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RESEARCH ARTICLE

Linear dependencies, polynomial factors in the Duke–Erdős forbidden sunflower problem†

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Abstract

We call a family of s sets $\{F_1, \dots, F_s\}$ a *sunflower with s petals* if, for any distinct $i, j \in [s]$, one has $F_i \cap F_j = \cap_{u=1}^s F_u$. The set $C = \cap_{u=1}^s F_u$ is called the *core* of the sunflower. It is a classical result of Erdős and Rado that there is a function $\phi(s,k)$ such that any family of k-element sets contains a sunflower with s petals. In 1977, Duke and Erdős asked for the size of the largest family $\mathcal{F} \subset \binom{[n]}{k}$ that contains no sunflower with s petals and core of size $t-1$. In 1987, Frankl and Füredi asymptotically solved this problem for $k \geqslant 2t+1$ and $n > n_0(s,k)$. Their paper is one of the pinnacles of the so-called Delta-system method.

In this paper, we extend the result of Frankl and Füredi to a much broader range of parameters: $n > poly(s,t)k$, when additionally $k > poly(s,t)\log n$ or $n > n_0(s,t)$. We also extend this result to other domains, such as $[n]^k$ and $\binom{n}{k/w}^w$ and obtain even stronger and more general results for forbidden sunflowers with core at most $t-1$ (including results for families of permutations and subfamilies of the k-th layer in a simplicial complex).

The methods of the paper, among other things, combine the spread approximation approach, introduced by Zakharov and the first author, with the Delta-system approach of Frankl and Füredi and the hypercontractivity for global functions approach, developed by Keller, Lifshitz, and coauthors. Previous works in extremal set theory relied on at most one of these approaches. Creating such a unified view was one of the goals for the paper.

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1. Introduction

We employ notation, standard for Extremal Set Theory: $[n]$ denotes the set $\{1,\dots,n\}$, and, given a set X and an integer k, we denote by 2^X ($\binom{X}{k}$) the family of all subsets of X (all subsets of X of size k). If a family $\mathcal{F} \subset \binom{[n]}{k}$ then we say that $\mathcal{F}$ is *k-uniform*. We call a family of s sets $\{F_1,\dots,F_s\}$ a *$\Delta(s)$-system* or a *sunflower with s petals* if, for any distinct $i, j \in [s]$, one has $F_i \cap F_j = \cap_{u=1}^s F_u$. The set $C = \cap_{u=1}^s F_u$ is called the *core* or *kernel* of the sunflower.

One of the founding papers for extremal set theory (and probably the second paper written on the subject) studies Δ-systems in families of sets. In 1960, Erdős and Rado [19] proved a result of Ramsey-theoretic flavor that states that a k-uniform family of size bigger than $k!(s-1)^k$ contains a $\Delta(s)$-system. The central object of their study plays one of the key roles in our paper.

Definition 1. Given positive integers s, k, define a function $\phi(s,k)$ as the maximum size of a k-uniform family $\mathcal{T}$ that does not contain a $\Delta(s)$-system.

Erdős and Rado also showed that $\phi(s,k) > (s-1)^{k+1}$ and stated that "it is not improbable that" $\phi(s,k) \leqslant (Cs)^{k+1}$, where C is some absolute constant. This is one of the most important unsolved problems in Extremal Combinatorics and one of Paul Erdős' 1000$ problems.

Over a long period of time, the best known upper bound on $\phi(s,t)$ was of the form $t^{t(1+o(1))}$, where the $o(1)$ term depends on s, until Alweiss, Lovett, Wu, and Zhang in a breakthrough paper [1] proved that $\phi(s,t) \leqslant (Cs^3(\ln t)\ln\ln t)^t$ for some absolute constant C. This result was pushed a bit further in several papers [53, 5] and blogposts.[1] The current record $\phi(s,t) \leqslant (Cs\ln t)^t$ with an absolute constant C is due to Bell, Chueluecha, and Warnke [5].

This paper is devoted to the study of the following closely related problem, which dates back to Duke and Erdős [15].

Probem 1 (Duke and Erdős, 1977). Suppose that a family $\mathcal{F} \subset \binom{[n]}{k}$ does not contain a sunflower with s petals and core of size $t-1$. What is the maximum size of $\mathcal{F}$?

In the 1970s, Erdős and coauthors posed several very general problems that became defining for the field of Extremal Set Theory. Problem 1 generalizes the famous forbidden one intersection problem of Erdős and Sós. In 1971, Erdős and Sós (see [21]) conjectured[2] that for $k \geqslant 2t$ and $n > n_0(k)$, the largest

[1] https://terrytao.wordpress.com/2020/07/20/the-sunflower-lemma-via-shannon-entropy/
[2] The original conjecture of Erdős and Sós is for families avoiding intersection size 1, the general case is due to Erdős.

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family $\mathcal{F} \subset \binom{[n]}{k}$ with no $A, B \in \mathcal{F}$ such that $|A \cap B| = t - 1$ has the form

$$\mathcal{F}_T = \left\{ F \in \binom{[n]}{k} : T \subset F \right\} \tag{1.1}$$

for some T of size t. The conjecture was resolved by Frankl and Füredi in a highly influential paper [27]. Great progress was made by Keller and Lifshitz [40] who solved the Erdős-Sós problem for *n linear* in k, but with unspecified dependence on t. Their result was further refined by Ellis, Keller, and Lifshitz [17], who managed to extend it to the regime $k \leqslant (1/2 - \varepsilon)n$ for arbitrarily small ε and $n > n_0(t, \varepsilon)$. Recently, Kupavskii and Zakharov [49] showed that any extremal example in the Erdős-Sós problem is isomorphic to (1.1) under *polynomial* dependencies between n, k, and t: that is, for $n = k^\alpha$ and $t = k^\beta$ with $0 < \beta < 1/2$ and $\alpha > 1 + 2\beta$, provided $k > k_0$. A part of the reason the problem of Erdős and Sós received a lot of attention is because of its numerous applications: in discrete geometry [29], communication complexity [55] and quantum computing [10]. We remark that, in the applications, it is important to allow t to grow with n.

In 1978, Deza, Erdős, and Frankl [13] studied the following generalization of the Erdős–Ko–Rado theorem [20] and the Erdős–Sós problem. For positive integers n, k and a set $L \subset \{0, \dots, k-1\}$, an (n, k, L)-*system* is a family $\mathcal{F} \subset \binom{[n]}{k}$ such that $|A \cap B| \in L$ for any $A, B \in L$. Deza, Erdős, and Frankl proved a general bound on the size of the largest (n, k, L)-system with $|L| = r$, provided $n > n_0(k)$. In this paper, they introduced the Delta-system method, which makes use of large sunflowers in the family in order to discern its lower-uniformity structure. The method was later developed in the works by Frankl and Füredi and had a huge impact on Extremal Set Theory, see [26, 27, 35, 50]. We discuss some aspects of the Delta-system method in Section 5.1. We note here that "forbidden intersections" or "allowed intersections" problems often have algebraic flavor. In particular, there are numerous results that use linear-algebraic methods [3], as well as many of these questions are directly related to the existence of designs of a certain type (cf., e.g., [23]).

Another particular case of Problem 1 of great importance is as follows. We say that sets $F_1, \dots, F_s$ form a *matching* of size s if they are pairwise disjoint. A matching is a sunflower with s petals and empty core. That is, if we substitute $t = 1$ in Problem 1, we get the following famous problem. The Erdős Matching Conjecture [18] states that any family $\mathcal{F} \subset \binom{[n]}{k}$ with no matching of size s satisfies

$$|\mathcal{F}| \leqslant \max\left\{ \binom{ks-1}{k}, \binom{n}{k} - \binom{n-(s-1)}{k} \right\},$$

provided $n \geqslant ks - 1$. Erdős [18] proved the conjecture for $n \geqslant f(k)s$. After numerous improvements [6, 36, 33, 24], it was shown [28] that the conjecture holds for $n \geqslant \frac{5}{3}sk$ and $s > s_0$. In [44] the conjecture was proved in the regime when n is only slightly bigger than sk.

Let us return to the question of Duke and Erdős. In their seminal paper [26], Frankl and Füredi derived the following asymptotic bound on a sunflower-free family $|\mathcal{F}|$ when s, k, t are fixed and n tends to infinity.

Theorem 1.1. *Let s, k, t be integers, $k \geqslant 2t + 1$. Suppose that s, k, t are fixed and $n \to \infty$. Then*

$$|\mathcal{F}| \leqslant (\phi(s,t) + o(1))\binom{n}{k-t}.$$

Frankl and Füredi also showed that this bound is asymptotically the best possible. The extremal example is given in Section 3. Theorem 1.1 holds even though we know little about the behavior of the function $\phi(s,t)$, which actually is a major complication for the study of the problem. The proof of the theorem is one of the pinnacles of the Delta-system method. (The authors use sunflowers in order to find sunflowers!) Previous to [26], partial cases $t \in \{2, 3\}$, $k = 3$ and arbitrary s were studied by Duke and Erdős [15], Frankl [22], Chung [11], and Frankl and Chung [12]. Recently, the case $k = 4$ was

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studied by Bucić et al. in [9]. Later, Bradač, Bucić, and Sudakov [8] proved that an extremal family in Problem 1 has size $O_k(n^{k-t}s^t)$ when n tends to infinity, s grows arbitrarily with n, and k is fixed. It is also worth mentioning some recent results on generalizations of the Duke–Erdős problem [37]. The authors of [37] address the case where sunflowers whose core sizes belong to a set L are forbidden.

In this paper, we deal with the case when n is potentially *linear* in k, while s is small. The following is a coarse version of our main result, stated and discussed in Section 3.

Theorem 1.2. *Fix integers n, k, t, $t \geqslant 2$. There exists a function $f_0(s,t)$, polynomial in both s and t, such that if $n \geqslant f_0(s,t)\cdot k$, $k \geqslant 2t+1$ and $n \geqslant n_0(s,t)$, then the following holds. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a family without a sunflower with s petals and the core of size $t-1$. Then, we have*

$$|\mathcal{F}| \leqslant \left(1 + \sqrt[3]{\frac{f_0(s,t)\cdot k}{n}}\right)\phi(s,t)\binom{n-t}{k-t},$$

Moreover, we prove similar results for subfamilies of permutations and subfamilies of $\binom{[n]}{k/w}^w$, see Section 7.

The following question has been raised as one of the initial problems in the proposal by Kalai for the Polymath project[3] on the Erdős–Rado Delta-system conjecture. In their paper [8], Bradač et al. reiterated it. How big could a family $\mathcal{F}$ be if we forbid sunflowers with the core *at most* $t-1$? While Problem 1 generalizes the Erdős Matching Conjecture and the Erdős-Sós forbidden one intersection problem, this question generalizes the Erdős Matching Conjecture and the Erdős–Ko–Rado Theorem [20]. In this paper, we answer this question.

Theorem 1.3. *Suppose that $n/k \geqslant 4s^2(t+1)^3\log_2^3(n/k)$ and $n \geqslant 2^{10}sk\log_2 k$. Suppose that a family $\mathcal{F} \subset \binom{[n]}{k}$ does not contain a sunflower with s petals and core of size at most $t-1$. Then, the following bound holds:*

$$|\mathcal{F}| \leqslant \left(1 + \frac{2^3 s^2(t+1)^3\log_2^3(n/k)}{n/k}\right)\phi(s,t)\cdot\binom{n-t}{k-t}.$$

In Section 4 we show that the theorem is tight up to second-order terms. Moreover, we prove that similar results hold if $\mathcal{F}$ is a subfamily of any of the following domains:

- $[n]^k$;
- $\prod_{i=1}^w \binom{[n]}{k_i}$;
- the set of permutations Σ_n on n elements;
- the k-th layer of a simplicial complex with rank at least n.

All these cases can be included in the general framework developed in [49, 45, 46], which we further develop in this paper. See Section 4 for the general statement and proofs. In the remaining part of the introduction, we discuss the methods of this and other works.

The starting point for us is the result of Alweiss, Lovett, Wu, and Zhang [1]. The core of their proof is the so-called spread lemma. Very roughly, the lemma states that a "pseudorandom" family $\mathcal{F} \subset \binom{[n]}{k}$ is likely to contain a matching of size s that with good probability respects a random partition, for an appropriate notion of pseudorandomness. The authors show that for any large enough family $\mathcal{F}$ there is a set A, $|A| < k$, such that the restriction

$$\mathcal{F}(A) = \{F \setminus A \,:\, F \in \mathcal{F} \text{ and } A \subset F\}$$

is "pseudorandom," and, therefore, contains a matching $F_1, \ldots, F_s$. Thus, $A \cup F_1, \ldots, A \cup F_s \in \mathcal{F}$ is a sunflower in $\mathcal{F}$ of size s. We discuss this argument and its applications in our setting in Section 4.1.1. Based on the spread lemma and the notion of τ-*homogeneity* that they introduced, Kupavskii and

[3] https://gilkalai.wordpress.com/2015/11/03/polymath10-the-erdos-rado-delta-system-conjecture

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Zakharov [49] developed the *spread approximation* technique and used it to obtain progress in several hypergraph Turán-type problems. The technique was refined in the follow-up papers [45, 46, 47]. That was the first method giving polynomial bounds in the Erdős–Sós problem. The method works when $n \geqslant \mathrm{poly}(t)k\log k$ and $k \geqslant \mathrm{poly}(t)\log n$. In this paper, we add several new structural ingredients and merge their approach with the Delta-system technique and the method of Bradač et al [8] to extend the spread approximation results beyond $k \geqslant \mathrm{poly}(t)\log n$, see Section 8. We should say that for $k < \sqrt{n}$, we no longer need to rely on the spread lemma, and develop other combinatorial techniques instead, which, nevertheless, are in line with the general spread approximations framework.

One fruitful point of view on the spread lemma is as a *sharp threshold* result; see Section 5.5 for the definition of sharp thresholds and its applications in our setting. This connection was used to settle Talagrand's conjecture on fractional-expectation thresholds in random graphs [34] and the well-known Kahn–Kalai conjecture [52]. The spread lemma, however, is not the only way to establish sharp thresholds in pseudorandom families. In [40], Keller and Lifshitz showed that a type of pseudorandom families called *uncapturable* admits a sharp threshold phenomenon. Based on this fact, they extended the *junta method* of Dinur and Friedgut [14], and using it they obtained significant progress in the famous Chvátal Simplex conjecture. Later, their method was refined by Keevash et al. [38] and applied to a number of hypergraph Turan-type problems. Finally, Keller et al. [41] using a variant of the notion of τ-homogeneity of Zakharov and the author as the notion of pseudorandomness, showed that such pseudorandom Boolean functions admit the *hypercontractivity* property (see Section 5.5 for definitions). This led to new results in sharp thresholds for pseudorandom families and has found several consequences in Extremal Combinatorics and beyond. We refer the reader to the follow-up papers of Keller et al. [42, 43]. We apply their results to study Problem 1 when $n = \mathrm{poly}(s,t)k$.

At this point of development of these methods, it seems that none of the Delta-system, spread approximation, and hypercontractivity approaches can be fully replaced by the others. One of the goals of this paper is to take the best of each of these methods and combine them to investigate Problem 1.

The paper is organized as follows. In Section 2, we introduce the notation used in our paper. In Section 3, we discuss Theorem 1.2 and present its version with the dependencies stated explicitly. In Section 4, we give a proof of Theorem 1.3 and present a key tool for the proof of Theorem 1.2. In Section 5, we introduce certain preliminary results necessary for the proof of Theorem 1.2. In Section 6, we present the reduction of Theorem 1.2 to several cases, which are treated in Section 8 and Section 9. In Section 7, we generalize our results for subfamilies of permutations and $[n]^k$, $\binom{[n]}{k/w}^w$. Section 10 contains some proofs omitted in Section 7.

2. Notation

Given a set X and an integer h, we denote by $\binom{X}{h}$ the family of subsets of X of size h. Families of tuples $\binom{[n]}{k/w}^w$ and $[n]^k$ are understood as subfamilies of $\binom{[wn]}{k}$ and $\binom{[kn]}{k}$ respectively.

For a family $\mathcal{F}$ and sets A, B, we define

$$\mathcal{F}(A,B) = \{F \setminus B \,:\, F \in \mathcal{F} \text{ and } F \cap B = A\}.$$

If $A = B$, we denote $\mathcal{F}(B) = \mathcal{F}(B,B)$. If B is a singleton $\{x\}$, we omit braces and use $\mathcal{F}(x) = \mathcal{F}(\{x\})$. Next, if $\mathcal{B}$ is a family of sets, then we define

$$\mathcal{F}[\mathcal{B}] = \{F \in \mathcal{F} \,:\, \exists B \in \mathcal{B} \text{ such that } B \subset F\}.$$

If $\mathcal{B} = \{B\}$ is a singleton of a set B, we omit braces and use $\mathcal{F}[B] = \mathcal{F}[\{B\}]$. We define the support of a family $\mathcal{F}$ as $\mathrm{supp}(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} F$. For families $\mathcal{F}$ and $\mathcal{B}$, we define

$$\mathcal{F} \vee \mathcal{B} = \{F \cup B \,:\, F \in \mathcal{F} \text{ and } B \in \mathcal{B}\}.$$

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A family $\mathcal{F}$ is *k-uniform* (*$\leqslant k$-uniform*), if all sets in $\mathcal{F}$ have size exactly k (at most k). Given a family $\mathcal{F}$ and a number h, we define its h-th layer $\mathcal{F}^{(h)}$ as follows:

$$\mathcal{F}^{(h)} = \{F \in \mathcal{F} \,:\, |F| = h\}.$$

The *shadow* of a family $\mathcal{F}$ on the layer h is defined as follows:

$$\partial_h \mathcal{F} = \bigcup_{F \in \mathcal{F}} \binom{F}{h}.$$

We put $\partial_{\leqslant h}\mathcal{F} = \bigcup_{h' \leqslant h} \partial_{h'}\mathcal{F}$. For a family $\mathcal{F} \subset 2^{[n]}$, we define the *upper-closure* of a family $\mathcal{F}$ as follows:

$$\mathcal{F}^{\uparrow} = \{S \subset [n] \,:\, \exists F \in \mathcal{F} \text{ such that } F \subset S\}.$$

We say that a set T is a *transversal* of a family $\mathcal{F}$, if each $F \in \mathcal{F}$ intersects T. The *covering number* $\tau(\mathcal{F})$ of a family $\mathcal{F}$ is the minimal cardinality of its transversal. This notation should not be confused with τ-homogeneity, defined below.

3. Main result

Here we present our main result with the parameter dependencies stated explicitly. Theorem 1.2 is its obvious corollary.

Theorem 3.1. *For integers n, k, t, $t \geqslant 2$, suppose that*

$$n \geqslant 2^{11} sk \cdot \min\{\log_2 k + 2^{22} s^9 t^{11}, 2^{379} s^{42} t^{82} + 2^{23000}\},$$

$n \geqslant n_0(s,t)$ and $k \geqslant 2t+1$, where $n_0(s,t)$ is some function depending on s,t only. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a family without a sunflower with s petals and the core of size $t-1$. Then it holds that

$$|\mathcal{F}| \leqslant \left(1 + \sqrt[3]{\frac{2^{54} s^{15} t^{18}}{n/k}}\right) \phi(s,t) \binom{n-t}{k-t}. \tag{3.1}$$

Moreover, if $k \geqslant 5t$, then $n_0(s,t)$ can be chosen to have the form $s^{O(t)} \cdot g_0(t)$, where $g_0(t)$ is some function depending on t only.

The minimum in the lower bound on n corresponds to two different strategies of obtaining bound (3.1). If $n = \Omega(sk \log k)$, then one can prove the result using the spread lemma. Otherwise, if $n \geqslant \mathrm{poly}(s,t) \cdot k$, the result can be proved using Boolean analysis. Note the latter leads to worse dependencies on s, t.

The proof is moved to Section 6. Intriguingly, the analysis is significantly simpler when $k \geqslant 33st^2 \ln n$. A similar effect was observed in [38] and [40]. In this regime, the result holds without the assumption $n \geqslant n_0(s,t)$, and the lower bound of the form $n \geqslant \mathrm{poly}(s,t) \cdot k$ is enough. In the follow-up of this paper [48], we will show that one can prove Theorem 3.1 in the regime when $n \geqslant \mathrm{poly}(s,t) \cdot k$ and $k \geqslant \mathrm{poly}(s,t)$ without the assumption that $n \geqslant n_0(s,t)$, that is, the dependencies between all parameters n, k, s, t are polynomial and there is no lower bound on n.

Compared to our theorem, Theorem 1.1 requires $n \geqslant n_0(s,k) = s^{2^k} \cdot \hat{g}_0(k)$ for some function $\hat{g}_0(k)$ depending on k only. We use Theorem 1.1 only if $k < 5t$, which requires $n \geqslant s^{2^{5t}} \cdot \hat{g}_0(t)$ in this case. We mentioned the case $k \geqslant 5t$ separately, since then we obtain a significant improvement in the lower bound on n. Unfortunately, both functions $g_0(t), \hat{g}_0(t)$ are doubly-exponential in t.

Note that Theorem 3.1 is tight up to second-order terms, as the following example demonstrates.

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Example 3.2 (Example 2.3 [26]). Suppose that $n > t\phi(s,t)$. Choose a family $\mathcal{T} \subset \binom{[n]}{t}$ that does not contain a sunflower with s petals and $|\mathcal{T}| = \phi(s,t)$. Then, define a family $\mathcal{F}$ as follows:

$$\mathcal{F} = \left\{ F \in \binom{[n]}{k} \,:\, F \cap \operatorname{supp}(\mathcal{T}) \in \mathcal{T} \right\}.$$

The family $\mathcal{F}$ does not contain a sunflower with s petals and the core of size $t-1$.

The family $\mathcal{T}$ has support of size at most $t\phi(s,t)$. Therefore, for the family $\mathcal{F}$ from Example 3.2, we have

$$\begin{aligned} |\mathcal{F}| &\geqslant \phi(s,t)\binom{n-t\phi(s,t)}{k-t} \geqslant \phi(s,t)\binom{n-t}{k-t} - t\phi^2(s,t)\binom{n-t-1}{k-t-1} \\ &\geqslant \left(1 - \frac{t\phi(s,t)(k-t)}{n-t}\right)\phi(s,t)\binom{n-t}{k-t}. \end{aligned}$$

Theorem 1.3 is closely related to Theorem 3.1, and even can be considered as an intermediate step in the proof of Theorem 3.1. Therefore, before we establish Theorem 3.1, we present the proof of Theorem 1.3.

4. Theorem 1.3 for spread domains

In order to state the promised generalizations of Theorem 1.3, we need some preliminaries.

4.1. Preliminaries I

4.1.1. Spread lemma

We start this section with the following definitions:

Definition 2. A probability measure μ on $2^{[n]}$ is said to be *R-spread*, if for any $X \in 2^{[n]}$ one has

$$\mu(X) \leqslant R^{-|X|}.$$

Definition 3. A random set $\mathbf{W}$ is called a *p-random subset of* $[n]$ if elements $j \in [n]$ belong to $\mathbf{W}$ independently with probability p.

One of the main reasons why the definition of R-spreadness is so important for us is the following theorem called "the Spread lemma":

Theorem 4.1 (Spread lemma). *If for some* $n, R \geqslant 1$ *a measure* $\mu : 2^{[n]} \to [0,1]$ *is R-spread and* $\mathbf{W}$ *is an* $m\delta$*-random subset of* $[n]$*, where* m *is an integer, then*

$$\mathbb{P}[\exists F \in \operatorname{supp}(\mu) \,:\, F \subset \mathbf{W}] \geqslant 1 - \left(\frac{2}{\log_2(R\delta)}\right)^m |\mu|.$$

The spread lemma was initially established in the seminal paper by Alweis et al. [1] with nonoptimal dependencies. The proof relies on the coding techniques in the spirit of Razborov's proof of the Håstad switching lemma (see Chapter 14.1 of [2]). It was refined by Rao in [53]. Later, Tao[4] provided a new proof based on entropy, but his derivations contain a mistake. Later, the proof was fixed and simplified by Hu.[5] Stoeckl[6] established Theorem 4.1 with $1 + h_2(\delta) \leqslant 2$ instead of constant 2, where $h_2(\delta) = -\delta \log_2 \delta - (1-\delta)\log_2(1-\delta)$ is the binary entropy of δ.

