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FIXED-ROTATION MINKOWSKI AVERAGING: CLASSIFICATION AND SHARP RATES

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Abstract

We study a discrete process on planar convex bodies in which, at each step, a body is replaced by a weighted Minkowski average of itself and its rotation by a fixed angle. Up to translation and uniform scaling, this produces a rigid averaging dynamical system. We give a complete classification of the limit shapes. If the angle is an irrational multiple of 2π, the iterates converge to a disk. If the angle is rational, they converge to the average of finitely many rotated copies of the initial body. We also obtain sharp convergence rates. In the rational case, the decay is uniform and exponential with an explicit constant depending only on the weight and the denominator of the angle. For irrational angles, we prove quantitative rates under a mild number-theoretic condition that holds for almost every angle: low regularity inputs have polynomial decay up to a logarithmic factor, while real analytic inputs have stretched exponential decay. For angles with bounded continued fraction coefficients, we give matching lower bounds along subsequences. These results describe the global attractors of the dynamics and indicate the absence of chaotic behaviour.

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1. Introduction

Iterating geometric transformations of convex bodies and understanding the limit shapes that emerge is a classical theme in convex geometry. A central example is Steiner symmetrisation: by repeatedly symmetrising, one drives a body towards the Euclidean ball, a mechanism underlying many isoperimetric principles and related inequalities (see, for instance, the survey [2]). This viewpoint naturally raises a quantitative question about the speed of convergence, for which sharp and near-sharp results are known in several settings [14]. Within this broader picture, the limiting

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symmetry is known to depend on arithmetic features of the directions: in planar Steiner symmetrisation with an axis that rotates by a fixed angle at each step, an irrational angle leads the iteration towards a disk, whereas a rational angle confines the process to finitely many directions, as in Klain's finite-direction framework [12]. Motivated by these phenomena, we study the rigid dynamics obtained by replacing a body with a weighted Minkowski average of itself and its rotation by a fixed angle. The resulting map is linear on support functions and amenable to spectral analysis, which yields a unified treatment, a complete classification of limit shapes and sharp quantitative convergence rates.

We fix parameters $0 < \lambda < 1$ and $\theta \in (0, 2\pi)$, and work on the phase space

$$\mathcal{K}_* = \{K \subset \mathbb{R}^2 : K \text{ is a convex body}, s(K) = 0, w(K) = 1\},$$

where $s(K)$ is the Steiner point and

$$w(K) = \frac{1}{\pi} \int_0^\pi (h_K(\varphi) + h_K(\varphi + \pi))\, d\varphi$$

is the mean width in terms of the support function $h_K(\varphi) = \max_{x \in K} \langle x, (\cos\varphi, \sin\varphi)\rangle$. Throughout, a *convex body* means a nonempty, compact, convex subset of $\mathbb{R}^2$ with nonempty interior. Standard background, including Minkowski addition and support functions, can be found in [17]. Support functions and convex bodies are in one-to-one correspondence (up to translation), which legitimises working at the level of support functions. For $K \in \mathcal{K}_*$, we define the one-step map

$$\Phi_{\lambda,\theta}(K) = (1 - \lambda)K \oplus \lambda R_\theta K,$$

where R_θ is rotation by θ and $\oplus$ denotes Minkowski addition. The step preserves the Steiner point and the mean width: both are Minkowski linear (and rotation equivariant/invariant, respectively). So, $\mathcal{K}_*$ is invariant and no further normalisation is needed (see Lemma 2.1 and [17]). Starting from $K_0 \in \mathcal{K}_*$, we iterate $K_{n+1} = \Phi_{\lambda,\theta}(K_n)$. Passing to support functions and writing $(S_\theta h)(\varphi) = h(\varphi - \theta)$, the update reads

$$h_{n+1} = (1 - \lambda)h_n + \lambda\, S_\theta h_n,$$

and $w(K_n) \equiv 1$ by invariance. We write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ for the circle; we freely pass between K_n and h_n, and use the $L^2(\mathbb{T})$ and $C^\alpha(\mathbb{T})$ topologies.

Classical symmetrisation procedures such as Steiner and Minkowski symmetrisations are known to round convex bodies under suitable choices of directions. Beyond qualitative convergence (see Schneider's monograph for Hadwiger's early result on Minkowski averages [17, Theorem 3.3.5]), a quantitative and structural theory has emerged: sharp and near-sharp rates for iterative symmetrisations [14], isomorphic rounding by a bounded number of Steiner steps [15], and dimension-efficient Minkowski symmetrisations towards near-Euclidean balls [13]. Refined proximity results for single symmetrals [4] and equality/stability for the perimeter inequality under Steiner symmetrisation [9] further anchor the isoperimetric applications.

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Foundational regularity facts such as continuity of Steiner symmetrisation in $W^{1,p}$ also support a functional-analytic viewpoint [6] (see the survey [2] for a broader account).

