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The Matroid Ramsey Number $n(6,6)$

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A tight upper bound on the number of elements in a connected matroid with fixed rank and largest cocircuit size is given. This upper bound is used to show that a connected matroid with at least thirteen elements contains either a circuit or a cocircuit with at least six elements. In the language of matroid Ramsey numbers, $n(6,6) = 13$: this is the largest currently known matroid Ramsey number.

1. Introduction

Extremal matroid theory is concerned with determining the maximum number of elements in matroids with fixed rank and prescribed properties (see [5] for a comprehensive survey). Our first result is an extremal matroid theorem where the prescribed property considered is bounded cocircuit size.

Theorem 1.1. *Let M be a matroid of rank $r > 0$ with ω connected components and largest cocircuit size b.*

(i) *If b is even, then M has at most $\frac{1}{2}(r+\omega)b$ elements.*
(ii) *If b is odd, then M has at most $\frac{1}{2}(r+\omega)(b-1)+\omega$ elements.*

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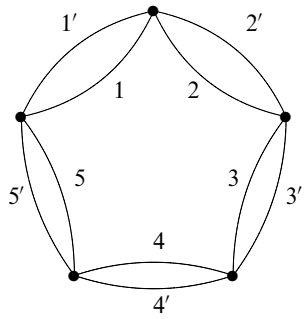


Figure 1

Several observations are in order. First, these upper bounds are tight. To see this, it suffices to focus on $\omega = 1$ (connected matroids) since, from examples of equality for $\omega = 1$, examples for arbitrary ω can be constructed using direct sums. First consider even integers b. Apply parallel extensions to the uniform matroid $U_{r,r+1}$ so all parallel classes in the resulting matroid $M_{r,b}$ have size $b/2$. (Figure 1 is a graphic representation of $M_{4,4}$.) There are $(r+1)b/2$ elements in $M_{r,b}$ and all cocircuits have b elements, illustrating equality in part (i). For odd integers b, equality holds in part (ii) for all connected rank-r single-element extensions of $M_{r,b-1}$.

Next observe that the upper bound in Theorem 1.1 is similar to an upper bound that is quite easy to prove for *simple* matroids. Indeed, let M be a simple matroid of rank r on n elements in which cocircuits have at most b elements. For a basis B of M, consider the B-fundamental-circuit incidence matrix D of M (see Oxley [7]). Since M is simple, each of the $n-r$ columns of D has at least two 1s. Since cocircuits have at most b elements, each of the r rows of D has at most $b-1$ entries that are 1. Thus $2(n-r) \leqslant r(b-1)$, so n, the number of elements of M, is at most $r(b+1)/2$.

Although similar, the upper bound of $r(b+1)/2$ in the simple case and the upper bound in Theorem 1.1 are considerably different. Comparing these expressions, we see that $r(b+1)/2$ generally fails to be an upper bound when we consider non-simple matroids, and when it is an upper bound, it is generally not tight. Furthermore, the upper bound of $r(b+1)/2$ is not tight in the simple case.

Corollary 1.1 follows immediately from Theorem 1.1 since a rank-r matroid has at most r connected components.

Corollary 1.1. *A matroid of rank $r > 0$ with largest cocircuit size b has at most rb elements.* □

The matroids for which equality holds in Corollary 1.1 are direct sums of r copies of the uniform matroid $U_{1,b}$. As with the upper bound of $r(b+1)/2$ for simple matroids, there are several easy counting proofs of Corollary 1.1. However, it seems unlikely that Theorem 1.1 could be proven with such techniques. In particular, the assumption of connectivity in the crucial case $\omega = 1$ creates significant difficulties that counting appears unable to address.

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The bound in Theorem 1.1 plays a key role in the proof of Theorem 1.2, which is about matroid Ramsey numbers. These numbers, defined below, are so named because their basic properties are strikingly similar to those of the Ramsey numbers of graph theory (see [8]). Considerable attention has been focused recently on matroid Ramsey numbers (see, for example, [1, 2, 3, 4, 6, 8, 10]).

Definition. Let a and b be positive integers. The *matroid Ramsey number* $n(a,b)$ is the least positive integer n such that every connected matroid with n elements contains either a circuit with at least a elements or a cocircuit with at least b elements.