[4]See footnote 1.
[5]https://theorydish.blog/2021/05/19/entropy-estimation-via-two-chains-streamlining-the-proof-of-the-sunflower-lemma/
[6]https://mstoeckl.com/notes/research/sunflower_notes.html

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Note that Theorem 4.1 can be applied to families of sets. We say that a family $\mathcal{F}$ is *R-spread* if the measure induced by uniform sampling from $\mathcal{F}$ is R-spread. In other words, $\mathcal{F}$ is R-spread if and only if for each $X \in 2^{[n]}$, one has

$$|\mathcal{F}(X)| \leqslant R^{-|X|}|\mathcal{F}|.$$

Note that for a k-uniform family $\mathcal{F}$, one may obtain a restriction X such that $\mathcal{F}(X)$ is R-spread (assuming that the family consisting only of the empty set is arbitrarily spread).

Observation 4.2 (Lemma 2 from T. Tao's blogpost[7]). Let X be the largest set such that $|\mathcal{F}(X)| \geqslant R^{-|X|}|\mathcal{F}|$ holds. Then, the family $\mathcal{F}(X)$ is R-spread.

The observation is nontrivial, provided $|\mathcal{F}| > R^k$. In that case, we have $|\mathcal{F}(X)| > R^{-|X|}R^k = R^{k-|X|}$. In particular, this implies $|X| < k$.

For families of sets, Theorem 4.1 can be efficiently combined with the following lemma by Keevash et al. [38]. We say that a family $\mathcal{G}$ is *upward closed* if for any $A \in \mathcal{G}$ and B, $A \subset B$, one has $B \in \mathcal{G}$.

Lemma 4.3. *Let $X_1, \dots, X_s$ be such that X_i is a p_i-random subset of $[n]$. Let $\mathcal{G}_1, \dots, \mathcal{G}_s$ be upward closed subfamilies of $2^{[n]}$ such that*

$$\mathbb{P}(X_i \in \mathcal{G}_i) \geqslant 3sp_i.$$

Then there exist disjoint sets $G_1 \in \mathcal{G}_1, \dots, G_s \in \mathcal{G}_s$.

Together, Theorem 4.1 and Lemma 4.3 imply the following proposition which is the main tool in our analysis.

Proposition 4.4. *Let $\mathcal{G}_1, \dots, \mathcal{G}_s$ be R-spread uniform families of uniformity at most k. Suppose that $R > 2^7 s\lceil \log_2 k \rceil$. Then, there exist pairwise disjoint sets $F_1 \in \mathcal{G}_1, \dots, F_s \in \mathcal{G}_s$.*

Proof. Put $p = 1/8s$ and consider a p-random set $\mathbf{W}$ of $[n]$. Choose $m = \lceil \log_2 k \rceil$. Due to Theorem 4.1, for each $i = 1, \dots, s$, we have

$$\mathbb{P}(\exists F \in \mathcal{G}_i \text{ such that } F \subset \mathbf{W}) \geqslant 1 - \left(\frac{2}{\log_2\left(\frac{pR}{m}\right)}\right)^m k > 1 - k2^{-m} > \frac{1}{2}.$$

We get

$$\mathbb{P}\left(\mathbf{W} \in \mathcal{G}_i^{\uparrow}\right) > \frac{1}{2} > 3sp.$$

By Lemma 4.3, there exist pairwise disjoint sets $G_i \in \mathcal{G}_i^{\uparrow}$, and so for some disjoint sets $F_i \subset G_i$, $i = 1, \dots, s$, we have $F_1 \in \mathcal{G}_1, \dots, F_s \in \mathcal{G}_s$. □

Proposition 4.4 yields the following bound on $\phi(s, t)$:

Claim 4.5. *We have $\phi(s, t) \leqslant (2^7 s \log_2 t)^t$.*

Proof. To obtain it, one should consider a family $|\mathcal{F}| > (2^7 s \log_2 t)^t$ of uniformity t, choose $R = 2^6 s\lceil \log_2 t \rceil$ and find a maximal X such that $|\mathcal{F}(X)| \geqslant R^{-|X|}|\mathcal{F}|$. Note that $|X| < t$. Observation 4.2 guarantees that $\mathcal{F}(X)$ is R-spread, and Proposition 4.4 implies that $\mathcal{F}(X)$ contains a matching $F_1, \dots, F_s$ of size s. Thus, $\mathcal{F}$ contains a sunflower $X \cup F_1, \dots, X \cup F_s$ of size s. □

[7]See footnote 1.

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Note that the constant 2^7 in Claim 4.5 is not optimal. For example, one may obtain $\phi(s,t) \leqslant (2^6 s \log_2 t)^t$ (see M. Stoeckl's blogpost[8]). For a short self-contained proof of Claim 4.5, we refer the reader to a recent paper of Rao [54].

In what follows, we will also use the following property of R-spread families.

Claim 4.6. *Let $\mathcal{G}$ be an R-spread family. Consider an arbitrary X of size less than R. Then, the family $\mathcal{G}(\varnothing, X)$ is $\hat{R}$-spread, where*

$$\hat{R} = R - |X|$$

and has size at least $\left(1 - \frac{|X|}{R}\right)|\mathcal{G}|$.

Proof. We have

$$|\mathcal{G}(\varnothing, X)| \geqslant |\mathcal{G}| - \sum_{x \in X} |\mathcal{G}(x)| \geqslant \left(1 - \sum_{x \in X} R^{-1}\right)|\mathcal{G}| \geqslant \left(1 - \frac{|X|}{R}\right)|\mathcal{G}|.$$

Let A be an arbitrary nonempty set disjoint from X. Then, we have

$$|\mathcal{G}(\varnothing, X)(A)| \leqslant |\mathcal{G}(A)| \leqslant R^{-|A|}|\mathcal{G}| \leqslant R^{-|A|}\left(1 - \frac{|X|}{R}\right)|\mathcal{G}(\varnothing, X)| \leqslant \hat{R}^{-|A|}|\mathcal{G}(\varnothing, X)|,$$

so $\mathcal{G}(\varnothing, X)$ is indeed $\hat{R}$-spread. □

Claim 4.6 yields the following simple but important corollary.

Claim 4.7. *Let $\mathcal{G}$ be an arbitrary R-spread family. Then the covering number of $\mathcal{G}$ is at least R.*

Proof. Let $X = \{x_1, \ldots, x_\ell\}$ be a transversal of $\mathcal{G}$. If $\ell < R$, then, by Claim 4.6, we have

$$|\mathcal{G}(\varnothing, X)| \geqslant \left(1 - \frac{\ell}{R}\right)|\mathcal{G}| > 0,$$

contradicting the definition of a transversal. □

4.1.2. Spread domains

An important topic in Extremal Set Theory is to study families $\mathcal{F}$, satisfying some restrictions in domains $\mathcal{A}$ different from $\binom{[n]}{k}$. There is a wide range of such works, for example, generalizations of the Erdős–Ko–Rado theorem for vector fields [31], integer sequences [30], layers of simplicial complexes [7] and permutations [25], generalizations of the Kruskal–Katona theorem for $[n]^k$ [32], generalizations of the Erdős-Sós problem for codes [39] and linear maps [16].

Zakharov and the first author [49] developed the machinery of spread approximations that can be used to deduce extremal set theory results for subfamilies of domains different from $\binom{[n]}{k}$. To discuss these results, we shall need the following definition.

Definition 4. We say that a family $\mathcal{A}$ is (r,t)-spread, if for any $T \in \partial_{\leqslant t}\mathcal{A}$, the family $\mathcal{A}(T)$ is r-spread.

Here, we illustrate the definition of (r,t)-spreadness with several examples, including $\binom{[n]}{k}$.

Proposition 4.8. *Fix integers n, k, t such that $k \geqslant t$ and $n \geqslant k$. Then the following holds:*

1. *The family $\binom{[n]}{k}$ is $\left(\frac{n}{k}, t\right)$-spread;*
2. *The family $[n]^k$ is (n,t)-spread;*
3. *Fix $w \leqslant k$ and $k_1, \ldots, k_w$ such that $\sum_i k_i = k$. The family $\binom{[n]}{k_1} \times \ldots \times \binom{[n]}{k_w}$ is (r,t)-spread for $r = \min_i \frac{n}{k_i}$.*

[8] See footnote 6.

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Proof. It is enough to prove 3, since 1 and 2 follow from it. Set

$$\mathcal{A} = \binom{I_1}{k_1} \times \ldots \times \binom{I_w}{k_w},$$

where $|I_i| = n$. Fix some set T of size at most t, and let $T_i = T \cap I_i$. Since $T \in \partial_t \mathcal{A}$, we have $|T_i| \leqslant k_i$. We aim to show that the family

$$\mathcal{A}(T) = \prod_{i=1}^{w} \binom{I_i \setminus T_i}{k - |T_i|}$$

is r-spread. Consider an arbitrary S disjoint from T, and let $S_i = S \cap (I_i \setminus T_i)$. We may assume that $|S_i| \leqslant k_i - |T_i|$, since otherwise $|\mathcal{A}(T)(S)| = 0$ and the spreadness definition is trivially satisfied. We have

$$|\mathcal{A}(T)(S)| = \prod_{i=1}^{w} \left| \binom{I_i \setminus T_i}{k - |T_i|}(S_i) \right| = \prod_{i=1}^{w} \left| \binom{I_i \setminus (T_i \cup S_i)}{k - |T_i| - |S_i|} \right|.$$

Then, we have

$$|\mathcal{A}(T)(S)| \leqslant \prod_i \left(\frac{n - |T_i|}{k_i - |T_i|} \right)^{-|S_i|} \binom{|I_i \setminus T_i|}{k_i - |T_i|} \leqslant \left(\min_i \frac{n}{k_i} \right)^{-|S|} |\mathcal{A}(T)|.$$

□

Next, we give some more sophisticated examples of families that are (r, t)-spread.

Consider the family Σ_n of permutations on $[n]$. One can consider Σ_n as a subfamily of $\binom{[n]^2}{n}$. For each $\sigma \in \Sigma_n$, consider the following set $F_\sigma \in \binom{[n]^2}{n}$:

$$F_\sigma = \{(i, j) \,:\, \sigma(i) = j\}.$$

We denote by $\mathfrak{S}_n$ the family of sets $\{F_\sigma \,:\, \sigma \in \Sigma_n\}$. Clearly, the family $\mathfrak{S}_n$ is n-uniform. The following proposition ensures that it is (r, t)-spread for a suitable r.

Proposition 4.9 (Section 3.4 from [49]). *Suppose that $n > 4t$. Then, the family $\mathfrak{S}_n$ is $(n/4, t)$-spread.*

The restrictions that a family $\mathcal{F} \subset \Sigma_n$ obeys are translated into the restrictions on the corresponding family $\mathfrak{F} = \{F_\sigma \,:\, \sigma \in \mathcal{F}\}$. For example, if for any $\sigma_1, \sigma_2 \in \mathcal{F}$ there exists x such that $\sigma_1(x) = \sigma_2(x)$, then the family $\mathfrak{F}$ is intersecting, that is, for any $F_1, F_2 \in \mathfrak{F}$ we have $|F_1 \cap F_2| \geqslant 1$. To formulate the counterpart of Theorem 1.3 for permutations, we need to elaborate the corresponding sunflower restriction for them.

Definition 5. Consider s permutations $\sigma_1, \ldots, \sigma_s$. If

1. there exists a set $X = \{x_1, \ldots, x_m\}$ such that $\sigma_1(x_i) = \ldots = \sigma_s(x_i)$ for each $i = 1, \ldots, m$,
2. for any $x \in [n] \setminus X$, all values $\sigma_1(x), \ldots, \sigma_s(x)$ are distinct,

then we say that $\sigma_1, \ldots, \sigma_s$ form a sunflower with s petals and the core of size m.

Note that $\mathcal{F} \subset \Sigma_n$ does not contain a sunflower with s petals and the core of size $t - 1$ if and only if the corresponding family of sets $\mathfrak{F} \subset \mathfrak{S}_n$ does not contain a sunflower with s petals and the core of size $t - 1$. Using this correspondence, we can establish the counterpart of Theorem 1.3 for permutations.

Another nonobvious example of an (r, t)-spread family is the k-th layer of a simplicial complex of large rank. The *rank* of a simplicial complex is the size of its smallest inclusion-maximal set. Consider

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a simplicial complex $\mathcal{C}$ of rank n. Let $n \geqslant k \geqslant t$. Consider the k-th layer $\mathcal{C}^{(k)}$ of $\mathcal{C}$:

$$\mathcal{C}^{(k)} = \{F \in \mathcal{C} \,:\, |F| = k\}.$$

The following lemma states that the family $\mathcal{C}^{(k)}$ is (r,t)-spread for a suitable r.

Lemma 4.10 (Lemma 4 from [46]). *The family $\mathcal{C}^{(k)}$ is $(n/k,t)$-spread.*

Thus, there is a wide range of (r,t)-spread families $\mathcal{A}$. In Section 4.2, we show that for each such $\mathcal{A}$ and $\mathcal{F} \subset \mathcal{A}$ some counterpart of Theorem 1.3 holds.

4.2. Theorem 1.3 for spread domains

In order to formulate a generalization of Theorem 1.3 for $\mathcal{F}$ being a subfamily of an (r,t)-spread domain (or ambient family) $\mathcal{A}$, we require some natural equivalents of the quantities used in the bound on $|\mathcal{F}|$. One such quantity replaces $\binom{n-t}{k-t}$. We denote by A_t the following quantity:

$$A_t = \max_{T \in \partial_t \mathcal{A}} |\mathcal{A}(T)|.$$

Using this notation, we present the main result of this section.

Theorem 4.11. *Let $\mathcal{A}$ be a k-uniform and (r,t)-spread ambient family. Assume that $r \geqslant 4s^2(t+1)^3 \log_2^3 r$ and $r \geqslant 2^{10} s \log_2 k$. Suppose that a family $\mathcal{F} \subset \mathcal{A}$ does not contain a sunflower with s petals and the core of size at most $t-1$. Then, there exists a family $\mathcal{T} \subset \partial_t \mathcal{A}$ that does not contain a sunflower with s petals such that*

$$|\mathcal{F} \setminus \mathcal{F}[\mathcal{T}]| \leqslant \phi(s,t) \cdot \frac{2^3 s^2 (t+1)^3 \log_2^3 r}{r} \cdot A_t.$$

We prove Theorem 4.11 in Section 4.3.

Obviously, the theorem yields the following upper bound on $\mathcal{F}$:

$$|\mathcal{F}| \leqslant \max_{\mathcal{T}} |\mathcal{A}[\mathcal{T}]| + \phi(s,t) \cdot \frac{2^3 s^2 (t+1)^3 \log_2^3 r}{r} \cdot A_t, \tag{4.1}$$

where the maximum is taken over all families $\mathcal{T} \subset \partial_t \mathcal{A}$ that do not contain a sunflower with s petals. If $\mathcal{A} = \binom{[n]}{k}$, then we have $r = n/k$ due to Proposition 4.8, and the following bound holds:

$$|\mathcal{F}| \leqslant \left(1 + \frac{2^3 s^2 (t+1)^3 \log_2^3 (n/k)}{n/k}\right) \phi(s,t) \cdot \binom{n-t}{k-t}.$$

Thus, Theorem 1.3 follows.

We claim that the bound (4.1) is tight up to remainder terms of order $\log_2^3 r / r$. The corresponding example below generalizes that of [26]. Consider a family $\mathcal{T}^* \subset \partial_t \mathcal{A}$ that maximizes $|\mathcal{A}[\mathcal{T}]|$ and does not contain a sunflower with s petals. Denote

$$\operatorname{supp} \mathcal{T}^* = \bigcup_{T \in \mathcal{T}^*} T.$$

Then, define

$$\mathcal{F}^* = \bigcup_{T \in \mathcal{T}^*} \mathcal{A}(T, \operatorname{supp} \mathcal{T}^*) \vee \{T\}.$$

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We claim that $\mathcal{F}^*$ does not contain a sunflower with s petals and the core of size at most $t-1$. Indeed, if such a sunflower $F_1,\dots,F_s$ exists, then $F_1\cap \mathrm{supp}\mathcal{T}^*,\dots,F_s\cap \mathrm{supp}\mathcal{T}^*$ form a sunflower with s petals and the core of size at most $t-1$. By construction, $F_1\cap \mathrm{supp}\mathcal{T}^*,\dots,F_s\cap \mathrm{supp}\mathcal{T}^*$ belongs to $\mathcal{T}^*$, a contradiction.

Next, we have

$$|\mathcal{A}[\mathcal{T}^*]|-|\mathcal{F}^*|\leqslant \sum_{T\in\mathcal{T}^*}|\mathcal{A}(T)|-|\mathcal{A}(T,\mathrm{supp}\mathcal{T}^*)|\leqslant \sum_{T\in\mathcal{T}^*}\sum_{x\in(\mathrm{supp}\mathcal{T}^*)\setminus T}|\mathcal{A}(T\sqcup\{x\})|.$$

Since $\mathcal{A}$ is (r,t)-spread, we can bound $|\mathcal{A}(T\sqcup\{x\})|\leqslant r^{-1}|\mathcal{A}(T)|$. Therefore, we have

$$|\mathcal{A}[\mathcal{T}^*]|-|\mathcal{F}^*|\leqslant \frac{|\mathcal{T}^*||\mathrm{supp}\mathcal{T}^*|}{r}\cdot A_t\leqslant \frac{t\phi^2(s,t)}{r}A_t,$$

so $|\mathcal{F}^*|\geqslant \max_{\mathcal{T}}|\mathcal{A}[\mathcal{T}]|-\frac{t\phi^2(s,t)}{r}A_t$, which matches bound (4.1) up to second-order terms.

Next, we present a corollary of Theorem 4.11 for subfamilies of a simplicial complex. Using Lemma 4.10, we obtain the following.

Corollary 4.11.1. *Consider a simplicial complex $\mathcal{C}$ with rank at least n. Fix numbers s,k,t such that $k\geqslant t$, $n/k\geqslant 4s^2(t+1)^3\log_2^3(n/k)$ and $n\geqslant 2^{10}sk\log_2 k$. Suppose that a family $\mathcal{F}\subset\mathcal{C}^{(k)}$ does not contain a sunflower with s petals and the core of size at most $t-1$. Then, there exists a family $\mathcal{T}\subset\mathcal{C}^{(t)}$ that does not contain a sunflower with s petals such that*

$$|\mathcal{F}\setminus\mathcal{F}[\mathcal{T}]|\leqslant\phi(s,t)\cdot\frac{2^3s^2(t+1)^3\log_2^3(n/k)}{n/k}\cdot\max_{T\in\mathcal{C}^{(t)}}|\mathcal{C}(T)|.$$

To state the corollary of Theorem 4.11 for permutations, we need one more definition. We say that a function ν is a *partial permutation* of size t, if ν is an injection from some set $T\subset[n]$ of size t to $[n]$. We define the family of partial permutations of size t by $\Sigma_n^{(t)}$. Definition 5 of a sunflower generalizes trivially to partial permutations. Since any such injection can be extended to a bijection from $[n]$ to $[n]$, there is a one-to-one correspondence between elements of $\Sigma_n^{(t)}$ and the shadow $\partial_t\mathfrak{S}_n$. Using it and Proposition 4.9, we deduce the following corollary (note that in our case the uniformity n and the spreadness $n/4$ are the same up to multiplicative constants, so we have only one inequality on n instead of two inequalities on r in Theorem 4.11).

Corollary 4.11.2. *Fix numbers n,s,t such that $n\geqslant 2^{20}s(t+1)\log_2 n$. Suppose that a family $\mathcal{F}\subset\Sigma_n$ of permutations does not contain a sunflower with s petals and the core of size at most $t-1$. Then, there exists a family $\mathcal{T}\subset\Sigma_n^{(t)}$ of partial permutations that does contain a sunflower with s petals and the core of size at most $t-1$, such that*

$$|\mathcal{F}\setminus\mathcal{F}[\mathcal{T}]|\leqslant\phi(s,t)\cdot\frac{2^5s^2(t+1)^3\log_2^3(n/4)}{n}\cdot(n-t)!$$

Here $\mathcal{F}[\mathcal{T}]$ is the family of all permutations σ in $\mathcal{F}$ such that there exists a partial permutation $\nu\in\mathcal{T}$ which coincides with σ on the support of ν.

4.3. *Proof of Theorem 4.11*

Set $w=(t+1)\log_2 r$. Consider two cases.

Case 1. We have $k\leqslant w$. In this case, we apply the following lemma with $\mathcal{F}$ in place of $\mathcal{S}$ with $q=k$.

Lemma 4.12. *Let $\mathcal{A}$ be a k-uniform (r,t)-spread family. Fix some $q\leqslant k$ and $\varepsilon\in(0,1)$. Assume that $\varepsilon r\geqslant s^2q^3$. Suppose that a family $\mathcal{S}\subset\partial_{\leqslant q}\mathcal{A}$*

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- *does not contain an s-sunflower with the core of size $\leqslant t-1$;*
- *consists of sets of cardinality at least t.*

Then, there exists a family $\mathcal{T} \subset \partial_t \mathcal{A}$ which does not contain any sunflower with s petals and

$$|\mathcal{A}[\mathcal{S}] \setminus \mathcal{A}[\mathcal{T}]| \leqslant |\mathcal{A}[\mathcal{S} \setminus \mathcal{S}[\mathcal{T}]]| \leqslant \phi(s,t) \cdot \frac{2\varepsilon}{1-\varepsilon} A_t.$$

We prove Lemma 4.12 in Section 4.4. We apply it with $\varepsilon = \frac{s^2q^3}{r} \leqslant \frac{s^2(t+1)^3 \log_2^3 r}{r}$. Note that the condition on r from the theorem implies that $\varepsilon < 1$.

The lemma gives suboptimal bounds if $k \geqslant (t+1)\log_2 r$. Indeed, if we apply it with k in place of q, it gives the result for $r > s^2k^3$, while we can establish the theorem provided $r \geqslant 2s^2(t+1)^3 \log^3 r$ and $r \geqslant 2^{10} s \log_2 k$.

Case 2. We have $k > (t+1)\log_2 r$. In this case, we use the following iterative process to obtain some structure within the family $\mathcal{F}$, cf. [45, 46].

Set $\mathcal{F}_0 = \mathcal{F}$. Given a family $\mathcal{F}_i$, we obtain a family $\mathcal{F}_{i+1}$ by the following procedure.

Set $R = r/2$. Let S_i be the maximal set such that $|\mathcal{F}_i(S_i)| \geqslant R^{-|S_i|}|\mathcal{F}_i|$. Note that $\mathcal{F}_i(S_i)$ is R-spread due to Observation 4.2. If $|S_i| > w$, then stop. Otherwise, set $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \mathcal{F}_i[S_i]$.

