Math. Proc. Camb. Phil. Soc. (1992), **112**, 165
Printed in Great Britain

Duals of vector-valued Köthe function spaces

BY MIGUEL FLORENCIO, PEDRO J. PAÚL AND CARMEN SÁEZ
E.S. Ingenieros Industriales, Avda, Reina Mercedes s/n, 41012 *Sevilla, Spain*

(*Received* 1 *August* 1991; *revised* 24 *October* 1991)

Abstract

Let Λ be a perfect Köthe function space in the sense of Dieudonné, and $\Lambda^\times$ its Köthe-dual. Let E be a normed space. Then the topological dual of the space $\Lambda(E)$ of Λ-Bochner integrable functions equals the corresponding $\Lambda^\times(E')$ if and only if E' has the Radon–Nikodým property. We also give some results concerning barrelledness for spaces of this kind.

1. *Introduction*

The purpose of this paper is to study several results concerning vector function spaces with values in a normed space. These spaces are obtained as a generalization of the Köthe function spaces studied by Dieudonné [**3**].

Let Λ be a Köthe space defined on a measure space X and $\Lambda^\times$ its Köthe-dual, and let E be a normed space with dual E'. We define the space $\Lambda(E)$ of Λ-Bochner integrable functions as the space of (classes of) strongly measurable functions $f: X \to E$ such that the composition $\|f(\cdot)\|$ is a function in Λ. There is a natural topology on $\Lambda(E)$ defined from the topology of Λ and the norm on E, and the locally convex structure of these spaces has been studied by Clauzure [**1**], Jordán Lluch and Torregrosa Sánchez [**6**], Macdonald [**8**], and Phuong-Các [**9**], among others.

Macdonald [**8**] provides a characterization of the dual of $\Lambda(E)$ when E is a locally convex Hausdorff space and the topology on Λ is compatible with the dual pair $(\Lambda, \Lambda^\times)$. This characterization is not easy to use even when E is a normed space, in which case it reads as follows: the topological dual of $\Lambda(E)$ can be identified with the set of all $\sigma(E', E)$-measurable functions $g: X \to E'$ satisfying that for every compact set $K \subset X$, the restriction $g|_K$ can be written as $g|_K = \phi \cdot h$ where h is a $\sigma(E', E)$-measurable function such that $\|h(\cdot)\| \leqslant 1$ almost everywhere and ϕ is in $\Lambda^\times$.

In some particular cases, e.g. when E is a reflexive Banach space or E is a Banach space with separable dual, the characterization above can be considerably simplified, the result reducing to $\Lambda(E)' = \Lambda^\times(E')$. This situation is much more manageable and comparable with the case of vector-valued sequence spaces studied by Rosier [**10**]. It seems, then, desirable to characterize those normed spaces E for which the dual of $\Lambda(E)$ equals $\Lambda^\times(E')$. For the case of Bochner integrable functions it is well-known that the characterization of the dual of $L^1(E)$ as $L^\infty(E')$ is equivalent to E' having the Radon–Nikodým property ([**2**], IV·1, theorem 1, p. 98). This is also the case when Λ is a Fréchet echelon space and E is a Banach space [**6**]. Here we prove that this characterization holds in general. In the last section, we study the permanence

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

of some properties related to barrelledness. Concerning the references given above, we think that this paper can be considered as a continuation of N. Phuong-Các's paper [9].

2. *Terminology and notation*

We refer the reader to Köthe's monograph [7] and to the book [2] of Diestel and Uhl for the terminology used in this paper.

Let X be a locally compact and Hausdorff topological space and μ be a non-negative Radon measure on X. We assume that X is countable at infinity, so that we can write $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is compact and $X_n \subset \operatorname{int}(X_{n+1})$. We shall always identify those functions that are equal almost everywhere with respect to μ (μ-a.e. in short).

We denote by Ω the space of all locally integrable real-valued functions defined on X, and by Φ the subspace of all measurable essentially bounded functions with compact support. When X is compact we obtain the spaces $L^1(\mu)$ and $L^\infty(\mu)$ respectively.

