde{M}$ we have*

$$\psi_t^\theta \left(B_D(p) \bigcap H^s(\theta) \right) \subset B_r(\psi_t^\theta(p)) \bigcap H^s(\gamma_\theta(t), \gamma'_\theta(t))$$

for every $t \geq T$, where $B_r(p)$ is the ball of radius r centered at p, and ψ_t^θ is, as in §1, the Busemann flow of θ.

Proof. By contradiction, suppose that there exist $D > 0$, $0 < r < D$ and sequences $\theta_n = (p_n, v_n)$ in $T_1\tilde{M}$, $t_n \to +\infty$ such that

$$d(\psi_{t_n}^{\theta_n}(x_n), \psi_{t_n}^{\theta_n}(p_n)) \geq r,$$

where $x_n \in B_D(p_n) \bigcap H_{\theta_n}(0)$. Since $\psi_{t_n}^{\theta_n}(p_n) = \gamma_{\theta_n}(t_n)$ it is not hard to see that

$$\inf_{t\in R} d(\psi_{t_n}^{\theta_n}(x_n), \gamma_{\theta_n}(t)) \geq r'$$

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for some $r' > 0$ depending only on r. Consider the geodesics defined by $\beta_n(t) = \psi_t^{\theta_n}(x_n)$. Notice that $\beta_n(t)$ is the asymptote of $\gamma_{\theta_n}(t)$ through the point x_n. So from Lemma 1.5 we get that

$$\begin{aligned} d(\beta_n(t), \gamma_{\theta_n}(t)) &\leq K d(\beta_n(0), \gamma_{\theta_n}(0)) + C \\ &\leq K d(x_n, p_n) + C \\ &\leq K D + C \end{aligned}$$

for every $t \geq 0$, where K, C are the quasi-convexity constants of $\tilde{M}$. Let $\bar{\beta}_n(t) = \beta_n(t + t_n)$, $\bar{\gamma}_n(t) = \gamma_n(t + t_n)$. According to the above inequalities these geodesics satisfy:

(1) $d(\bar{\beta}_n(s), \bar{\gamma}_n(s)) \leq K D + C$ for every $s \geq -t_n$;

(2) $d(\bar{\beta}_n(0), \bar{\gamma}_n(0)) \geq r'$.

Up to the action of the fundamental group we can assume that $\bar{\gamma}_n(0)$ belongs to a certain compact subset of $\tilde{M}$ for every n. So letting n go to $+\infty$ we get limit geodesics $\gamma(t)$, $\beta(t)$ of certain convergent subsequences satisfying:

(1) $d(\beta(s), \gamma(s)) \leq K D + C$ for every $s \in R$;

(2) $d(\beta(0), \gamma(0)) \geq r'$.

This implies that γ and β are different bi-asymptotic geodesics contradicting Corollary 2.1. This finishes the proof of Corollary 2.2. □

Next, we have to analyse the divergence of geodesic rays in $\tilde{M}$, i.e. we want to show that for every pair of geodesics $\gamma, \beta \subset \tilde{M}$ with $\gamma(0) = \beta(0)$ and $\gamma \neq \beta$, then

$$\lim_{t \to +\infty} d(\gamma(t), \beta(t)) = +\infty.$$

Notice that this fact is not straightforward from the divergence of asymptotic geodesics stated in Lemma 2.1 and in Corollary 2.2. We shall need this property to deduce the uniqueness of asymptotes in $\tilde{M}$, and this fact will play an important role in the proof of Proposition 2.1. Observe that Pesin in [**8**] states that if $\tilde{M}$ satisfies the axiom of asymptoticity, then the asymptote of any geodesic through a given point is unique. In his proof, the divergence of geodesic rays in $\tilde{M}$ for compact M with no conjugate points is used as a result due to Green [**4**]. However, according to Eberlein [**2**, appendix], Green's proof of this lemma has a gap. I would like to thank the referee for pointing out this fact to me. So we start with the following lemma from [**3**] describing the asymptotic behaviour of Jacobi fields having a zero.

LEMMA 2.2. *Let M be a compact Riemannian manifold with no conjugate points. There exists $B > 0$ such that for every geodesic $\gamma \subset M$ and every non-zero, perpendicular Jacobi field $J(t)$ of γ with $J(0) = 0$ we have that $|J(t)| \geq B|J'(0)|$ for every $t \geq 1$.*

In other words, a non-trivial, perpendicular Jacobi field $J(t)$ cannot be arbitrarily close to zero for $t \geq 1$. It is not hard to deduce the following corollary from this result.