Ramsey numbers are of fundamental importance in graph theory because cliques and independent vertex sets are basic substructures of graphs. Likewise, matroid Ramsey numbers are of interest since they reflect fundamental substructures of matroids, namely circuits and cocircuits. Several matroid Ramsey numbers have been determined: for example, that $n(5,6)$ is 11 was proved by Hurst and Reid [3]. The next result gives the largest currently known matroid Ramsey number.

Theorem 1.2. $n(6,6) = 13$.

The best upper bound known for matroid Ramsey numbers is $n(a,b) \leqslant \binom{a+b-4}{a-2} - 1$ for $a, b \geqslant 4$. This bound is exponential. The fundamental question is whether $n(a,b)$ is polynomial in a and b. Theorems 1.1 and 1.2 support the conjecture that $n(a,b) \leqslant \frac{1}{2}(a-1)(b-1)+1$ for $(a,b) \neq (2,2)$. Wu [10] proves this last inequality in the case of connected graphic matroids, thereby providing further support for the conjecture.

We follow the terminology of Oxley [7]. In particular, the ground set of a matroid M is denoted by $E(M)$, the rank of M is denoted by $r(M)$, and the closure of a set $X \subseteq E(M)$ is denoted by $cl_M(X)$. The deletion and contraction of $e \in E(M)$ from M are denoted by $M\backslash e$ and M/e, respectively. The fundamental circuit in M of an element x with respect to a basis B is denoted $C_M(x,B)$.

Recall that a matroid M is connected if, for each pair of distinct elements of M, some circuit contains both elements. If M is connected and, for all $e \in E(M)$, the contraction M/e is not connected, then we say M is *contraction-minimally connected*.

We say a set $X' \subset E(M)$ is a *parallel set* of $X \subset E(M)$ if X contains no parallel elements and there is a bijection $x \mapsto x'$ of X onto X' such that each $x \in X$ is parallel to its image x'. In particular, X and X' are disjoint. We will make use of the following obvious result.

Lemma 1.1. *Let M be a connected matroid with basis B such that B' is a parallel set of B and $E(M) - (B \cup B')$ consists of a single parallel class P. Then, for each x in P, $C_M(x,B) = B \cup x$.* □

2. The proofs

We begin this section by recalling some results we will use in the proofs of Theorems 1.1 and 1.2. The fundamental result on matroid connectivity is due to Tutte [9].

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Lemma 2.1. *Let e be an element of a connected matroid M. Then $M\backslash e$ or M/e is connected.*

The following result of Oxley [7, Corollary 10.2.3] is a generalization of an earlier result of Seymour.

Lemma 2.2. *Let M be a contraction-minimally connected matroid that is not a cocircuit. Then M has a basis B and an element $f \notin B$ not parallel to any element of B, so that $B \cup f$ has a parallel set $B' \cup f'$.*

The next result of Kahn and Seymour (see Oxley [7, Lemma 10.2.4]) is a key part of the proof of Theorem 1.1.

Lemma 2.3. *Let M be a connected simple matroid with rank $r \geqslant 2$. Let $X = \{x \in E(M) : M/x \text{ is disconnected}\}$. Then $|X| \leqslant r - 2$.*

Proof of Theorem 1.1. The result for disconnected matroids (*i.e.*, $\omega \geqslant 2$) follows immediately from the result for connected matroids. Thus assume M is connected. We induct on r.

If $r = 1$, then $M \cong U_{1,b}$, so $|E(M)| = b = \frac{1}{2}(r+1)b = \frac{1}{2}(r+1)(b-1) + 1$. Thus the result holds. Note that, since M is connected, b can be 1 only when r is 1, so we may now assume b exceeds 1.

Suppose that r exceeds one and the result holds for connected matroids of rank $r-1$. Assume first that there is an element $e \in E(M)$ for which M/e is connected. Note that cocircuits of M/e have at most b elements. If b is even, then $b \geqslant 2$ and

$$\begin{aligned} |E(M)| &= |E(M/e)| + 1 \\ &\leqslant \frac{1}{2}rb + 1 \\ &\leqslant \frac{1}{2}(r+1)b. \end{aligned}$$

If b is odd, then, since $b \geqslant 3$,

$$\begin{aligned} |E(M)| &= |E(M/e)| + 1 \\ &\leqslant \frac{1}{2}r(b-1) + 1 + 1 \\ &\leqslant \frac{1}{2}(r+1)(b-1) + 1. \end{aligned}$$

Hence we may assume that M is contraction-minimally connected. From Lemma 2.2, there is a basis B of M, each element of which is in a non-trivial parallel class.