For a subset $\Lambda \subset \Omega$ its Köthe-dual is the set $\Lambda^\times$ of all $f \in \Omega$ such that $f \cdot g$ is integrable for each $g \in \Lambda$. We have $\Lambda \subset \Lambda^{\times\times} = (\Lambda^\times)^\times$, and Λ is said to be a perfect space if $\Lambda = \Lambda^{\times\times}$. When Λ is a subspace of Ω containing Φ, in particular if Λ is perfect, then Λ and $\Lambda^\times$ are put into duality by the canonical bilinear form on $\Lambda \times \Lambda^\times$:

$$(f, g) \mapsto \langle f, g \rangle = \int_X f(t) \cdot g(t)\, d\mu(t).$$

A subset H of Ω is said to be normal (or solid) if whenever H contains f it also contains all $g \in \Omega$ satisfying $|g(t)| \leqslant |f(t)|$ μ-a.e. The strong topology $\beta(\Lambda, \Lambda^\times)$ on Λ equals the polar topology on Λ generated by the family $\mathscr{B}$ of all normal and $\sigma(\Lambda^\times, \Lambda)$-bounded subsets of $\Lambda^\times$, and it is given by the seminorms

$$p_M : f \in \Lambda \mapsto p_M(f) := \sup\left\{ \int_X |f(t) \cdot g(t)|\, d\mu(t) : g \in M \right\} \quad (M \in \mathscr{B}).$$

In what follows, Λ remains a perfect Köthe space endowed with its strong topology $\beta(\Lambda, \Lambda^\times)$. We refer the reader to Dieudonné's paper [3] for additional information about Köthe function spaces.

Let E be a normed space with dual E'. We define the space $\Lambda(E)$ of Λ-Bochner integrable functions as the space of (classes of) strongly measurable functions $f : X \to E$ such that the composition $\|f(\cdot)\|$ is a function in Λ.

Starting from the strong topology $\beta(\Lambda, \Lambda^\times)$ on Λ and the norm on E, we can generate in $\Lambda(E)$ a natural topology defined by the seminorms

$$q_M(f) := p_M(\|f(\cdot)\|) = \sup\left\{ \int_X \|f(t)\| \cdot |g(t)|\, d\mu(t) : g \in M \right\} \quad (M \in \mathscr{B}).$$

We will refer to this topology as the natural topology of $\Lambda(E)$. When $\Lambda = L^p(\mu)$ with $1 \leqslant p \leqslant +\infty$, we obtain the space $L^p(\mu, E)$ of p-Bochner integrable functions (see [2], IV·1, p. 97). In particular, $L^\infty(\mu, E)$ is the space of strongly measurable, essentially bounded functions from X into E with the ess-sup norm. Denote by B the unit ball of $L^\infty(\mu, E)$; thus a function $h : X \to E$ is in B if it is μ-measurable

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

and $\|h(t)\| \leqslant 1$ μ-a.e. Note that if $\phi \in \Lambda$ and $h \in B$ then the function f given by $f(t) := \phi(t).h(t)$ is in $\Lambda(E)$. On the other hand, every function $f \in \Lambda(E)$ can be decomposed in the same way, namely $\phi(t) = \|f(t)\|$ and

$$h(t) = \begin{cases} f(t).\|f(t)\|^{-1} & \text{if } \|f(t)\| \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using this decomposition and the form of the seminorms q_M, it is easy to check that the bounded sets of $\Lambda(E)$ are obtained from the bounded sets of Λ by multiplication with B, i.e. a set $S \subset \Lambda(E)$ is bounded if and only if there is a bounded set $R \subset \Lambda$ such that $S \subset R.B$.

Two properties of these spaces that we shall use are the fact that the space of compact support simple functions is dense on $\Lambda(E)$ for the natural topology ([**9**], proposition 1), and that if E is a Banach space, then $\Lambda(E)$ is complete ([**9**], proposition 2).

3. *A characterization of the dual of vector-valued Köthe function spaces*

We say that a Banach space E has the Radon–Nikodým property with respect to μ if whenever $G: X \to E$ is a σ-additive vector measure that has bounded variation and vanishes on sets of measure zero, there exists a Bochner integrable function $g \in L^1(\mu, E)$ such that

$$G(A) = \int_A g \, d\mu.$$

In the sequel, we will use the following lemma which was stated in [**4**].