COROLLARY 2.3. *Let M be compact without conjugate points. Given $\epsilon > 0$ there exists $D = D(\epsilon)$ such that if γ, β are two different geodesics in $\tilde{M}$ with $p = \gamma(0) = \beta(0)$ which make an angle $\geq \epsilon$ at p, we have*

$$\liminf_{t \geq 1} d(\gamma(t), \beta(t)) \geq D.$$

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LEMMA 2.3. *Let M be compact without conjugate points satisfying the horosphere uniform contraction hypotheses, i.e. given $0 < r < r'$ there exists $T = T(r, r') > 0$ such that for every $\theta = (p, v) \in T_1\tilde{M}$ we have*

$$\psi_t^\theta \left(B_{r'}(p) \bigcap H^s(\theta) \right) \subset B_r(\psi_t^\theta(p)) \bigcap H^s(\gamma_\theta(t), \gamma'_\theta(t))$$

for every $t \geq T$, where $B_r(p)$ is the ball of radius r centered at p, and ψ_t^θ is, as in §1, the Busemann flow of θ. Then:

(1) *the sets $H^s(\theta)$, $H^u(\theta)$ depend continuously on $\theta \in T_1\tilde{M}$ in the compact open topology;*

(2) *given $\theta \in T_1\tilde{M}$, a geodesic $\psi_t^\theta(q)$ is bi-asymptotic to γ_θ if and only if $\gamma_\theta = \psi_t^\theta(q)$.*

Proof. To begin with the proof of (1), notice that the uniform contraction of horospheres implies that for every $\theta \in T_1\tilde{M}$ and $q \in \tilde{M}$ there exists an asymptote of γ_θ containing q, namely, the geodesic $\psi_t^\theta(q)$. Moreover, given $0 < \epsilon < r$ there exists $T = T(r, \epsilon) > 0$ such that if $d(\gamma_\theta(0), q) \leq r$, then

$$d(\gamma_\theta(t), \psi^\theta_{t+b_\theta(q)}(q)) \leq \epsilon,$$

where $b_\theta(q)$ is the value of the Busemann function of θ at q. Since the continuity of horospheres (in the compact open topology) will follow from the continuity of ∇b_θ we shall show this last assertion.

So let $\theta_n \to \theta$ and fix $r > 0$. From the hypotheses, every convergent sequence of asymptotes of γ_{θ_n} through points in $H^s(\theta_n) \cap B_r(\gamma_\theta(0))$ will converge to an asymptotic geodesic γ_η of γ_θ satisfying

$$\lim_{t \to +\infty} d(\gamma_\theta(t), \gamma_\eta(t)) = 0.$$

Therefore, if the Busemann asymptote $\psi_t^\theta(\gamma_\eta(0))$ does not coincide with γ_η we would obtain two different geodesic rays in $\tilde{M}$ whose distance goes to zero as $t \to +\infty$, contradicting Corollary 2.3. Assertion (1) in the lemma is straightforward from this fact.

To prove assertion (2), let $\gamma_\eta = \psi_t^\theta(q)$ and assume that $d(\gamma_\eta, \gamma_\theta)$ is finite. Then we have (up to parametrization) that

$$\gamma_\eta(t) \in H^s(\gamma_\theta(t), \gamma'_\theta(t)) \quad \text{and} \quad d(\gamma_\eta(t), \gamma_\theta(t)) \leq C$$

for every $t \in R$ and some constant C. From the uniform contraction of horospheres we have that for every $\epsilon < D$ there exists $T = T(C, \epsilon)$ such that

$$\begin{aligned} d(\eta, \theta) &= d(\psi_t^\theta(\gamma_\eta(-t)), \psi_t^\theta(\gamma_\theta(-t))) \\ &\leq \epsilon \end{aligned}$$

for every $t \geq T$. Since ϵ is arbitrary we conclude that $\eta = \theta$ and assertion (2) in the lemma holds. □