Iteration schemes and the choice of directions have been studied along several axes. Convergence in shape and the existence of universal orderings of directions are known [1, 20], with convergence established for finite and countable direction sets [5, 12]. Randomised models exhibit almost-sure convergence for sets and functions, and provide complementary rate phenomena [7, 10, 16, 19]. More recently, general criteria unify many of these processes across families of symmetrisations [3, 18]. Against this background, we fix a single rotation and iterate a weighted Minkowski average, yielding rigid one-parameter averaging dynamics on support functions; its linear, Fourier-diagonalisable structure enables a complete limit-shape classification together with sharp quantitative rates.

The asymptotics is completely rigid. If $\theta/2\pi$ is irrational, then K_n converges to the unit mean width disk; if $\theta = 2\pi p/q$ is rational, then K_n converges to the explicit q-fold rotational average. Quantitatively, in the resonant case, the decay is uniform and exponential with a spectral rate. In the nonresonant case with Diophantine θ, we obtain polynomial decay (up to a logarithmic loss) for C^α data, while analytic data exhibit stretched-exponential decay; in the bounded-type case, we obtain matching subsequence lower bounds. Consequently, the induced map is nonexpansive on the natural function spaces, the topological entropy is zero and, in the irrational case, the global attractor is a single point.

For terminology: by a Diophantine angle, we follow the classical conventions from Diophantine approximation (see Cassels and Khinchin [8, 11]) and for topological entropy, we refer to Walters [21].

2. Limit-shape classification

We now determine the limit shape selected by the iteration on $\mathcal{K}_*$. Throughout, we assume $0 < \lambda < 1$. On support functions, we write $T = (1-\lambda)I + \lambda S_\theta$, as the normalisation is immaterial on $\mathcal{K}_*$.

LEMMA 2.1. *Fix $0 < \lambda < 1$ and $\theta \in (0, 2\pi)$. For $K \in \mathcal{K}_*$, set*

$$K' = (1-\lambda)K \oplus \lambda R_\theta K.$$

Then, $s(K') = 0$ and $w(K') = 1$. In particular, the one-step map on $\mathcal{K}_*$ may be written without normalisation as $K \mapsto (1-\lambda)K \oplus \lambda R_\theta K$. Equivalently, on support functions, the update is

$$h \longmapsto (1-\lambda)h + \lambda S_\theta h, \quad (S_\theta h)(\varphi) = h(\varphi - \theta),$$

and this map preserves mean width $w \equiv 1$.

PROOF. We use standard identities for support functions. For convex bodies $A, B \subset \mathbb{R}^2$ and $t \geq 0$, $h_{A\oplus B} = h_A + h_B, h_{tA} = t h_A, h_{R_\theta A}(\varphi) = h_A(\varphi - \theta)$. These identities and the representation formulae below are classical (see Schneider [17]). The Steiner point admits the representation

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$$s(A) = \frac{1}{\pi}\int_0^{2\pi} u(\varphi)\, h_A(\varphi)\, d\varphi, \quad u(\varphi) = (\cos\varphi, \sin\varphi),$$

hence, it is Minkowski linear and rotation equivariant, and

$$s(A \oplus B) = s(A) + s(B), \quad s(tA) = t\, s(A), \quad s(R_\theta A) = R_\theta s(A).$$

Likewise, the mean width can be written as

$$w(A) = \frac{1}{\pi}\int_0^{\pi} (h_A(\varphi) + h_A(\varphi + \pi))\, d\varphi,$$

so w is Minkowski linear and rotation invariant:

$$w(A \oplus B) = w(A) + w(B), \quad w(tA) = t\, w(A), \quad w(R_\theta A) = w(A).$$

Apply these to $K' = (1-\lambda)K \oplus \lambda R_\theta K$. Since $K \in \mathcal{K}_*$ has $s(K) = 0$ and $w(K) = 1$,

$$s(K') = (1-\lambda)s(K) + \lambda\, s(R_\theta K) = (1-\lambda)\cdot 0 + \lambda\, R_\theta s(K) = 0,$$
$$w(K') = (1-\lambda)w(K) + \lambda\, w(R_\theta K) = (1-\lambda)\cdot 1 + \lambda \cdot 1 = 1.$$

Thus, $K' \in \mathcal{K}_*$. Passing to support functions, the step is exactly $h \mapsto (1-\lambda)h + \lambda S_\theta h$ and the above computation shows that it preserves $w \equiv 1$. □