LEMMA. *Let $g: X \to E'$ and $\epsilon: X \to \mathbb{R}$ be μ-measurable functions, ϵ in addition strictly positive. Then there exists a μ-measurable function $y: X \to E$ such that*

(1) *y is countably valued with values in the unit ball of E, and*

(2) *$\|g(t)\| \leqslant \langle g(t), y(t)\rangle + \epsilon(t)$ μ-a.e. in X.*

THEOREM 1. *Assume that the dual of $\Lambda(\beta(\Lambda, \Lambda^\times))$ is $\Lambda^\times$. Then $\Lambda(E)' = \Lambda^\times(E')$ if and only if E' has the Radon–Nikodým property with respect to μ.*

Proof. We start by noting that $\Lambda^\times(E')$ can always be identified with a subspace of $\Lambda(E)'$. Indeed, for $g \in \Lambda^\times(E')$ consider the element

$$f \mapsto \langle g, f\rangle := \int_X \langle g(t), f(t)\rangle \, d\mu(t)$$

of $\Lambda(E)'$. We have

$$|\langle g, f\rangle| \leqslant \int_X |\langle g(t), f(t)\rangle| \, d\mu(t) \leqslant \int_X \|g(t)\| \, \|f(t)\| \, d\mu(t) \leqslant q_M(f),$$

where M is a set in $\mathscr{B}$ such that $\|g(\cdot)\| \in M$.

For the implication in one direction, assume that E' has the Radon–Nikodým property with respect to μ. First, we deal with the case when X is compact. Let u be a continuous linear form on $\Lambda(E)$. Then there is some $M \in \mathscr{B}$ such that $|u(f)| \leqslant q_M(f)$ for all $f \in \Lambda(E)$. For each measurable set A, define $G(A)$ by

$$\langle G(A), x\rangle := u(x.\chi_A), \quad \text{for } x \in E.$$

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

Since $|\langle G(A), x\rangle| = |u(x.\chi_A)| \leqslant q_M(x.\chi_A) = p_M(\chi_A).\|x\|$, G defines a vector measure on E'. Clearly $G(A) = 0$ whenever $\mu(A) = 0$. We prove now that G is σ-additive. Let (A_n) be a sequence of disjoint measurable sets with union A. For each $n \in \mathbb{N}$, the function

$$\phi_n(t) = \chi_A(t) - \sum_{i=1}^{n} \chi_{A_i}(t)$$

is in Λ because X is compact and $\Phi \subset \Lambda$. Moreover, the sequence (ϕ_n) decreases to zero. Now, by hypothesis, $\beta(\Lambda, \Lambda^\times)$ equals the Mackey topology $\tau(\Lambda, \Lambda^\times)$ and $\phi_n \to 0$ in this topology (see [9], lemma 1). Therefore for all $x \in E$ we have

$$\left|\left\langle G(A) - \sum_{i=1}^{n} G(A_i), x \right\rangle\right| = |u(\phi_n.x)| \leqslant q_M(\phi_n.x) = p_M(\phi_n).\|x\|.$$

As $n \to \infty$, the latter expression converges to zero uniformly on the unit ball of E. Hence $G(A) = \sum_{n=1}^{\infty} G(A_n)$ in the dual norm of E'.

Let us show now that G has bounded variation. Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of X and take arbitrary points $x_1, x_2, \ldots, x_n$ in the unit ball of E. Take scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ of modulus 1 such that

$$|\langle G(A_i), x_i\rangle| = \langle G(A_i), \alpha_i x_i\rangle \quad (1 \leqslant i \leqslant n).$$

Then every $z_i = \alpha_i x_i (1 \leqslant i \leqslant n)$ is in the unit ball of E and we have

$$\sum_{i=1}^{n} |\langle G(A_i), x_i\rangle| = \left|\sum_{i=1}^{n} \langle G(A_i), z_i\rangle\right| = \left|u\left(\sum_{i=1}^{n} z_i.\chi_{A_i}\right)\right|$$

$$\leqslant q_M\left(\sum_{i=1}^{n} z_i.\chi_{A_i}\right) \leqslant p_M\left(\sum_{i=1}^{n} \chi_{A_i}\right) = p_M(\chi_X).$$

Taking appropriate suprema shows that G has bounded variation. Since E' has the Radon–Nikodým property with respect to μ, there is $g \in L^1(\mu, E')$ such that

$$G(A) = \int_A g\, d\mu.$$

Let (B_n) be an increasing sequence of measurable sets such that $X = \bigcup_{n=1}^{\infty} B_n$ and g is bounded on each B_n. For each $n \in \mathbb{N}$, consider the functional u_n defined by

$$u_n(f) = \int_{B_n} \langle g(t), f(t)\rangle\, d\mu(t) \quad \text{for } f \in \Lambda(E).$$