LEMMA 2.4. *Let M be compact without conjugate points satisfying the horosphere uniform contraction hypotheses. Then for every two different geodesics γ, β in $\tilde{M}$ with $\gamma(0) = \beta(0)$ we have that*

$$\limsup_{t \to +\infty} d(\gamma(t), \beta(t)) = +\infty.$$

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Proof. Assume by contradiction that there exist two geodesic rays γ, β such that $\gamma(0) = \beta(0)$ and $d(\gamma(t), \beta(t)) \leq C$. From Corollary 2.3 there exists $D > 0$ depending on the angle at $p = \gamma(0) = \beta(0)$ such that

$$D \leq d(\gamma(t), \beta(t)) \leq C$$

for every $t \geq 1$. Let M_0 be a compact fundamental domain of M, and for $n \in N,\ n > 0$ let $\gamma_n(t) = T_n(\gamma(n-t))$, $\beta_n(t) = T_n(\beta(n-t))$, where T_n is a sequence of deck transformations such that $T_n(\gamma(n)) \in M_0$. This means that $\gamma_n(0) = \gamma(n) \in M_0$ and

$$D \leq d(\beta_n(t), \gamma_n(t)) \leq C$$

for every $t \leq n-1$. Passing to a convergent subsequence of the γ_n, β_n (that we still denote by γ_n, β_n) we obtain geodesics

(1) $\gamma_0 = \lim_{n\to+\infty} \gamma_n$

(2) $\beta_0 = \lim_{n\to+\infty} \beta_n$

(3) $D \leq d(\gamma_0(t), \beta_0(t)) \leq C$ for every $t \in R$.

It is clear that β_0 is perpendicular to the stable horosphere $H^s(\gamma_0(0), \gamma_0'(0))$ at the point $\beta_0(0)$, because on one hand, the rays β_n were always perpendicular to the spheres $S_n(\gamma_n(n))$ of radius n and centered at $\gamma_n(0)$, and on the other hand, spheres in $\tilde{M}$ approach horospheres uniformly on compact sets (see the first part of the proof of Theorem 6.1 in [**8**] for instance). So from the continuity of horospheres (Lemma 2.3(1)) we get that the rays β_n actually approach an orbit of $-\nabla b_{(\gamma_0(0),\gamma_0'(0))}$. This means that both γ_0 and β_0 are bi-asymptotic orbits of $-\nabla b_{(\gamma_0(0),\gamma_0'(0))}$ contradicting Lemma 2.3. This finishes the proof of Lemma 2.4. □

COROLLARY 2.4. *Let M be a compact manifold without conjugate points satisfying the uniform contraction condition for horospheres. Then every asymptotic geodesic of γ_θ is an orbit of $-\nabla b_\theta$. Therefore, given $\theta \in T_1\tilde{M}$ and $q \in \tilde{M}$ there exists a unique geodesic asymptotic to γ_θ containing q.*

Proof. This is a direct consequence of Lemma 2.4, since the uniform contraction of horospheres already implies that given $\theta \in T_1\tilde{M}$ and any $q \in \tilde{M}$ the orbit of $-\nabla b_\theta$ at q is asymptotic to γ_θ. □

Proof of Proposition 2.1. Let $\theta \in T_1M$ and let $\bar{W}^s(\theta)$ be a lifting of $W^s(\theta)$ in $T_1\tilde{M}$ by the projection $\bar{\pi} : T_1\tilde{M} \longrightarrow T_1M$ given by $\bar{\pi}(p, v) = (\pi(v), d_p\pi(v))$ (recall that $\pi : \tilde{M} \longrightarrow M$ is the covering map of M). To simplify the notation we still denote by θ some lifting of θ in $T_1\tilde{M}$ contained in $\bar{W}^s(\theta)$. We want to show that $\bar{W}^s(\theta) = \mathcal{F}^s(\theta)$ and that $\bar{W}^u(\theta) = \mathcal{F}^u(\theta)$.