LEMMA 2.2. *Let $T = (1-\lambda)I + \lambda S_\theta$ act on 2π-periodic functions on the circle, where $(S_\theta h)(\varphi) = h(\varphi - \theta)$ and $0 < \lambda < 1$, $\theta \in (0, 2\pi)$. For the Fourier modes $e_k(\varphi) = e^{ik\varphi}$, $k \in \mathbb{Z}$,*

$$S_\theta e_k(\varphi) = e^{ik(\varphi-\theta)} = e^{-ik\theta} e_k(\varphi), \quad Te_k = \mu_k\, e_k, \quad \mu_k = (1-\lambda) + \lambda e^{-ik\theta},$$

so $\{e_k\}_{k\in\mathbb{Z}}$ is a basis of eigenfunctions and T is diagonal in the Fourier basis. Consequently, if $h(\varphi) = \sum_{k\in\mathbb{Z}} c_k e^{ik\varphi}$ in $L^2(\mathbb{T})$, then

$$Th = \sum_{k\in\mathbb{Z}} \mu_k\, c_k e^{ik\varphi}, \quad T^n h = \sum_{k\in\mathbb{Z}} \mu_k^n c_k e^{ik\varphi} \quad (n \geq 1).$$

In particular, $\mu_0 = 1$, and for $k \neq 0$, one has $|\mu_k| < 1$ if and only if $k\theta \notin 2\pi\mathbb{Z}$, while $\mu_k = 1$ if and only if $k\theta \in 2\pi\mathbb{Z}$.

PROOF. For $e_k(\varphi) = e^{ik\varphi}$, we compute directly

$$S_\theta e_k(\varphi) = e_k(\varphi - \theta) = e^{ik(\varphi-\theta)} = e^{-ik\theta} e^{ik\varphi} = e^{-ik\theta} e_k(\varphi),$$

hence,

$$Te_k = (1-\lambda)e_k + \lambda S_\theta e_k = ((1-\lambda) + \lambda e^{-ik\theta})e_k = \mu_k e_k.$$

By linearity and completeness of the Fourier basis in $L^2(\mathbb{T})$, if $h = \sum c_k e^{ik\varphi}$, then $Th = \sum \mu_k c_k e^{ik\varphi}$ and iterating yields $T^n h = \sum \mu_k^n c_k e^{ik\varphi}$.

It remains to characterise $|\mu_k|$. For any real α,

$$|(1-\lambda) + \lambda e^{-i\alpha}|^2 = (1-\lambda)^2 + \lambda^2 + 2\lambda(1-\lambda)\cos\alpha = 1 - 2\lambda(1-\lambda)(1-\cos\alpha) \leq 1,$$

with equality if and only if $\cos\alpha = 1$, that is, $\alpha \in 2\pi\mathbb{Z}$. Taking $\alpha = k\theta$ gives $|\mu_k| < 1$ exactly when $k\theta \notin 2\pi\mathbb{Z}$, and $\mu_k = 1$ exactly when $k\theta \in 2\pi\mathbb{Z}$. Since $e^{0\cdot i\varphi} \equiv 1$, we also

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have $\mu_0 = (1-\lambda) + \lambda = 1$, which corresponds to preservation of the mean (and hence of the mean width). □

THEOREM 2.3. *Assume $0 < \lambda < 1$ and $\theta \in (0, 2\pi)$. Let $K_{n+1} = (1-\lambda)K_n \oplus \lambda R_\theta K_n$ be the iteration on $\mathcal{K}_*$ and let h_n be the support function of K_n. Then, with the map $T = (1-\lambda)I + \lambda S_\theta$ acting on support functions,*

$$h_n = T^n h_0 \quad \text{in } L^2(\mathbb{T}).$$

Moreover, we have the following.

(i) *Nonresonant case. If $\theta/2\pi \notin \mathbb{Q}$, then $h_n \to \bar{h}$ in $L^2(\mathbb{T})$, where*

$$\bar{h} \equiv \frac{1}{2\pi}\int_0^{2\pi} h_0(\varphi)\, d\varphi = \frac{1}{2}$$

is the constant function corresponding to the unit mean width disk. In particular, K_n converges, in the L^2-support-function sense, to the unit mean width disk.

(ii) *Resonant case. If $\theta = 2\pi p/q$ with coprime $p, q \in \mathbb{N}$, then $h_n \to P_q h_0$ in $L^2(\mathbb{T})$, where*

$$P_q = \frac{1}{q}\sum_{j=0}^{q-1} S_{j\theta}$$

is the averaging projector onto the S_θ-invariant subspace. Equivalently,

$$K_n \longrightarrow K_\infty = \frac{1}{q}\sum_{j=0}^{q-1} R_{j\theta} K_0 \in \mathcal{K}_*,$$

whose support function is $h_\infty = P_q h_0$ and which has q-fold rotational symmetry.

The limit is unique and depends on K_0 only through its projection onto the S_θ-invariant subspace.