Let N be a simplification of M whose ground set contains B. By Lemma 2.3, there are two elements, say b_1 and b_2, of B such that N/b_1 and N/b_2 are connected. It follows that $M/b_i\backslash(cl_M(b_i) - b_i)$ is connected for $i \in \{1, 2\}$.

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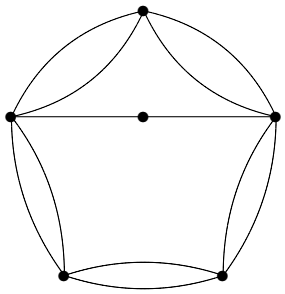


Figure 2

Suppose b is even. Assume first that $|cl_M(b_1)| \leqslant \frac{b}{2}$. Then

$$\begin{aligned} |E(M)| &= |E(M/b_1\backslash(cl_M(b_1)-b_1))| + |cl_M(b_1)| \\ &\leqslant \frac{1}{2}rb + |cl_M(b_1)| \\ &\leqslant \frac{1}{2}rb + \frac{b}{2} \\ &= \frac{1}{2}(r+1)b. \end{aligned}$$

Hence we may assume that $|cl_M(b_1)| > \frac{b}{2}$, and likewise $|cl_M(b_2)| > \frac{b}{2}$. Since M is connected, there is a cocircuit C^* of M containing b_1 and b_2. Since no circuit and cocircuit can intersect in a single element, $cl_M(b_1) \cup cl_M(b_2) \subseteq C^*$. Thus

$$\begin{aligned} |C^*| &\geqslant |cl_M(b_1)| + |cl_M(b_2)| \\ &> \frac{b}{2} + \frac{b}{2} \\ &= b, \end{aligned}$$

a contradiction.

Suppose b is odd. Assume first that $|cl_M(b_1)| \leqslant \frac{b-1}{2}$. Then

$$\begin{aligned} |E(M)| &= |E(M/b_1\backslash(cl_M(b_1)-b_1))| + |cl_M(b_1)| \\ &\leqslant \frac{1}{2}r(b-1) + 1 + \frac{1}{2}(b-1) \\ &= \frac{1}{2}(r+1)(b-1) + 1. \end{aligned}$$

Hence we may assume that $|cl_M(b_1)| \geqslant \frac{1}{2}(b+1)$, and likewise $|cl_M(b_2)| \geqslant \frac{1}{2}(b+1)$. It follows, as in the previous paragraph, that M has a cocircuit containing at least $\frac{1}{2}(b+1) + \frac{1}{2}(b+1) = b+1$ elements, a contradiction. Hence the result holds by induction. □

Proof of Theorem 1.2. The cycle matroid of the graph in Figure 2 has 12 elements; its circuits and cocircuits have at most 5 elements. Thus $n(6, 6) > 12$.

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Suppose M is a connected matroid with 13 elements and all circuits and cocircuits have at most 5 elements. We will obtain a contradiction, thereby showing that $n(6,6) \leqslant 13$, and completing the proof. We may assume that $r(M) \leqslant r(M^*)$ by duality. From Theorem 1.1(ii), we have $|E(M)| \leqslant \frac{1}{2}(r(M)+1)(5-1)+1$, or $13 \leqslant 2r(M)+3$. Thus $r(M) \geqslant 5$. Hence $r(M) \in \{5,6\}$.

Assume $r(M) = 5$. Suppose there is an element $e \in E(M)$ such that M/e is connected. By Theorem 1.1(ii), $|E(M/e)| \leqslant \frac{1}{2}(4+1)(5-1)+1$, or $12 \leqslant 11$, a contradiction. Thus M is contraction-minimally connected. By Lemma 2.2, there is a basis B of M and an element $f \in E(M)\backslash B$ so that $B \cup f$ has a parallel set $B' \cup f'$. Let g be the unique element of M not in $B \cup f$ or this parallel set. Since M/g is not connected, $M\backslash g$ is connected. By Lemma 1.1, $C_{M\backslash g}(f,B) = B \cup f$. Hence $B \cup f$ is a 6-element circuit of M, a contradiction.

We may therefore assume $r(M) = 6$. If M were contraction-minimally connected, then, by Lemma 2.2, there would be a 7-element set in M with a parallel set, contradicting $|E(M)| = 13$. Hence there is an element $e \in E(M)$ such that M/e is connected.