From Corollary 2.2 we have that

$$\mathcal{F}^s(\theta) \subset \bar{W}^s(\theta) \quad \text{and} \quad \mathcal{F}^u(\theta) \subset \bar{W}^u(\theta).$$

Indeed, if $\eta \in \mathcal{F}^s(\theta)$, Corollary 2.2 tells us that

$$\lim_{t\to+\infty} d(\gamma_\theta(t), \gamma_\eta(t)) = 0,$$

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where γ_θ and η_η are the underlying geodesics in $\tilde{M}$. It is not difficult to show that the corresponding orbits of θ and η under the geodesic flow satisfy

$$\lim_{t\to+\infty} d(\phi_t(\theta), \phi_t(\eta)) = 0$$

(here we identify $\eta \in T_1\tilde{M}$ with its projection in T_1M). The same reasoning with $t \to -\infty$ holds for $\mathcal{F}^u(\theta)$. So it remains to show that $\bar{W}^s(\theta) \subset \mathcal{F}^s(\theta)$ for every $\theta \in T_1\tilde{M}$. Let $\eta \in \bar{W}^s(\theta)$, then $\lim_{t\to+\infty} d(\phi_t(\theta), \phi_t(\eta)) = 0$ in $T_1\tilde{M}$. As before, the geodesics γ_θ and γ_η in $\tilde{M}$ satisfy

$$\lim_{t\to+\infty} d(\gamma_\theta(t), \gamma_\eta(t)) = 0.$$

Or, in other words, γ_θ and γ_η are asymptotic. Since the geodesic flow of M is expansive, the Busemann flows contract uniformly the stable horospheres (Corollary 2.2) so the orbit $\psi_t^\theta(\gamma_\eta(0))$ is also asymptotic to γ_θ and

$$\lim_{t\to+\infty} d(\psi_t^\theta(\gamma_\eta(0)), \gamma_\theta(t)) = 0 \Rightarrow \lim_{t\to+\infty} d(\psi_t^\theta(\gamma_\eta(0)), \gamma_\eta(t)) = 0.$$

Thus, by Corollary 2.3, γ_η and $\psi_t^\theta(\gamma_\eta(0))$ must coincide since they have a point in common. In other words, γ_η has to be an orbit of $-\nabla b_\theta$. Now, from the definition of $\mathcal{F}^s(\theta)$ it follows that $\eta \in \mathcal{F}^s(\theta)$. □

We conclude §2 with the following characterization of expansive geodesic flows in manifolds with no conjugate points, which is essentially a summary of the results obtained in the section.

PROPOSITION 2.2. *Let M be a compact Riemannian manifold with no conjugate points. Then the following statements are equivalent:*

(1) *the geodesic flow of $\tilde{M}$ is expansive;*
(2) *the geodesics in $\tilde{M}$ diverge in the sense of Lemma 2.1;*
(3) *stable horospheres are uniformly contracted (in the sense of Corollary 2.1) by the positive iterates of the Busemann flow, and unstable horospheres are uniformly contracted by negative iterates.*

Proof. In fact, from Lemma 2.1 we have that (1) ⇔ (2), and from Lemma 1.2 and Corollary 2.1 we have that (1) ⇒ (3).

To prove (3) ⇒ (1), notice that (3) implies that given $\theta \in T_1\tilde{M}$ and $q \in \tilde{M}$ there exists an asymptote of γ_θ containing q, namely, the orbit of the Busemann flow of θ through q. By Lemma 2.4 and Corollary 2.4 we have that every bi-asymptotic geodesic of γ_θ is an orbit of $-\nabla b_\theta$, so from Lemma 2.3 we conclude that there are no geodesics bi-asymptotic to γ_θ which are different from γ_θ. This clearly implies the expansiveness of the geodesic flow. □

3. *The local product structure*

In this section we shall conclude the proof of Theorem 1 given in the Introduction. Notice that the existence of invariant sets, which is assertion (1) of this theorem, is

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already given by Proposition 2.1, so it remains to show the existence of the local product structure induced by the invariant foliations.

Let M be, as before, a compact Riemannian manifold having expansive geodesic flow ϕ_t. Let us say that two elements $\theta, \theta' \in T_1\tilde{M}$ are forward (resp. backward) asymptotic if the geodesics γ_θ and $\gamma_{\theta'}$ are forward (resp. backward) asymptotic in $\tilde{M}$. In §1 we proved that both relations are equivalence relations in $T_1\tilde{M}$ and, moreover, from Lemma 2.1 we have the following.