PROOF. By Lemma 2.1, on $\mathcal{K}_*$, the one-step map is $K \mapsto (1-\lambda)K \oplus \lambda R_\theta K$ and, on support functions, $h \mapsto Th = (1-\lambda)h + \lambda S_\theta h$. Write the Fourier expansion $h_0(\varphi) = \sum_{k\in\mathbb{Z}} c_k e^{ik\varphi}$ in $L^2(\mathbb{T})$. Lemma 2.2 yields $Te_k = \mu_k e_k$, $\mu_k = (1-\lambda) + \lambda e^{-ik\theta}$, hence,

$$T^n h_0 = \sum_{k\in\mathbb{Z}} \mu_k^n c_k e^{ik\varphi} \quad \text{in } L^2(\mathbb{T}).$$

For case (i), if $\theta/2\pi \notin \mathbb{Q}$, then $k\theta \notin 2\pi\mathbb{Z}$ for all $k \neq 0$, so $|\mu_k| < 1$ by Lemma 2.2. Therefore, $\mu_k^n \to 0$ for $k \neq 0$, while $\mu_0 = 1$. It follows that $T^n h_0 \to c_0 e^{i0\varphi} = \bar{h}$ in $L^2(\mathbb{T})$. Since the iteration preserves the mean width $w \equiv 1$ (Lemma 2.1), we have $\bar{h} = \frac{1}{2}$, which is the support function of the unit mean width disk. This identifies the limit shape.

For case (ii), let $\theta = 2\pi p/q$ with $(p, q) = 1$. Then, $\mu_k = 1$ exactly when $k \equiv 0 \pmod q$, and $|\mu_k| < 1$ otherwise (Lemma 2.2). Hence, $T^n h_0 \to \sum_{m\in\mathbb{Z}} c_{mq} e^{i(mq)\varphi}$ in $L^2(\mathbb{T})$. The operator

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$$P_q = \frac{1}{q}\sum_{j=0}^{q-1} S_{j\theta}$$

acts on Fourier modes by $P_q e_k = e_k$ if $k \equiv 0 \pmod q$ and $P_q e_k = 0$ otherwise, so indeed, $T^n h_0 \to P_q h_0$ in $L^2(\mathbb{T})$. By Minkowski linearity of support functions and rotation covariance,

$$h_{\left(\frac{1}{q}\sum_{j=0}^{q-1} R_{j\theta}K_0\right)} = P_q h_0,$$

while s and w are preserved under this averaging (as in Lemma 2.1), so the geometric limit is exactly

$$K_\infty = \frac{1}{q}\sum_{j=0}^{q-1} R_{j\theta}K_0 \in \mathcal{K}_*.$$

Uniqueness follows from the L^2 convergence of $T^n h_0$ to the indicated limit and linearity. □

3. Sharp rates and dynamics

In this section, we quantify the convergence established above. For resonant angles $\theta = 2\pi p/q$, we obtain uniform exponential decay towards $P_q h_0$ with an explicit spectral constant $\rho \in (0, 1)$. For nonresonant angles with Diophantine θ, we obtain quantitative polynomial decay (up to a logarithmic loss) for C^α data and, under the same assumption, analytic data yield stretched-exponential decay; for bounded-type angles, the exponent is optimal along subsequences. We then record dynamical consequences: the averaging map is nonexpansive, the topological entropy is zero, the global attractor is a single point in the irrational case, and in the rational case, it is the S_θ-invariant subspace.

THEOREM 3.1. *Assume $0 < \lambda < 1$ and $\theta \in (0, 2\pi)$, and let $T = (1-\lambda)I + \lambda S_\theta$ act on support functions as above.*

(i) *Resonant case ($\theta = 2\pi p/q$ with $(p, q) = 1$). Let*

$$P_q = \frac{1}{q}\sum_{j=0}^{q-1} S_{j\theta}, \quad \rho = \max_{1\le r\le q-1} |(1-\lambda) + \lambda e^{-2\pi i r/q}| \in (0, 1).$$

Then, for all $h_0 \in L^2(\mathbb{T})$,

$$\|T^n h_0 - P_q h_0\|_{L^2} \le \rho^n \|h_0 - P_q h_0\|_{L^2} \quad (n \ge 1),$$

and if $h_0 \in C^\alpha(\mathbb{T})$ for some $\alpha > 0$, there exists $C_\alpha = C_\alpha(\alpha, \lambda, q)$ such that

$$\|T^n h_0 - P_q h_0\|_{L^\infty} \le C_\alpha \rho^n \|h_0\|_{C^\alpha} \quad (n \ge 1).$$

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(ii) *Nonresonant (Hölder data) ($\theta/2\pi \notin \mathbb{Q}$). Assume θ is Diophantine of type (c,τ), that is,*

$$\operatorname{dist}(k\theta, 2\pi\mathbb{Z}) \;\geq\; c\,|k|^{-\tau} \quad (k \in \mathbb{Z}\setminus\{0\}),$$

and $h_0 \in C^\alpha(\mathbb{T})$ for some $\alpha > 0$. Then, there exists $C > 0$ depending only on (α,τ,λ,c) and $\|h_0\|_{C^\alpha}$ such that, for all $n \geq 2$,

$$\|T^n h_0 - \bar{h}\|_{L^\infty} \;\leq\; C\,n^{-\alpha/2\tau}(\log n)^{\alpha/2\tau},$$

where $\bar{h} \equiv 1/(2\pi)\int_0^{2\pi} h_0(\varphi)\,d\varphi = \frac{1}{2}$. The implicit constants depend only on (α,τ,λ,c) and $\|h_0\|_{C^\alpha}$.