Suppose that the procedure above stops at step m. It means that $|S_m| \geqslant w+1$ and

$$|\mathcal{F}_m| \leqslant R^{|S_m|}|\mathcal{F}(S_m)| \leqslant R^{|S_m|}|\mathcal{A}(S_m)| \overset{(\mathcal{A} \text{ is } (r,t)\text{-spread})}{\leqslant} R^{|S_m|} r^{t-|S_m|} |\mathcal{A}(T)| \leqslant r^t \left(\frac{R}{r}\right)^{w+1} |\mathcal{A}(T)| \leqslant r^t 2^{-w-1} |\mathcal{A}(T)| \leqslant \frac{1}{r}|\mathcal{A}(T)|.$$

for some $T \subset S_m$ of size t. Define $\mathcal{S} = \{S_1, \dots, S_{m-1}\}$ and $\mathcal{F}_{S_i} = \mathcal{F}_i(S_i)$. Note that $\mathcal{F}$ admits the following decomposition:

$$\mathcal{F} = \mathcal{F}_m \sqcup \bigsqcup_{S \in \mathcal{S}} \mathcal{F}_S \vee \{S\},$$

and each $\mathcal{F}_S$ is R-spread.

We claim the following.

Claim 4.13. *The family $\mathcal{S}$ possesses two properties:*

1. *it does not contain a sunflower with s petals and the core of size at most $t-1$;*
2. *it consists of sets of cardinality at least t.*

Proof. Arguing indirectly, assume that $\mathcal{S}$ contains a sunflower $S_1, \dots, S_s$ with s petals and the core of size at most $t-1$. Note that one may consider s copies of a set S of size at most $t-1$ as a sunflower with s empty petals and the core of size at most $t-1$. The proof below does not require $S_1, \dots, S_s$ to be distinct, so it implies both 1 and 2.

The families $\mathcal{F}_{S_1}, \dots, \mathcal{F}_{S_s}$ are R-spread. Put $S = \cup_j S_j$. Consider the families $\mathcal{G}_i = \mathcal{F}_{S_i}(\varnothing, S \setminus S_i)$, $j \in [s]$. Due to Claim 4.6, $\mathcal{G}_i$ are $\hat{R}$-spread for

$$\hat{R} = R - |S| \geqslant R - sw \geqslant R/2,$$

since $R \geqslant 2sw$. Next, $\hat{R} \geqslant R/2 \geqslant 2^8 s \log_2 k \geqslant 2^7 s \lceil \log_2 k \rceil$, so Proposition 4.4 implies that there are disjoint $G_1 \in \mathcal{G}_1, \dots, G_s \in \mathcal{G}_s$. Therefore, $\mathcal{F}$ contains a sunflower $S_1 \cup G_1, \dots, S_s \cup G_s$ with s petals and the core of size at most $t-1$, a contradiction. □

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Note that $\mathcal{S} \subset \partial_{\leqslant w}\mathcal{A}$ by definition. Then, we apply Lemma 4.12 with $q = w$ and $\varepsilon = \frac{s^2w^3}{r} = \frac{s^2(t+1)^3\log_2^3 r}{r}$ to $\mathcal{S}$, and obtain

$$|\mathcal{F}[\mathcal{S}] \setminus \mathcal{A}[\mathcal{T}]| \leqslant |\mathcal{A}[\mathcal{S}] \setminus \mathcal{A}[\mathcal{T}]| \leqslant \phi(s,t) \cdot \frac{2\varepsilon}{1-\varepsilon} A_t \overset{(\varepsilon \leqslant 0.5)}{\leqslant} \phi(s,t) \cdot \frac{4s^2(t+1)^3\log_2^3 r}{r} A_t.$$

Therefore, we have

$$|\mathcal{F} \setminus \mathcal{F}[\mathcal{T}]| \leqslant |\mathcal{F}_m| + |\mathcal{F}[\mathcal{S}] \setminus \mathcal{A}[\mathcal{T}]| \leqslant \phi(s,t) \cdot \frac{2^3 s^2(t+1)^3\log_2^3 r}{r} A_t.$$

This completes the proof of Theorem 4.11 modulo Lemma 4.12.

4.4. Proof of Lemma 4.12: a simplification argument

Lemma 4.12 is the key ingredient in the proofs of Theorems 3.1 and 4.11. As a reminder, let us restate the lemma.

Lemma 4.12. *Let $\mathcal{A}$ be a k-uniform (r,t)-spread family. Fix some $q \leqslant k$ and $\varepsilon \in (0,1)$. Assume that $\varepsilon r \geqslant s^2q^3$. Suppose that a family $\mathcal{S} \subset \partial_{\leqslant q}\mathcal{A}$*

- *does not contain an s-sunflower with the core of size $\leqslant t-1$;*
- *consists of sets of cardinality at least t.*

Then, there exists a family $\mathcal{T} \subset \partial_t\mathcal{A}$ which does not contain any sunflower with s petals and

$$|\mathcal{A}[\mathcal{S}] \setminus \mathcal{A}[\mathcal{T}]| \leqslant |\mathcal{A}[\mathcal{S} \setminus \mathcal{S}[\mathcal{T}]]| \leqslant \phi(s,t) \cdot \frac{2\varepsilon}{1-\varepsilon} A_t.$$

We will obtain $\mathcal{T}$ at the end of the following process. Set $\mathcal{T}_0 = \mathcal{S}$. Given a family $\mathcal{T}_i$, we will construct a family $\mathcal{T}_{i+1}$ by Procedure 1 starting from $i = 0$.

Procedure 1 Simplification procedure

1: Set $\mathcal{W}_i \leftarrow \mathcal{T}_i^{(q-i)}$, $\mathcal{T}'_{i+1} \leftarrow \varnothing$;
2: Set
$$\alpha \leftarrow sq;$$
3: Find a maximal set T such that $|\mathcal{W}_i(T)| \geqslant \alpha^{-|T|}|\mathcal{W}_i|$;
4: **while** $|T| \leqslant q-i-1$ **do**
5: Define $\mathcal{T}_{i,T} := \mathcal{W}_i(T)$;
6: Remove sets containing T from $\mathcal{W}_i$:
$$\mathcal{W}_i \leftarrow \mathcal{W}_i \setminus \mathcal{W}_i[T];$$
7: Add T to $\mathcal{T}'_{i+1}$:
$$\mathcal{T}'_{i+1} \leftarrow \mathcal{T}'_{i+1} \cup \{T\};$$
8: Find a maximal set T' such that $|\mathcal{W}_i(T')| \geqslant \alpha^{-|T'|}|\mathcal{W}_i|$;
9: Set $T \leftarrow T'$;
10: **end while**
11: Set $\mathcal{T}_{i+1} \leftarrow \mathcal{T}'_{i+1} \cup \mathcal{T}_i^{(\leqslant q-i-1)}$.

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In what follows, we define families $\mathcal{T}_{i+1}, \mathcal{W}_i, \mathcal{T}_{i,T}, \mathcal{T}'_{i+1}$ as obtained after this procedure. The desired family $\mathcal{T}$ is $\mathcal{T}_{q-t}$. Note that each set in $\mathcal{T}_i$ has size at most $q-i$. Finally, for any $T \in \mathcal{T}'_{i+1}$, there exists an α-spread family $\mathcal{T}_{i,T}$ such that

$$\mathcal{T}_i^{(q-i)} \setminus \mathcal{W}_i = \bigsqcup_{T \in \mathcal{T}'_{i+1}} \{T\} \vee \mathcal{T}_{i,T}.$$

In order to establish Lemma 4.12, we need the following three claims.

Claim 4.14. *For each $i = 0, \dots, q-t$, $\mathcal{T}_i$ admits two properties:*

- *does not contain a sunflower with the core of size $\leqslant t-1$ and s petals;*
- *each set of $\mathcal{T}_i$ has size at least t.*

Proof. The proof is by induction on i. For $i = 0$ the statement holds by the assumptions of Lemma 4.12. Assume that $\mathcal{T}_i$ satisfies both properties. We prove that $\mathcal{T}_{i+1}$ does also.

Assume that $\mathcal{T}_{i+1}$ contains a sunflower with the core of size $\leqslant t-1$ and s petals. Consider such a sunflower $T_1, \dots, T_s \in \mathcal{T}_{i+1}$. Recall that $\mathcal{T}_{i+1} = \mathcal{T}_i^{(\leqslant q-i-1)} \cup \mathcal{T}'_{i+1}$. Thus, we may assume that $T_1, \dots, T_\ell \in \mathcal{T}'_{i+1}$ for some ℓ, $0 \leqslant \ell \leqslant s$, and the remaining sets $T_{\ell+1}, \dots, T_s \in \mathcal{T}_i^{(\leqslant q-i-1)}$.

Families $\mathcal{T}_{i,T_w}$, $w = 1, \dots, \ell$, defined by the procedure, are α-spread. Claim 4.7 implies that their covering number is at least $\alpha = sq$. We claim that there are disjoint $F_1 \in \mathcal{T}_{i,T_1}, \dots, F_\ell \in \mathcal{T}_{i,\ell}$ that do not intersect $\bigcup_{p=1}^s T_p$. Assume that the opposite holds and let j be the maximum integer such that there are disjoint $F_1 \in \mathcal{T}_{i,T_1}, \dots, F_j \in \mathcal{T}_{i,T_j}$ that do not intersect $\bigcup_{p=1}^s T_p$. It means that $\bigcup_{p=1}^j (F_j \cup T_j) \cup \bigcup_{p=j+2}^s T_p$ is a transversal for $\mathcal{T}_{i,T_{j+1}}$, so the covering number of $\mathcal{T}_{i,T_j}$ is at most $j(q-i)+(s-1-j)(q-i) < s(q-i)$, a contradiction. Therefore, $\mathcal{T}_i$ contains sets $T_1 \cup F_1, \dots, T_\ell \cup F_\ell, T_{\ell+1}, \dots, T_s$, which form a sunflower with the core of size $\leqslant t-1$, a contradiction.

Next, assume that $\mathcal{T}_{i+1}$ contains a set T of size at most $t-1$. By the induction hypothesis, we may assume that $T \in \mathcal{T}'_{i+1}$. Then, the family $\mathcal{T}_{i,T}$ is α-spread, and, thus, its covering number is at least α_i. Arguing as before, we find disjoint sets $F_1, \dots, F_s \in \mathcal{T}_{i,T}$. Thus, $\mathcal{T}_i$ contains a sunflower $T \cup F_1, \dots, T \cup F_s$ with the core T, $|T| \leqslant t-1$, a contradiction. □

Claim 4.15. *We have $|\mathcal{W}_i| \leqslant 2\binom{q-i}{t}\phi(s, q-i)\alpha^{q-i-t}$.*

Proof. We define a graph $G = (\mathcal{W}_i, E)$ such that two sets W_1, W_2 are connected by an edge if they intersect in at least t elements. We prove below that the maximal degree in this graph is at most $\binom{q-i}{t}\alpha^{q-i-t}$. Consider a set $W \in \mathcal{W}_i$. Then all neighbors of W in the graph G belong to a set

$$\bigcup_{T \in \binom{W}{t}} \mathcal{W}_i[T].$$

We claim that $|\mathcal{W}_i(T)| \leqslant \alpha^{q-i-t}$. Consider two cases.

Case 1. We have $|\mathcal{W}_i(T)| \leqslant \alpha^{-|T|}|\mathcal{W}_i|$. By the construction of $\mathcal{W}_i$, the bound $\alpha^{-|X|}|\mathcal{W}_i| \leqslant |\mathcal{W}_i(X)|$ holds for some X of size $q-i$. But $|\mathcal{W}_i(X)| = 1$, therefore, $|\mathcal{W}_i| \leqslant \alpha^{q-i}$, and so $|\mathcal{W}_i(T)| \leqslant \alpha^{q-i-t}$.

Case 2. We have $|\mathcal{W}_i(T)| > \alpha^{-|T|}|\mathcal{W}_i|$. Consider the maximal X disjoint from T such that $|\mathcal{W}_i(X \sqcup T)| \geqslant \alpha^{-|X|}|\mathcal{W}_i(T)|$. Since we have $|\mathcal{W}_i(X \sqcup T)| > \alpha^{-|X|-|T|}|\mathcal{W}_i|$ by the assumption of the case, the size of X should be $q-i-t$ by the construction of $\mathcal{W}_i$. Hence, we have $1 = |\mathcal{W}_i(X \sqcup T)| \geqslant \alpha^{-(q-i-t)}|\mathcal{W}_i(T)|$, and so $|\mathcal{W}_i(T)| \leqslant \alpha^{q-i-t}$.

Therefore, the set W has at most $\sum_{T \in \binom{W}{t}} |\mathcal{W}_i(T)| \leqslant \binom{q-i}{t}\alpha^{q-i-t}$ neighbors. It implies that the maximal degree of the graph G is at most $\binom{q-i}{t}\alpha^{q-i-t}$. Using a standard greedy argument, one can show that the size of the largest independent set in G is at least

$$\frac{|\mathcal{W}_i|}{1 + \binom{q-i}{t}\alpha^{q-i-t}} \geqslant \frac{|\mathcal{W}_i|}{2\binom{q-i}{t}\alpha^{q-i-t}}. \tag{4.2}$$

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Therefore, there exists a family $\mathcal{I} \subset \mathcal{W}_i$ of size at least the RHS of (4.2), such that any two distinct sets $F_1, F_2 \in \mathcal{I}$ intersect in at most $t-1$ elements.

We claim that $\mathcal{I}$ does not contain a sunflower with s petals. Suppose that such a sunflower $F_1, \dots, F_s \in \mathcal{I}$ exists. Then, the core of this sunflower must have size at most $t-1$. Since $\mathcal{W}_i$ is a subset of $\mathcal{T}_i$, the sets $F_1, \dots, F_s$ belong to $\mathcal{T}_i$, contradicting Claim 4.14. Therefore, (4.2) $\leqslant |\mathcal{I}| \leqslant \phi(s, q-i)$, and so

$$|\mathcal{W}_i| \leqslant \phi(s, q-i) \cdot 2\binom{q-i}{t}\alpha^{q-i-t}.$$

The claim follows. □

Next, we adapt the proof of the Erdős–Rado Theorem [19] to bound $\phi(s, q-i)$ via $\phi(s,t)$.

Claim 4.16. *For any $a \leqslant b$, we have $\phi(s,b) \leqslant \phi(s,a) \cdot (sb)^{b-a}$.*

Proof. Let $\mathcal{G}$ be a b-uniform family without a sunflower with s petals and such that $|\mathcal{G}| = \phi(s,b)$. Consider the maximal matching $F_1, \dots, F_m$ in $\mathcal{G}$. Since $\mathcal{G}$ does not contain a sunflower with s petals, we have $m < s$. Since the matching is maximal, each set $G \in \mathcal{G}$ intersects $M = \bigcup_{j=1}^{m} F_m$, where $|M| \leqslant (s-1)b$. Therefore, the size of $\mathcal{G}$ is at most $\sum_{x \in M} |\mathcal{G}(x)|$. Each family $\mathcal{G}(x)$ in the latter sum does not contain a sunflower with s petals, so $|\mathcal{G}(x)| \leqslant \phi(s, b-1)$. It yields the bound

$$\phi(s,b) = |\mathcal{G}| \leqslant (s-1)b \cdot \phi(s, b-1).$$

Iterating the inequality above, we obtain

$$\phi(s,b) \leqslant (s-1)^{b-a} \cdot \phi(s,a) \cdot \prod_{j=a+1}^{b} j \leqslant (sb)^{b-a} \cdot \phi(s,a).$$

□

Finally, we are ready to prove Lemma 4.12.

Proof of Lemma 4.12. Decomposing $\mathcal{A}[\mathcal{S} \setminus \mathcal{S}[\mathcal{T}]]$ into the union of $\mathcal{A}[\mathcal{W}_i], i = 0, \dots, q-t-1$, we obtain

$$\begin{aligned}
|\mathcal{A}[\mathcal{S} \setminus \mathcal{S}[\mathcal{T}]]| &\leqslant \sum_{i=0}^{q-t-1} |\mathcal{W}_i| \max_{W \in \mathcal{W}_i} |\mathcal{A}(W)| \\
&\leqslant \sum_{i=0}^{q-t-1} |\mathcal{W}_i| A_{q-i} \\
&\overset{\text{Claim 4.15}}{\leqslant} 2 \sum_{i=0}^{q-t-1} \phi(s, q-i)\binom{q-i}{q-i-t}\left(\frac{\alpha}{r}\right)^{q-i-t} A_t \\
&\overset{\text{Claim 4.16}}{\leqslant} 2 \sum_{i=0}^{q-t-1} \phi(s,t)(s \cdot (q-i))^{q-i-t} \cdot (q-i)^{q-i-t} \cdot \left(\frac{\alpha}{r}\right)^{q-i-t} A_t.
\end{aligned}$$

Since $\varepsilon r \geqslant \alpha s q^2$, the last sum is at most

$$2\phi(s,t) \sum_{i=0}^{q-t-1} \varepsilon^{q-i-t} A_t \leqslant \frac{2\varepsilon}{1-\varepsilon} \cdot \phi(s,t) A_t.$$

□

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5. Preliminaries II

5.1. Delta-systems

The original method of Frankl and Füredi heavily relies on the following lemma (see Lemma 7.4 from [26]).

Lemma 5.1. *Let $\mathcal{F} \subset \binom{n}{k}$. Fix numbers p, t, such that $p \geqslant k \geqslant 2t+1$. Suppose that $\mathcal{F}$ does not contain a sunflower with p petals and the core of size $t-1$. Then, there exists a subfamily $\mathcal{F}^* \subset \mathcal{F}$ that admits the following properties:*

1. *for any $F \in \mathcal{F}^*$, there exists a subset $T \subset F$ of size t such that each set E, $T \subset E \subsetneq F$, is a core of a sunflower with p petals in $\mathcal{F}^*$;*
2. *$|\mathcal{F} \setminus \mathcal{F}^*| \leqslant c(p,k)\binom{n}{k-t-1}$,*

where $c(p,k)$ is some function that can be bounded by $p^{2^k} \cdot 2^{2^{Ck}}$ for some absolute constant C.

While the paper [26] omits precise numerical dependencies between n, k and t, we need them stated explicitly. For the sake of completeness, we present here a proof of the sunflower theorem by Frankl and Füredi. Moreover, the core idea will be used in the proof of our result.

Theorem 5.2. *Let $k \geqslant 2t+1$. Suppose that a family $\mathcal{F}$ does not contain a sunflower with s petals and the core of size $t-1$. Then*

$$|\mathcal{F}| \leqslant \phi(s,t)\binom{n}{k-t} + \frac{k \cdot c(sk,k)}{n-k}\binom{n}{k-t}.$$

Proof. Applying Lemma 5.1 with $p = sk$, we infer that there exists $\mathcal{F}^* \subset \mathcal{F}$, $|\mathcal{F} \setminus \mathcal{F}^*| \leqslant c(sk,k)\binom{n}{k-t-1}$, such that for any $F \in \mathcal{F}^*$ there is a subset $T \subset F$ of size t such that any E, $T \subset E \subsetneq F$, is a core of a sunflower with sk petals in $\mathcal{F}^*$. For each $F \in \mathcal{F}^*$, fix one such $T = T(F)$. Then, we have

$$|\mathcal{F}^*| = \sum_{D \in \binom{[n]}{k-t}} |\{F \in \mathcal{F}^* \,:\, F \setminus D = T(F)\}|.$$

Define $\mathcal{F}_D = \{T(F) \,:\, F \in \mathcal{F}^* \text{ and } F \setminus D = T(F)\}$. We claim that $\mathcal{F}_D$ does not contain a sunflower with s petals and arbitrary core. Otherwise choose such a sunflower $T_1, \dots, T_s \in \mathcal{F}_D$. Choose $X \subset D$, $|X| = t - 1 - |\cap_i T_i|$. Each $T_i \cup X$ is the core of a sunflower in $\mathcal{F}^*$ with sk petals. Let m be the maximum number such that there are disjoint $F_1 \in \mathcal{F}^*(T_1 \cup X), \dots, F_m \in \mathcal{F}^*(T_m \cup X)$ that do not intersect $\cup_{i=1}^m T_i$. Since $\mathcal{F}^*$ does not contain a sunflower with s petals and the core of size $t-1$, m is strictly less than s. Consider a family $\mathcal{F}^*(T_{m+1} \cup X)$. By the definition of $\mathcal{F}^*$, it contains pairwise disjoint sets $G_1, \dots, G_{sk}$. At least one such set does not intersect $\cup_{i=1}^m (T_i \cup X \cup F_i)$ since $|\cup_{i=1}^m (T_i \cup X \cup F_i)| \leqslant (s-1)k < sk$. Hence, we can choose $F_{m+1} \in \mathcal{F}^*(T_i \cup X)$, that does not intersect $\cup_{i=1}^{m+1} T_i \cup X$, contradicting the maximality of m.

Therefore, $\mathcal{F}_D \subset \binom{n}{t}$ does not contain a sunflower with s petals, and so $|\mathcal{F}_D| \leqslant \phi(s,t)$. Since

$$|\mathcal{F}^*| \leqslant \sum_{D \in \binom{[n]}{k-t}} |\mathcal{F}_D|,$$

we have

$$|\mathcal{F}| \leqslant \phi(s,t)\binom{n}{k-t} + c(sk,k)\binom{n}{k-t-1} \leqslant \phi(s,t)\binom{n}{k-t} + \frac{k \cdot c(sk,k)}{n-k}\binom{n}{k-t}.$$

□

We shall use Theorem 5.2 for $k < 5t$. For larger k, we use different techniques, which are built on the results presented in the following sections.

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5.2. Sunflower Turán number: the result of Bradač et al.

Theorem 5.2 is quite restrictive, as it requires the lower bound on n of the form $(st)^{2^{\Omega(t)}}$. To relax this restriction, we employ the following result due to Bradač et al. [8].

Theorem 5.3. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a family that does not contain a sunflower with s petals and the core of size exactly $t-1$. Then, for each k there is some constant C_k depending on k only, such that*

$$|\mathcal{F}| \leqslant \begin{cases} C_k n^{k-t} s^t & \text{if } k \geqslant 2t-1 \\ C_k n^{t-1} s^{k-t+1} & \text{if } t \leqslant k < 2t-1. \end{cases}$$

The bounds provided by Theorem 5.3 are tight up to the dependence on k. The example can be constructed using a nice probabilistic argument, see Lemmas 10 and 11 of [8].

We will use this theorem for $k \leqslant 2t+1$. In that regime, $C_k \sim t! \cdot t^{\binom{2t-1}{t}}$, so C_k is double-exponential in k. In what follows, we assume that C_k is increasing in k and larger than 1.

5.3. Homogeneity and general domains

While the main theorem of our paper deals with subfamilies of $\binom{[n]}{k}$, some parts of the analysis can be extended to other domains. As in Theorem 4.11, we consider families $\mathcal{F}$ that are subfamilies of some k-uniform family $\mathcal{A}$ which satisfies a number of assumptions.

When dealing with the uniform families $\mathcal{F} \subset \mathcal{A}$, it is convenient to replace the notion of R-spreadness with the notion of τ-homogeneity, introduced in [49].

Definition 6. A family $\mathcal{F}$ is *τ-homogeneous* if for each X, we have

$$\frac{|\mathcal{F}(X)|}{|\mathcal{A}(X)|} \leqslant \tau^{|X|} \frac{|\mathcal{F}|}{|\mathcal{A}|}.$$

In what follows, we use the notation $\mu(\mathcal{F}) = |\mathcal{F}|/|\mathcal{A}|$. Using this, we can rewrite the condition above as follows: $\mu(\mathcal{F}(X)) \leqslant \tau^{|X|}\mu(\mathcal{F})$. A variant of the definition of τ-homogeneity, called τ-globalness, was used in [41], see Definition 8. If the domain $\mathcal{A}$ is r-spread, a τ-homogeneous family $\mathcal{F}$ is r/τ-spread. Indeed, we have

$$|\mathcal{F}(X)| \leqslant \tau^{|X|}|\mathcal{F}|\frac{|\mathcal{A}(X)|}{|\mathcal{A}|} \leqslant (r/\tau)^{-|X|}|\mathcal{F}|.$$

In general, we require $\mathcal{A}(S)$ to be r-spread for any S that has size smaller than some integer q, $q \geqslant t$. That implies the first assumption on the domain $\mathcal{A}$, which strengthens the corresponding assumption of Theorem 4.11.