LEMMA 3.1. *Let $\theta, \theta' \in T_1\tilde{M}$. Then the set of vectors which are simultaneously forward asymptotic to θ and backward asymptotic to θ' is either empty or a single orbit of the geodesic flow.*

Let us define the maps $\Psi^s : T_1\tilde{M} \times \tilde{M} \longrightarrow T_1\tilde{M}$ and $\Psi^u : T_1\tilde{M} \times \tilde{M} \longrightarrow T_1\tilde{M}$ by

$$\Psi^s((p, v), q) = (q, -\nabla_q b_{(p,v)}), \quad \Psi^u((p, v), q) = (q, \nabla_q b_{(p,-v)}).$$

Notice that from Proposition 1.1 and Lemma 1.2 these maps are continuous. Moreover, $\Psi^s(\theta, q) \in \mathcal{F}^s(\gamma_\theta(t), \gamma'_\theta(t))$ and $\Psi^u(\theta, q) \in \mathcal{F}^u(\gamma_\theta(r), \gamma'_\theta(r))$ for some $t, r \in R$. Therefore, from Proposition 2.1, $\Psi^s(\theta, q)$ is forward asymptotic to θ and $\Psi^u(\theta, q)$ is backward asymptotic to θ.

For a given $\theta = (p, v) \in T_1\tilde{M}$ let $D_\theta(r)$ be the disc in $\tilde{M}$ which is the image under the exponential map of $\{(p, w) : w \perp v \text{ and } |w| < r\}$. From the continuity of Ψ^s, Ψ^u and the compactness of M we can choose $\delta > 0$ and $r > 0$ small enough so that if $d(\eta, \theta) < \delta$, then any geodesic which is forward or backward asymptotic to γ_η intersects $D_\theta(r)$ in at most one point. Now, define $\Psi^{su}_\theta, \Psi^{us}_\theta : D_\theta(r) \times D_\theta(r) \longrightarrow T_1\tilde{M}$ by

$$\Psi^{su}_\theta(x, y) = \Psi^s(\Psi^u(\theta, x), y), \quad \Psi^{us}_\theta(x, y) = \Psi^u(\Psi^s(\theta, x), y).$$

LEMMA 3.2. *The maps Ψ^{su}_θ and Ψ^{us}_θ are continuous and injective.*

Proof. The continuity follows from the fact that Ψ^{su}_θ and Ψ^{us}_θ are compositions of continuous maps. In order to check that, for instance, Ψ^{su}_θ is injective, assume that

$$\Psi^s(\Psi^u(\theta, x), y) = \Psi^s(\Psi^u(\theta, x'), y') = \eta.$$

Then, by definition we have that $y = y'$, and $\Psi^u(x), \Psi^u(x')$ are forward asymptotic to η. But $\Psi^u(x), \Psi^u(x')$ are also backward asymptotic to θ, so Lemma 3.1 tells us that they must lie in the same orbit of the geodesic flow. Since this orbit is backward asymptotic to θ, the choice of r forces the orbit to intersect $D_\theta(r)$ in at most one point, which implies that $x = x'$. Thus Ψ^{su}_θ is injective. An analogous argument leads to the injectivity of Ψ^{us}_θ. □

COROLLARY 3.1. *The images*

$$\mathrm{Im}(\Psi^{su}_\theta) = \Psi^{su}_\theta(D_\theta(r) \times D_\theta(r))$$

and

$$\mathrm{Im}(\Psi^{us}_\theta) = \Psi^{us}_\theta(D_\theta(r) \times D_\theta(r))$$

are open subsets of $T_1\tilde{M}$.

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Proof. This follows from Lemma 3.2 and Brouwer's invariance of domain theorem, since the dimension of $T_1 D_\theta(r)$ equals the dimension of $D_\theta(r) \times D_\theta(r)$. □

Corollary 3.1 implies that given $\theta \in T_1\tilde{M}$ there exists $0 < \delta(\theta) < \delta$ such that $G(\theta) = \{\eta \in T_1 D_\theta(r) : d(\eta, \theta) \leq \delta(\theta)\}$ lies in $\mathrm{Im}(\Psi_\theta^{su}) \cap \mathrm{Im}(\Psi_\theta^{us})$.

Define

$$g_\theta = G(\theta) \longrightarrow D_\theta(r) \times D_\theta(r)$$

by setting

$$g_\theta(\eta) = (x, y)$$

if $\eta = \Psi_\theta^{su}(x, y') = \Psi_\theta^{us}(x', y)$.