This functional u_n is linear. It is also continuous because g is bounded on B_n so that $\|\chi_{B_n}(\cdot)g(\cdot)\| \in \Phi \subset \Lambda^\times$. Clearly u_n agrees with u on all simple functions supported on B_n. Fix $f \in \Lambda(E)$. Using again [9], lemma 1 to ensure that $f.\chi_{B_n} \to f$ in $\Lambda(E)$, and the fact that simple functions are dense in $\Lambda(E)$, we have

$$u(f) = \lim_n u(f.\chi_{B_n}) = \lim_n u_n(f.\chi_{B_n}) = \lim_n \int_X \langle g(t), f(t).\chi_{B_n}(t)\rangle\, d\mu(t).$$

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

Since Λ is solid, we can apply the equality above to $f.\phi$ for each scalar function $\phi \in L^\infty(\mu)$, and this enables us to use the Monotone Convergence Theorem to deduce that the function $|\langle g(\cdot), f(\cdot)\rangle|$ is integrable and, in fact,

$$u(f) = \int_X \langle g(\cdot), f(\cdot)\rangle \, d\mu(t).$$

We now consider the σ-compact case. Write $X = \bigcup_{n=1}^\infty X_n$ where each X_n is compact and $X_n \subset \operatorname{int}(X_{n+1})$. Given $u \in \Lambda(E)'$ and $n \in \mathbb{N}$, let u_n be the restriction of u to the space $\Lambda_{X_n}(E) := \{f.\chi_{X_n} : f \in \Lambda(E)\}$. This is a complemented subspace in $\Lambda(E)$. Then $u_n \in \Lambda_{X_n}(E)'$. Therefore, as proved above, there is a function $g_n \in L^1_{X_n}(E')$ such that

$$u_n(f.\chi_{X_n}) = \int_{X_n} \langle g_n(t), f(t)\rangle \, d\mu(t), \quad \text{for } f \in \Lambda(E).$$

Define $g := \sum_{n=1}^\infty g_n \cdot \chi_{Y_n}$ where $Y_1 = X_1$ and $Y_n = X_n \backslash X_{n-1}$ for $n \geqslant 2$. This function g is μ-measurable. For $f \in \Lambda(E)$ we have, reasoning as before, that $|\langle g(\cdot), f(\cdot)\rangle|$ is in L^1 and

$$u(f) = \int_X \langle g(t), f(t)\rangle \, d\mu(t).$$

Finally, take an arbitrary function $h \in \Lambda$, and apply the Lemma above to $h.g$ and the positive function $\epsilon(t) = \sum_{n=1}^\infty 2^{-n} \chi_{Y_n}(t) \, (\mu(Y_n) + 1)^{-1}$. Then there is a μ-measurable function y from X into the unit ball of E such that

$$\|h(t).g(t)\| \leqslant \langle h(t).g(t), y(t)\rangle + \epsilon(t) \quad \mu\text{-a.e.}$$

Consequently, we have

$$\int_X |h(t)| . \|g(t)\| \, d\mu(t) \leqslant \int_X \langle g(t), y(t).h(t)\rangle \, d\mu(t) + \mu(X) < +\infty.$$

This proves that $\|g(\cdot)\| \in \Lambda^\times$ and, therefore, that $g \in \Lambda^\times(E')$.

For the opposite implication, suppose that $\Lambda(E)' = \Lambda^\times(E')$. Again we assume, for the first part of the proof, that X is compact. Let G be a σ-finite vector measure, having bounded variation on E' that vanishes on μ-null sets. Its variation $|G|$ is σ-additive, finite and also vanishes on μ-null sets. By the classical Radon–Nikodým Theorem, there is $h \in L^1(\mu)$ such that h is non-negative and such that for each measurable set A we have

$$|G|(A) = \int_A h \, d\mu.$$

Let (B_n) be a sequence of disjoint measurable sets covering X and such that h is bounded on each B_n. For each $n \in \mathbb{N}$, define u_n as follows:

$$u_n\left(\sum_{i=1}^k x_i \cdot \chi_{A_i}\right) = \sum_{i=1}^k \langle G(A_i \cap B_n), x_i\rangle.$$