Assumption 1. Let q be an integer. The family $\mathcal{A}$ is k-uniform and $(r, q+t-1)$-spread.

The necessary q will be chosen later.

Solutions of hypergraph Turan-type problems in $\binom{[n]}{k}$ often involve the binomial coefficient $\binom{n-t}{k-t}$, which is the number of sets in $\binom{[n]}{k}$ containing some set of size t. We recall a natural equivalent for this quantity in arbitrary domains:

$$A_t = \max_{T \in \partial_t \mathcal{A}} |\mathcal{A}(T)|.$$

The second assumption on $\mathcal{A}$ requires A_t to be not too small.

Assumption 2. Let $\eta \geqslant 1$ be a real number. We have $A_t \geqslant r^{-\eta t}|\mathcal{A}|$.

The next two assumptions concern the shadow of $\mathcal{A}$.

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Assumption 3. For any set $S \in \partial_{\leqslant q}\mathcal{A}$, a family $\mathcal{F} \subset \mathcal{A}(S)$, and $h \leqslant t-1$, we have

$$\frac{|\mathcal{F}|}{|\mathcal{A}(S)|} = \frac{1}{|\partial_h(\mathcal{A}(S))|}\sum_{H\in\partial_h(\mathcal{A}(S))}\frac{|\mathcal{F}(H)|}{|\mathcal{A}(H\cup S)|},$$

or, equivalently, for a random set $\mathbf{H}$ uniformly distributed on $\partial_h(\mathcal{A}(S))$, we have $\mu(\mathcal{F}) = \mathbb{E}\mu(\mathcal{F}(\mathbf{H}))$.

Assumption 4. For any set $R \in \partial_{\leqslant q}\mathcal{A}$, an integer h, $h \leqslant t$, and some $\mu > 1$, it holds

$$\frac{|\partial_h(\mathcal{A}(R))|}{|\partial_h\mathcal{A}|} \geqslant \left(1-\frac{q}{\mu k}\right)^h.$$

The natural question is when these assumptions are satisfied. The following propositions show that some interesting domains fit these assumptions.

Proposition 5.4. *Suppose that $n, k \geqslant q+t$, $q \geqslant t$ and $n > 8k$. Then, the following holds:*

1. *Suppose additionally that $n \geqslant 2kt$. Then $\binom{n}{k}$ satisfies Assumptions 1-4 with $r = n/k, \mu = n/2k, \eta = 2$;*
2. *Suppose that $n \geqslant 2t, k \geqslant 2q$. Then $[n]^k$ satisfies Assumptions 1-4 with $r = n, \mu = 1/2, \eta = 2$;*
3. *Fix an integer w, such that $n \geqslant 2kt/w$ and $w|k$. Then $\binom{[n]}{k/w}^w$ satisfies Assumptions 1-3 with $r = \frac{wn}{k}$ and $\eta = 2$. Moreover, if $k \geqslant w(q+t)$, then $\binom{[n]}{k/w}^w$ satisfies Assumption 4 with $\mu = n/2k$.*

Proof. By taking $w = 1$ and $w = k$ respectively, satisfiability of Assumptions 1-3 for $[n]^k$, $\binom{[n]}{k}$ follows from the third statement, so we consider only the family $\binom{[n]}{k/w}^w$ for these assumptions. Assumption 1 is satisfied due to Proposition 4.8, which is used with $q+t-1$ in place of t.

Next, we check Assumption 2. Consider a set T of size t, and set $t_i = |T \cap I_i|$. Then, we have

$$\frac{|\mathcal{A}(T)|}{|\mathcal{A}|} \geqslant \prod_{i=1}^{w}\frac{\binom{n-t_i}{k/w-t_i}}{\binom{n}{k/w}} = \prod_{i=1}^{w}\frac{\binom{k/w}{t_i}}{\binom{n}{t_i}} \geqslant \prod_{i=1}^{w}\left(\frac{k/w+1-t_i}{n}\right)^{t_i} \tag{5.1}$$

If $t \leqslant k/2w$, then the latter product is at least

$$\prod_{i=1}^{w}\left(\frac{k}{2wn}\right)^{t_i} = \left(\frac{k}{2wn}\right)^{t} \geqslant r^{-2t},$$

provided $n \geqslant 2k/w$. In particular, this bound implies that Assumption 2 holds for $\binom{[n]}{k}$, since $k \geqslant 4q \geqslant 2t$. If $t \geqslant k/2w$, then we bound the RHS of (5.1) as follows:

$$\prod_{i=1}^{w}\left(\frac{k/w+1-t_i}{n}\right)^{t_i} \geqslant n^{-t} = (2t)^{-t}(2t)^t n^{-t} \geqslant (2t)^{-t}\left(\frac{k}{wn}\right)^t \geqslant \left(\frac{k}{wn}\right)^{2t} = r^{-2t},$$

where we used $wn \geqslant 2tk$ in the last inequality. Hence, Assumption 2 holds with $\eta = 2$.

Similarly to Proposition 4.8, we assume that $\mathcal{A}$ has the form

$$\mathcal{A} = \binom{I_1}{k/w} \times \ldots \binom{I_w}{k/w},$$

where disjoint sets $I_1, \ldots, I_w$ have size n. The following claim ensures that Assumption 3 holds for any domain of the form

$$\mathcal{A} = \binom{I_i}{k_i} \times \ldots \binom{I_w}{k_w}.$$

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We need this claim stated separately, because we use it in what follows.

Claim 5.5. *Let $I_1, \dots, I_w$ be arbitrary disjoint sets and $k_1 \leqslant |I_1|, \dots, k_w \leqslant |I_w|$ be integer numbers. Then the family*

$$\mathcal{A} = \binom{I_i}{k_i} \times \dots \binom{I_w}{k_w}$$

satisfies Assumption 3.

Proof. Fix S of size at most q, and consider a set $\mathbf{H}$ uniformly distributed on $\partial_h(\mathcal{A}(S))$. Let $\mathbf{h}_1 = |\mathbf{H} \cap I_1 \setminus S|, \dots, \mathbf{h}_w = |\mathbf{H} \cap I_w \setminus S|$ and $\mathbf{h} = (\mathbf{h}_1, \dots, \mathbf{h}_w)$. Let $\partial_{\mathbf{h}}(\mathcal{A}(S))$ be a family of sets H from $\partial_h(\mathcal{A}(S))$ such that $|H \cap I_i| = \mathbf{h}_i$. Then, for a family $\mathcal{F} \subset \mathcal{A}(S)$, we have

$$\mathbb{E}[\mu(\mathcal{F}(\mathbf{H})) \mid \mathbf{h}] = \frac{1}{|\partial_{\mathbf{h}}(\mathcal{A}(S))|} \sum_{H \in \partial_{\mathbf{h}}(\mathcal{A}(S))} \frac{|\mathcal{F}(H)|}{|\mathcal{A}(H \cup S)|}.$$

Each set $F \in \mathcal{F}$ is counted $\prod_{i=1}^{w} \binom{k_i - |S \cap I_i|}{\mathbf{h}_i}$ times, so we have

$$\mathbb{E}[\mu(\mathcal{F}(\mathbf{H})) \mid \mathbf{h}] = \frac{|\mathcal{A}(S)| \prod_{i=1}^{w} \binom{k_i - |S \cap I_i|}{\mathbf{h}_i}}{|\partial_{\mathbf{h}} \mathcal{A}(S)| \prod_{i=1}^{w} \binom{n - \mathbf{h}_i - |I_i \cap S|}{k_i - \mathbf{h}_i - |S \cap I_i|}} \cdot \mu(\mathcal{F}),$$

where $\mu(\mathcal{F}) = |\mathcal{F}|/|\mathcal{A}(S)|$. The multiplicative factor of $\mu(\mathcal{F})$ on the right-hand side equals

$$\frac{\prod_{i=1}^{w} \binom{n - |I_i \cap S_i|}{k_i - |S \cap I_i|} \binom{k_i - |S \cap I_i|}{\mathbf{h}_i}}{\prod_{i=1}^{w} \binom{n - |S \cap I_i|}{\mathbf{h}_i} \binom{n - \mathbf{h}_i - |S \cap I_i|}{k_i - \mathbf{h}_i - |S \cap I_i|}} = 1,$$

so $\mathbb{E}[\mu(\mathcal{F}(\mathbf{H})) \mid \mathbf{h}] = \mu(\mathcal{F})$, and, by taking the expectation of both sides, Assumption 3 follows. □

Then, we check Assumption 4 for $\binom{[n]}{k/w}^w$. By taking $w = 1$, it implies the first statement of the proposition. Consider arbitrary $R, |R| \leqslant q$. If $k/w \geqslant q + t$, then $k \geqslant w(|R| + t)$. For a partition of h into nonnegative ordered summands $h_1, \dots, h_w$, define $\overline{h} = (h_1, \dots, h_w)$, and note that $h_i \leqslant t \leqslant k/w - |R|$. Thus, we have

$$\begin{aligned}
\frac{|\partial_h(\mathcal{A}(R))|}{|\partial_h \mathcal{A}|} &= \frac{\sum_{\overline{h}} \prod_{i=1}^{w} \binom{n - |R \cap I_i|}{h_i}}{\sum_{\overline{h}} \prod_{i=1}^{w} \binom{n}{h_i}} \geqslant \min_{\overline{h}} \prod_{i=1}^{w} \frac{\binom{n - |R \cap I_i|}{h_i}}{\binom{n}{h_i}} \\
&\geqslant \min_{(h_1, \dots, h_w), \sum_i h_i = h} \prod_{i=1}^{w} \left(\frac{n - |R| - h}{n}\right)^{h_i} \geqslant \left(1 - \frac{|R| + h}{n}\right)^h \\
&\geqslant \left(1 - \frac{q + t}{n}\right)^h \geqslant \left(1 - \frac{2q}{n}\right)^h,
\end{aligned}$$

so Assumption 4 is satisfied with $\mu = n/2k$.

Thus, the first and the third statements of the proposition are proved. It remains to check that $[n]^k$ satisfies Assumption 4 with $\mu = 1/2$. For any R, we have

$$\left|\partial_h([n]^k(R))\right| = \binom{k - |R|}{h} n^h,$$

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so

$$\frac{\left|\partial_h([n]^k(R))\right|}{|\partial_h[n]^k|} \geqslant \frac{\binom{k-|R|}{h}}{\binom{k}{h}} \geqslant \left(\frac{k-|R|-h}{k}\right)^h \geqslant \left(1-\frac{q+h}{k}\right)^h \geqslant \left(1-\frac{2q}{k}\right)^h,$$

provided $k \geqslant 2q$. It implies the second statement of the proposition. □

It turns out that under mild restrictions on n, t, q the family of permutations Σ_n as the family of sets $\mathfrak{S}_n$ (see Section 4.1.2) also satisfies Assumptions 1-4. Note that n plays the role of uniformity of the family Σ_n.

Proposition 5.6. *Suppose that $n > 4(q+t)$, $n \geqslant 16$ and $q \geqslant t$. Then, $\mathfrak{S}_n$ satisfies Assumptions 1-4 with $k = n, r = n/4$, $\eta = 2$ and $\mu = 1/4$.*

Proof. Since $n > 4(q+t)$, Proposition 4.9 implies that $\mathfrak{S}_n$ is $(r, q+t-1)$-spread for $r = n/4$, so Assumption 1 is satisfied.

Next, we check Assumption 2. We have

$$(n-t)! \geqslant n^{-t}n! = 4^{-t}(n/4)^{-t}n! \geqslant (n/4)^{-2t}n!,$$

since $n \geqslant 16$. Therefore, Assumption 2 holds with $\eta = 2$.

Then, we check Assumption 3. We may consider a set $S \in \partial_{\leqslant q}\mathfrak{S}_n$ as an injection that maps some set $\mathrm{Dom}S$ of size $|S|$ to a subset $\mathrm{Im}S$ of $[n]$. Thus, $\mathfrak{S}_n(S)$ can be considered as a family of such permutations that coincide with S on the domain $\mathrm{Dom}S$ of S. The shadow $\partial_h(\mathfrak{S}_n)$ can be considered as a number of bijections from a subset of $[n] \setminus \mathrm{Dom}S$ of size h to a subset of $[n] \setminus \mathrm{Im}S$. It yields

$$\begin{aligned}|\mathfrak{S}_n(S)| &= (n-|S|)!\\ |\partial_h(\mathfrak{S}_n(S))| &= \binom{n-|S|}{h}^2 h!\\ |\mathfrak{S}_n(S\cup H)| &= (n-|S|-h)!\end{aligned}$$

for any $H \in \partial_h(\mathfrak{S}_n(S))$. Finally, we have

$$\sum_{H\in\partial_h(\mathfrak{S}_n(S))} |\mathcal{F}(H)| = \binom{n-|S|}{h}|\mathcal{F}|.$$

It implies

$$\begin{aligned}\frac{1}{|\partial_h(\mathfrak{S}_n(S))|}\sum_{H\in\partial_h(\mathfrak{S}_n(S))}\frac{|\mathcal{F}(H)|}{|\mathfrak{S}_n(S\cup H)|} &= \frac{1}{\binom{n-|S|}{h}^2 h!}\cdot\frac{\binom{n-|S|}{h}}{(n-|S|-h)!}\cdot|\mathcal{F}| = \frac{|\mathcal{F}|}{(n-|S|)!}\\ &= \frac{|\mathcal{F}|}{|\mathfrak{S}_n(S)|},\end{aligned}$$

and Assumption 3 follows.

Finally, we check Assumption 4. Fix a set $R \in \partial_{\leqslant q}\mathfrak{S}_n$, and bound

$$\frac{|\partial_h(\mathfrak{S}_n(R))|}{|\partial_h\mathfrak{S}_n|} = \frac{\binom{n-|R|}{h}^2 h!}{\binom{n}{h}^2 h!} \geqslant \left(1-\frac{|R|+h}{n}\right)^{2h} \geqslant \left(1-\frac{4q}{n}\right)^h,$$

so Assumption 4 holds with $\mu = 1/4$. □

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We complete this section with three statements about τ-homogeneous families. The first one shows that most restrictions of a τ-homogeneous family $\mathcal{F}$ are also homogeneous.

Proposition 5.7. *Suppose that Assumption 3 holds. Let $\mathcal{F} \subset \mathcal{A}(S)$, $S \in \partial_{\leqslant q}\mathcal{A}$, be a τ-homogeneous family. Let h be a positive integer less than k and $\alpha \in (0,1)$. Suppose that $\tau \leqslant (1-\alpha\rho)^{-1/h}$. Then, for all but $\alpha|\partial_h\mathcal{A}|$ sets $H \in \partial_h\mathcal{A}$, the family $\mathcal{F}(H)$ is $\hat{\tau}$-homogeneous for $\hat{\tau} = \tau/(1-\rho)$ and $\mu(\mathcal{F}(H)) \geqslant (1-\rho)\cdot\tau^h\cdot\mu(\mathcal{F})$.*

Proof. If $\mathcal{F}(H)$ is not $\hat{\tau}$-homogeneous, then there exists Z such that

$$\hat{\tau}^{-|Z|}\mu(\mathcal{F}(H\cup Z)) \geqslant \mu(\mathcal{F}(H)).$$

Since $\mathcal{F}$ is τ-homogeneous, we have

$$\tau^{|H|+|Z|}\hat{\tau}^{-|Z|}\mu(\mathcal{F}) \geqslant \hat{\tau}^{-|Z|}\mu(\mathcal{F}(H\cup Z)) \geqslant \mu(\mathcal{F}(H)),$$

so $\mu(\mathcal{F}(H)) \leqslant \left(\frac{\tau}{\hat{\tau}}\right)^{|Z|}\cdot\tau^{|H|}\mu(\mathcal{F}) \leqslant \tau/\hat{\tau}\cdot\tau^{|H|}\mu(\mathcal{F})$. Define

$$\mathcal{H} = \{H \in \partial_h(\mathcal{A}(S)) \,:\, \mathcal{F}(H) \text{ is not } \hat{\tau}\text{-homogeneous or } \mu(\mathcal{F}(H)) \leqslant \frac{\tau}{\hat{\tau}}\cdot\tau^h\mu(\mathcal{F})\}$$

Due to Assumption 3, we have

$$\frac{1}{|\partial_h(\mathcal{A}(S))|}\sum_{H\in\partial_h(\mathcal{A}(S))}\mu(\mathcal{F}(H)) = \mu(\mathcal{F}),$$

For each $H \in \mathcal{H}$, we have $\mu(\mathcal{F}(H)) \leqslant \frac{\tau}{\hat{\tau}}\cdot\tau^h\mu(\mathcal{F})$ by the reasoning above. If $H \notin \mathcal{H}$, we bound $\mu(\mathcal{F}(H)) \leqslant \tau^{|H|}\mu(\mathcal{F})$ by the τ-homogeneity of $\mathcal{F}$, and obtain

$$\frac{1}{|\partial_h(\mathcal{A}(S))|}\sum_{H\in\partial_h\mathcal{A}}\left(\frac{\tau}{\hat{\tau}}\cdot\tau^{|H|}\mu(\mathcal{F})\mathbb{1}\{H\in\mathcal{H}\} + \tau^{|H|}\mu(\mathcal{F})\mathbb{1}\{H\notin\mathcal{H}\}\right) \geqslant \mu(\mathcal{F}),$$

$$\mu(\mathcal{H})\cdot\frac{\tau}{\hat{\tau}}\cdot\tau^h + \tau^h(1-\mu(\mathcal{H})) \geqslant 1,$$

where $\mu(\mathcal{H}) = |\mathcal{H}|/|\partial_h(\mathcal{A}(S))|$. Rearranging the terms, we get

$$\mu(\mathcal{H}) \leqslant \frac{1-\tau^{-h}}{1-\tau/\hat{\tau}} \leqslant \frac{1-(1-\alpha\rho)}{1-(1-\rho)} = \alpha.$$

□

Next, we show that τ-homogeneous families have large shadows.

Proposition 5.8. *Suppose that Assumption 3 holds. Let $\mathcal{F} \subset \mathcal{A}(S)$, $S \in \partial_{\leqslant q}\mathcal{A}$, be a nonempty τ-homogeneous family. Then, we have $\mu(\partial_h\mathcal{F}) \geqslant \tau^{-h}$ for any $h \leqslant k$.*

Proof. Due to Assumption 3, we have

$$\mu(\mathcal{F}) = \frac{1}{|\partial_h(\mathcal{A}(S))|}\sum_{P\in\partial_h\mathcal{F}}\mu(\mathcal{F}(P)) \leqslant \frac{|\partial_h\mathcal{F}|}{|\partial_h(\mathcal{A}(S))|}\cdot\tau^h\mu(\mathcal{F}).$$

Rearranging the terms, we obtain the claim. □

Finally, we show that homogeneity can be preserved after removing elements from the support of a τ-homogeneous family.

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Claim 5.9. *If $\mathcal{G} \subset \mathcal{A}$ is τ-homogeneous for some r-spread $\mathcal{A}$, and $|X| < r/\tau$, then $\mathcal{G}(\varnothing, X)$ is nonempty and τ'-homogeneous for*

$$\tau' = \frac{\tau}{1 - \frac{|X|\tau}{r}}.$$

Proof. If $\mathcal{G}$ is τ-homogeneous, then it is r/τ-spread, so

$$|\mathcal{G}(\varnothing, X)| \geqslant \left(1 - \frac{|X|\tau}{r}\right)|\mathcal{G}| \tag{5.2}$$

due to Claim 4.6. It yields

$$\begin{aligned}\frac{|\mathcal{G}(\varnothing, X)(Y)|}{|\mathcal{A}(Y)|} = \frac{|\mathcal{G}(Y, X \cup Y)|}{|\mathcal{A}(Y)|} &\leqslant \frac{|\mathcal{G}(Y)|}{|\mathcal{A}(Y)|} \\ &\overset{(\mathcal{G} \text{ is } \tau\text{-homogeneous})}{\leqslant} \tau^{|Y|}\frac{|\mathcal{G}|}{|\mathcal{A}|} \overset{(5.2)}{\leqslant} \tau^{|Y|}\frac{|\mathcal{G}(\varnothing, X)|}{(1 - |X|\tau/r)|\mathcal{A}|} \\ &\leqslant (\tau/(1 - |X|\tau/r))^{|Y|}\frac{|\mathcal{G}(\varnothing, X)|}{|\mathcal{A}|}.\end{aligned}$$

□

5.4. Spread approximation

There is a natural counterpart of Observation 4.2 for the τ-homogeneity property.

Observation 5.10 (Observation 1 from [49]). Let $\mathcal{F} \subset \mathcal{A}$ be an arbitrary subfamily of a k-uniform family $\mathcal{A}$. Fix some $\tau > 1$. Let $\mathcal{S}$ be the maximal set such that $\mu(\mathcal{F}(S)) \geqslant \tau^{-|S|}\mu(\mathcal{F})$. Then, the family $\mathcal{F}(S)$ is τ-homogeneous as a subfamily of $\mathcal{A}(S)$.

Using it, one may obtain the following lemma that allows applying the τ-homogeneity machinery in an arbitrary Turan-type problem.

Lemma 5.11 [49]. *Fix an integer q and a real number $\tau > 1$. Let $\mathcal{A}$ be a k-uniform r-spread family. Let $\mathcal{F} \subset \mathcal{A}$. Then there exists a family $\mathcal{S} \subset \partial_{\leqslant q}\mathcal{A}$ and a partition of $\mathcal{F}$ into families $\mathcal{R}$ and $\mathcal{F}_S \subset \mathcal{A}(S)$, $S \in \mathcal{S}$:*

$$\mathcal{F} = \mathcal{R} \sqcup \bigsqcup_{S \in \mathcal{S}} \mathcal{F}_S \vee \{S\},$$

such that each $\mathcal{F}_S$ is τ-homogeneous as a subfamily of $\mathcal{A}(S)$ and $|\mathcal{R}| \leqslant \tau^{-(q+1)}|\mathcal{A}|$.

Informally, we refer to the family $\mathcal{S}$ as the spread approximation of $\mathcal{F}$. Depending on the problem, $\mathcal{S}$ possesses numerous useful properties. For example, if $\mathcal{F}$ does not contain two sets intersecting in exactly $t - 1$ elements, then each two elements of $\mathcal{S}$ intersect in at least t elements under some assumptions on r, k and t [49]. The accurate analysis of $\mathcal{S}$ depending on n, k and t is the core of our approach.

However, Lemma 5.11 is not the only way to obtain a spread approximation of $\mathcal{F}$. Later, we present other ways, so we require a precise definition of such families $\mathcal{S}$.

Definition 7. A family $\mathcal{S}$ is an R-spread (τ-homogeneous) core of $\mathcal{F}$, if $\mathcal{F}$ admits a decomposition

$$\mathcal{F} = \bigsqcup_{S \in \mathcal{S}} \mathcal{F}_S \vee \{S\},$$

where families $\mathcal{F}_S \subset \mathcal{F}(S)$ are R-spread (τ-homogeneous).

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5.5. Boolean analysis

When k is extremely large (namely, $k \geqslant 2^{(st)^C}$ for some large constant C), techniques based on τ-homogeneous approximation give suboptimal results. The reason is that the spread lemma works for the families without any assumption on their measure. In general, the factor $\log k$ in applications of Theorem 4.1 is unavoidable, as was shown in [1], but in our special case it can be removed.

The idea is to use recent results in Boolean Analysis from papers [38] and [41]. Before we start, we give a number of definitions.

