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Isomorphic measures on compact groups

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1. *Introduction*

This paper deals with the problem of point realizations of isomorphisms of measure algebras.

Investigations in these directions started with the well known paper of von Neumann [**20**], in which it was proved that any automorphism of the measure algebra of a σ-finite Borel measure on a Polish space can be induced by an invertible (Borel) point transformation. Subsequent generalizations of the above were given by Maharam [**15**], for products of probability measures on Polish spaces, and by Choksi [**2**], for an arbitrary σ-finite measure on the completion of the product σ-algebra of a product of Polish spaces. Also Choksi in [**3**], and Choksi and Simha in [**8**] established similar results for measures on compact groups and homogeneous spaces. Note that Maharam also proved a number of unpublished results on the realization of σ-homomorphisms by point maps; see e.g. [**5**, **10**].

In 1979, Choksi and Fremlin wrote an article [**6**] in which a large class of results in topological measure theory, usually proved for Polish spaces, was shown to hold in any uncountable product $X = \prod_{\gamma\in\Gamma} X_\gamma$ of compact metric spaces. This paper examined results about isomorphic measure algebras. Two (Radon probability) measures on X, whose measure algebras are homogeneous of the same Maharam type, were shown to be completion Baire isomorphic. Also the authors, combining this result with a classical theorem of Maharam [**16**], arrived at some surprising and rather unexpected results, some of whose proofs do not assume any set-theoretic hypotheses. The main of the latter are included in the following statement (see [**6**], p. 122, theorem 5):

(*) For a large, in fact cofinal, class of cardinals Γ, any two completion regular measures on $\prod_{\gamma\in\Gamma} X_\gamma$ (each X_γ compact metric with at least two points) are completion Baire isomorphic.

In a later survey article [**4**], Choksi asks whether analogues of the results of [**6**], without additional set-theoretic hypotheses, are valid for compact groups (and possibly even for homogeneous spaces). In the present paper it is shown, by a measure-theoretic method, that this is exactly the case, at least for the results just mentioned. In particular, '$\prod_{\gamma\in\Gamma} X_\gamma$, with X_γ compact metric spaces' can be replaced by 'compact group G of topological weight $|\Gamma|$' in (*).

In analogy to the paper of Choksi and Fremlin, the discussion is centred around the isomorphisms of measure algebras on compact groups and their inducing point transformations. The classical Weil–Pontrjagin theorem ([**21**], p. 88) is basic for this study. It should be noticed that our method is applied to those results of [**6**]

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involving additional set-theoretic assumptions as well. Thus, roughly speaking, the main theorems of [**6**] about products of compact metric spaces can be translated to ones on compact groups (for the precise connexion see our discussion in Sections 4, 5).

We mention finally that to carry out this extension in the area of topological groups we use in a crucial way the fact, incidentally established here, that the (normalized) Haar measure on any uncountable compact group is homogeneous of Maharam type equal to the topological weight of the group.

2. *Preliminaries and notation*

All measures mentioned in this paper will be (positive) Radon measures on compact (Hausdorff) topological spaces.

For any compact space X, $\mathscr{B}_X$ and $\mathscr{B}_X^\circ$ denote its Borel and Baire σ-algebras respectively. For any Radon measure μ on X, let μ_0 denote the completion of the Baire restriction of μ. We recall that the measure space (X,μ) (or just μ) is said to be completion regular if every Borel set is μ_0-measurable, i.e. $\bar{\mathscr{B}}_X^{\circ\mu} = \bar{\mathscr{B}}_X^{\mu}$, where $\bar{\mathscr{B}}^\mu$ denotes the completion of $\mathscr{B}$ whenever $(X,\mathscr{B},\mu)$ is a measure space.

Next let I be a directed set and $\{X_i, p_{ij}, i,j\in I\}$ a projective system of compact spaces. Suppose that, for each $i\in I$, μ_i is a Radon probability measure on X_i. We shall say that the family $\{X_i, p_{ij}, \mu_i, i,j\in I\}$ (or just $\{\mu_i, i\in I\}$) constitutes a projective system of measure spaces (or, simply, of measures) if $p_{ij}(\mu_j) = \mu_i$ for $i \leqslant j$.

Now let G be an arbitrary compact topological group and let R_G be the set of closed normal subgroups H of G and Q_G (respectively Q_G°) the subsets of all H such that G/H is metrizable (respectively, a Lie group). Then $Q_G^\circ \subset Q_G \subset R_G$ and all three sets are directed by the relation in which $H_1 < H_2$ if and only if $H_2 \subset H_1$. Also $\{G/H, \pi_{H,F}, H\in Q_G\}$ and $\{G/H, \pi_{H,F}, H\in Q_G^\circ\}$, where for $H,F\in R_G$ and $H<F$, $\pi_{H,F}$ denotes the projection from G/F onto G/H, are both projective systems of compact groups and

$$G \cong \operatorname{proj}_{Q_G}\lim G/H \cong \operatorname{proj}_{Q_G^\circ}\lim G/H;$$

see [**3**, **18**]. Let ν be a Radon measure on G and $H\in R_G$. Put $\nu_H = \pi_H\nu$ and $G(H) = \pi_H^{-1}(\mathscr{B}_{G/H}^\circ)$, where π_H is the projection from G to G/H. Now let (ν,H) denote the measure algebra of $(G/H, \mathscr{B}_{G/H}, \nu_H)$ and $H(\nu)$ the subalgebra of the measure algebra E_ν of ν consisting of equivalence classes of elements of $G(H)$. Set

$$B(\nu_H) = \bigcup\{\pi_{F,H}^{-1}(\bar{\mathscr{B}}_{G/H}^{\nu_H}),\ F\in Q_G, H>F\}, \quad G(\nu,H) = \pi_H^{-1}B(\nu_H).$$