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

A straightforward computation shows that u_n is a continuous linear functional on the space of simple functions. By the Hahn–Banach Theorem, u_n has a continuous linear extension to the whole of $\Lambda(E)$. Since the dual of $\Lambda(E)$ is $\Lambda^\times(E')$, for this extension, that we still call u_n, there is $g_n \in \Lambda^\times(E') \subset L^1(\mu, E')$ such that

$$u_n(f) = \int_X \langle g_n(t), f(t) \rangle \, d\mu(t)$$

for every $f \in \Lambda(E)$; hence for every measurable set A, we have

$$G(A \cap B_n) = \int_{A \cap B_n} g_n \, d\mu.$$

Applying the Lemma above to $h_n := \sum_{i=1}^n g_i \cdot \chi_{B_i}$ and $\epsilon \equiv 1$, we obtain a countably valued μ-measurable function y from X into the unit ball of E, and we write $y = \sum_{j=1}^\infty x_j \cdot \chi_{C_j}$, such that

$$\|h_n(t)\| \leqslant \langle h_n(t), y(t) \rangle + \epsilon(t), \quad \mu\text{-a.e.}$$

Writing $B = \bigcup_{i=1}^n B_i$, we have

$$\begin{aligned} \sum_{i=1}^n \int_X \|g_i(t) \cdot \chi_{B_i}(t)\| \, d\mu(t) = \int_B \|h_n(t)\| \, d\mu(t) &\leqslant \sum_{j=1}^\infty \int_{B \cap C_j} \langle h_n(t), x_j \rangle \, d\mu(t) + \mu(B) \\ &= \sum_{j=1}^\infty \langle G(B \cap C_j), x_j \rangle + \mu(B) \leqslant |G|(B) + \mu(B) \\ &\leqslant \int_X h(t) \, d\mu(t) + \mu(X). \end{aligned}$$

Now, define the function g from X to E' by

$$g(t) = \sum_{n=1}^\infty g_n(t) \cdot \chi_{B_n}(t) \in E' \quad \text{for all } t \in X.$$

Then g is μ-measurable, and, by applying the Monotone Convergence Theorem to the chain of inequalities above, we have $\|g(\cdot)\| \in L^1(\mu)$. Moreover, given a measurable set A, we have

$$G(A) = \sum_{n=1}^\infty G(A \cap B_n) = \sum_{n=1}^\infty \int_{A \cap B_n} g \, d\mu$$

and by the Dominated Convergence Theorem

$$G(A) = \int_A g \, d\mu.$$

Finally, let $X = \bigcup_{n=1}^\infty X_n$ where each X_n is compact and $X_n \subset \operatorname{int}(X_{n+1})$. Given a σ-finite vector measure G on E' that vanishes on μ-null sets and has bounded

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

variation, let G_n be its restriction to the corresponding σ-algebra in X_n. Since it is clear from our hypothesis that $(\Lambda_{X_n}(E))' = (\Lambda_{X_n})^\times(E')$, we can apply the case above to obtain $g \in L^1_{X_n}(E')$ such that

$$G_n(A) = \int_A g_n \, d\mu.$$

Write $Y_1 = X_1$ and $Y_n = X_n \backslash X_{n-1}$ for $n \geqslant 2$. Define $g := \sum_n g_n \cdot \chi_{Y_n}$. Then g is μ-measurable and

$$G(A) = \sum_{n=1}^{\infty} G_n(A \cap Y_n) = \sum_{n=1}^{\infty} \int_{A \cap Y_n} g_n \, d\mu = \sum_{n=1}^{\infty} \int_{A \cap Y_n} g \, d\mu.$$

Moreover

$$\int_{X_n} \|g(t)\| \, d\mu(t) = |G|(X_n) \leqslant |G|(X) < +\infty.$$

Then, by the Monotone Convergence Theorem, $\|g(\cdot)\| \in L^1$ and, in fact,

$$G(A) = \int_A g \, d\mu.$$

Thus $g \in L^1(\mu, E')$ is the integrable function representing G. ∎

THEOREM 2. *Assume that the dual of $\Lambda(\beta(\Lambda, \Lambda^\times))$ is $\Lambda^\times$. If E' has the Radon–Nikodým property with respect to μ then*

(1) *a set $H \subset \Lambda(E)' = \Lambda^\times(E')$ is equicontinuous if and only if there is $M \in \mathscr{B}$, the family of all normal and $\sigma(\Lambda^\times, \Lambda)$-bounded subsets of $\Lambda^\times$, such that $H \subset M.U$, where U is the unit ball of $L^\infty(\mu, E')$, and*

(2) *the strong topology $\beta(\Lambda^\times(E'), \Lambda(E))$ is the natural topology of the space $\Lambda^\times(E')$ when $\Lambda^\times$ is endowed with the strong topology $\beta(\Lambda^\times, \Lambda)$ and E' carries its dual norm.*

Proof. By Theorem 1 we know that $\Lambda(E)' = \Lambda^\times(E')$.