(iii) *Nonresonant (analytic data, Diophantine) ($\theta/2\pi \notin \mathbb{Q}$ and θ is Diophantine of type (c,τ)). If h_0 extends analytically to the strip $\{|\Im\varphi| < \sigma\}$ for some $\sigma > 0$ (equivalently $|c_k| \leq C_0 e^{-\sigma|k|}$ for its Fourier coefficients), then there exist constants $C, c' > 0$ depending only on (σ,τ,λ,c) and $\|h_0\|_{L^\infty}$ such that*

$$\|T^n h_0 - \bar{h}\|_{L^\infty} \;\leq\; C\,\exp(-c'\,n^{1/(2\tau+1)}) \quad (n \geq 1).$$

PROOF. Write the Fourier expansion $h_0(\varphi) = \sum_{k\in\mathbb{Z}} c_k e^{ik\varphi}$ in $L^2(\mathbb{T})$. By Lemma 2.2, $\mu_k = (1-\lambda) + \lambda e^{-ik\theta}$, $Te_k = \mu_k e_k$, hence, $T^n h_0 = \sum_k \mu_k^n c_k e^{ik\varphi}$.

(i) *Resonant.* If $\theta = 2\pi p/q$, then $\mu_k = 1$ exactly when $k \equiv 0 \pmod q$ and otherwise, $|\mu_k| \leq \rho < 1$. Since $P_q e_k = e_k$ for $k \equiv 0 \pmod q$ and 0 otherwise,

$$T^n h_0 - P_q h_0 = \sum_{k\not\equiv 0\,(q)} \mu_k^n c_k e^{ik\varphi}.$$

Parseval gives the L^2 estimate. If $h_0 \in C^\alpha$, then $|c_k| \leq C_\alpha \|h_0\|_{C^\alpha}|k|^{-(1+\alpha)}$, so $\sum_k |c_k| < \infty$ and

$$\|T^n h_0 - P_q h_0\|_{L^\infty} \;\leq\; \sum_{k\not\equiv 0\,(q)} |\mu_k|^n |c_k| \;\leq\; \rho^n \sum_{k\not\equiv 0\,(q)} |c_k| \;\leq\; C_\alpha\,\rho^n\,\|h_0\|_{C^\alpha}.$$

(ii) *Hölder.* For any real x, set $t := 2\lambda(1-\lambda)(1-\cos x) \in [0, 4\lambda(1-\lambda)]$. Then,

$$|(1-\lambda) + \lambda e^{-ix}|^2 = 1 - t, \quad |(1-\lambda) + \lambda e^{-ix}| = \sqrt{1-t} \leq e^{-t/2},$$

and hence,

$$|(1-\lambda) + \lambda e^{-ix}| \leq \exp(-\lambda(1-\lambda)(1-\cos x))$$

and $1 - \cos x \geq (2/\pi^2)\operatorname{dist}(x, 2\pi\mathbb{Z})^2$. With the Diophantine bound, this yields

$$|\mu_k|^n \;\leq\; \exp(-a_0\,n\,|k|^{-2\tau}), \quad a_0 = a_0(\lambda,c) > 0.$$

Hence,

$$\|T^n h_0 - \bar{h}\|_{L^\infty} \;\leq\; \sum_{k\neq 0} e^{-a_0 n|k|^{-2\tau}}\,|c_k|.$$

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Split at a cutoff $K \geq 1$:

$$\sum_{1\leq|k|\leq K} e^{-a_0 n|k|^{-2\tau}}|c_k| \;\leq\; e^{-a_0 nK^{-2\tau}} \sum_{1\leq|k|\leq K} |c_k| \;\leq\; C_1\, e^{-a_0 nK^{-2\tau}}\|h_0\|_{C^\alpha},$$

and using $|c_k| \leq C_\alpha \|h_0\|_{C^\alpha}|k|^{-(1+\alpha)}$,

$$\sum_{|k|>K} |c_k| \;\leq\; C_2\, \|h_0\|_{C^\alpha}\, K^{-\alpha}.$$

Choose $K = K(n)$ so that $e^{-a_0 nK^{-2\tau}} \approx K^{-\alpha}$, that is, $a_0 nK^{-2\tau} \sim \alpha \log K$. For $n \geq 2$, one may take

$$K \asymp \left(\frac{n}{\log n}\right)^{1/(2\tau)},$$

which gives

$$\|T^n h_0 - \bar{h}\|_{L^\infty} \;\leq\; C\,\|h_0\|_{C^\alpha}\, K^{-\alpha} \;\leq\; C\,\|h_0\|_{C^\alpha}\, n^{-\alpha/2\tau}(\log n)^{\alpha/2\tau}.$$