Finally, the following notation will be used in the sequel. For any topological space X, ${}^1X$, ${}^2X$ denote two copies of X and $\langle X\rangle$ their topological sum, i.e. $X = {}^1X \oplus {}^2X$. In the particular case when $X = G$ is a compact group, one can easily verify that the family $\{\langle G/H\rangle = {}^1G/H \oplus {}^2G/H, p_{F,H}, F,H\in Q_G\}$ (where for $F<H$, ${}^i\pi_{F,H}$ is the projection of ${}^iG/H$ to ${}^iG/F$ and $p_{F,H} = {}^1\pi_{F,H} \oplus {}^2\pi_{F,H}$ i.e. $p_{F,H}: \langle G/H\rangle \to \langle G/F\rangle$ is defined by $x \mapsto {}^i\pi_{F,H}x$, for $x\in {}^iG/H$ and for $i = 1,2$) is a projective system with $\langle G\rangle \cong \operatorname{proj}_{Q_G}\lim\langle G/H\rangle$ (as topological spaces). Note that, for $H\in R_G$, the canonical projection from $\langle G\rangle$ onto $\langle G/H\rangle$ is the mapping $p_H = {}^1\pi_H \oplus {}^2\pi_H$, where ${}^i\pi_H = {}^i\pi_{H,G}$ for $i = 1,2$.

3. *The homogeneity of the non-atomic Haar measure*

In this section we collect together a few technical results and some comments which will be used in the sequel.

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We recall that a measure algebra (E, μ) (or, simply, μ) of finite magnitude is called homogeneous if every non-zero principal ideal has a minimal σ-basis of the same cardinal; this cardinal is called the Maharam type of the measure algebra.

We now consider a compact group G of uncountable topological weight $w(G)$. Let λ denote the normalized (left) Haar measure on G and $e = \mathrm{id}_G$. The following result is not surprising.

THEOREM 3·1. *The measure algebra (L, λ) of λ is homogeneous of Maharam type $w(G)$.*

For the proof of this theorem we need a lemma. For any measurable subset D of G, we write λ_D for the restriction of λ to D.

LEMMA 3·2. *If K is a measurable set in G with $0 < \lambda K < 1$, then there exists $z \in G$ such that $\lambda(zK \cap (G \backslash K)) > 0$.*

Proof. The result follows from [**12**], p. 296, 20·17 corollary. ∎

Proof of Theorem 3·1. Firstly notice that, for $x \in G$ and A any λ-measurable set, λ_A and λ_{xA} have isomorphic measure algebras (because they are homeomorphic measures).

Step 1. (L, λ) *is homogeneous.* Suppose otherwise. Then, by a theorem of Maharam ([**8**, **16**]), L can be decomposed into the direct sum of a countable number of homogeneous measure algebras (L_i, λ_i), where distinct L_i have distinct Maharam types. Without loss of generality we can assume that each (L_i, λ_i) is of the form $(L_i, \lambda_i) = (L_i, \lambda_{K_i})$, where K_i is some λ-measurable set with $0 < \lambda K_i < 1$ and L_i is the associated measure algebra. Fix $n \in \mathbb{N}$. By Lemma 3·2, there is $z \in G$ such that $\lambda(zK_n \cap K_{n+1}) > 0$. Then, since λ_{zK_n} and $\lambda_{K_{n+1}}$ are both homogeneous, $\lambda_{zK_n \cap K_{n+1}}$, λ_{zK_n}, $\lambda_{K_{n+1}}$ all have the same Maharam type, and this is a contradiction.

Step 2. (L, λ) *is of Maharam type* $w(G)$. Suppose that (L, λ) is of Maharam type γ. It suffices to prove that $w(G) \leqslant \gamma$. Take $\Gamma_0 \subset \mathscr{B}_G$ of cardinal γ, dense for the measure-pseudometric d of $\mathscr{B}_G$. Then for every $E \in \mathscr{B}_G$ and $\epsilon > 0$ there is some $B \in \Gamma_0$ such that $d(E, B) < \epsilon/4$, so that

$$d(xE, E) \leqslant d(xE, xB) + d(xB, B) + d(B, E) < \epsilon/2 + d(xB, B), \quad \text{for } x \in G. \tag{3·3}$$

By (3·3), since the class of sets of the form $\{y : d(yE, E) < 1/n\}$ with $E \in \mathscr{B}_G$ and $n \in \mathbb{N}$ is a base at e (see e.g. [**11**], p. 267, theorem B), the class of sets of the form $\{y : d(yB, B) < 1/n\}$ with $B \in \Gamma_0$ and $n \in \mathbb{N}$ is also a base of neighbourhoods of e and therefore $w(G) \leqslant \gamma$. ∎

Remarks 3·4. (i) It is clear from the above proof that the non-atomic Haar measure on an arbitrary compact group has an homogeneous measure algebra.

(ii) The second part of Theorem 3·1 follows also by using techniques of abstract harmonic analysis (combine (16·12) in [**12**] and theorem 28·2 in [**13**]).

4. *Measurable transformations on compact groups*

In this section we first give some results concerning measurable transformations on projective limits. We prove a series of lemmas and end this section with two results inspired by the work of Choksi and Fremlin [**6**].

Throughout, G is an arbitrary compact topological group and μ a Radon measure on $\langle G \rangle = {}^1G \oplus {}^2G$ (in the notation of Section 2). Clearly $\mu = \mu 1 \oplus \mu 2$, where μi is the restriction of μ to ${}^iG$, for $i = 1, 2$.

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We fix more terminology.

4·1. *Terminology and notation.* Let $(X, \mathscr{B}, \lambda)$ be a measure space and $T: X \to X$ a measurable bijection. We say that T is an invertible measurable transformation of X if T^{-1} is also measurable and $\lambda(TZ) = 0$ if and only if $\lambda(Z) = 0$.