(1) Suppose that $H \subset M.U$ for some $M \in \mathscr{B}$. Then, of course, $H \subset \Lambda^\times(E')$ and for $f \in \Lambda(E)$ and $g \in H$ we have

$$|\langle g, f \rangle| \leqslant \int_X |\langle g(t), f(t) \rangle| \, d\mu(t) \leqslant \int_X \|g(t)\| \cdot \|f(t)\| \, d\mu(t) \leqslant q_M(f).$$

Conversely, if $H \subset \Lambda(E)'$ is an equicontinuous set, then there is an absolutely convex and closed set $M \in \mathscr{B}$ such that $|\langle g, f \rangle| \leqslant q_M(f)$ for all $f \in \Lambda(E)$ and $g \in H$. As we pointed out, every $g \in H$ can be written as $g = \|g\| \cdot h$ where $h \in U$, the unit ball of $L^\infty(\mu, E')$. Therefore it suffices to show that $\{\|g(\cdot)\| : g \in H\} \subset M$. Let $g \in H$, let ϕ be in the polar M° of M in Λ, and let $\alpha > 0$. By applying the Lemma above to $\phi \cdot g$ and $\epsilon(t) = \sum_{n=1}^{\infty} \alpha 2^{-n} \chi_{Y_n}(t) \, (\mu(Y_n) + 1)^{-1}$, where $Y_1 = X_1$ and $Y_n = X_n \backslash X_{n-1}$ for $n \geqslant 2$, we obtain a μ-measurable function y from X into the unit ball of E such that

$$\|\phi(t) \cdot g(t)\| \leqslant \langle \phi(t) \cdot g(t), y(t) \rangle + \epsilon(t), \quad \mu\text{-a.e.}$$

Now, by integration on X

$$\begin{aligned} \int_X |\phi(t)| \cdot \|g(t)\| \, d\mu(t) &\leqslant \int_X \langle g(t), \phi(t) \cdot y(t) \rangle \, d\mu(t) + \alpha \\ &\leqslant q_M(\phi \cdot y) + \alpha = p_M(\phi) + \alpha \leqslant 1 + \alpha. \end{aligned}$$

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

Since α was an arbitrary positive number, it follows that $\|g(\cdot)\| \in M^{\circ\circ} = M$.

(2) Let S be $\sigma(\Lambda(E), \Lambda^\times(E'))$-bounded. Since the dual of $\Lambda(E)$ equals $\Lambda^\times(E')$, we have that S is bounded for the natural topology of $\Lambda(E)$. Then $R = \{\|h(\cdot)\| : h \in S\}$ is $\sigma(\Lambda, \Lambda^\times)$-bounded and for $g \in \Lambda^\times(E')$ we have

$$\sup\left\{\left|\int_X \langle g(t), h(t)\rangle \, d\mu(t)\right| : h \in S\right\} \leqslant \sup\left\{\int_X \|g(t)\| \, . \, \|h(t)\| \, d\mu(t) : h \in S\right\} = q_R(g).$$

This proves that every strong seminorm in $\Lambda^\times(E')$ is continuous for the natural topology.

Conversely, given $R \subset A$ normal and $\sigma(\Lambda, \Lambda^\times)$-bounded, define $S := R.B$, where B is the unit ball of $L^\infty(\mu, E)$. Then S is $\sigma(\Lambda(E), \Lambda^\times(E'))$-bounded. Now, given $g \in \Lambda^\times(E')$ and $\alpha > 0$ there is some $\phi \in R$ such that

$$q_R(g) \leqslant \alpha + \int_X |\phi(t)| \, . \, \|g(t)\| \, d\mu(t).$$

By application of the Lemma above to $\phi.g$ and $\epsilon(t) = \sum_{n=1}^\infty \alpha 2^{-n} \chi_{Y_n}(t) \, (\mu(Y_n) + 1)^{-1}$, where $Y_1 = X_1$ and $Y_n = X_n \backslash X_{n-1}$ for $n \geqslant 2$, we obtain a μ-measurable function y from X into the unit ball of E such that

$$\|\phi(t) \, . \, g(t)\| \leqslant \langle \phi(t) \, . \, g(t), y(t)\rangle + \epsilon(t), \quad \mu\text{-a.e.}$$