We consider the Boolean cube $\{0,1\}^n$ with the p-biased measure μ_p, $\mu_p(x) = p^{\sum_i x_i}(1-p)^{n-\sum_i x_i}, x \in \{0,1\}^n$. Next, for the linear space of functions $f : \{0,1\}^n \to \mathbb{R}$, we define the scalar product

$$\langle f, g\rangle = \mathbb{E}_{x\sim\mu_p} f(x)g(x).$$

We denote the obtained Hilbert space by $L_2(\{0,1\}^n, \mu_p)$. We define the q-norm of the function f as

$$\|f\|_q = \mathbb{E}^{1/q}_{x\sim\mu_p} |f(x)|^q.$$

We denote expectation $\mathbb{E}_{x\sim\mu_p} f(x)$ of f by $\mu_p(f)$. There is a natural correspondence between the Boolean functions $f : \{0,1\}^n \to \{0,1\}$ and families of subsets $\mathcal{F}$ of the Boolean cube, defined as follows: $f(x) = 1$ if and only if for the corresponding family $\mathcal{F}$ it holds that $\mathrm{supp}(x) \in \mathcal{F}$. Then, the measure $\mu_p(\mathcal{F})$ is just the expectation $\mu_p(f)$ of the corresponding Boolean function and can be obtained as follows:

$$\mu_p(\mathcal{F}) = \sum_{F\in\mathcal{F}} p^{|F|}(1-p)^{n-|F|}.$$

One can interpret $\mu_p(\mathcal{F})$ as the probability that $\mathcal{F}$ contains a p-random subset $\mathbf{X}$ of $[n]$.

When dealing with the uniform families $\mathcal{F} \subset \binom{[n]}{k}$, one can use techniques from Boolean Analysis, by applying them to the upper-closure $\mathcal{F}^\uparrow$ of the family $\mathcal{F}$. There is a simple inequality, which relates the p-biased measure of $\mathcal{F}^\uparrow$ and the size of $\mathcal{F}$:

Claim 5.12 (Lemma IV.2.2. [38]). *Let $\mathcal{F}$ be a subfamily of $\binom{[n]}{k}$ and set $p = \frac{k}{n}$, then $\mu_p(\mathcal{F}^\uparrow) \geqslant \frac{|\mathcal{F}|}{4\binom{n}{k}}$.*

When dealing with the restrictions $\mathcal{F}(A,B)$ of a family $\mathcal{F}$, one should be careful with the p-biased measure of $\mathcal{F}(A,B)$. Indeed, for a set B, one can treat $\mathcal{F}(\varnothing, B)$ either as a subfamily of the Boolean cube $2^{[n]}$ or as a subfamily of the Boolean cube $2^{[n]\setminus B}$. Both cases arise in our proofs, so we define μ_p^{-B} for the p-biased measure on $2^{[n]\setminus B}$ and $\mu_p^{[n]}$ for $2^{[n]}$. We may omit the superscript if the measure and the domain are clear from the context. For restrictions of the form $\mathcal{F}(B)$, we assume that the measure $\mu_p^{-B}(\mathcal{F}(B))$ is considered by default.

The next definition, which is a variant of τ-homogeneity, is the core notion of this section.

Definition 8. A family $\mathcal{F} \subset 2^{[n]}$ is τ-global with respect to the measure $\mu_p^{[n]}$, if for any set B and a subset $A \subset B$, the following holds:

$$\mu_p^{-B}(\mathcal{F}(A,B)) \leqslant \tau^{|B|}\mu_p^{[n]}(\mathcal{F}).$$

In terms of the Boolean functions $f : \{0,1\}^n \to \{0,1\}$, τ-globalness is defined as follows:

$$\mu_p^{-S}(f_{S\to x}) \leqslant \tau^{|S|}\mu_p(f)$$

for any $S \subset [n]$ and any $x \in \{0,1\}^S$, where the function $f_{S\to x} : \{0,1\}^{[n]\setminus S} \to \{0,1\}$ is the restriction of f to S, that is, it equals $f(x,y)$ for any $y \in \{0,1\}^{[n]\setminus S}$. In the set families notation, $f_{S\to x}$ vs f corresponds to $\mathcal{F}(A,S)$ vs $\mathcal{F}$, where $A = \{i \in S : x_i = 1\}$.

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Importantly, given a family $\mathcal{F}$, it is relatively simple to obtain a restriction B, $A \subset B$, such that $\mathcal{F}(A, B)$ is τ-global.

Observation 5.13 (Section 2.3 [42]). For a family $\mathcal{F} \subset 2^{[n]}$, consider a pair of sets A, B, $A \subset B$, that maximizes the quantity $\tau^{-|B|}\mu_p^{-B}(\mathcal{F}(A, B))$. Then, $\mathcal{F}(A, B)$ is τ-global with respect to μ_p. Additionally, if $\frac{1}{1-p} < \tau$, then the maximum is attained when $A = B$.

Before we state the main theorem of this section, we need several other definitions. For a vector $x \in \{0,1\}^n$, we say a random vector y is ρ-correlated with x, if it can be obtained as follows: independently for all i, with probability ρ set $y_i = x_i$ and with probability $1 - \rho$ sample y_i from the measure μ_p on $\{0,1\}$. We denote the resulting distribution as $N_\rho(x)$.

Next, we define the following linear operator on $L_2(\{0,1\}^n, \mu_p)$. For a function $f \in L_2(\{0,1\}^n, \mu_p)$, set $(T_\rho f)(x) = \mathbb{E}_{y \sim N_\rho(x)} f(y)$. The operator T_ρ is called the *noise operator*.

In general, for small enough ρ, the operator T_ρ admits the following property: $\|T_\rho f\|_q \leqslant \|f\|_2$ for $q > 2$. Such a property is referred to as *hypercontractivity*. For an extensive introduction to Boolean Analysis and hypercontractivity on the Boolean cube, we refer the reader to the seminal book by O'Donnel [51]. For general hypercontractivity and its connections to Markov chains and functional analysis, we recommend the book [4] by Bakry, Gentil, and Ledoux.

For a function f, one may also consider the following quantity, which is called the *stability* of f:

$$\mathrm{Stab}_\rho(f) = \langle T_\rho f, f \rangle.$$

In general, if the q-th norm of $T_\rho f$ is efficiently bounded, one can control $\mathrm{Stab}_\rho(f)$ via the Hölder inequality, see the proof of Lemma 5.16.

The main result of the paper [41] is that the hypercontractivity phenomenon happens for τ-global functions f.

Theorem 5.14 (Theorem 1.4 [41]). *Let $f : \{0,1\}^n \to \{0,1\}$ be a τ-global Boolean function, that is,*

$$\mu_p(f_{S \to x}) \leqslant \tau^{|S|}\mu_p(f)$$

for all $S \subset [n]$ and all $x \in \{0,1\}^S$. Then for any $q > 2$ and $\rho \leqslant \frac{\ln q}{16\tau q}$ we have $\|T_\rho f\|_q \leqslant \|f\|_2$.

The theorem holds for any function $f : \{0,1\}^n \to \mathbb{R}$, but we do not use this fact. The result has numerous applications in combinatorics, as it was shown by the authors in the follow-up papers [42, 43]. Intriguingly, in some cases, techniques based on the spread approximation and techniques based on the hypercontractivity can be used to obtain almost the same result, see papers [49, 42, 47].

To apply Theorem 5.14 in our special case, we notice the following. We use the spread Lemma 4.1 to prove that R-spread families have large $\mu_{1/8s}$ measure, and, therefore, they contain a matching. In Boolean Analysis, the phenomenon when the p-biased measure $\mu_p(f)$ of a function f is rapidly increasing when p is slightly increasing is known as the *sharp threshold* phenomenon. The fact that pseudorandom families (such as spread or global) admit sharp thresholds was exploited before to obtain the Chvátal Simplex conjecture in the seminal paper [40] by Ellis, Keller, and Lifshitz. Later, this tool was further developed in paper [38], where the following proposition is presented.

Proposition 5.15 (Proposition III.3.4 [38]). *Let $f : \{0,1\}^n \to \{0,1\}$ be a monotone Boolean function. Let $0 < p < \widetilde{p} < 1$ and $\rho = \frac{p(1-\widetilde{p})}{\widetilde{p}(1-p)}$. Then*

$$\mu_{\widetilde{p}}(f) \geqslant \frac{\mu_p^2(f)}{\mathrm{Stab}_\rho(f)},$$

where Stab_ρ is w.r.t. the measure μ_p.

This result together with the hypercontractivity bound will be used in Section 5.6 to reprove the results of Sections 5.3 and 8.1 for μ_p in the case when n is linear in k.

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5.6. Sharp thresholds and matchings

We start with the following lemma, which will be a building block for the μ_p-counterpart of Theorem 4.1.

Lemma 5.16. *Suppose that an upper-closed family $\mathcal{F}$ is τ-global. For a number $p \in (0; 2^{-7}/\tau)$, define $\widetilde{p} = 2^6\tau p$. Then, the following holds:*

$$\mu_{\widetilde{p}}(\mathcal{F}) \geqslant \mu_p^{3/4}(\mathcal{F}).$$

Proof. We consider $\mathcal{F}$ as a monotone Boolean function. Set $\rho = \frac{p(1-\widetilde{p})}{\widetilde{p}(1-p)}$. Since $p \leqslant \widetilde{p}$, we have $1-\widetilde{p} \leqslant 1-p$, and $\rho \leqslant \frac{p}{\widetilde{p}} = \frac{1}{2^6\tau}$. Hence, we may apply Theorem 5.14 with $q = 4$ and obtain $\|T_\rho f\|_4 \leqslant \|f\|_2$. An application of Hölder's inequality implies

$$\mathrm{Stab}_\rho(f) = \langle T_\rho f, f\rangle \leqslant \|T_\rho f\|_4 \|f\|_{4/3} \leqslant \|f\|_2 \|f\|_{4/3}.$$

Since f is Boolean, $\|f\|_2 = \mu_p^{1/2}(f)$ and $\|f\|_{4/3} = \mu_p^{3/4}(f)$. Therefore, $\mathrm{Stab}_\rho(f) \leqslant \mu_p^{5/4}(f)$. By our choice of ρ, we can apply Proposition 5.15 and obtain

$$\mu_{\widetilde{p}}(f) \geqslant \frac{\mu_p^2(f)}{\mathrm{Stab}_\rho(f)} \geqslant \frac{\mu_p^2(f)}{\mu_p^{5/4}(f)} = \mu_p^{3/4}(f).$$

□

The main idea is to iterate Lemma 5.16. Applying Lemma 5.16 several times to a family $\mathcal{F}$, one may obtain that $\mu_{\widetilde{p}}(\mathcal{F})$ is large for $\widetilde{p}$ much greater than p.

Corollary 5.16.1. *Fix integers m, z and $\tau \geqslant 2$. Let $\mathcal{F}$ be an upward closed family such that $\mu_p(\mathcal{F}) \geqslant \tau^{-z}$ and $p \in (0; (2^6\tau)^{-m}/2)$. Then, there exists a set R of size at most $4z$ such that, for $\widetilde{p} = (2^6\tau)^m p$, the following holds*

$$\mu_{\widetilde{p}}^{-R}(\mathcal{F}(R)) \geqslant \left(\mu_p(\mathcal{F})\right)^{(\frac{3}{4})^m},$$

and $\mathcal{F}(R)$ is τ-global w.r.t the measure $\mu_{\widetilde{p}}^{-R}$.

Proof. We obtain the result by an iterative application of Lemma 5.16. Set $\mathcal{F}_0 = \mathcal{F}$, $p_0 = p$. Note that $\mu_{p_0}(\mathcal{F}_0) \geqslant \tau^{-z}$. At step i, find R_i that maximizes the quantity $\tau^{-|S|}\mu_{p_i}(\mathcal{F}_i(R_i))$ and set $\mathcal{F}_{i+1} = \mathcal{F}_i(R_i)$. We prove by induction that $\mu_{p_i}(\mathcal{F}_i) \geqslant \mu_p(\mathcal{F})^{(\frac{3}{4})^i} \geqslant \tau^{-z(\frac{3}{4})^i}$. Clearly, the base $i = 0$ holds.

Due to Observation 5.13, $\mathcal{F}_i(R_i)$ is τ-global. By the inductive assumption and the choice of R, we have

$$\tau^{-z(\frac{3}{4})^i} \leqslant \mu_{p_i}(\mathcal{F}_i) \leqslant \mu_{p_i}(\mathcal{F}_i(R_i))\tau^{-|R_i|} \leqslant \tau^{-|R_i|},$$

and, therefore, the set R_i has size at most $z\left(\frac{3}{4}\right)^i$. Define $p_{i+1} = 2^6\tau p_i$ and $\mathcal{F}_{i+1} = \mathcal{F}_i(R_i)$. Due to Lemma 5.16, we have

$$\mu_{p_{i+1}}(\mathcal{F}_{i+1}) \geqslant \mu_{p_i}^{3/4}(\mathcal{F}_i) \geqslant (\mu_p(\mathcal{F}_0))^{(3/4)^{i+1}},$$

and the induction step is proved.

Stop the procedure after m steps and set $R = \bigsqcup_{i=1}^m R_i$. We have

$$|R| = \sum_{i=0}^{m} |R_i| \leqslant z\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i = 4z.$$

We have $\mathcal{F}_m = \mathcal{F}(R)$, and the corollary follows. □

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Corollary 5.16.1 is a counterpart of Theorem 4.1. To obtain a statement corresponding to Proposition 4.4, we need the following claim, which coincides with Claim 5.9 up to the measure used.

Claim 5.17. *Fix numbers τ, p, $p < 1 < \tau$. Let X be a set of size strictly less than $1/\tau p$. Suppose that a family $\mathcal{F} \subset 2^{[n]}$ is τ-global w.r.t. measure $\mu_p^{[n]}$. Then the family $\mathcal{G} = \mathcal{F}(\varnothing, X)$ is $\hat{\tau}$-global w.r.t. $\mu_p^{[n]}$ for*

$$\hat{\tau} = \frac{\tau}{1 - |X|p\tau},$$

and $\mu_p(\mathcal{G}) \geqslant (1 - |X|p\tau)\mu_p(\mathcal{F})$.

Proof. Let $\mathcal{G} = \mathcal{F}(\varnothing, X)$. Consider an arbitrary nonempty B and $A \subset B$. We have

$$\mu_p^{-B}(\mathcal{G}(A, B)) \leqslant \mu_p^{-B}(\mathcal{F}(A, B)) \leqslant \tau^{|B|}\mu_p^{[n]}(\mathcal{F}), \tag{5.3}$$

since $\mathcal{F}$ is τ-global. We claim that $\mu_p^{[n]}(\mathcal{F}[x]) \leqslant p\tau\mu_p(\mathcal{F})$ for any $x \in [n]$. Indeed, we have

$$\mu_p^{[n]}(\mathcal{F}[x]) = \sum_{F \in \mathcal{F}(x)} p^{|F|+1}(1-p)^{n-|F|-1} = p\mu_p^{-\{x\}}(\mathcal{F}(x)) \leqslant p\tau\mu_p(\mathcal{F}).$$

It yields $\mu_p^{[n]}(\mathcal{G}) \geqslant \mu_p(\mathcal{F}) - \sum_{x \in X} \mu_p^{[n]}(\mathcal{F}[x]) \geqslant (1 - p\tau|X|)\mu_p(\mathcal{F})$. Substituting it into (5.3), we get

$$\mu_p^{-B}(\mathcal{G}(A, B)) \leqslant \tau^{|B|}\mu_p(\mathcal{F}) \leqslant \left(\frac{\tau}{1 - |X|p\tau}\right)^{|B|} \mu_p^{[n]}(\mathcal{G}),$$

so $\mathcal{G}$ is indeed $\hat{\tau}$-global. □

Finally, we obtain a counterpart of Proposition 4.4. The difference is that we require the families to have not too small measure, so that we can upscale it by Corollary 5.16.1.

Lemma 5.18. *Let z be an integer and $\tau \geqslant 4$ some positive number. Let $\mathcal{F}_1 \subset \binom{[n]}{k_1}, \dots, \mathcal{F}_s \subset \binom{[n]}{k_s}$ be $\tau/2$-homogeneous families with $|\mathcal{F}_i| \geqslant 8\tau^{-z}\binom{[n]}{k_i}$. Set $p_i = k_i/n$ and $m = \left\lceil \log_{4/3}\left(\frac{z \log \tau}{\log 2}\right)\right\rceil$. Suppose that $sz(2^6\tau)^m \tau p_i < 1/64$ for each $i = 1, \dots, s$. Then, there exist pairwise disjoint sets $F_1 \in \mathcal{F}_1, \dots, F_s \in \mathcal{F}_s$.*

Proof. We apply Corollary 5.16.1 to the upper closures of families $\mathcal{F}_1, \dots, \mathcal{F}_s$ step by step. By our choice of m, we have

$$\left(\frac{3}{4}\right)^m z \log \tau \leqslant \log 2.$$

Set $p = k_i/n$ and $\widetilde{p}_i = (2^6\tau)^m p_i$. Consider the upper closure $\mathcal{Q}_1 = \mathcal{F}_1^{\uparrow}$ of the family $\mathcal{F}_1$. Due to Claim 5.12, we have $\mu_{p_1}(\mathcal{Q}_1) \geqslant \mu(\mathcal{F}_1)/4 \geqslant \tau^{-z}$. Applying Corollary 5.16.1 to $\mathcal{Q}_1$, we obtain a set R_1, $|R_1| \leqslant 4z$, such that

$$\mu_{\widetilde{p}_1}^{-R_1}(\mathcal{Q}_1(R_1)) \geqslant (\mu_{p_1}(\mathcal{Q}_1))^{(3/4)^m} \geqslant (\tau^{-z})^{(3/4)^m} \geqslant e^{-\log 2} = \frac{1}{2},$$

and $\mathcal{Q}_1(R_1)$ is τ-global.

Then we proceed inductively. At step $i+1$, suppose that we have obtained disjoint sets $R_1, \dots, R_i$. Set $R_{\leqslant i} = \bigsqcup_{j=1}^{i} R_j$, $|R_{\leqslant i}| \leqslant 4iz$. By Proposition 4.8, the family $\binom{[n]}{k_i}$ is n/k_i-spread. Put $r_i = n/k_i$. By Claim 5.9, the family $\mathcal{F}_{i+1}(\varnothing, R_{\leqslant i})$ is $\hat{\tau}$-homogeneous for

$$\hat{\tau} = \frac{\tau/2}{1 - \frac{2|R_{\leqslant i}|\tau}{r_i}} \leqslant \frac{\tau/2}{1 - \frac{8szk_i\tau}{n}} = \frac{\tau/2}{1 - 8szp_i\tau} \leqslant \tau.$$

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and

$$\mu(\mathcal{F}_{i+1}(\varnothing, R_{\leqslant i})) \geqslant \left(1 - \frac{2|R_{\leqslant i}|\tau}{r_i}\right)\mu(\mathcal{F}_{i+1}) \geqslant (1 - 8sz p_i \tau)\mu(\mathcal{F}_{i+1}) \geqslant 4\tau^{-z}.$$

Consider the upper closure $\mathcal{Q}_{i+1} = [\mathcal{F}_{i+1}(\varnothing, R_{\leqslant i})]^{\uparrow}$ of the family $\mathcal{F}_{i+1}(\varnothing, R_{\leqslant i})$ w.r.t $[n] \setminus R_{\leqslant i}$. Due to Claim 5.12, we have $\mu_{p_{i+1}}^{-R_{\leqslant i}}(\mathcal{Q}_{i+1}) \geqslant \tau^{-z}$. Apply Corollary 5.16.1 and obtain a set $R_{i+1}, |R_{i+1}| \leqslant 4z$, such that $\mu_{\widetilde{p}_{i+1}}^{-(R_{\leqslant i} \sqcup R_{i+1})}(\mathcal{Q}_{i+1}(R_{i+1})) \geqslant 1/2$ and $\mathcal{Q}_{i+1}(R_{i+1})$ is τ-global.

When the procedure stops, define $R = \bigsqcup_{i=1}^{s} R_i$. Set $\mathcal{G}_i = \mathcal{Q}_i(R_i, R)$. Due to Claim 5.17, we have

$$\begin{aligned}\mu_{\widetilde{p}_i}^{-R_{\leqslant i}}(\mathcal{G}_i) &\geqslant (1 - \widetilde{p}_i \tau |R|)\mu_{\widetilde{p}_i}^{-R_{\leqslant i}}(\mathcal{Q}_i(R_i)) \\ &\geqslant (1 - 4(2^6\tau)^m p_i \tau s z)\mu_{\widetilde{p}_i}^{-R_{\leqslant i}}(\mathcal{Q}_i(R_i)) \geqslant \frac{1}{4},\end{aligned}$$

with the convention $R_{\leqslant 0} = \varnothing$. Since $\mu_{\widetilde{p}_i}^{-R}(\mathcal{G}_i) \geqslant \mu_{\widetilde{p}_i}^{-R_{\leqslant i}}(\mathcal{G}_i)$, we have $\mu_{\widetilde{p}_i}^{-R}(\mathcal{G}_i) \geqslant 1/4$. Since $s\widetilde{p}_i \leqslant 1/16$, we have $\mu_{\widetilde{p}_i}^{-R}(\mathcal{G}_i) \geqslant 3s\widetilde{p}_i$. Therefore, there exist disjoint sets $G_1 \in \mathcal{G}_1, \dots, G_s \in \mathcal{G}_s$ due to Lemma 4.3. Thus, there exist disjoint sets $R_1 \cup G_1 \in \mathcal{F}_s, \dots, R_s \cup G_s \in \mathcal{F}_s$, and the lemma follows. □

6. Proof of Theorem 3.1

Proof. We consider several cases depending on the behavior of k as a function of n.

Case 1: $\mathbf{k \geqslant 33st^2 \ln n}$. Here $\ln(\cdot)$ stands for the natural logarithm. We apply the following theorem, which is proved in Section 8. Moreover, using this theorem, we will derive analogs of Theorem 3.1 for subfamilies of Σ_n and $[n]^k$. The statement of the theorem is given in the notation of Assumptions 1-4.

Theorem 6.1. *Suppose that $\mathcal{A}$ satisfies Assumptions 1-4 with $\varepsilon r \geqslant s^2q^3$ for some $\varepsilon \in (0, 1/2]$ and $q = \lceil 16\eta s t^2 \ln r \rceil$. Suppose additionally, that at least one of the following two conditions holds:*

i) $r \geqslant 2^{11} s \lceil \log_2 k \rceil$;
ii) $\mathcal{A} = \binom{[n]}{k}$ *(so $\eta = 2, r = n/k, q = \lceil 32st^2 \ln \frac{n}{k} \rceil, \mu = n/k$ due to Proposition 5.4) and $n/k \geqslant 2^{390} s^{43} t^{82} + 2^{23000}$.*

Assume that $k \geqslant 2t + q$, $\mu k \geqslant 2stq$. Let $\mathcal{F} \subset \mathcal{A}$ be a family without a sunflower with s petals and the core of size exactly $t - 1$. Then, there exists a family $\mathcal{T} \subset \partial_t \mathcal{A}$ of uniformity t without a sunflower with s petals (and the core of arbitrary size), such that

$$|\mathcal{F} \setminus \mathcal{A}[\mathcal{T}]| \leqslant 32r^{-\eta t} A_t + \phi(s,t) \cdot 4\varepsilon A_t.$$

The proof of Theorem 6.1 is postponed to Section 9.