Now let $K \in R_G$ and $\langle G \rangle$, Q_G, Q_G°, $E_{\mu i}$, $K(\mu i)$, $G(K)$, $G(\mu i, K)$ $(i = 1, 2)$ be as before. Then the measure algebra E of μ is $E = E_{\mu 1} \oplus E_{\mu 2}$ and that of $p_K \mu$ is $E_K = (\mu 1, K) \oplus (\mu 2, K)$. For convenience in writing relations involving direct sums, some abbreviated notation is introduced. Set

$$K_\mu^\times = K(\mu 1) \oplus K(\mu 2), \quad B_\mu^\times = B(\mu 1) \oplus B(\mu 2), \quad B_\mu^K = B(\mu 1_K) \oplus B(\mu 2_K)$$

$$\text{and} \quad G_\mu^k = G(\mu 1, K) \oplus G(\mu 2, K).$$

Definition. (*a*) Let Φ be an automorphism of the measure algebra E of μ. A subgroup $K \in R_G$ is said to be invariant under ϕ if $\phi(K_\mu^\times) = K_\mu^\times$; then ϕ induces an automorphism ϕ_K of E_K.

(*b*) Let T be a $B_\mu^\times$ measurable point transformation of $\langle G \rangle$. A subgroup $K \in R_G$ is said to be invariant under T if $T(G_\mu^K) = G_\mu^K$.

The following auxiliary result is, essentially, a modification of [**3**], theorem 1.

LEMMA 4·2. *Every automorphism Φ of the measure algebra E of $(\langle G \rangle, \mu)$ is induced by an invertible completion Baire point transformation T of $\langle G \rangle$; in fact T and T^{-1} are $B_\mu^\times$ measurable.*

The tools and techniques that we will use to prove Lemma 4·2 are those introduced by Choksi in [**3**]. We omit details of those proofs which are similar to proofs of [**3**].

LEMMA 4·3. *If $H, K \in R_G$ such that $H \cap K = \{\mathrm{id}_G\}$, then*

(1) *$\theta: g \mapsto (p_H g, p_K g)$ defines an homeomorphism of $\langle G \rangle$ into $\langle G/H \rangle \times \langle G/K \rangle$,*

(2) *two cosets xH and yK meet in at most one element of $\langle G \rangle$,*

(3) *xH and yK meet if and only if $p_{HK,H}(xH) = p_{HK,K}(xK)$,*

(4) *let T_H, T_K be bijections of $\langle G/H \rangle$ and $\langle G/K \rangle$ respectively, and $\tilde{T}$ the bijection of $\langle G/H \rangle \times \langle G/K \rangle$ given by $\tilde{T} = (T_H, T_K)$. If*

$$p_{HK,H} \circ T_H \circ p_H = p_{HK,K} \circ T_K \circ p_K \quad \text{and} \quad p_{HK,H} \circ T_H^{-1} \circ p_H = p_{HK,K} \circ T_K^{-1} \circ p_K,$$

then $\tilde{T}(\theta\langle G \rangle) = \theta\langle G \rangle$ and so $\theta^{-1} \circ \tilde{T} \circ \theta$ gives a bijection of $\langle G \rangle$ onto itself.

Proof. This proceeds exactly as that of [**3**], lemma 2, making use of the properties of a direct sum. ∎

LEMMA 4·4. (*a*) *If Φ is an automorphism of the measure algebra E of μ, then for any $H \in Q_G$ there exists $\bar{H} \in Q_G$ with $H < \bar{H}$ and $\bar{H}$ invariant under Φ.*

(*b*) *If T is an invertible $B_\mu^\times$ measurable point transformation of $\langle G \rangle$, then for any $H \in Q_G$ there exists $\bar{H} \in Q_G$ with $H < \bar{H}$ and $\bar{H}$ invariant under T.*

Proof. This follows from the proof of [**3**], lemma 4, using the above terminology. ∎

COROLLARY 4·5. (*a*) *If $K \in R_G$ and if Φ_K is an automorphism of E_K, then for any $H \in Q_G$ with $H < K$ there exists $\bar{H} \in Q_G$ with $H < \bar{H} < K$ and $\bar{H}$ invariant under Φ_K.*

(*b*) *If $K \in R_G$ and if T_K is a B_μ^K measurable invertible point transformation of $\langle G/K \rangle$, then for any $H \in Q_G$ with $H < K$ there exists $\bar{H} \in Q_G$ with $H < \bar{H} < K$ and $\bar{H}$ invariant under T_K.*

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Proof. This proceeds as in [3], p. 111, corollary. ∎

LEMMA 4·6. *Suppose that G is metrizable. If $H \in R_G$ and if H has cardinal c, then there exists a Borel set $Z \subset \langle G \rangle$ with $\mu(Z) = 0$ and such that Z meets the set $p_H x = xH \subset \langle G \rangle$ in a set of cardinal c, for each $x \in \langle G \rangle$.*

Proof. By [3], lemma 6, for $i = 1, 2$, there exists a Borel set Z_i in ${}^1G$ with $\mu i(Z_i) = 0$ and such that Z_i meets every coset xH in a set of cardinal c. Then $Z = Z_1 \cup Z_2$ is a Borel subset of $\langle G \rangle$ with the required properties. ∎

LEMMA 4·7. *Suppose that G is metrizable and $H \in Q_G$. Let Φ be an automorphism of the measure algebra E of $(\langle G \rangle, \mu)$ such that H is invariant under Φ and so Φ induces an automorphism Φ_H of E_H. Let T_H be an invertible completion Borel measurable point transformation of $\langle G/H \rangle$ inducing Φ_H. Then there is an invertible completion Borel point transformation T of $\langle G \rangle$ inducing Φ and such that $p_H \circ T = T_H \circ p_H$.*

Proof. This is identical to that of [3], lemma 7, except that Lemma 4·6 replaces [3], lemma 6. ∎

LEMMA 4·8. *Let Φ be an automorphism of E. Suppose that $K \in R_G$ is invariant under Φ, so that Φ induces an automorphism Φ_K of E_K. Suppose further that Φ_K is induced by an invertible B_μ^K measurable point transformation T_K of $\langle G/K \rangle$. Suppose, finally, that there exists $H \in Q_G$ such that $K \cap H = \{\mathrm{id}_G\}$. Then there is an invertible $B_\mu^\times$ measurable point transformation T of $\langle G \rangle$ which induces Φ and is such that $p_K \circ T = T_K \circ p_K$.*

Proof. This is analogous to that of [3], lemma 8 and uses the above lemmas.