(iii) *Analytic.* If $|c_k| \leq C_0 e^{-\sigma|k|}$, then

$$\|T^n h_0 - \bar{h}\|_{L^\infty} \;\leq\; C_0 \sum_{k\geq 1} \exp(-a_0 nk^{-2\tau} - \sigma k).$$

Let $k_* \asymp (n)^{1/(2\tau+1)}$ be the minimiser of $a_0 nk^{-2\tau} + \sigma k$. Then, for $k \leq k_*$, we have $a_0 nk^{-2\tau} \geq c\, n^{1/(2\tau+1)}$, while for $k \geq k_*$, $\sigma k \geq c\, n^{1/(2\tau+1)}$. Hence, for all $k \geq 1$,

$$\exp(-a_0 nk^{-2\tau} - \sigma k) \;\leq\; \exp(-c\, n^{1/(2\tau+1)})\, e^{-\sigma k/2},$$

and summing the geometric tail $\sum_{k\geq 1} e^{-\sigma k/2}$ yields

$$\|T^n h_0 - \bar{h}\|_{L^\infty} \;\leq\; C\, \exp(-c'\, n^{1/(2\tau+1)}).$$ □

PROPOSITION 3.2. *Assume $0 < \lambda < 1$ and $\theta/2\pi \notin \mathbb{Q}$ is of bounded type (equivalently, Diophantine with exponent $\tau = 1$). Then, there exist $h_0 \in \mathcal{K}_*$, analytic in a strip $\{|\Im\varphi| < \sigma_0\}$, and an increasing sequence $n_j \to \infty$ such that*

$$\|T^{n_j} h_0 - \bar{h}\|_{L^\infty} \geq C\, \exp(-C' n_j^{1/3}) \quad (j \geq 1)$$

for some constants $C, C' > 0$ depending only on λ, on the bounded-type constants of θ and on the analyticity width of h_0. The stretched-exponential exponent $1/3$ in Theorem 3.1(iii) cannot be improved along subsequences for bounded-type angles.

PROOF. Let (q_ℓ) be the convergents of $\theta/2\pi$. Since θ is of bounded type, there exists $M \geq 1$ such that $q_{\ell+1} \leq Mq_\ell$ for all ℓ. Choose a sparse subsequence (k_j) by setting $k_j := q_{m_j}$ with m_{j+1} large enough that $k_{j+1} \geq k_j^2$ and $k_1 \geq 2$. Define

$$h_0(\varphi) = \frac{1}{2} + \sum_{j\geq 1} b_j \cos(k_j\varphi), \quad b_j := \min\left\{e^{-\sigma k_j},\; \frac{1}{4(k_j^2-1)}\right\},$$

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and fix $\sigma > 0$ so that $\sum_j b_j k_j^2 \le \frac{1}{4}$. Then, h_0 is analytic in $\{|\Im\varphi| < \sigma\}$, $h_0 + h_0'' \ge \frac{1}{4} > 0$, and since the first harmonic is absent and the mean is $\frac{1}{2}$, we have $K_0 \in \mathcal{K}_*$.

For convergents, one has

$$\operatorname{dist}(q_\ell\theta, 2\pi\mathbb{Z}) \le \frac{2\pi}{q_{\ell+1}} \le \frac{2\pi M}{q_\ell}, \quad 1 - \cos(q_\ell\theta) \;\le\; \frac{1}{2}\operatorname{dist}(q_\ell\theta, 2\pi\mathbb{Z})^2 \;\le\; \frac{C_1}{q_\ell^2}.$$

Set $t := 2\lambda(1-\lambda)(1-\cos x)$. For $t \in [0, \frac{1}{2}]$, we have $\sqrt{1-t} \ge e^{-t}$. Thus, for j large enough,

$$|\mu_{k_j}| = \sqrt{1 - 2\lambda(1-\lambda)(1-\cos(k_j\theta))} \;\ge\; \exp\Big(-\frac{C_2}{k_j^2}\Big),$$

with $C_2 = C_2(\lambda, M)$. Choose

$$n_j := \Big\lfloor \frac{\sigma}{2C_2}\, k_j^3 \Big\rfloor.$$

Then, $|\mu_{k_j}|^{n_j} \ge \exp(-\sigma k_j/2)$, so the principal mode contributes at least $b_j e^{-\sigma k_j/2}$ at n_j.