Calling $h = \phi \, . \, y \in S$, we obtain

$$q_R(g) \leqslant 2\alpha + \int_X \langle g(t), h(t)\rangle \, d\mu(t) \leqslant 2\alpha + \sup\{|\langle g, f\rangle| : f \in S\}.$$

Since α was arbitrary, it follows that the seminorm q_R is dominated by the strong seminorm corresponding to S. ∎

4. *Other permanence properties*

In this last section, we study some properties of the spaces $\Lambda(E)$ related to barrelledness. Note that our first result requires weaker hypotheses than the theorems above.

THEOREM 3. (1) *If $\Lambda(\beta(\Lambda, \Lambda^\times))$ is quasi-barrelled, then $\Lambda(E)$ is also quasi-barrelled. Moreover, if E is a Banach space, then $\Lambda(E)$ is barrelled.*

(2) *If $\Lambda(\beta(\Lambda, \Lambda^\times))$ is bornological then $\Lambda(E)$ is also bornological.*

Proof. (1) Let N be a $\beta(\Lambda(E)', \Lambda(E))$-bounded subset of $\Lambda(E)'$. We have to prove that N is equicontinuous. For every $u \in N$ and every function $h \in B$, the unit ball of $L^\infty(\mu, E)$, consider the composition

$$T(u, h) : \phi \mapsto \phi \, . \, h \mapsto \langle u, \phi \, . \, h\rangle$$

from Λ to $\mathbb{R}$. It is clear that $T(u, h)$ is a continuous linear form on Λ. Now, if R is a bounded set in Λ, then $\{\phi \, . \, h : \phi \in R, h \in B\}$ is bounded in $\Lambda(E)$, and since N is strongly bounded, we have

$$\sup\{|\langle T(u, h), \phi\rangle| = |\langle u, \phi \, . \, h\rangle| : u \in N, h \in B, \phi \in R\} < +\infty.$$

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

This shows that

$$N^* := \{T(u,h) : u \in N, h \in B\}$$

is a $\beta(\Lambda', \Lambda)$-bounded subset of Λ', the topological dual of $\Lambda(\beta(\Lambda, \Lambda^\times))$. Since Λ is quasi-barrelled, it follows that N^* is equicontinuous. Therefore there is a set $M \in \mathscr{B}$ such that

$$\sup\{|\langle T(u,h), \phi\rangle| : u \in N, h \in B\} \leqslant p_M(\phi), \text{ for all } \phi \in \Lambda.$$

Finally, given $f \in \Lambda(E)$ write $f = \|f(\cdot)\| . h$ with $h \in B$. We have

$$\sup\{|\langle u, f\rangle| : u \in N\} \leqslant \sup\{|\langle T(u,h), \|f(\cdot)\|\rangle| : u \in N, h \in B\} \leqslant q_M(f)$$

and this proves that N is equicontinuous.

If E is a Banach space then $\Lambda(E)$ is complete from [9], proposition 2 and therefore it is a barrelled space from [7], §27·1·(1).

(2) Since Λ is bornological, it is quasi-barrelled and, as we have just proved, $\Lambda(E)$ is quasi-barrelled. Now, let u be a locally bounded linear form on $\Lambda(E)$. Using that the bounded sets of $\Lambda(E)$ can be lifted from the bounded sets in Λ multiplying by B, the unit ball of $L^\infty(\mu, E)$, it is easy to see that the set

$$C := \{\phi \in \Lambda : |u(\phi . h)| \leqslant 1 \text{ for all } h \in B\}$$

is absolutely convex and bornivorous in Λ. Since Λ is bornological, C is a neighbourhood of zero in Λ. Hence there is $M \in \mathscr{B}$ such that $M^\circ \subset C$. Therefore, for all $h \in B$ and $\phi \in \Lambda$, we have

$$|u(\phi . h)| \leqslant p_M(\phi).$$

Finally, for $f \in \Lambda(E)$ write $f = \|f(\cdot)\| . h$ with $h \in B$. Then

$$|u(f)| = |u(\|f(\cdot)\| . h)| \leqslant p_M(\|f(\cdot)\| . h) = q_M(f),$$

so that u is continuous. By [7], §28·1·(3), $\Lambda(E)$ is bornological. ∎

For a study of when $\Lambda(E)$ is a barrelled space when E is a normed space, not necessarily complete, we refer the interested reader to [5]. For our next result, recall that a locally convex space is said to be distinguished if its strong dual is a barrelled space.