The last lemma is entirely analogous to [3], lemma 9.

LEMMA 4·9. *Let Φ be an automorphism of E and $\{K_j, j \in J\}$ a totally ordered set of groups in R_G such that*

(i) *each K_j is invariant under Φ with Φ_j the induced automorphism of E_{K_j},*

(ii) *each Φ_j is induced by a B_μ^K measurable invertible point transformation T_j of $\langle G/K_j \rangle$, and*

(iii) *if $j_1, j_2 \in J$ with $K_{j_1} < K_{j_2}$ then $p_{K_{j_1}, K_{j_2}} \circ T_{j_2} = T_{j_1} \circ p_{K_{j_1}, K_{j_2}}$.*

If $K = \bigcap_{j \in J} K_j$, then K is invariant under Φ and if Φ_K is the induced automorphism of E_K, then there exists a B_μ^K measurable invertible point transformation T_K of $\langle G/K \rangle$ inducing Φ_K with $p_{K_j, K} \circ T_K = T_j \circ p_{K_j, K}$, for all $j \in J$.

Proof of Lemma 4·2. This proceeds exactly as the proof of [3], theorem 1, with the above lemmas replacing those used there. The main steps of the proof are as follows.

1. Consider the family $\mathbb{F}$ of ordered pairs (K, T_K), where (i) $K \in R_G$ and K is invariant under Φ, (ii) T_K is an invertible B_μ^K measurable point transformation of $\langle G/K \rangle$ and (iii) the automorphisms of E_K induced by $\Phi/K_\mu^\times$ and by T_K are the same.

2. Define, in $\mathbb{F}$, an ordering by setting $(K, T_K) < (K_1, T_{K_1})$ if $K < K_1$ and $p_{K, K_1} \circ T_{K_1} = T_K \circ p_{K, K_1}$ on $\langle G/H \rangle$.

3. Making use of Zorn's lemma, we easily verify the existence of a (maximal) member (L, T) of $\mathbb{F}$ such that $L = \{\mathrm{id}_G\}$ and so $G/L = G$. ∎

Let ρ and ξ be measures on compact spaces X and Y respectively. We recall that the measure spaces (X, ρ), (Y, ξ) (or just the measures ρ, ξ) are said to be completion

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Baire isomorphic if there is a bijection R of X to Y such that RZ is ξ_0-measurable if and only if Z is ρ_0-measurable and then $\rho(Z) = \xi(RZ)$ (cf. [6]).

Suppose now that $w(G)$ is uncountable. For the proof of the next result, which is given only for completeness, we use several of the techniques of the paper [6].

THEOREM 4·10. *If μ and ν are Radon probability measures on G and if there exists a measure-preserving isomorphism Ψ of the measure algebras E_μ, E_ν of μ and ν respectively, then μ, ν are completion Baire isomorphic; and in fact Ψ can be realized by a (measurable) bijection.*

Proof. Fix $H_0 \in Q_G$. Then, in the notation of Sections 2 and 3,

$$i_G = \mathrm{proj}_{H \in Q_G} \lim {}^iG/H \quad \text{for } i = 1, 2 \quad \text{and} \quad \langle G \rangle = {}^1G \oplus {}^2G.$$

Put μ on ${}^1G$, ν on ${}^2G$, so that $\mu \oplus \nu = \tau$ is a Radon measure on $\langle G \rangle$, whose measure algebra is $E_\mu \oplus E_\nu$. Put $\Phi = \Psi$ on E_μ, $\Phi = \Psi^{-1}$ on E_ν, so that Φ extends naturally to a measure-preserving automorphism of $E_\mu \oplus E_\nu$.

Now let $H_C \in Q_G$ satisfy $H_0 < H_C \neq \mathrm{id}_G$ and be invariant under Φ, which therefore induces an automorphism Φ_C of $(\mu, H_C) \oplus (\nu, H_C)$, which in turn gives a measure-preserving isomorphism Ψ_C of the measure algebras (μ, H_C) and (ν, H_C). By von Neumann's classical theorem [20], there is an invertible Borel isomorphism S_C of the compact metric spaces ${}^1G/H_C$ and ${}^2G/H_C$, inducing Ψ_C. Therefore there exists a Borel isomorphism T_C of $\langle G/H_C \rangle = {}^1G/H_C \oplus {}^2G/H_C$ onto itself, which induces Φ_C and which is such that $T_C({}^1G/H_C) = {}^2G/H_C$.

Now, following the proof of Lemma 4·2, consider the family $\mathbb{F}$ of ordered pairs (K, T_K), where (i) $K \in R_G$ and K is invariant under Φ, (ii) T_K is an invertible $B(\mu_K) \oplus B(\nu_K)$ measurable point transformation of $\langle G/K \rangle$, (iii) the automorphisms of $(\mu, K) \oplus (\nu, K)$ induced by $\Phi/K_\tau^\times$ and by $T_{K_\tau}^\times$ are the same, (iv) $H_C < K$ and T_K extends to T_C. The (maximal) transformation T inducing Φ obtained from the proof of Lemma 4·2 necessarily also extends T_C and satisfies $T({}^1G) = {}^2G$. This completes the proof of Theorem 4·10. ∎

Remark 4·11. Note that results in [2, 17] and particularly [9], §2·23 suggest that there may be results much more general than Theorem 4·10 applying to automorphisms between algebras of the form $\mathscr{B}_G^\circ/\mathscr{J}$ for σ-ideals $\mathscr{J}$.