It remains to show that the contribution of the other modes is negligible. Since bounded type implies a Diophantine lower bound $\operatorname{dist}(k\theta, 2\pi\mathbb{Z}) \ge c_0/k$ for all $k \ge 1$,

$$|\mu_k| \le \exp(-c_1\, k^{-2}), \quad c_1 = c_1(\lambda, c_0) > 0.$$

For indices $i < j$,

$$|\mu_{k_i}|^{n_j} \le \exp\Big(-c_1\,\frac{n_j}{k_i^2}\Big) \le \exp\Big(-c_3\,\frac{k_j^3}{k_{j-1}^2}\Big) \le \exp(-c_4\, k_j^2),$$

using $k_{j-1} \le \sqrt{k_j}$. For $i > j$, we simply use $|\mu_{k_i}|^{n_j} \le 1$ and the tail of the analytic coefficients,

$$\sum_{i>j} b_i |\mu_{k_i}|^{n_j} \le \sum_{i>j} e^{-\sigma k_i} \le C\, e^{-\sigma k_{j+1}} \le C\, e^{-\sigma k_j^2},$$

$$\sum_{i\ne j} b_i\, |\mu_{k_i}|^{n_j} \le C\, e^{-\sigma k_j^2} \ll b_j e^{-\sigma k_j/2} \quad \text{for large } j,$$

$$\|T^{n_j} h_0 - \bar h\|_{L^\infty} \ge \tfrac{1}{2}\, b_j e^{-\sigma k_j/2} \ge c\, e^{-Ck_j}.$$

Since $n_j \asymp k_j^3$, we conclude $\|T^{n_j} h_0 - \bar h\|_{L^\infty} \ge C \exp(-C'\, n_j^{1/3})$, as claimed. □

PROPOSITION 3.3. *Assume $0 < \lambda < 1$ and $\theta \in (0, 2\pi)$.*

(i) *Resonant case ($\theta = 2\pi p/q$ with $(p, q) = 1$). Let*

$$\rho = \max_{1\le r\le q-1} |(1-\lambda) + \lambda e^{-2\pi i r/q}| \in (0, 1).$$

There exists $h_0 \in C^\infty(\mathbb{T})$ which is the support function of some $K_0 \in \mathcal{K}_*$ such that

$$\|T^n h_0 - P_q h_0\|_{L^\infty} \ge c\rho^n \quad \textit{for all } n \ge 1,$$

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for some $c > 0$ depending on h_0 and λ, q. Hence, the exponential rate ρ in Theorem 3.1(i) cannot be improved in general.

(ii) *Nonresonant case ($\theta/2\pi \notin \mathbb{Q}$). For every $\rho_0 \in (0,1)$ and every $\alpha > 0$, there exists $h_0 \in C^\infty(\mathbb{T})$ which is the support function of some $K_0 \in \mathcal{K}_*$ such that for all $C > 0$, the bound*

$$\|T^n h_0 - \bar{h}\|_{L^\infty} \le C\rho_0^n \quad (n \ge 1)$$

fails. In particular, no uniform exponential rate holds in the irrational case.

PROOF. We work on support functions and use Lemma 2.2. Throughout, we ensure that the constructed h_0 is a valid support function in $\mathcal{K}_*$ by taking a small Fourier perturbation of the constant $\frac{1}{2}$, which fixes the mean width and, if needed, avoids the first harmonic to keep the Steiner point at the origin.

(i) *Resonant.* Choose $r_* \in \{1, \dots, q-1\}$ attaining the maximum in the definition of ρ. Pick an integer $k \ge 2$ with $k \equiv r_* \pmod q$. Let

$$h_0(\varphi) = \frac{1}{2} + \varepsilon \cos(k\varphi), \quad 0 < \varepsilon \le \frac{1}{2\,(k^2-1)}.$$

Then, $h_0 \in C^\infty$ and $h_0 + h_0'' = \frac{1}{2} + \varepsilon(1-k^2)\cos(k\varphi) \ge 0$, so h_0 is the support function of a convex body; its mean is $\frac{1}{2}$ and the $k=1$ mode is absent, and hence, $K_0 \in \mathcal{K}_*$. Since $k \not\equiv 0 \pmod q$, we have $P_q h_0 = \frac{1}{2}$. By Lemma 2.2,

$$T^n h_0 - P_q h_0 = \varepsilon\, \Re(\mu_k^n e^{ik\varphi}), \quad \mu_k = (1-\lambda) + \lambda e^{-ik\theta}.$$

Because $k \equiv r_* \pmod q$ and $\theta = 2\pi p/q$, we have $e^{-ik\theta} = e^{-2\pi i r_*/q}$ and thus, $|\mu_k| = \rho$. Taking L^∞ norms gives $\|T^n h_0 - P_q h_0\|_{L^\infty} = \varepsilon\,|\mu_k|^n = \varepsilon\,\rho^n$, so the claim holds with $c = \varepsilon$. In particular, no smaller base than ρ can work uniformly.