Let us discuss the case $\mathcal{A} = \binom{[n]}{k}$. Note that $\mathcal{A} = \binom{[n]}{k}$ can satisfy both 6.1 and 6.1, depending on n and k. First, we check when the conditions of the theorem are satisfied. Due to Proposition 5.4, we have $r = n/k$, $\eta = 2$, $\mu = n/k$ and $q = \lceil 32st^2 \ln r \rceil \leqslant 33st^2 \ln(n/k)$. Thus, the condition $\varepsilon r \geqslant s^2q^3$ for some $\varepsilon \leqslant 1/2$ is satisfied if

$$n/k \geqslant 2s^2(33st^2 \ln(n/k))^3 = 2 \cdot 33^3 s^5 t^6 \ln^3(n/k).$$

If $n/k \geqslant 2^{25}$, then $\ln(n/k) \leqslant (n/k)^{1/6}$. Therefore, it is enough to guarantee the following two inequalities

$$n \geqslant 2^{25} k \quad \text{and } n/k \geqslant (2 \cdot 33^3)^2 s^{10} t^{12}.$$

Both of them are satisfied if $n \geqslant 2^{33} s^{10} t^{12} k$. Condition 6.1 is satisfied if $n \geqslant 2^{11} sk \log_2 k$. Condition 6.1 is satisfied if $n \geqslant (2^{390} s^{43} t^{82} + 2^{23000})k$. The condition $\mu k \geqslant 2stq$ is satisfied, since $\mu = n/k$ and $n \geqslant 2s^2q^3 \geqslant 2stq$. Finally, we have $k \geqslant 33st^2 \ln n \geqslant 2t + q$, and so Theorem 6.1 is applicable.

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By the definition, we have $|\mathcal{T}| \leqslant \phi(s,t)$. Choosing $\varepsilon = \frac{s^2q^3}{r}$, we obtain

$$|\mathcal{F}| \leqslant \phi(s,t)\binom{n-t}{k-t} + 32\left(\frac{k}{n}\right)^{2t}\binom{n-t}{k-t} + \phi(s,t)\cdot\frac{4\cdot s^2(33st^2\ln(n/k))^3}{n/k}\cdot\binom{n-t}{k-t}$$
$$\leqslant \left(1+\frac{2^{18}s^5t^6\ln^3(n/k)}{n/k}\right)\phi(s,t)\binom{n-t}{k-t} \leqslant \left(1+\sqrt[3]{\frac{2^{54}s^{15}t^{18}}{n/k}}\right)\phi(s,t)\binom{n-t}{k-t}.$$

The above argument gives us the following lower bound on n:

$$n \geqslant 2^{11}sk\cdot\min\{\log_2 k + 2^{22}s^9t^{11}, 2^{379}s^{42}t^{82}+2^{23000}\}.$$

It remains to study the case $k \leqslant 33st^2\ln n$, or, equivalently, $n \geqslant e^{k/33st^2}$.

Case 2. We have $\mathbf{n \geqslant e^{k/33st^2}}$ and $\mathbf{k \geqslant 2^{10}st^3\ln k}$. In this case, we use the fact that $n \geqslant s^{40t}\cdot g_0(t)$ for some large enough function $g_0(\cdot)$ depending on t. For such k, we have $k^{20t} \leqslant e^{k/33st^2}$, and thus, we may apply the following theorem.

Theorem 6.2. *Let $\mathcal{A} = \binom{[n]}{k/w}^w$ for some $w|k$. Suppose that $n \geqslant k^{20t}$, $n \geqslant s^{40t}\cdot g_0(t)$ and $k \geqslant 5t$, where $g_0(t) \geqslant (2^{20}t^6)^{20t}$. Let $\mathcal{F} \subset \mathcal{A}$ be a family without a sunflower with s petals and the core of size exactly $t-1$. Then, we have*

$$|\mathcal{F}| \leqslant \phi(s,t)|\partial_{k-t}\mathcal{A}| + n^{-1/3}\cdot A_t.$$

The proof of Theorem 6.2 is postponed to Section 9.

For the ambient family $\mathcal{A} = \binom{[n]}{k}$, we have $|\partial_{k-t}\mathcal{A}| = \binom{n}{k-t}$. Therefore, the family $\mathcal{F}$ as in Theorem 6.2 has size at most

$$|\mathcal{F}| \leqslant \phi(s,t)\binom{n}{k-t} + n^{-1/3}\binom{n-t}{k-t}.$$

Bounding

$$\frac{\binom{n}{k-t}}{\binom{n-t}{k-t}} \leqslant \frac{n^{k-t}}{(n-k)^{k-t}} = \left(1+\frac{k}{n-k}\right)^{k-t} \leqslant 1+\frac{2k^2}{n-k},$$

we obtain

$$|\mathcal{F}| \leqslant \left(1+\frac{k^2}{n-k}+n^{-1/3}\right)\phi(s,t)\binom{n-t}{k-t} \leqslant \left(1+\sqrt[3]{\frac{2^{54}s^{15}t^{18}}{n/k}}\right)\phi(s,t)\binom{n-t}{k-t}.$$

Cases 1 and 2 together cover the regime

$$n \geqslant 2^{11}sk\cdot\min\{\log_2 k + 2^{22}s^9t^{11}, 2^{379}s^{42}t^{82}+2^{23000}\}, \quad n \geqslant s^{40t}\cdot g_0(t),$$

and $k \geqslant 2^{10}st^3\log k$. It remains to consider the case $2t+1 \leqslant k \leqslant 2^{13}st^3\log k$.

Case 3. We have $\mathbf{5t \leqslant k \leqslant 2^{10}st^3\log k}$. In particular, we have $k \leqslant 2^{20}s^2t^6$, and, since $n \geqslant s^{40t}\cdot g_0(t) \geqslant k^{20t}$, we may again apply Theorem 6.2 and obtain

$$|\mathcal{F}| \leqslant \left(1+\sqrt[3]{\frac{2^{54}s^{15}t^{18}}{n/k}}\right)\phi(s,t)\binom{n-t}{k-t}$$

as before.

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Case 4. We have $2t+1 \leqslant k < 5t$. In this case, we apply Theorem 5.2, and, obtain

$$|\mathcal{F}| \leqslant \phi(s,t)\binom{n}{k-t} + \frac{(5st)^{2^{Ct}}}{n}\binom{n}{k-t}$$

for some absolute constant C. The desired bound holds provided $n \geqslant n_0(s,t)$. □

7. Erdős–Duke problem for spread domains

Before we move on to proofs of Theorems 6.1,6.2, let us discuss how results of Section 6 can be generalized to other domains.

7.1. Permutations

Let us show that Theorem 6.1 can be applied to families of permutations. Due to Proposition 5.6, the family $\mathfrak{S}_n$ satisfies Assumptions 1-4 with $k = n$, $r = n/4$, $\eta = 2$ and $\mu = 4$. Hence, the condition $\varepsilon r \geqslant s^2q^3$ for $\varepsilon \leqslant 0.5$ of Theorem 6.1 requires $n/4 \geqslant 2s^2q^3 = 2s^2\lceil 32st^2\ln(n/4)\rceil^3$. This inequality is fulfilled if $n \geqslant 2^7 s^5t^6\ln^3 n$. Next, the condition $r \geqslant 2^{11}s\lceil\log_2 k\rceil$ is implied by $n/4 \geqslant 2^{12}s\ln(n/4)/\ln(2)$. Thus, if $n \geqslant 2^{15}s^5t^6\ln^3 n$, both conditions are satisfied. Under this assumption, the inequalities $k = n \geqslant 2t+q$ and $\mu k = n/2 \geqslant 2stq$ hold, so we can deduce the following corollary of Theorem 6.1 with $\varepsilon = 4s^2q^3/n$.

Corollary 7.0.1. *Suppose that $n \geqslant 2^{15}s^5t^6\ln^3 n$. Let $\mathcal{F} \subset \Sigma_n$ be a family of permutations that does not contain a sunflower with s petals and the core of size $t-1$ in the sense of Definition 5. Then, we have*

$$|\mathcal{F}| \leqslant \left(1 + \frac{2^{21}s^5t^6\ln^3 n}{n}\right)\phi(s,t)(n-t)!.$$

Proof. Consider the subfamily $\mathfrak{F} \subset \mathfrak{S}_n$ of sets corresponding to the family $\mathcal{F} \subset \Sigma_n$, see Section 4.1.2. Then, $\mathfrak{F}$ meets conditions of Theorem 6.1 by the reasoning above. It implies

$$\begin{aligned}|\mathfrak{F}| &\leqslant \phi(s,t)\cdot(n-t)! + 32\cdot(n/4)^{-2t}(n-t)! + \frac{16s^2(33st^2\ln(n/4))^3}{n}\cdot(n-t)!\\ &\leqslant \left(1 + \frac{2^{21}s^5t^6\ln^3 n}{n}\right)\phi(s,t)\cdot(n-t)!.\end{aligned}$$

□

One can construct an example similar to Example 3.2, which shows that Corollary 7.0.1 is tight up to second-order terms.

Proposition 7.1. *Let $n > t\phi(s,t) + t$. Then, there exists a family $\mathcal{F} \subset \Sigma_n$ without a sunflower with s petals and the core of size $t-1$, such that*

$$|\mathcal{F}| \geqslant \left(1 - \frac{t\phi(s,t)}{n-t}\right)\cdot\phi(s,t)(n-t)!.$$

Proof. Let $\mathcal{G}$, $|\mathcal{G}| = \phi(s,t)$, be an extremal family of uniformity t without a sunflower with s petals. Assume that $n \geqslant t^2\phi^2(s,t) \geqslant |\mathrm{supp}\mathcal{G}|^2$ and $\mathrm{supp}\mathcal{G} \subset [n]$. We will construct a family $\mathcal{F} \subset \Sigma_n$ without a sunflower with s petals as the family of all permutations σ for which there exists $G \in \mathcal{G}$ with the following properties:

1. for any $x \in G$, we have $\sigma(x) = x$;
2. for any $x \in (\mathrm{supp}\mathcal{G}) \setminus G$, we have $\sigma(x) \neq x$.

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The number of permutations such that for some set Y we have $\sigma(y) = y$ for any $y \in Y$, and for some set X we have $\sigma(x) \neq x$ for any $x \in X$, is at least

$$(n - |Y|)! - |X| \cdot (n - |Y| - 1)! = (n - |Y|)!\left(1 - \frac{|X|}{n - |Y|}\right).$$

Hence, we can bound the size of $\mathcal{F}$ as follows:

$$|\mathcal{F}| \geqslant \left(1 - \frac{t\phi(s,t)}{n-t}\right) \cdot \phi(s,t)(n-t)!.$$

It remains to show that $\mathcal{F}$ does not contain a sunflower with s petals. Suppose that $\sigma_1, \dots, \sigma_s$ form a sunflower with the core of size $t-1$. Define G_i as the set of all $x \in \operatorname{supp}\mathcal{G}$ such that $\sigma_i(x) = x$. Note that by the construction of $\mathcal{F}$, we have $G_i \in \mathcal{G}$. We claim that $G_1, \dots, G_s$ form a sunflower with the core of size at most $t-1$. Indeed, if $x \in G_i \cap G_j$, then $\sigma_i(x) = \sigma_j(x) = x$. Then, we have $\sigma_1(x) = \sigma_2(x) = \ldots = \sigma_s(x) = x$ by Definition 5 of sunflowers for permutations. By the construction of the family $\mathcal{F}$, we have $x \in G_\ell$ for all $\ell \in [s]$, and so $G_i \cap G_j = \cap_\ell G_\ell$. Thus, $G_1, \dots, G_s$ form a sunflower with s petals and the core of size at most $t-1$, contradicting the definition of $\mathcal{G}$. □

7.2. Subfamilies of product domains

In contrast to the case of permutations, one should study the regime when $k \leqslant \ln n$ to obtain a counterpart of Theorem 3.1 for $\mathcal{A} = \binom{[n]}{k/w}^w$. Theorem 6.2 gives suboptimal bounds. For example, for $\mathcal{A} = [n]^k$, we have $|\partial_{k-t}\mathcal{A}| = \binom{k}{t}n^{k-t}$ and so, for an extremal family $\mathcal{F}$, Theorem 6.2 implies $|\mathcal{F}| \leqslant (1 + o(1))\binom{k}{t}\phi(s,t)n^{k-t}$ as n tends to infinity. It turns out that with additional effort, this bound can be improved to $|\mathcal{F}| \leqslant \left(1 + O(\sqrt{t/k})\right)\phi(s,t)n^{k-t}$. The next two theorems show that the result of Theorem 6.2 can be made essentially sharp for subfamilies of $\binom{[n]}{k/w}^w$.

Theorem 7.2. *Let $\mathcal{F} \subset [n]^k$ be a family without a sunflower with s petals and the core of size $t-1$. Suppose that $k \geqslant 5t$, $n \geqslant s^{80t} \cdot \tilde{g}_0(t)$ for some large enough function $\tilde{g}_0(t)$, and*

$$n \geqslant 2^{33}s^{10}t^{12} + 2^{11}s\log_2 k.$$

Then, we have

$$|\mathcal{F}| \leqslant (1+\zeta)\phi(s,t) \cdot n^{k-t}, \quad \text{where} \quad \zeta = \max\left\{7\sqrt{\frac{t}{k}}, \frac{2^{18}s^5t^6\ln^3 n}{n}\right\}.$$

Theorem 7.3. *Consider an ambient family $\mathcal{A} = \binom{[n]}{k/w}^w$ for $k \geqslant 2^{12}w^6t$. Suppose that $n \geqslant s^{60t} \cdot \tilde{g}_0(t)$ for a large enough function $\tilde{g}_0(t)$, and*

$$wn \geqslant (2^{33}s^{10}t^{12} + 2^{11}s\log_2 k) \cdot k.$$

Then, for any family $\mathcal{F} \subset \mathcal{A}$ that does not contain a sunflower with s petals and the core of size exactly $t-1$, we have

$$|\mathcal{F}| \leqslant (1+\zeta')\phi(s,t) \cdot A_t, \quad \text{where} \quad \zeta' = \max\left\{7\sqrt{\frac{t}{k}}, \frac{2^{18}s^5t^6\ln^3(wn/k)}{wn/k}\right\}.$$

We prove both theorems simultaneously. Here $\mathcal{A}$ stands for the ambient family, which is $[n]^k$ for Theorem 7.2 and $\binom{[n]}{k/w}^w$ for Theorem 7.3.

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Proof. As in the proof of Theorem 3.1, we consider several cases.

Case 1: $\mathbf{k \geqslant 33wst^2 \ln n}$ if $\mathcal{A} = \binom{[n]}{k/w}^w$ and $\mathbf{k \geqslant 33st^2 \ln n}$ if $\mathcal{A} = [\mathbf{n}]^{\mathbf{k}}$. In this case, we apply Theorem 6.1. We check its conditions for both families.

- If $\mathcal{A} = \binom{[n]}{k/w}^w$ for $k \geqslant 33wst^2 \ln n$, then $\mathcal{A}$ satisfies Assumptions 1-4 with $r = \frac{wn}{k}$, $\eta = 2$, $\mu = n/k$ and $q = \lceil 32st^2 \ln r \rceil$ due to Proposition 5.4. The condition $\varepsilon r \geqslant s^2 q^3$ for some $\varepsilon \leqslant 1/2$ is satisfied if

$$\frac{wn}{k} \geqslant 2s^2(33st^2 \ln(n/k))^3 = 2 \cdot 33^2 s^5 t^6 \ln^3\Big(\frac{wn}{k}\Big).$$

If $\frac{wn}{k} \geqslant 2^{25}$, then $\ln\big(\frac{wn}{k}\big) \leqslant \big(\frac{wn}{k}\big)^{1/6}$. Thus, it is enough to guarantee the following two inequalities:

$$\frac{wn}{k} \geqslant 2^{25} \quad \text{and} \quad \frac{wn}{k} \geqslant (2 \cdot 33^3)^2 s^{10} t^{12}.$$

Both are satisfied if $wn \geqslant 2^{33} s^{10} t^{12} k$. The condition $r \geqslant 2^{11} s \lceil \log_2 k \rceil$ holds if $wn \geqslant 2^{12} sk \log_2 k$. Assumptions $k \geqslant 2t + q$ and $n = \mu k \geqslant 2stq$ are satisfied provided $k \geqslant 33swt^2 n \ln n$ and $n \geqslant \frac{wn}{k} \geqslant 2s^2 q^3$. The conclusion of Theorem 6.1 yields

$$\begin{aligned} |\mathcal{F}| &\leqslant \phi(s,t) A_t + 32\Big(\frac{wn}{k}\Big)^{-2t} A_t + \frac{4s^2(33st^2 \ln(wn/k))^3}{wn/k} \phi(s,t) A_t \\ &\leqslant \left(1 + \frac{2^{18} s^5 t^6 \ln^3(wn/k)}{wn/k}\right) \phi(s,t) \cdot A_t. \end{aligned}$$

We summarize the observations above in the following lemma.

Lemma 7.4. *Let $\mathcal{A} = \binom{[n]}{k/w}^w$ for $k \geqslant 33wst^2 \ln n$ and $wn \geqslant (2^{33} s^{10} t^{12} + 2^{12} s \log_2 k) \cdot k$. Then, for any family $\mathcal{F} \subset \mathcal{A}$ without a sunflower with s petals and the core of size $t-1$, we have*

$$|\mathcal{F}| \leqslant \left(1 + \frac{2^{18} s^5 t^6 \ln^3(wn/k)}{wn/k}\right) \phi(s,t) \cdot A_t.$$

- If $\mathcal{A} = [n]^k$ for $k \geqslant 2^8 s^2 t^3 \ln n$, then $\mathcal{A}$ satisfies Assumptions 1-4 with $r = n$, $\eta = 2$, $\mu = 1/2$ and $q = \lceil 32st^2 \ln n \rceil$ due to Proposition 5.4. The condition $\varepsilon r \geqslant s^2 q^3$ for some $\varepsilon \leqslant 1/2$ is satisfied if

$$n \geqslant 2s^2(33st^2 \ln n)^3 = 2 \cdot 33^3 s^5 t^6 \ln^3 n.$$

If $n \geqslant 2^{25}$, then $\ln n \leqslant n^{1/6}$. Thus, it is enough to guarantee the following two inequalities:

$$n \geqslant 2^{25} \quad \text{and} \quad n \geqslant (2 \cdot 33^3)^2 s^{10} t^{12}.$$

Both are satisfied if $n \geqslant 2^{33} s^{10} t^{12}$. The condition $r \geqslant 2^{11} s \lceil \log_2 k \rceil$ holds if $n \geqslant 2^{12} s \log_2 k$. Assumptions $k \geqslant 2t + q$ and $k/2 = \mu k \geqslant 2stq$ are satisfied since $k \geqslant 2^8 s^2 t^3 \ln n \geqslant 4stq$. The conclusion of Theorem 6.1 implies

$$\begin{aligned} |\mathcal{F}| &\leqslant \phi(s,t) n^{k-t} + 32 n^{-2t} n^{k-t} + \frac{4s^2(33st^2 \ln n)^3}{n} \phi(s,t) n^{k-t} \\ &\leqslant \left(1 + \frac{2^{18} s^5 t^6 \ln^3 n}{n}\right) \phi(s,t) \cdot n^{k-t}. \end{aligned}$$

We summarize the above analysis in the following lemma.

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Lemma 7.5. *Let n, k, s, t be integers such that $n \geqslant 2^{33}s^{10}t^{12} + 2^{12}s\log_2 k$ and $k \geqslant 2^8 s^2 t^3 \ln n$. Let $\mathcal{F} \subset [n]^k$ be a family without a sunflower with s petals and the core of size $t-1$. Then, we have*

$$|\mathcal{F}| \leqslant \left(1 + \frac{2^{18}s^5t^6\ln^3 n}{n}\right)\phi(s,t)\cdot n^{k-t}.$$

We conjecture that the lower bound $n \geqslant 2^{33}s^{10}t^{12}+2^{12}s\log_2 k$ in the above lemma can be replaced with $n \geqslant \mathrm{poly}(s,t)$, using the semigroup technique introduced in [39].

Thus, the case is resolved by Lemmas 7.4 and 7.5.

Case 2: $\max\{\mathbf{2^{10}wst^3\ln k}, \mathbf{2^{12}w^6t}\} \leqslant \mathbf{k} \leqslant \mathbf{33wst^2\ln n}$ for $\mathcal{A} = \binom{[n]}{k/w}^{\mathbf{w}}$ and $\mathbf{2^{13}s^2t^4\ln k} \leqslant \mathbf{k} \leqslant \mathbf{2^8s^2t^3\ln n}$ for $\mathcal{A} = [\mathbf{n}]^{\mathbf{k}}$. In this case, we apply the following theorem.

Theorem 7.6. *Assume that one of the following two conditions holds:*

1. $\mathcal{A} = \binom{[n]}{k/w}^w$ *for* $k \geqslant 2^{12}w^6t$;
2. $\mathcal{A} = [n]^k$.

Suppose that $n \geqslant k^{20t}$, $n \geqslant s^{40t}\cdot g_0(t)$ and $k \geqslant 5t$, where $g_0(t) \geqslant (2^{20}t^6)^{20t}$. Let $\mathcal{F} \subset \mathcal{A}$ be a family without a sunflower with s petals and the core of size exactly $t-1$. Then, we have

$$|\mathcal{F}| \leqslant \left(1 + 7\sqrt{\frac{t}{k}}\right)\phi(s,t)\cdot A_t.$$

The proof of Theorem 7.6 is sketched in Section 10.

- If $\mathcal{A} = \binom{[n]}{k/w}^w$ and $k \leqslant 33wst^2\ln n$, then $n \geqslant e^{k/33st^2}$. Since $k \geqslant 2^{10}wst^3\ln k$, $n \geqslant k^{20t}$ and since $k \geqslant 2^{12}w^6t$, we can apply Theorem 7.6, and obtain

$$|\mathcal{F}| \leqslant \left(1 + 7\sqrt{\frac{t}{k}}\right)\phi(s,t)A_t,$$

provided $n \geqslant s^{40t}\cdot g_0(t)$.

- If $\mathcal{A} = [n]^k$ and $k \leqslant 2^8s^2t^3\ln n$, then $n \geqslant e^{k/2^8s^2t^3}$. Since $k \geqslant 2^{13}st^4\ln k$, we have $n \geqslant k^{20t}$, so we can apply Theorem 7.6, and obtain

$$|\mathcal{F}| \leqslant \left(1 + 7\sqrt{\frac{t}{k}}\right)\phi(s,t)A_t,$$

provided $n \geqslant s^{40t}g_0(t)$.

Case 3. We have $\mathbf{2^{12}w^6t} \leqslant \mathbf{k} \leqslant \mathbf{2^{10}wst^3\ln k}$ if $\mathcal{A} = \binom{[n]}{k/w}^{\mathbf{w}}$ and $\mathbf{5t} \leqslant \mathbf{k} \leqslant \mathbf{2^{13}s^2t^4\ln k}$ if $\mathcal{A} = [\mathbf{n}]^{\mathbf{k}}$. Consider two cases:

- If $\mathcal{A} = \binom{[n]}{k/w}^w$ and $2^{12}w^6t \leqslant k \leqslant 2^{10}wst^3\ln k$, then

$$\begin{aligned} k &\leqslant 2^{10}\left(\frac{k}{2^{12}t}\right)^{1/6} st^3\ln k \leqslant 2^8 k^{1/6}st^3\ln k \\ &\leqslant 2^8 st^3 k^{2/3}, \end{aligned}$$

so $k \leqslant 2^{24}s^3t^9$. Since $n \geqslant s^{60t}\cdot\tilde{g}_0(t)$ for large enough $\tilde{g}_0(t)$, we have $n \geqslant k^{20t}$. Thus, we can apply Theorem 7.6, and obtain Theorem 7.3.

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- If $\mathcal{A} = [n]^k$ and $5t \leqslant k \leqslant 2^{10}s^2t^4 \ln k$, then, in particular, we have $k \leqslant 2^{20}s^4t^8$. Since $n \geqslant s^{80t} \cdot \tilde{g}_0(t)$ for a large enough function $\tilde{g}_0(t)$, we have $n \geqslant k^{20t}$. Thus, we can apply Theorem 7.6, and obtain Theorem 7.2.