Combining now Maharam's famous theorem [16] and Theorem 4·10, we immediately obtain

COROLLARY 4·12. (*a*) *If μ is a Radon probability measure on G whose measure algebra is homogeneous of Maharam type $w(G)$, then μ is completion Baire isomorphic to the Haar measure on G.*

(*b*) *If μ and ν are two Radon probability measures on G whose measure algebras are homogeneous of the same Maharam type, then μ and ν are completion Baire isomorphic.*

5. *Completion regular measures on compact groups*

In this final section we discuss results on completion regularity. It is proved that for every compact group G of uncountable topological weight $w(G)$ the least Maharam type of any completion regular measure on $\{0, 1\}^{w(G)}$ is equal to the least Maharam type of any completion regular measure on G. As a consequence, the main

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results of [6] about completion regular measures on products of compact metric spaces can be translated to ones on compact groups.

Throughout this section G is a compact topological group with $w(G) = \alpha$ uncountable. We need some auxiliary notation.

If X is a compact space, (Y, Σ, μ) any measure space and g a mapping from Y to X such that $g^{-1}C \in \Sigma$ for every $C \in \mathscr{B}_X^\circ$, then $g(\mu)$ will denote the (Baire) measure on X defined by $g(\mu)D = \mu(g^{-1}D)$, for $D \in \mathscr{B}_X^\circ$.

In the sequel we shall identify a cardinal γ with the initial ordinal of cardinal γ. Also, when no ambiguity arises, we shall identify a Radon measure with its Baire contraction (i.e. its restriction to the Baire σ-algebra).

Remarks 5·1. Consider a compact group G and $H \in R_G$. Let S be a Borel set in G such that:

(*a*) $G = S.H$,

(*b*) $\pi_H^{-1}\{\dot{x}\} \cap S$ is a single point $\{r(\dot{x})\}$ for every $\dot{x} \in G/H$,

(*c*) the bijection $q: G/H \times H \to G$ defined by $q(\dot{x}, t) := r(\dot{x}).t$ is a Borel isomorphism (i.e. $q(\mathscr{B}_{G/H \times H}) = \mathscr{B}_G$).

Let $p := q^{-1}$. If λ, β and μ denote (normalized) Haar measure on G, H and G/H respectively, then the Radon product measure $\nu = \mu \otimes \beta$ on $G/H \times H$ satisfies

$$\nu = p(\lambda), \tag{5·2}$$

and so the measure ν defined by (5·2) is a strictly positive product measure (i.e. $\nu(U) > 0$ for every open set U and $\nu(A \times B) = \nu(A \times H).\nu(G/H \times B)$ for $A \in \mathscr{B}_{G/H}$ and $B \in \mathscr{B}_H$).

Clearly (5·2) is an immediate consequence of the (Fubini type) relation

$$\lambda(C) = \int_{G/H} d(\dot{x}) \int_H \chi_C(x.t)\, d\beta(t), \quad \text{for } C \in \mathscr{B}_G,$$

where χ denotes, as usual, the characteristic function (see e.g. [1, 4]).

It should be noticed that in the particular case when G is metrizable, there always exists such a Borel set S (i.e. a Borel subset S of G satisfying (*a*), (*b*) and (*c*); see the proof of [3], lemma 6 and the references given there).

We start with the following result which states that, in the particular case when $w(G) = \aleph_1$, the Haar measure is Baire isomorphic to a special product measure.

PROPOSITION 5·3. *Let λ be the (normalized) Haar measure on G. If $w(G) = \aleph_1$, then there exists a family $(\mu_\gamma)_{\gamma < \omega_1}$ of Baire probability measures, each μ_γ supported on a compact metric space X_γ with at least two points, and a Baire isomorphism f from $X = \Pi_{\gamma < \omega_1} X_\gamma$ onto G such that*

$$f(\times_{\gamma < \omega_1} \mu_\gamma) = \lambda. \tag{5·4}$$

(Here, $\mu = \times_{\gamma < \omega_1} \mu_\gamma$ denotes the simple product of the μ_γ on $\mathscr{B}_X^\circ$, i.e. the Baire contraction of the Radon product measure $\otimes_{\gamma < \omega_1} \mu_\gamma$.)

Proof. Because $w(G) = \aleph_1$, there is a directed set $\Gamma = \{F_j, j \in J\}$ of groups in Q_G of cardinal $\aleph_1$ such that $\bigcap_{j \in J} F_j = \{\mathrm{id}_G\}$. By transfinite induction on $\gamma < \omega_1$, we construct a strictly increasing family $(H_\gamma)_{\gamma < \omega_1}$ in Q_G such that $\bigcap_{\gamma < \omega_1} H_\gamma = \{\mathrm{id}_G\}$. (Enumerating Γ as $\Gamma = \{F_\xi, \xi < \omega_1\}$ and taking $H_0 = F_0$, we set $H_\gamma := H_\delta \cap F_\gamma$ if γ is a successor ordinal of the form $\gamma = \delta + 1$ for some $\delta < \omega_1$ and $H_\gamma := \bigcap_{\xi < \gamma} H_\xi$ otherwise.)