THEOREM 4. *Assume that the dual of $\Lambda(\beta(\Lambda, \Lambda^\times))$ is $\Lambda^\times$. If Λ is distinguished and E' has the Radon–Nikodým property with respect to μ, then $\Lambda(E)$ is distinguished.*

Proof. By Theorems 1 and 2, the strong dual of $\Lambda(E)$ is the space $\Lambda^\times(E')$ endowed with its natural topology, when we consider in $\Lambda^\times$ the strong topology $\beta(\Lambda^\times, \Lambda)$. Since $\Lambda^\times(\beta(\Lambda^\times, \Lambda))$ is barrelled and E' is a Banach space, the space $\Lambda^\times(E')$ is barrelled, by Theorem 3. ∎

THEOREM 5. *If Λ and E are reflexive then $\Lambda(E)$ is reflexive.*

Proof. Assume that Λ and E are reflexive. Then, in particular, Λ is barrelled and E' has the Radon–Nikodým property with respect to μ. By Theorems 1 and 2, the strong dual of $\Lambda(E)$ is the space $\Lambda^\times(E')$ endowed with its natural topology, when $\Lambda^\times$ carries the strong topology $\beta(\Lambda^\times\Lambda)$.

Now the dual of $\Lambda^\times(\beta(\Lambda^\times, \Lambda))$ equals Λ because Λ is reflexive, and $E'' = E$ has the Radon–Nikodým property with respect to μ because it is reflexive. Therefore we can

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press

apply again Theorems 1 and 2 to $\Lambda^{\times}(E')$ and deduce that the dual of this space is precisely $\Lambda(E)$. Therefore $\Lambda(E)$ coincides with its bidual and, by Theorem 3, is barrelled. Hence $\Lambda(E)$ is reflexive. ∎

This research was partially supported by La Consejería de Educación y Ciencia de la Junta de Andalucía.

REFERENCES

[1] P. CLAUZURE. Contribution a l'étude topologique et geometrique d'espaces de type Köthe–Orlicz. These, Université des Ciencies et Techniques du Languedoc, Montpellier (1985).

[2] J. DIESTEL and J. J. UHL, JR. *Vector Measures*. Mathematical Surveys and Monographs no. 15 (American Mathematical Society, 1977).

[3] J. DIEUDONNÉ. Sur les espaces de Köthe. *J. Analyse Math.* **1** (1951), 81–115.

[4] M. FLORENCIO, P. J. PAÚL and C. SÁEZ. A note on the Köthe dual of Banach-valued echelon spaces. *Collect. Math.* **39** (1988), 257–261.

[5] L. DREWNOWKI, M. FLORENCIO and P. J. PAÚL. Barrelled function spaces. In *Progress in Functional Analysis. Proceedings of the Peñíscola Meeting* 1990, *on the occasion of the* 60*th birthday of Professor M. Valdivia* (editors K. D. Bierstedt *et al.*), Mathematical Studies no. 170 (North-Holland, 1992), pp. 191–199.

[6] C. JORDÁN LLUCH and J. R. TORREGROSA SÁNCHEZ. A characterization of the duals of some echelon Köthe spaces. *Collect. Math.* **38** (1987), 229–248.

[7] G. KÖTHE. *Topological Vector Spaces*, vol. 1 (Springer-Verlag, 1969).

[8] A. L. MACDONALD. Vector valued Köthe function spaces. I. *Illinois J. Math.* **17** (1973), 533–545. II. *Illinois J. Math.* **17** (1973), 546–557. III. *Illinois J. Math.* **18** (1974), 136–146.

[9] N. PHUONG-CÁC. Generalized Köthe function spaces. I. *Proc. Cambridge Philos. Soc.* **65** (1969), 601–611.

[10] R. C. ROSIER. Dual spaces of certain vector sequence spaces. *Pacific J. Math.* **46** (1973), 487–501.

https://doi.org/10.1017/S0305004100070845 Published online by Cambridge University Press