(ii) *Nonresonant.* Fix $\rho_0 \in (0,1)$. Since $\theta/2\pi$ is irrational, the set $\{e^{-ik\theta} : k \in \mathbb{Z}\}$ is dense on the unit circle, and hence there exists $k \ge 2$ such that

$$|\mu_k| = |(1-\lambda) + \lambda e^{-ik\theta}| > \rho_0.$$

Define $h_0(\varphi) = \frac{1}{2} + \varepsilon\cos(k\varphi)$ with $0 < \varepsilon \le 1/(2(k^2-1))$ as above, so h_0 is the support function of a body in $\mathcal{K}_*$. Then, exactly as in part (i), $\|T^n h_0 - \bar{h}\|_{L^\infty} = \varepsilon\,|\mu_k|^n$. Given any $C > 0$, since $|\mu_k|/\rho_0 > 1$, the ratio

$$\frac{\varepsilon|\mu_k|^n}{C\rho_0^n} = \frac{\varepsilon}{C}\Big(\frac{|\mu_k|}{\rho_0}\Big)^n \to \infty \quad (n \to \infty),$$

so the putative bound $\|T^n h_0 - \bar{h}\|_{L^\infty} \le C\rho_0^n$ fails for all large n. Therefore, no uniform exponential estimate can hold over all admissible initial data in the irrational case. □

PROPOSITION 3.4. *Assume $0 < \lambda < 1$ and $\theta \in (0, 2\pi)$, and let $T = (1-\lambda)I + \lambda S_\theta$ act on support functions.*

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(i) *Nonexpansiveness. For* $X = L^2(\mathbb{T})$ *or* $X = C^\alpha(\mathbb{T})$ *(with* $\alpha \geq 0$*), we have* $\|Th - Tg\|_X \leq \|h - g\|_X$ *(for all* $h, g \in X$*). In particular,* T *is* 1*-Lipschitz on these spaces. Equivalently, in the geometric picture,* $K \mapsto (1-\lambda)K \oplus \lambda R_\theta K$ *is nonexpansive in the* L^∞ *support-function metric; for compact convex bodies, one has* $d_H(K, L) = \|h_K - h_L\|_{L^\infty}$ *(no centring assumption is needed).*

(ii) *Zero topological entropy. The Bowen topological entropy of* T *is* 0*: for every compact* $K \subset X$ *and every* $\varepsilon > 0$*, the maximal cardinality of* (n, ε)*-separated sets in* K *is bounded independently of* n*, and hence,* $h_{\mathrm{top}}(T|_K) = 0$ *and so* $h_{\mathrm{top}}(T) = 0$.

(iii) *Attractor / limit set. If* $\theta/2\pi \notin \mathbb{Q}$*, then* $\mathrm{Fix}(T) = \{$*constants*$\}$ *and, within the normalised class,* $\mathrm{Fix}(T) = \{\bar{h}\}$ *with* $\bar{h} \equiv \frac{1}{2}$*. For every* h_0*, one has* $T^n h_0 \to \bar{h}$*; thus, the global (point) attractor is the single point* $\bar{h}$*. If* $\theta = 2\pi p/q$*, then* $\mathrm{Fix}(T) = \mathrm{Range}(P_q)$*. For every* h_0*, one has* $T^n h_0 \to P_q h_0 \in \mathrm{Fix}(T)$*. Hence, the invariant set* $A_q = \mathrm{Range}(P_q) \cap \{s = 0, w = 1\}$ *attracts every initial datum, and each orbit converges to a single point in* A_q*.*

PROOF.

(i) For any normed space X among L^2, L^∞, C^α, the shift S_θ is an isometry: $\|S_\theta f\|_X = \|f\|_X$. Hence,

$$\begin{aligned}\|Th - Tg\|_X &= \|(1-\lambda)(h-g) + \lambda(S_\theta h - S_\theta g)\|_X \\ &\leq (1-\lambda)\|h-g\|_X + \lambda\|S_\theta(h-g)\|_X = \|h-g\|_X,\end{aligned}$$

showing 1-Lipschitz. For convex bodies, $\|h_K - h_L\|_{L^\infty}$ equals the Hausdorff distance $d_H(K, L)$, so nonexpansiveness transfers to the geometric map.

(ii) Let $K \subset X$ be compact and fix $\varepsilon > 0$. If $\{x_1, \ldots, x_M\} \subset K$ is (n, ε)-separated (in Bowen's sense), then for some $0 \leq j < n$, we must have $\|T^j x_a - T^j x_b\|_X \geq \varepsilon$ whenever $a \neq b$. Since T is 1-Lipschitz, $\|x_a - x_b\|_X \geq \varepsilon$ already at time 0. Thus, M is bounded by the ε-packing number of K, which depends on K, ε but not on n. Therefore, the exponential growth rate of (n, ε)-separated sets is zero, that is, $h_{\mathrm{top}}(T|_K) = 0$. Taking the supremum over compact K gives $h_{\mathrm{top}}(T) = 0$.

(iii) The description of $\mathrm{Fix}(T)$ follows from Lemma 2.2: $Te_k = \mu_k e_k$ with $\mu_k = 1$ if and only if $k = 0$ in the irrational case, and if and only if $k \equiv 0 \pmod q$ in the resonant case. The convergence statements are exactly Theorem 2.3. In particular, every orbit converges to a single point in the relevant fixed set: to $\bar{h}$ when $\theta/2\pi \notin \mathbb{Q}$ and to $P_q h_0$ when $\theta = 2\pi p/q$. This proves the stated attractor properties. □

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