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Set

$$X_0 := G/H_1 \quad \text{and} \quad X_\gamma = (G/H_{\gamma+1})/(G/H_\gamma)\,(\cong H_\gamma/H_{\gamma+1}) \quad \text{for } \gamma > 0.$$

By induction on $\gamma < \omega_1$, using standard arguments involving projective limits of compact spaces and Remark 5·1, for every $\gamma < \omega_1$ we find a Borel isomorphism q_γ from $\hat{X}_\gamma = \Pi_{\delta<\gamma} X_\delta$ onto G/H_γ such that

$$q_{\gamma_1} \circ p_{\gamma_2,\gamma_1} = \pi_{H_{\gamma_2},H_{\gamma_1}} \circ q_{\gamma_2} \quad \text{for } \gamma_1 < \gamma_2 < \omega_1,$$

where p_{γ_2,γ_1} is the canonical projection from $\hat{X}_{\gamma_2}$ onto $\hat{X}_{\gamma_1}$. (If γ is a successor ordinal, say $\gamma = \delta+1$, then Remark 5·1 yields a Borel isomorphism $\hat{q}_\gamma : G/H_\delta \times X_\delta \to G/H_\gamma$. We take

$$q_\gamma = \hat{q}_\gamma \circ (q_\delta \times I_{X_\delta}) : \hat{X}_\gamma \to G/H_\gamma,$$

where $q_\delta \times I_{X_\delta}$ denotes the mapping

$$\hat{X}_\delta \times H_\delta = \hat{X}_\delta \to G/H_\delta \times X_\delta : (x,y) \mapsto (q_\delta(x), y).$$

In the case when γ is a limit ordinal we take $q_\gamma : \hat{X}_\gamma \to G/H_\gamma$ to be the (unique) mapping satisfying

$$q_\gamma \circ p_{\gamma,\delta} = \pi_{H_\gamma,H_\delta} \circ q_\gamma, \quad \text{for } \delta < \gamma < \omega_1.)$$

Again there exists a unique mapping

$$f : \Pi_{\gamma<\omega_1} X_\gamma\,(\cong \operatorname{proj}_\beta \lim \Pi_{\delta<\beta} X_\delta) \to G(\cong \operatorname{proj}_\gamma \lim G/H_\gamma)$$

such that

$$\pi_{H_\gamma} \circ f = q_\gamma \circ p_\gamma, \quad \text{for } \gamma < \omega_1 \tag{5·5}$$

(where p_γ denotes the canonical projection from $\Pi_{\delta<\omega_1} X_\delta$ onto $\Pi_{\delta<\gamma} X_\delta$). Then f, thus defined, is $1-1$ by construction. Since $\Pi_{\gamma<\omega_1} X_\gamma$ is a cartesian product, it is immediate from (5·5) that f is surjective and in view of (5·2) and (5·5), is a Baire isomorphism satisfying (5·4). This completes the proof of the proposition. ∎

The condition $w(G) = \aleph_1$ is essential to the above proof. The question that naturally arises is what happens if $w(G)$ is an arbitrary cardinal. In the sequel a side result is given for arbitrary $w(G)$. This enables us to prove analogues of the principal results of [**6**] for compact groups.

LEMMA 5·6. *Let $H \in R_G$ be a Lie group. Then there exists a Borel set S in G satisfying the conditions (a), (b) and (c) of the Remarks 5·1.*

Proof. By [**14**], theorem 3·1, there exist a compact neighbourhood Q of id_G and a compact subset F of Q such that (i) $Q = F.H$ and (ii) $\pi_H^{-1}\{\dot{x}\} \cap F$ is a single point for $\dot{x} \in \pi_H Q$.

Let $\{Q_1 = x_1 Q, Q_2 = x_2 Q, \ldots, Q_N = x_N Q\}$ be any finite cover of G. Because $\{\pi_H(x_1 F), \ldots, \pi_H(x_N F)\}$ is a finite cover of G/H, we can easily construct a finite cover $\{S^1, S^2, \ldots, S^M\}$ of Borel, pairwise disjoint sets in G/H such that

for each $m \in \{1,2,..,M\} = I_M$ there is some $k \in \{1,2,\ldots,N\} = I_N$ such that $S^m \subset \pi_H(x_k F)$.

For $m \in I_M$ set

$$i_m := \min\{k \in I_N : S^m \subset \pi_H(x_k F)\}, \quad S_m := \{y \in x_{i_m} F : \pi_H y \in S^m\} \quad \text{and} \quad S := \bigcup_{m=1}^{M} S_m.$$

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It is clear that $\{S_m, m \in I_M\}$ is a family of pairwise disjoint Borel sets in G such that for each $\dot{x} \in S^m$, $\pi_H^{-1}\{\dot{x}\} \cap S_m$ is a single point. So the Borel set S thus defined satisfies (a) and (b) of 5·1. Let

$$\pi_H^{-1}\{\dot{x}\} \cap S = \{r(\dot{x})\}, \quad \text{for} \quad \dot{x} \in G/H.$$

To complete the proof, we define $q: G/H \times H \to G$ by $q(\dot{x}, y) := r(\dot{x}).y$. Then, by the definition, it follows that q is a Borel isomorphism. ▌

LEMMA 5·7. *Let $(H_\gamma)_{\gamma<\alpha}$ be an increasing family of groups in R_G and $(K_\gamma)_{\gamma<\alpha}$ a family of compact (non-trivial) Lie groups such that, for every $\gamma < \alpha$*

(i_1) $(G/H_{\gamma+1})/(G/H_\gamma) \cong K_\gamma$, *and*

(i_2) $H_\gamma = \bigcap_{\beta<\gamma} H$ *if γ is a limit ordinal.*

Let

$$H := \bigcap_{\gamma<\alpha} H_\gamma, \quad G_* := G/H \,(\cong \mathrm{proj}_{\gamma<\alpha} \lim G/H_\gamma).$$

If λ_* is the (normalized) Haar measure on G_*, then there exist a family $(\mu_\gamma)_{\gamma<\alpha}$ of Baire probability measures, each μ_γ supported on a compact metric space X_γ with at least two points, and a bijection f from $\Pi_{\gamma<\alpha} X_\gamma$ onto G_*, with the following properties:

(P_1) $f(\mathscr{B}^\circ_X) \subset \mathscr{B}_{G_*}$ *and* $f^{-1}(\mathscr{B}^\circ_{G_*}) \subset \mathscr{B}_X$,

(P_2) $f(\mu) = \lambda_*$,

where $\mu = \times_{\gamma<\alpha} \mu_\gamma$, i.e. μ is the Baire contraction of the Radon product measure $\bigotimes_{\gamma<\alpha} \mu_\gamma$.