□

8. Proof of Theorem 6.1

We start by presenting the proof sketch. First, using a bit modified variant of Lemma 5.11 with $\tau = e^{1/8st}$, we find a family $\mathcal{S} \subset \partial_{\leqslant q}\mathcal{A}$, such that

$$|\mathcal{F} \setminus \mathcal{A}[\mathcal{S}]| \leqslant 32\left(e^{1/(8st)}\right)^{-q}|\mathcal{A}| \leqslant r^{-\eta t}A_t.$$

We prove that $\mathcal{S}$ admits the following two properties:

- it does not contain a sunflower with s petals and the core of size at most $t-1$;
- it consists of sets of cardinality at least t.

This is done in Lemma 8.1.

Next, using Lemma 4.12, we obtain a simplification of $\mathcal{S}$, that is, a family $\mathcal{T}$ that possesses the same properties as $\mathcal{S}$, consists of sets of size exactly t, and is such that

$$|\mathcal{A}[\mathcal{S}] \setminus \mathcal{A}[\mathcal{T}]| \leqslant \phi(s,t) \cdot \frac{2\varepsilon}{1-\varepsilon}A_t \overset{(\varepsilon \leqslant 0.5)}{\leqslant} \phi(s,t) \cdot 4\varepsilon A_t.$$

While $\mathcal{S}$ does not contain a sunflower with the core of size at most $t-1$, $\mathcal{T}$ does not contain a sunflower with an arbitrary core. Thus, the theorem follows.

8.1. Spread approximation

Lemma 8.1. *Let $\mathcal{F}, \mathcal{A}, q$ be from the statement of Theorem 6.1. Then, there exists a family $\mathcal{S}$ such that*

1. *it does not contain a sunflower with s petals and the core of size at most $t-1$;*
2. *it consists of sets of size at least t;*
3. *the following inequality holds:*

$$|\mathcal{F} \setminus \mathcal{A}[\mathcal{S}]| \leqslant 32 \cdot r^{-\eta t}A_t.$$

Note that property 2 follows from property 1, but we state it explicitly. Indeed, as we have already mentioned, s copies of a set of size at most $t-1$ form a sunflower with s empty petals and the core of size at most $t-1$.

Proof of Lemma 8.1. For our purposes, we modify Lemma 5.11 a bit.

Lemma 8.2. *Fix an integer q and a real number $\tau > 1$. Let $\mathcal{A}$ be a k-uniform r-spread family. Let $\mathcal{F} \subset \mathcal{A}$. Then there exists a family $\mathcal{S} \subset \partial_{\leqslant q}\mathcal{A}$ and a partition of $\mathcal{F}$ into families $\mathcal{R}$ and $\mathcal{F}_S \subset \mathcal{A}(S)$, $S \in \mathcal{S}$:*

$$\mathcal{F} = \mathcal{R} \sqcup \bigsqcup_{S \in \mathcal{S}} \mathcal{F}_S \vee \{S\},$$

such that each $\mathcal{F}_S$ is τ-homogeneous as a subfamily of $\mathcal{A}(S)$, $\mu(\mathcal{F}_S) = \frac{|\mathcal{F}_S|}{|\mathcal{A}(S)|} \geqslant 32\tau^{-q}$ and $|\mathcal{R}| \leqslant 32\tau^{-q}|\mathcal{A}|$.

Proof. We construct the spread approximation $\mathcal{S}$ iteratively. Set $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{S} = \varnothing$. Given a family $\mathcal{F}_i$, we construct a family $\mathcal{F}_{i+1}$ as follows. Let S_i be the maximal set such that $\mu(\mathcal{F}_i(S_i)) > \tau^{|S_i|}\mu(\mathcal{F}_i)$ holds. If $|S_i| \geqslant q+1$ or $\mu(\mathcal{F}_i(S_i)) < 32\tau^{-q}$, we stop and put $\mathcal{R} = \mathcal{F}_i$. Otherwise, we add S_i to $\mathcal{S}$, set $\mathcal{F}_{S_i} = \mathcal{F}_i(S_i)$ and put $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \mathcal{F}_i[S_i]$. Note that $\mathcal{F}_{S_i}$ is τ-homogeneous by Observation 5.10.

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Suppose that the procedure stops at step m. Then, we have $|S_m| \geqslant q+1$ or $\mu(\mathcal{F}_m(S_m)) \leqslant 32\tau^{-q}$. In the former case, we have

$$\mu(\mathcal{R}) \leqslant \tau^{-(q+1)}\mu(\mathcal{F}_m(S_m)) \leqslant 32\tau^{-q}.$$

In the latter case, we have

$$\mu(\mathcal{R}) \leqslant \tau^{-|S_m|}\mu(\mathcal{F}_m(S_m)) \leqslant 32\tau^{-q}.$$

In either case, we obtain the desired bound on $\mu(\mathcal{R})$. This completes the proof. □

Applying Lemma 8.2 with $\tau = e^{1/(8st)}$ and $q = \lceil 16\eta st^2 \ln r \rceil$, we obtain a family $\mathcal{S} \subset \partial_{\leqslant q}\mathcal{A}$ and families $\mathcal{F}_S$ such that

$$\mathcal{F} = \mathcal{R} \sqcup \bigsqcup_{S\in\mathcal{S}} \mathcal{F}_S \vee \{S\}, \tag{8.1}$$

and $\mathcal{F}_S$ are τ-homogeneous.

First, we have

$$\begin{aligned}|\mathcal{R}| &\leqslant 32\tau^{-q}|\mathcal{A}| \leqslant 32e^{-q/8st}|\mathcal{A}| \leqslant 32\exp\left\{-\frac{16\eta st^2\ln r}{8st}\right\}|\mathcal{A}|\\ &= 32r^{-2\eta t}|\mathcal{A}| \overset{\text{Assumption 1}}{\leqslant} 32r^{-\eta t}A_t.\end{aligned}$$

Since each $\mathcal{F}_S$ is a subset of $\mathcal{A}(S)$, it implies that $|\mathcal{F}\setminus\mathcal{A}[S]| \leqslant |\mathcal{R}| \leqslant 32r^{-\eta t}A_t$, and property 3 holds.

Next, we check property 1. Suppose that $S_1,\dots,S_s \in \mathcal{S}$ form a sunflower with the core of size $t-1-h$, where $h \leqslant t-1$. Each family $\mathcal{F}_{S_i}$ is τ-homogeneous. Define $\mathcal{G}_k = \mathcal{F}_{S_k}(\varnothing, \bigcup_{i=1}^s S_i)$. According to Claim 5.9, they are $\tilde{\tau}$-homogeneous for

$$\begin{aligned}\tilde{\tau} &= \tau\Big/\left(1-\frac{\tau}{r}\sum_{i=1}^s |S_i|\right) \leqslant \tau\Big/\left(1-\frac{sq\tau}{r}\right)\\ &= e^{1/8st}\Big/\left(1-\frac{sq\tau}{r}\right) \overset{(\tau\leqslant 2)}{\leqslant} \exp\left(\frac{1}{8st}+\frac{4sq}{r}\right) \leqslant e^{1/(4st)},\end{aligned}$$

where we use $-\log(1-x) \leqslant 2x$ for any $x < 0.5$ and $r \geqslant 2s^2q^3 \geqslant 32s^2tq$. Moreover, we have

$$|\mathcal{G}_i| \geqslant \left(1-\frac{\tau|\cup_i S_i|}{r}\right)|\mathcal{F}_{S_i}| \geqslant \left(1-\frac{s\tau q}{r}\right)|\mathcal{F}_{S_i}| \geqslant \frac{1}{2}|\mathcal{F}_{S_i}| \geqslant 16\tau^{-q}\cdot|\mathcal{A}(S_i)|, \tag{8.2}$$

where we used the property $\mu(\mathcal{F}_{S_i}) \geqslant 32\tau^{-q}$ guaranteed by Lemma 8.2.

Consider two cases.

Case 1. We have h = 0. Consider two cases again.

Case 1a. Condition 6.1 of Theorem 6.1 holds. Subfamilies $\mathcal{A}(S)$ are r-spread, so $\mathcal{G}_i \subset \mathcal{A}(S_i), i = 1,\dots,s$, are $r/\tilde{\tau}$-spread. Since $r \geqslant 2^8 s\lceil\log_2 k\rceil \geqslant \tilde{\tau}\cdot 2^7 s\lceil\log_2 k\rceil$, Proposition 4.4 ensures that there are s disjoint sets $F_1 \in \mathcal{G}_1,\dots,F_s \in \mathcal{G}_s$ that do not intersect $\bigcup_{i=1}^s S_i$ by the definition of $\mathcal{G}_i$. From decomposition (8.1), $\mathcal{F}$ contains a sunflower $S_1\cup F_1,\dots,S_s\cup F_s$ with the core of size exactly $t-1$, a contradiction.

Case 1b. Condition 6.1 of Theorem 6.1 holds. To prove that families $\mathcal{G}_i$, $i = 1,\dots,s$, contain disjoint sets $F_i \in \mathcal{G}_i$, we are going to use Lemma 5.18. First, we check that conditions of the lemma are satisfied.

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Claim 8.3. *Suppose that* $n/k \geqslant 2^{390}s^{43}t^{82} + 2^{23000}$. *Fix numbers* $n' \geqslant n/2$, $k_i \leqslant k$, $i = 1, \dots, s$ *and* $\tau \leqslant 16$. *Consider arbitrary* τ*-homogeneous families* $\mathcal{F}_1 \subset \binom{[n']}{k_1}, \dots, \mathcal{F}_s \subset \binom{[n']}{k_s}$, $\mu(\mathcal{F}_i) \geqslant 8\tau^{-q}$. *Then, conditions of Lemma 5.18 hold with* $z = q$ *and* n' *in place of* n.

Proof. In terms of Lemma 5.18, we have $\tau \leqslant 32$. Therefore, it holds that

$$(2^6\tau)^m \leqslant 2^{11(1+\log_{4/3}(5z))} = 2^{11} \cdot 2^{11\log_{4/3}(5q)} = 2^{11}(5q)^{\frac{11}{\log_2(4/3)}} \leqslant 2^{11}(5q)^{39} \leqslant 2^{128}q^{39},$$

where $m = \left\lceil \log_{4/3}\left(\frac{z\log\tau}{\log 2}\right)\right\rceil$ by the condition of Lemma 5.18. Thus, it is enough to ensure that

$$sq2^{128}q^{39} \cdot 32p_i = 2^{133}q^{40}sp_i < 1/64,$$

where $p_i = n'/k_i \geqslant n/2k$. Since $q = 2^5st^2 \ln\frac{n}{k} \leqslant 2^6st^2 \ln\frac{1}{p_i}$, the above is implied by $2^{139}s(2^6st^2)^{40}p_i \cdot \ln^{40}\frac{1}{p_i} < 1$. Numerically, we checked that provided $p_i \leqslant 2^{-22999}$, we have $\ln^{40}\frac{1}{p_i} \leqslant p_i^{-1/41}$. Therefore, it is enough to ensure $2^{379}s^{41}t^{80}p_i^{40/41} < 1$ and $p_i \leqslant 2^{-22999}$, or $n' \geqslant 2^{389}t^{82}s^{43}k$ and $n' \geqslant 2^{22999}k$. Both inequalities are satisfied, since $n' \geqslant n/2$ and $n/k \geqslant 2^{390}s^{43}t^{82} + 2^{23000}$. □

It is easy to check that $\mathcal{G}_1, \dots, \mathcal{G}_s$ meet the conditions of Claim 8.3 with $n' = n - |\cup_i S_i|$ and $k_i = k - |S_i|$. Thus, there are disjoint sets $G_1 \in \mathcal{G}_1, \dots, G_s \in \mathcal{G}_s$, and so $\mathcal{F}$ contains a sunflower $G_1 \cup S_1, \dots, G_s \cup S_s$ with the core of size exactly $t-1$, a contradiction.

Case 2. We have h > 0. Set $\alpha = 1/2s$ and $\rho = 1/2$. Recall that $\mathcal{G}_1, \dots, \mathcal{G}_s$ are $\tilde{\tau}$-homogeneous with $\tilde{\tau} = e^{1/4st}$. We have

$$\tilde{\tau} \leqslant \left(1 - \frac{1}{4s}\right)^{-1/h} = (1-\alpha\rho)^{-1/h},$$

thus, due to Proposition 5.7, for each $S \in \mathcal{S}$ and for all but $\alpha|\partial_h(\mathcal{A}(S))|$ sets H families $\mathcal{G}_i(H)$ are $2\tilde{\tau}$-homogeneous and $\mu(\mathcal{G}_i(H)) \geqslant \mu(\mathcal{G}_i)/2 \geqslant 8 \cdot (2\tilde{\tau})^{-q}$, where we used the lower bound (8.2). Thus, for a random $\mathbf{H}$ uniformly distributed on $\partial_h\mathcal{A}$, we have

$$\mathbb{P}(\mathcal{G}_i(\mathbf{H}) \text{ is } 2\tilde{\tau}\text{-homogeneous and } \mu(\mathcal{G}_i(\mathbf{H})) \geqslant 8 \cdot (2\tilde{\tau})^{-q} \mid \mathbf{H} \in \partial_h(\mathcal{A}(S))) \geqslant 1-\alpha,$$
$$\mathbb{P}(\mathcal{G}_i(\mathbf{H}) \text{ is } 2\tilde{\tau}\text{-homogeneous and } \mu(\mathcal{G}_i(\mathbf{H})) \geqslant 8 \cdot (2\tilde{\tau})^{-q}) \geqslant (1-\alpha)\mathbb{P}[\mathbf{H} \in \partial_h(\mathcal{A}(S))],$$

where $\mu(\mathcal{G}_i(\mathbf{H}))$ is defined w.r.t. $\mathcal{A}(S_i \cup \mathbf{H}, \cup_i S_i \cup \mathbf{H})$, that is, $\mu(\mathcal{G}_i(\mathbf{H})) = \frac{|\mathcal{G}_i(\mathbf{H})|}{|\mathcal{A}(S_i \cup \mathbf{H}, \cup_i S_i \cup \mathbf{H})|}$. Due to Assumption 4 and the condition $\mu k \geqslant 2stq$, we have

$$\mathbb{P}[\mathbf{H} \in \partial_h(\mathcal{A}(S))] \geqslant \left(1 - \frac{q}{\mu k}\right)^h \geqslant 1 - \frac{1}{2s}.$$

Hence, we have

$$\begin{aligned}\mathbb{P}(\mathcal{G}_i(\mathbf{H}) \text{ is } 2\tilde{\tau}\text{-homogeneous and } \mu(\mathcal{G}_i(\mathbf{H})) \geqslant 8 \cdot (2\tilde{\tau})^{-q}) &\geqslant (1-\alpha)(1-\frac{1}{2s}) \\ &> 1 - \alpha - \frac{1}{2s} \\ &= 1 - \frac{1}{s},\end{aligned}$$

which by the union bound implies

$$\mathbb{P}(\forall i \in [s]\ \mathcal{G}_i(\mathbf{H}) \text{ is } 2\tilde{\tau}\text{-homogeneous and } \mu(\mathcal{G}_i(\mathbf{H})) \geqslant 8 \cdot (2\tilde{\tau})^{-q}) > 0.$$

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Here we have $2\tilde{\tau} \leqslant 4$. Therefore, there exists a set $H \in \partial_h \mathcal{A}$ such that all $\mathcal{G}_i(H)$, $i = 1, \dots, s$, are 4-homogeneous and $\mu(\mathcal{G}_i(H)) \geqslant 8 \cdot 4^{-q}$. Again, we consider two different cases.

Case 2a. Condition 6.1 of Theorem 6.1 holds. Since each $\mathcal{G}_i(H)$ is 4-homogeneous, they are $r/4$-spread due to Assumption 1. Proposition 4.4 implies that there exist disjoint sets $F_1 \in \mathcal{G}_i(H), \dots, F_s \in \mathcal{G}_i(H)$. These sets do not intersect $H \cup \bigcup_{i=1}^{s} S_i$, so decomposition (8.1) implies that $\mathcal{F}$ contains a sunflower $H \cup S_1 \cup F_1, \dots, H \cup S_s \cup F_s$ with s petals and the core $H \cup \bigcap_{i=1}^{s} S_i$ of size exactly $t-1$, a contradiction.

Case 2b. Condition 6.1 of Theorem 6.1 holds. Since $\mathcal{G}_i(H)$ are 4-homogeneous and $\mu(\mathcal{G}_i(H)) \geqslant 8 \cdot 4^{-q}$, they meet the conditions of Claim 8.3 with $n' = n - |\cup_i S_i| - |H|$ and $k_i = k - |S_i| - |H|$, and, therefore, there are disjoint sets $G_1 \in \mathcal{G}_1(H), \dots, G_s \in \mathcal{G}_s(H)$. Thus, $\mathcal{F}$ contains a sunflower $S_1 \cup H \cup G_1, \dots, S_s \cup H \cup G_s$ with the core of size exactly $t-1$, a contradiction. □

8.2. *Proof of Theorem 6.1*

Proof of Theorem 6.1. Combining Lemma 8.1 and Lemma 4.12, we get

$$\begin{aligned} |\mathcal{F} \setminus \mathcal{A}[\mathcal{T}]| &\leqslant |\mathcal{F} \setminus \mathcal{A}[\mathcal{S}]| + |\mathcal{A}[\mathcal{S}] \setminus \mathcal{A}[\mathcal{T}]| \\ &\leqslant 32 \cdot r^{-\eta t} A_t + \phi(s,t) \cdot \frac{2\varepsilon}{1-\varepsilon} A_t \cdot \\ &\overset{(\varepsilon \leqslant 0.5)}{\leqslant} 32 \cdot r^{-\eta t} A_t + \phi(s,t) \cdot 4\varepsilon A_t \cdot \end{aligned}$$

□

9. Proof of Theorem 6.2

We divide the proof into two parts, the first one, Section 9.1, can be applied to $\mathcal{A}$ that satisfies Assumptions 1,3, and the second one (Sections 9.2-9.3) is specific for $\mathcal{A} = \binom{[n]}{k/w}^w$, where w is an arbitrary integer such that $w|k$.

The reason why the case of small k requires a different approach is that it is much harder, if even possible, to obtain stability results, that is, find a family $\mathcal{T}$ of uniformity t and with no s-sunflower, such that most of $\mathcal{F}$ is contained in $\mathcal{A}[\mathcal{T}]$. In Section 10, we will see that a direct approach based on the Kruskal–Katona theorem gives the desired result up to a remainder that depends on k instead of n. Thus, there is little chance to obtain a result like Theorem 6.1 by our methods in general. However, for $k \geqslant 5t$, we can get some structural information on the extremal example. As we saw in Section 5.1, the case $k < 5t$ can be resolved by Theorem 5.2.

Section 9.1 develops tools to obtain the aforementioned structural results. Applying them with a number of tricks, we show in Section 9.2 that there exists a family $\mathcal{C}^* \in \partial_t \mathcal{A}$, such that $|\mathcal{C}^*| \leqslant 70t2^t \phi(s,t)\sqrt{n}\log n$ and

$$|\mathcal{F} \setminus \mathcal{F}[\mathcal{C}^*]| \leqslant \frac{C_t s^t \log n}{\sqrt{n}} \cdot A_t,$$

where C_t is some constant depending on t only.

The rest of the proof is pretty simple and similar to what Frankl and Füredi did in [26]. One can apply to each $\mathcal{F}(C)$, $C \in \mathcal{C}^*$ a technique similar to Lemma 5.1, and then repeat the proof of Theorem 5.2.

9.1. *Intersection reduction*

When k is much smaller than $\log r$, we cannot obtain $\mathcal{S}$ with properties as strong as in Lemma 5.11. Fortunately, $\mathcal{S}$ inherits restrictions imposed on $\mathcal{F}$, namely, it does not contain a sunflower with s petals and the core of size $t-1$. Additionally, all sets of $\mathcal{S}$ have cardinality at least t.

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Proposition 9.1. *Suppose that Assumptions 1,3 hold for some q and r. Let $\mathcal{S} \subset \partial_{\leqslant q}\mathcal{A}$ be a τ-homogeneous core of a family $\mathcal{F} \subset \mathcal{A}$, $\mathcal{F}$ does not contain a sunflower with s petals and the core of size $t-1$. Assume that $r > 2^{10}\tau^t s\lceil\log_2 k\rceil$ and $r \geqslant 2qs\tau$. Then, it holds that*

1. *$\mathcal{S}$ does not contain a sunflower with s petals and the core of size $t-1$;*
2. *each $S \in \mathcal{S}$ has cardinality at least t.*

Proof. Suppose that $\mathcal{S}$ contains a sunflower $S_1, \dots, S_s$ with the core of size $t-1$. For each $\mathcal{F}_{S_i}$, define $\mathcal{G}_i = \mathcal{F}_{S_i}\left(\varnothing, \bigcup_{p=1}^s S_p\right)$. Since $\mathcal{F}_{S_i}$ is τ-homogeneous, $\mathcal{G}_i$ is τ'-homogeneous for

$$\tau' = \frac{\tau}{1-\frac{qs\tau}{r}} \leqslant 2\tau$$

due to Claim 5.9. Hence, $\mathcal{G}_i, i \in [s]$, are R-spread for

$$R = \frac{r}{2\tau} > 2^7 s\lceil\log_2 k\rceil.$$

Proposition 4.4 guarantees that there exist disjoint $G_1 \in \mathcal{G}_1, \dots, G_s \in \mathcal{G}_s$. By the definition of $\mathcal{G}_i$, they do not intersect $\bigcup_{p=1}^s S_p$. Hence, $S_1 \cup G_1, \dots, S_s \cup G_s$ form a sunflower with the core of size $t-1$, a contradiction. That proves property 1.

Next, assume that there exists $S \in \mathcal{S}$ of size at most $t-1$. Set $h = t-1-|S|$ and consider two cases.

Case 1. We have h = 0. Since $\mathcal{F}_S$ is τ-homogeneous, it is R-spread for $R = r/\tau > 2^7 s\lceil\log_2 k\rceil$. Due to Proposition 4.4, it contains a matching $F_1, \dots, F_s$. Thus, $\mathcal{F}$ contains a sunflower $S \cup F_1, \dots, S \cup F_s$, a contradiction.

Case 2. We have h > 0. We are going to apply Proposition 5.7 with $\alpha = \rho = 1 - \tau^{-(t-1)}/2$. Let us check that the conditions of the proposition hold. We have

$$(1-\alpha\rho)^{-1/h} = (1-(1-\tau^{t-1}/2)^2)^{-1/h} = (\tau^{-(t-1)} - \tau^{-2t-2}/4)^{-1/h} \geqslant \tau^{(t-1)/h} \geqslant \tau,$$

and so there exists $H \in \partial_h\mathcal{A}(S)$ such that $\mathcal{F}_S(H)$ is $2\tau^t$-homogeneous. In particular, it is R-spread for

$$R = \frac{r}{2\tau^t} > 2^7 s\lceil\log_2 k\rceil.$$

Thus, $\mathcal{F}_S(H)$ contains s disjoint sets $F_1, \dots, F_s$, and $F_1 \cup S \cup H, \dots, F_s \cup S \cup H$ belong to $\mathcal{F}$ and form a sunflower with the core of size $t-1$, a contradiction.

Thus, property 2 holds. □

We can impose some additional restrictions on $\mathcal{S}$. The key element for doing that is the following simple lemma.