LEMMA 3.3. *The map g_θ is well defined, continuous and injective.*

Proof. The map g_θ is defined since Ψ_θ^{su} and Ψ_θ^{us} are injective. Moreover, from the injectivity of Ψ_θ^{su} and Ψ_θ^{us}, and the fact that $G(\theta)$ is a compact, Hausdorff subset of the images of these maps, we deduce that Ψ_θ^{su} and Ψ_θ^{us} have continuous inverses on $G(\theta)$. The continuity of g_θ is straightforward from previous arguments. To see that g_θ is injective, observe that if $g_\theta(\eta) = (x, y)$ then:

(1) $x, y \in D_\theta(r)$;

(2) $\Psi^u(\theta, x)$ is forward asymptotic to η and backward asymptotic to θ;

(3) $\Psi^s(\theta, y)$ is backward asymptotic to η and forward asymptotic to θ.

So, if $g_\theta(\alpha) = (x, y)$ for some $\alpha \in G_\theta$ we would have, from item (2) above, that $\Psi^u(\theta, x)$ is forward asymptotic to both η and α, and since the relation of being asymptotic is an equivalence relation in $T_1\tilde{M}$ we would get that η, α are forward asymptotic. Similarly, from item (3) above we would get that η and α are backward asymptotic. Therefore, by Lemma 3.1 they have to belong to the same orbit, and since $\eta, \alpha \in G(\theta) \subset T_1 D_\theta(r)$, the choices of r and δ imply $\eta = \alpha$. This finishes the proof of the lemma. □

The map g_θ is a natural candidate to obtain a coordinate system for $T_1\tilde{M}$ satisfying the conclusions of Theorem 1. Indeed, let $\rho(\theta)$ be small enough so that $D_\theta(\rho(\theta)) \times D_\theta(\rho(\theta))$ lies in the image of g_θ. Let $f_\theta : D_\theta(\rho(\theta)) \times D_\theta(\rho(\theta)) \longrightarrow T_1 D_\theta(r)$ be the inverse of g_θ. Then from items (2) and (3) above it is not hard to see that f_θ satisfies condition (2) in the statement of Theorem 1. This concludes the proof of the Theorem. □

Theorem 1 has a series of remarkable consequences concerning the topological dynamics of expansive geodesic flows in manifolds with no conjugate points. From this result it is possible to construct Markov partitions for such flows and therefore the features of hyperbolic symbolic dynamics have counterparts in our case. As an application of Theorem 1 we shall show the density of closed orbits and the existence of a dense orbit. We remark that these results have already been proved in [**10**] in the case of expansive geodesic flows of compact surfaces.

Proof of Theorem 2. We just give the ideas of the proof without details because they are the same as in the hyperbolic case (see [**12**] for instance).

To show the density of closed orbits recall that the set of recurrent points in $T_1 M$ is dense, so it is enough to show that recurrent points are accumulated by closed orbits. So

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let $\theta \in T_1M$ be (forward and backward) recurrent and let S_θ be the product section of Theorem 1. There is a sequence of Poincare maps $\mathcal{P}_n$ for $n \in Z$ of S_θ where positive (negative) n means that the return times are positive (respectively negative) such that

$$\lim_{n \to \mp\infty} \mathcal{P}_n(\theta) = \theta.$$

There are projections $P_s : S_\theta \longrightarrow W^s(\theta) \cap S_\theta$, $P_u : S_\theta \longrightarrow W^u(\theta) \cap S_\theta$ given by the local product structure, i.e. $P_s(\eta)$ is the intersection of the local unstable set of η with the local stable set of θ, and P_u is the intersection of the local stable set of η with the local unstable set of θ. But since the stable horocycles are contracted by $\mathcal{P}_n$ for n positive, according to Corollary 2.2 we get that the restrictions of $P_s \circ \mathcal{P}_n$ to $W^s(\theta)$ for n positive are contracting for n big enough. This implies that for a given n suitably large, there exists a fixed unstable fibre near θ for the map $\mathcal{P}_n$. An analogous argument shows that there is a fixed stable fibre near θ for $\mathcal{P}_n$ and n negative with large absolute value. Thus the intersection—which is non-empty from the local product structure of the flow—of these fixed fibers provides a periodic orbit of the geodesic flow intersecting S_θ.

The proof of the transitivity is obtained in the same way as for basic sets of Axiom A systems, using the local product structure and the fact that every point is non-wandering for the geodesic flow of M. □

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