Suppose M/e is contraction-minimally connected. By Lemma 2.2, there is a basis B of M/e and an element $g \in E(M/e)\backslash B$ such that $B \cup g$ has a parallel set $B' \cup g'$. Hence $E(M/e) = B \cup B' \cup \{g, g'\}$. By Lemma 1.1, $C_{M/e}(g,B) = B \cup g$. Hence $B \cup g$ or $B \cup \{e,g\}$ is a circuit of M with more than 5 elements, a contradiction. Thus there is an element $f \in E(M/e)$ such that $M/e,f$ is connected.

Suppose there is an element $g \in E(M/e,f)$ with $M/e,f,g$ connected. The connectivity of M/e and $M/e,f$ imply that $\{e,f,g\}$ is independent in M. Hence $M/e,f,g$ has rank 3. By Theorem 1.1(ii), $10 = |E(M/e,f,g)| \leqslant \frac{1}{2}(3+1)(5-1)+1$, a contradiction. Thus $M/e,f$ is contraction-minimally connected.

Lemma 2.2 implies that there is a basis $B = \{1,2,3,4\}$ of $M/e,f$ and element $5 \in E(M)\backslash B$ such that $B \cup 5$ has a parallel set $B' \cup 5'$. Let g be the unique element of $M/e,f$ not in $B \cup B' \cup \{5,5'\}$. By the previous paragraph, $M/e,f,g$ is not connected, so by Lemma 2.1, $M/e,f\backslash g$ is connected. From Lemma 1.1, $C_{M/e,f\backslash g}(5,B) = B \cup 5$. Hence $M/e,f\backslash g$ is the cycle matroid of the graph given in Figure 1.

Suppose $M/f\backslash g$ is connected. Let C be a circuit of $M/f\backslash g$ containing 1 and e. Then $C\backslash e$ is a circuit of $M/e,f\backslash g$ containing 1. Since $|C| \leqslant 5$, we have $|C\backslash e| \leqslant 4$. Since $\{1,1'\}$ is the only circuit of $M/e,f\backslash g$ containing 1 with fewer than five elements, $C\backslash e = \{1,1'\}$. Thus $\{1,1',e\}$ is a circuit of $M/f\backslash g$. By symmetry, $\{i,i',e\}$ is a circuit of $M/f\backslash g$ for $1 \leqslant i \leqslant 5$. Let C^* be a cocircuit of $M/f\backslash g$ containing 1 and e. Since the circuit $\{i,i',e\}$ and the cocircuit C^* both contain e, we have $|C^* \cap \{i,i',e\}| \geqslant 2$ for $1 \leqslant i \leqslant 5$. Thus C^* is a cocircuit of $M/f\backslash g$ with at least 6 elements, a contradiction. Thus $M/f\backslash g$ is disconnected. Likewise, $M/e\backslash g$ is disconnected. But $M/e,f\backslash g$ is connected. Hence $\{e,g\}$ and $\{f,g\}$ are cocircuits of M.

Let C_1 be a circuit of M containing g and 1. Since the circuit C_1 has nonempty intersection with both cocircuits $\{e,g\}$ and $\{f,g\}$, we have $\{e,f,g,1\} \subseteq C_1$. Likewise, for $1 \leqslant i \leqslant 5$ there are circuits C_i of M with $\{e,f,g,i\} \subseteq C_i$. Each circuit C_i has 4 or 5 elements. Suppose $|C_1| = 4$. Then $M/e,f$ is the cycle matroid of the graph in Figure 1 with edge g added in parallel to edges 1 and 1′. Hence all circuits of $M/e,f$ containing g and 2 have five elements. Thus $|C_2| = 7$, a contradiction. Thus $|C_1| = 5$. Likewise, $|C_i| = 5$ for $1 \leqslant i \leqslant 5$. Moreover, $\{g\}$ is a rank-1 flat of $M/e,f$ and $i' \notin C_i$. Therefore $C_i - \{e,f\}$

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spans a line L_i of $M/e,f$ containing at least three rank-1 flats. Since $L_i - g$ is a line of $M/e,f\backslash g$, it follows that $L_i = \{g,i,i',j,j'\}$ for some $j \neq i$ in $\{1,2,3,4,5\}$. Thus the lines through g pair off the five sets $\{i,i'\}$, which is impossible since 5 is odd. This contradiction completes the proof of Theorem 1.2. □

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