Proof. Note that Lemma 5·6 asserts the existence of Borel sections, instead of the existence of Baire sections used in the proof of Proposition 5·3. The proof of Lemma 5·7 is, from there on, identical with that of Proposition 5·3. ▌

Let X be a compact space and η a cardinal. Following [**6**] put define $\zeta(X)$ to be the least Maharam type of any completion regular measure on X, $\tau(\eta) = \zeta(\{0,1\}^\eta)$.

Lemma 5·7 is used to prove the main result of this section:

THEOREM 5·8. *If G is a compact group with $w(G) = \alpha$ uncountable, then $\tau(\alpha) = \zeta(G)$.*

Proof. We prove the result for $\leqslant$; the proof for the reverse inequality $\geqslant$ is similar.

First we claim that there exist two families $(H_\gamma)_{\gamma<\alpha}$ and $(K_\gamma)_{\gamma<\alpha}$ in R_G as in Lemma 5·7 such that $H = \{\mathrm{id}_G\}$. (Then, following the above notation, we have $G \cong G_*$.)

Because $w(G) = \alpha$, there is a directed set $\Gamma = \{F_j, j \in J\}$ of groups in Q°_G of cardinal α such that $\bigcap_{j\in J} F_j = \{\mathrm{id}_G\}$. The family $(H_\gamma)_{\gamma<\alpha}$ will be defined by transfinite induction on $\gamma < \alpha$. Enumerating Γ as $\Gamma = \{F_\xi, \xi < \alpha\}$ and taking $H_0 = F_0$, we set $H_\gamma = H_\delta \cap F_\gamma$ if $\gamma = \delta + 1$ for some $\delta < \alpha$ and $H_\gamma = \bigcap_{\delta<\alpha} H_\delta$ otherwise.

Let $K_\gamma = H_\gamma/H_{\gamma+1}\,(\cong (G/H_{\gamma+1})/(G/H_\gamma))$, where $\gamma < \alpha$. Then the K_γ are Lie groups (see e.g. the references given for remark 2 on p. 85 in [**19**]) and by construction, they satisfy the conditions (i_1), (i_2). This ends the proof of the claim.

Lemma 5·7 yields a compact space $X = \Pi_{\gamma<\alpha} X_\gamma$ and a bijection $q: X \to G$ satisfying the conclusion of the lemma. Let ρ be any probability completion regular homogeneous measure on G of Maharam type β. It suffices to prove that

there exists a Radon probability completion regular measure ρ' on X of Maharam type at most β such that ρ and ρ' are completion Baire isomorphic. (5·9)

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Because the probability measure $\nu' = \frac{1}{2}(\lambda + \rho)$ is completion regular, there is a homogeneous completion regular probability measure ν on G of Maharam type α such that $\nu \leqslant \epsilon . \nu'$, for some $\epsilon > 0$. Therefore by Corollary 4·12, there exists a completion Baire isomorphism $\psi : (G, \lambda_*) \to (G, \nu)$ with $\psi(\lambda_*) = \nu$ (where λ_* denotes Haar measure on G). Then clearly $T := (\psi \circ q)^{-1} : (G, \nu) \to (X, \mu)$ is a completion Baire isomorphism such that $T(\nu) = \mu$. Now it is clear that the measure $\rho' = T(\rho)$ satisfies (5·9), completing the proof of the theorem. ▮

Remark 5·10. The above theorem shows, in particular, that the existence of a completion regular measure on G of Maharam type $\tau(w(G))$ is purely a property of $w(G)$ and independent of what group G is.

COROLLARY 5·11. *For every compact group G with $w(G)$ uncountable, there exist a family $(X_\gamma)_{\gamma < w(G)}$ of compact metric spaces, each X_γ having at least two points, and a family $(\mu_\gamma)_{\gamma < w(G)}$ of Radon probability measures, each μ_γ supported on X_γ, with the following properties*:

(1) *the Haar measure on G is completion Baire isomorphic to the Radon product measure $\bigotimes_{\gamma < w(G)} \mu_\gamma$ on $X = \prod_{\gamma < w(G)} X_\gamma$;*

(2) *every completion regular measure on G (resp. on X) is completion Baire isomorphic to some completion regular measure on X (resp. on G).*

Proof. This follows immediately from the arguments used in the proof of Theorem 5·8. ▮

COROLLARY 5·12. *If $\alpha = \tau(\alpha)$, then $\zeta(G) = \alpha$.*

Let $\mathscr{K}$ denote the class of all cardinals β such that $\gamma < \beta$ implies $\gamma \aleph_0 < \beta$. This class was considered in [**6**]. The following are now clear for an arbitrary compact group G of uncountable topological weight (compare with theorems 5, 7, 9 and 10 of [**6**]).

THEOREM 5·A. *If $w(G) \in \mathscr{K}$ then $\zeta(G) = w(G)$; hence every completion regular probability measure on G is homogeneous of Maharam type $w(G)$ and any two such measures are completion Baire isomorphic.*

THEOREM 5·B. *If $w(G) = \aleph_1$, then, subject to the continuum hypothesis, $\zeta(G) = \aleph_0$.*

THEOREM 5·C (assuming Martin's Axiom and the negation of the continuum hypothesis). (*a*) *If $w(G) = \aleph_\xi$ (where $\xi \leqslant \omega$), then $\zeta(G) = \aleph_\xi$.*

(*b*) *If $w(G) = c = \aleph_2$, then $\zeta(G) = c$.*

Remarks 5·13. (1) Combining Lemma 5·7 with the claim in the proof of Theorem 5·8, we immediately have (with a different proof) that the Haar measure on G is homogeneous of Maharam type $w(G)$.

(2) The author has been able to show that the following is valid: 'If (X, μ) is any completion regular Radon measure space, then for every completion regular Radon measure ν on G, the Radon product measure $\mu \otimes \nu$ is again completion regular.' In the particular case when ν is the Haar measure, this result is, in some sense, the 'compact group analogue' of [**6**], theorem 3.

(3) We give a short alternative proof of Theorem 5·A only using approximation by Lie groups.

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Using similar notation, assume that μ is any completion regular measure on G. First we claim that for every closed supporting $F \subset G$ there is $H \in Q_G$ such that $F = \pi_H^{-1}\pi_H F$. To prove this, it suffices to show that there is $H \in Q_G$ such that $U = G\backslash F = \pi_H^{-1}\pi_H U$. Let B be a Baire set containing U with $\mu(U) = \mu(B)$. There exists $H \in Q_G$ such that $B = \pi_H^{-1}\pi_H B$; see [**4**], p. 84. Then, since $U \subset \pi_H^{-1}\pi_H U \subset B$ and F is supporting, we must have $U = \pi_H^{-1}\pi_H U$ and so $F = \pi_H^{-1}\pi_H F$.

Our second claim is that if A is any μ-measurable set with $\mu(A) > 0$, then there exists $H = H_A \in Q_G$ such that $\mu(A \cap \pi_H^{-1}\pi_H W) = 0$ for all open sets W with $\mu(A \cap W) = 0$. Since $\mu(A) > 0$, without loss of generality, we can assume that A is supporting and closed. By the first claim, there is some $H \in Q_G$ with $A = \pi_H^{-1}\pi_H A$. We claim that this H has the properties in the second claim. For let W be an open set with $\mu(A \cap W) = 0$. Then, since A is supporting, we have $A \cap W = \varnothing$, so that $W \subset G\backslash A = \pi_H^{-1}\pi_H(G\backslash A)$ which implies that $\pi_H^{-1}\pi_H W \subset G\backslash A$ and that $A \cap \pi_H^{-1}\pi_H W = \varnothing$. This completes the proof of the second claim.

Suppose now, if possible, that μ is of Maharam type $\gamma < \alpha$. Its measure algebra E is then of the form $E = \{b_i, i \in I\}$, $|I| \leqslant \gamma\aleph_0$, where each b_i is the equivalence class of a set $B_i = \pi_{H_i}^{-1}\pi_{H_i} B_i$ for some $H_i \in Q_G$. Since $\alpha > |I|$, we have $H_0 = \bigcap_{i \in I} H_i \neq \{\mathrm{id}_G\}$. By [**19**], p. 85, remark 2, there exists $K \in R_G$ with $K \subset H_0$ such that $H = H_0/K$ is a Lie group and $(G/K)/H \cong G/H_0$. Now let p be the projection of G/K to $(G/K)/H$, β the Haar measure on H and let $\mu' = \pi_K \mu, \nu' = p\mu'$. By theorem 1 in section 5·4 of [**18**] (see also [**14**], theorem 3·1), there exist an open neighbourhood Q of $e = \mathrm{id}_{G/K}$, $F \subset (G/K)/H$ compact, $U \subset F$ open (in F) and $V \subset H$ open (in H) such that (i) $\beta V < 1$ and $Q = U.V$, and (ii) $\pi_H^{-1}\{z\} \cap F$ is a single point whenever $z \in pQ$. Since Q is a neighbourhood of e, we can find $y \in G/K$ such that $\mu'(yQ) > 0$. Set $R := yQ$.

Our third claim is that $\mu(\pi_K^{-1} R \Delta B_i) > 0$, for all $i \in I$. Because the B_i depend on K (as well as on H_0), it suffices to show that $\mu'(R \Delta \pi_K B_i) \neq 0$, for all $i \in I$. Suppose otherwise. Then there must exist a compact set A in G/K with $A = p^{-1}pA$ such that $\mu'(R \Delta A) = 0$. Set $U_1 = yUH, V_1 = FV$. Then, by the choice of Q, it is simple to verify the following relations:

(1) $R = U_1 \cap V_1$, $pR = pU_1$, $p^{-1}pU_1 = U_1$,

(2) $p(R \cap A) = p(A \cap U_1) = pA \cap pU_1$.

We consider two cases.

Case 1. *V is a single point.* Let $V = \{z\} \subset H$. Then, since $R = yUz$, the set yU is open in G/K so, by our second claim above, $\mu'(p^{-1}pR \cap (G/K\backslash A)) = 0$ implies that $R \sim_{\mu'} p^{-1}pR = U_1$, where $\sim_{\mu'}$ denotes equivalence modulo μ'-negligible sets.

Fix any $w \in H, w \neq z$. Then, similarly, $yUw \sim_{\mu'} U_1$. We deduce that $yUz \sim_{\mu'} yUw$, which contradicts the fact that $yUz \cap yUw = \varnothing$. This ends the proof of the third claim in Case 1.

Case 2. *V is not a single point.* In this case we can find two pairwise disjoint open sets Z, W in H such that (*a*) $Z \cup W \subset V$ and (*b*) yUZ, yUW are both open in G/K. We easily deduce that $U_1 \sim_{\mu'} yUZ \sim_{\mu'} yUW$, which is a contradiction. This ends the proof of the claim and accordingly contradicts the fact that $\gamma < \alpha$, completing the proof of the theorem. ∎

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