Lemma 9.2. *Let $\mathcal{F} \subset \mathcal{A}$ be a τ-homogeneous family. Then, for arbitrary $\alpha \leqslant 1/2k$, there exists a subfamily $\mathcal{G} \subset \mathcal{F}$, such that*

- *$|\mathcal{G}| \geqslant (1-2\alpha k)|\mathcal{F}|$;*
- *for any P of size at most $t-1$, if $\mathcal{G}(P)$ is not empty, then $\mathcal{F}(P)$ is $\alpha(\tau/\alpha)^t$-homogeneous.*

Proof. Define

$$\mathcal{P} = \left\{P \in \partial_{\leqslant t-1}\mathcal{A} : \mu(\mathcal{F}(P)) < \alpha^{|P|}\mu(\mathcal{F})\right\}.$$

Then, let $\mathcal{G} = \mathcal{F} \setminus \mathcal{F}[\mathcal{P}]$. We have

$$|\mathcal{G}| \geqslant |\mathcal{F}| - \sum_{P\in\mathcal{P}} \alpha^{|P|}\frac{|\mathcal{A}(P)|}{|\mathcal{A}|}|\mathcal{F}| \geqslant |\mathcal{F}|\left(1 - \sum_{p=1}^{t-1}\alpha^p\binom{k}{p}\right).$$

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Since

$$\sum_{p=1}^{t-1}\alpha^p\binom{k}{p}\leqslant\sum_{p=1}^{t-1}(\alpha k)^p\leqslant\frac{\alpha k}{1-\alpha k}\leqslant 2\alpha k,$$

we have $|\mathcal{G}|\geqslant(1-2\alpha k)|\mathcal{F}|$.

Clearly, if $\mathcal{G}(P)$ is not empty, then $\mu(\mathcal{F}(P))\geqslant\alpha^{|P|}\mu(\mathcal{F})$. Then, it is $\alpha(\tau/\alpha)^t$-homogeneous, since otherwise we get

$$\mu(\mathcal{F})\tau^{|P|+|S|}\geqslant\mu(\mathcal{F}(P\cup S))>\alpha^{|S|}(\tau/\alpha)^{t|S|}\mu(\mathcal{F}(P))\geqslant\alpha^{|P|+|S|}(\tau/\alpha)^{t|S|}\mu(\mathcal{F}),$$
$$(\tau/\alpha)^{|P|}>(\tau/\alpha)^{(t-1)|S|}$$

for some nonempty S and P of size $\leqslant t-1$, a contradiction. □

The idea is to apply Lemma 9.2 to each $\mathcal{F}_S, S\in\mathcal{S}$, and obtain a large and well-structured subfamily of $\mathcal{F}$. The established property is somewhat similar to the one given in Lemma 5.11, but much weaker.

Lemma 9.3. *Suppose that Assumptions 1,3 hold for some q and r, $k\geqslant q+(t-1)$. Let $\mathcal{F}$ be a family that does not contain a sunflower with s petals and the core of size $t-1$. Let $\mathcal{S}\subset\partial_{\leqslant q}\mathcal{A}$ be its τ-homogeneous core. Assume that $r>\max\{2^{10}\lceil\log_2 k\rceil,2q\}\cdot s\alpha^{1-t}\tau^t$ for some $\alpha\in(0,(4k)^{-1}]$. Then there exist families $\mathcal{U}_S\subset\mathcal{F}_S, S\in\mathcal{S}$, such that*

1. *$|\mathcal{U}_S|\geqslant(1-2\alpha k)|\mathcal{F}_S|$ for each $S\in\mathcal{S}$,*
2. *$\mathcal{S}$ does not contain a sunflower with s petals and the core of size $t-1$; if $S_1,\dots,S_s\in\mathcal{S}$ form a sunflower with s petals and the core of size at most $t-2$, then for any $F_1\in\mathcal{U}_{S_1},\dots,F_s\in\mathcal{U}_{S_s}$ we have $|\bigcap_{p=1}^s F_p|\leqslant t-|\bigcap_{p=1}^s S_s|-2$,*
3. *let $\tau'\leqslant\tau$ be the minimal homogeneity of $\mathcal{F}_S$, then $|\partial_h(\mathcal{U}_S)|\geqslant\left(\frac{\tau'}{1-2\alpha k}\right)^{-h}|\partial_h\mathcal{A}(S)|$ for any $h<k$.*

Proof. We have

$$\mathcal{F}=\bigsqcup_{S\in\mathcal{S}}\mathcal{F}_S\vee\{S\},$$

and for each $S\in\mathcal{S}$, $\mathcal{F}_S$ is τ-homogeneous. Define $\mathcal{U}_S$ as a family $\mathcal{G}$ obtained via Lemma 9.2 for $\mathcal{F}_S$. If τ' is the minimal homogeneity of $\mathcal{F}_S$, then $\mathcal{U}_S$ is $\tau'/(1-2\alpha k)$-homogeneous, and so $\mu(\partial_h\mathcal{U}_S)\geqslant\left(\frac{\tau'}{1-2\alpha k}\right)^{-h}$ for any h due to Proposition 5.8.

Due to Proposition 9.1, $\mathcal{S}$ does not contain a sunflower with s petals and the core of size $t-1$. Consider s sets $S_1,\dots,S_s\in\mathcal{S}$ that form a sunflower with the core C of size at most $t-1$. Suppose that there exist $F_i\in\mathcal{U}_{S_i}, i\in[s]$ such that $\left|\bigcap_{p=1}^s F_p\right|\geqslant t-|C|-1$. Choose $P\subset\bigcap_{p=1}^s F_p$ of size exactly $t-|C|-1$. Lemma 9.2 implies that $\mathcal{F}_{S_i}(P), i\in[s]$, are $\alpha^{1-t}\tau^t$-homogeneous. Note that we need the assumption $k\geqslant q+(t-1)$ to ensure that $\mathcal{F}_{S_i}(P)$ is not empty. Due to Claim 5.9, $\mathcal{F}_{S_i}\left(P,P\cup\bigcup_{p=1}^s S_p\right)$ is τ'-homogeneous for

$$\tau'=\frac{\alpha^{1-t}\tau^t}{1-\frac{sq\alpha^{1-t}\tau^t}{r}}\leqslant 2\alpha^{1-t}\tau^t.$$

Thus, each $\mathcal{G}_i=\mathcal{F}_{S_i}\left(P,P\cup\bigcup_{p=1}^s S_p\right)$ is R-spread, for

$$R\geqslant\frac{r}{2\tau^t}\alpha^{t-1}>2^7 s\lceil\log_2 k\rceil.$$

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Proposition 4.4 implies that there are s disjoint sets $G_1 \in \mathcal{G}_1, \dots, G_s \in \mathcal{G}_s$. By the definition of $\mathcal{G}_i$, they do not intersect $P \cup \bigcup_{p=1}^{s} S_p$, hence, sets $P \cup S_1 \cup G_1, \dots, P \cup S_s \cup G_s$ are contained in $\mathcal{F}$ and form a sunflower with s petals and the core $C \cup P$ of size $t-1$, a contradiction. □

Definition 9. Let $\mathcal{S} \subset \bigcup_{p=t}^{q} \partial_p \mathcal{A}$ be a family that does not contain a sunflower with s petals and the core of size $t-1$. We say that a family $\mathcal{B} \subset \mathcal{A}$ is *an* $(\mathcal{S}, s, t)$*-system*, if $\mathcal{B}$ can be decomposed into disjoint families $\mathcal{B}_S \vee \{S\}$, $S \in \mathcal{S}$ and $\mathcal{B}_S \subset \mathcal{A}(S)$, such that for any s sets $S_1, \dots, S_s \in \mathcal{S}$ the following properties hold.

- $S_1, \dots, S_s$ do not form a sunflower with the core of size exactly $t-1$.
- if $S_1, \dots, S_s$ form a sunflower with the core of size at most $t-2$, then for any $F_i \in \mathcal{B}_{S_i}, i \in [s]$ we have $\left|\bigcap_{i=1}^{s} F_i\right| \leqslant t - |S_1 \cap S_2| - 2$.

Given a τ-homogeneous core $\mathcal{S}$ of a family $\mathcal{F}$ without a sunflower with s petals and the core of size $t-1$, define

$$\mathcal{U} = \bigsqcup_{S \in \mathcal{S}} \mathcal{U}_S \vee \{S\},$$

where $\mathcal{U}_S, S \in \mathcal{S}$, are families obtained from Lemma 9.3. Then, Proposition 9.1 guarantees that the family $\mathcal{S}$ does not contain a sunflower with s petals and the core of size $t-1$, and each of its sets has size at least t. Applying Lemma 9.3, we infer that $\mathcal{U}$ is an $(\mathcal{S}, s, t)$-system.

We shall use reduction to $(\mathcal{S}, s, t)$-systems to prove that $\mathcal{F}$ admits some nice decomposition up to a polynomial-size remainder.

Proposition 9.4. *Let $\mathcal{A}$ be a k-uniform (r,t)-spread family and $\mathcal{U}$ be an $(\mathcal{S}, s, t)$-system such that $|\partial_{t-1}\mathcal{U}_S| \geqslant \lambda |\partial_{t-1}\mathcal{A}|$ for every $S \in \mathcal{S} \subset \partial_{\leqslant q}\mathcal{A}$. Suppose that $r \geqslant 2s^2q^3$. Then, there exists a family $\hat{\mathcal{T}} \subset \partial_t \mathcal{A}$, such that*

$$|\hat{\mathcal{T}}| \leqslant \frac{1}{\lambda}\left(1 + 2\phi(s,t)\ln(\lambda|\partial_{\leqslant q}\mathcal{A}|)\right) \tag{9.1}$$

and

$$\sum_{S \in \mathcal{S} \setminus \mathcal{S}[\hat{\mathcal{T}}]} |\mathcal{U}_S| \leqslant \frac{8\phi(s,t)s^2q^3}{\lambda r} \cdot \ln(\lambda|\partial_{\leqslant q}\mathcal{A}|) \cdot A_t. \tag{9.2}$$

Proof. The set $\hat{\mathcal{T}}$ is formed by the following procedure. Set $\mathcal{S}^0 = \mathcal{S}$. For a defined $\mathcal{S}^i$, consider two cases. If $|\mathcal{S}^i| \leqslant \frac{1}{\lambda}$, then put $m := i$ and stop. Otherwise, we claim that there exists $H_i \in \partial_{t-1}\mathcal{A}$, such that

$$\mathcal{S}^i_{H_i} = \{S \in \mathcal{S}^i \,:\, H_i \in \partial_{t-1}\mathcal{U}_S\}$$

has size at least $\lambda|\mathcal{S}^i|$. This follows from a double-counting argument. Consider a bipartite graph with vertices $\mathcal{S}^i$ and $\partial_{t-1}\mathcal{A}$, and connect $S \in \mathcal{S}^i$ and $H \in \partial_{t-1}\mathcal{A}$, if $H \in \partial_{t-1}(\mathcal{U}_S)$. The assumptions of the lemma imply that the number of edges is at least $\sum_{S \in \mathcal{S}^i} |\partial_{t-1}\mathcal{U}_S| \geqslant \lambda|\partial_{t-1}\mathcal{A}||\mathcal{S}^i|$. Thus, there exists $H_i \in \partial_{t-1}\mathcal{A}$ that is connected to at least $\lambda|\partial_{t-1}\mathcal{A}||\mathcal{S}^i|/|\partial_{t-1}\mathcal{A}| = \lambda|\mathcal{S}^i| > 1$ indices S in $\mathcal{S}^i$.

By Definition 9 of an $(\mathcal{S}^i, s, t)$-system, the family $\mathcal{S}^i_{H_i}$ does not contain a sunflower with s petals and the core of size at most $t-1$. Due to Lemma 4.12, there exists a family $\mathcal{T}_i$ of uniformity t such that

$$\sum_{S \in \mathcal{S}^i_{H_i} \setminus \mathcal{S}^i_{H_i}[\mathcal{T}_i]} |\mathcal{U}_S| \leqslant |\mathcal{A}[\mathcal{S}^i_{H_i} \setminus \mathcal{S}^i_{H_i}[\mathcal{T}_i]]| \leqslant \frac{4\phi(s,t)s^2q^3}{r} A_t, \tag{9.3}$$

and $\mathcal{T}_i$ does not contain a sunflower with s petals. Define $\mathcal{S}^{i+1} = \mathcal{S}^i \setminus \mathcal{S}^i_{H_i}$ and repeat the procedure.

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Once the procedure stops, put $\mathcal{T}_m$ to be the smallest family of uniformity t such that $\mathcal{S}^m = \mathcal{S}^m[\mathcal{T}_m]$. Clearly, $|\mathcal{T}_m| \leqslant |\mathcal{S}^m| \leqslant \frac{1}{\lambda}$. Then, define

$$\hat{\mathcal{T}} = \bigcup_{i=1}^{m} \mathcal{T}_i.$$

Next, we bound the number of steps m. Since $|\mathcal{S}^i_H| \geqslant \lambda|\mathcal{S}^i|$, we have $|\mathcal{S}^{i+1}| \leqslant (1-\lambda)|\mathcal{S}^i|$. If $m > 0$ and so $\lambda|\mathcal{S}| > 1$, let $\tilde{m}$ be defined as the least integer such that $e^{-\lambda\tilde{m}}|\mathcal{S}| \leqslant \frac{1}{\lambda}$, that is $\tilde{m} = \lceil \frac{1}{\lambda}\ln(\lambda|\mathcal{S}|)\rceil$. We claim that $m \leqslant \tilde{m}$. Otherwise, we have $\tilde{m} < m$ and, therefore, $\mathcal{S}^{\tilde{m}}$ is defined, $|\mathcal{S}^{\tilde{m}}| > \frac{1}{\lambda}$ and

$$|\mathcal{S}^{\tilde{m}}| \leqslant (1-\lambda)^{\tilde{m}}|\mathcal{S}| \leqslant e^{-\lambda\tilde{m}}|\mathcal{S}| \leqslant \frac{1}{\lambda},$$

a contradiction. Thus, we have

$$m \leqslant \frac{2}{\lambda}\ln(\lambda|\mathcal{S}|)\mathbb{1}\{\lambda|\mathcal{S}| > 1\} \leqslant \frac{2}{\lambda}\ln(\lambda|\partial_{\leqslant q}\mathcal{A}|),$$

and $|\hat{\mathcal{T}}| \leqslant m\phi(s,t) + \frac{1}{\lambda} \leqslant \frac{1}{\lambda}\big(1 + 2\phi(s,t)\ln(\lambda|\partial_{\leqslant q}\mathcal{A}|)\big)$, which proves (9.1).

Finally, we bound the sum $\sum_{S\in\mathcal{S}\setminus\mathcal{S}[\hat{\mathcal{T}}]}|\mathcal{U}_S|$:

$$\begin{aligned}\sum_{S\in\mathcal{S}\setminus\mathcal{S}[\hat{\mathcal{T}}]}|\mathcal{U}_S| &\leqslant \sum_{i=0}^{m-1}\sum_{S\in\mathcal{S}^i_H\setminus\mathcal{S}^i_H[\mathcal{T}_i]}|\mathcal{U}_S| \overset{(9.3)}{\leqslant} m\cdot\frac{4\phi(s,t)s^2q^3}{r}A_t\\ &\leqslant \frac{8s^2q^3\ln(\lambda|\partial_{\leqslant q}\mathcal{A}|)}{\lambda r}\cdot\phi(s,t)\cdot A_t.\end{aligned}$$

□

9.2. Simplification argument II

The main result of this section is the following lemma:

Lemma 9.5. *Assume that $\mathcal{A} = \binom{[n]}{k/w}^w$ for some $w|k$. Suppose that $n \geqslant 2^{30}(sk^2)^{4t+2}$ and $k \geqslant 5t$. Let $\mathcal{F} \subset \binom{[n]}{k/w}^w$ be a family without a sunflower with s petals and the core of size $t-1$. Then there exists a family $\mathcal{C}^* \subset \partial_t\mathcal{A}$, such that $|\mathcal{C}^*| \leqslant 70t\cdot 2^t\phi(s,t)\sqrt{n}\ln n$ and*

$$|\mathcal{F}\setminus\mathcal{F}[\mathcal{C}^*]| \leqslant \frac{C_t s^t k\ln n}{\sqrt{n}}\cdot A_t,$$

where C_t is some constant depending on t only.

We will construct $\mathcal{C}^*$ via iterative simplification of $\mathcal{F}$. First, we find a family $\mathcal{T}$ that has small uniformity and covers $\mathcal{F}$ almost completely.

Lemma 9.6. *Assume that $\mathcal{A} = \binom{[n]}{k/w}^w$ for some $w|k$. Suppose that $wn \geqslant 2(sk^2)^{4t+2}$ and $k \geqslant 2t+1$. Let $\mathcal{F} \subset \binom{[n]}{k/w}^w$ be a family without a sunflower with s petals and the core of size $t-1$. Then, there exists a family $\mathcal{T} \subset \partial_{2t}\mathcal{A}\cup\partial_{2t+1}\mathcal{A}$ such that $\mathcal{T}$ does not contain a sunflower with s petals and the core of size $t-1$, and*

$$|\mathcal{F}\setminus\mathcal{F}[\mathcal{T}]| \leqslant \frac{\tilde{C}_t s^{t-1}}{\sqrt{n}}\cdot A_t,$$

where $\tilde{C}_t$ is some constant depending on t only.

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Proof. The simplification argument is a modification of that of Lemma 4.12. We will obtain $\mathcal{T}$ at the end of the following iterative process. Set $\mathcal{T}_0 = \mathcal{F}$. Given a family $\mathcal{T}_i$, we will construct a family $\mathcal{T}_{i+1}$ by the following procedure:

Procedure 2

Set $\mathcal{W}_i \leftarrow \mathcal{T}_i^{(k-i)}$, $\alpha = sk$, $\mathcal{U}_{i+1} \leftarrow \varnothing$;
while there exists a set T, such that $|T| \leqslant k-i-1$ and $\mathcal{W}_i(T)$ is α-spread **do**
  Find a maximal set T such that $|T| \leqslant k-i-1$ and $\mathcal{W}_i(T)$ is α-spread;
  Define $\mathcal{T}_{i+1,T} := \mathcal{W}_i(T)$;
  Remove sets containing T from $\mathcal{W}_i$:

$$\mathcal{W}_i \leftarrow \mathcal{W}_i \setminus \mathcal{W}_i[T];$$

  if $|T| > 2t-1$ **then**
    Add T to $\mathcal{T}_{i+1}$:

$$\mathcal{T}_{i+1} \leftarrow \mathcal{T}_{i+1} \cup \{T\};$$

  else
    Add T to $\mathcal{U}_{i+1}$:

$$\mathcal{U}_{i+1} \leftarrow \mathcal{U}_{i+1} \cup \{T\};$$

  end if
end while

We define $\mathcal{T}$ as $\mathcal{T}_{k-(2t+1)}$. Since any set from $\mathcal{T}_i$ has cardinality between $2t$ and $k-i$ by the construction, $\mathcal{T}$ consists of sets of size either $2t$ or $2t+1$.

First, we describe several properties of the obtained system.

Claim 9.7. *The family $\bigcup_{i=0}^{k-(2t+1)}(\mathcal{T}_i \cup \mathcal{U}_i)$ does not contain a sunflower with s petals and the core of size $t-1$.*

Proof. Let $T_1 \in \mathcal{T}_{i_1} \cup \mathcal{U}_{i_1}, \dots, T_s \in \mathcal{T}_{i_s} \cup \mathcal{U}_{i_s}$ be such a sunflower that $(i_1, \dots, i_s)$ is minimal in the lexicographical order. If $i_1, \dots, i_s = 0$, then $T_1, \dots, T_s$ belong to the initial family $\mathcal{F}$, a contradiction. Thus, there is at least one $\ell \in [s], i_\ell > 0$. It means that for some $j < i_\ell$, $\mathcal{T}_{i_\ell,T_\ell} \subset \mathcal{T}_j(T_\ell)$ is α-spread. Due to Claim 4.7, the covering number of $\mathcal{T}_j(T_\ell)$ is at least $\alpha = sk$. Thus, we may find $F \in \mathcal{T}_j(T_{i_\ell})$ that does not intersect $\bigcup_{p=1}^{s} T_p$. Note that $T_1 \in \mathcal{T}_{i_1} \cup \mathcal{U}_{i_1}, \dots, F \cup T_\ell \in \mathcal{T}_j, \dots, T_s \in \mathcal{T}_{i_s} \cup \mathcal{U}_{i_s}$ form a sunflower with s petals and the core of size $t-1$, while $(i_1, \dots, i_{\ell-1}, j, i_{\ell+1}, \dots, i_s)$ is strictly less than $(i_1, \dots, i_s)$ in the lexicographical order, contradicting the minimality of the latter tuple. □

Thus, $\mathcal{T}$ does not contain such a sunflower, as stated. The second property we establish allows us to bound the size of $\mathcal{W}_i$ and $\mathcal{T}_i[\mathcal{U}_{i+1}]$.

Claim 9.8. *We have $|\mathcal{W}_i| \leqslant \alpha^{k-i}$ and*

$$|\mathcal{T}_i[\mathcal{U}_{i+1}]| \leqslant \tilde{C}_t \alpha^{k-i} w^t n^t s^{t-1},$$

where a constant $\tilde{C}_t$ depends on t only.

Proof. In order to bound $|\mathcal{W}_i|$, note that there is no set T with $|T| \leqslant k-i-1$ such that $\mathcal{W}_i(T)$ is α-spread. In this case, choose maximal X such that $|\mathcal{W}_i(X)| \geqslant \alpha^{|X|}|\mathcal{W}_i|$. We have $|X| = k-i$, and so $|\mathcal{W}_i| \leqslant \alpha^{k-i}$.

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Next, we bound $|\mathcal{T}_i[\mathcal{U}_{i+1}]|$. For some $j \in [2t-1]$, consider the layer $\mathcal{U}_{i+1}^{(j)}$:

$$\mathcal{U}_{i+1}^{(j)} = \{F \in \mathcal{U}_{i+1} \,:\, |F| = j\}.$$

Define the upper shadow $\partial^u \mathcal{U}_{i+1}^{(j)}$ of the family $\mathcal{U}_{i+1}^{(j)}$ as follows:

$$\partial^u \mathcal{U}_{i+1}^{(j)} = \{F \in \partial_{j+1}\mathcal{A} \,:\, \exists F' \in \mathcal{U}_{i+1}^{(j)} \text{ such that } F' \subset F\}.$$

Consider a set T' that belongs to the upper shadow and some set $T \in \mathcal{U}_{i+1}^{(j)}$ such that $T \subset T'$. Due to the third line of the procedure above, the family $\mathcal{T}_{i+1,T}(T' \setminus T)$ is not α-spread. Choose a maximal X disjoint from T' such that $|\mathcal{T}_{i+1,T}((T' \setminus T) \cup X)| \geqslant \alpha^{|X|}|\mathcal{T}_{i+1,T}(T' \setminus T)|$. Since $\mathcal{T}_{i+1,T}(T' \setminus T)$ is not α-spread, X is not empty. It implies that $|X| = k - i - |T'|$, since otherwise, $\mathcal{T}_{i+1,T}(T' \cup X)$ is α-spread due to Observation 4.2, and we may choose $X \cup T'$ in the third line of the procedure instead of T. Therefore, $|\mathcal{T}_{i+1,T}(T' \setminus T)| \leqslant \alpha^{k-i-(j+1)}$.

Hence, we may bound $|\mathcal{T}_i[\mathcal{U}_{i+1}^{(j)}]|$. Indeed, it admits a decomposition

$$|\mathcal{T}_i[\mathcal{U}_{i+1}^{(j)}]| = \sum_{T \in \mathcal{U}_{i+1}^{(j)}} |\mathcal{T}_{i+1,T}| \leqslant \sum_{T \in \mathcal{U}_{i+1}^{(j)}} \sum_{x \in \operatorname{supp}(\mathcal{A}) \setminus T} |\mathcal{T}_{i+1,T}(x)| \leqslant \alpha^{k-i-(j+1)} \cdot wn \cdot |\mathcal{U}_{i+1}^{(j)}|. \tag{9.4}$$

If $j < t$, then we can bound $|\mathcal{U}_{i+1}^{(j)}| \leqslant \binom{wn}{t-1} \leqslant w^{t-1}n^{t-1}$. Otherwise, we use Theorem 5.3. Claim 9.7 ensures that $\mathcal{U}_{i+1}^{(j)}$ does not contain a sunflower with s petals and the core of size $t-1$. We consider $\mathcal{U}_{i+1}^{(j)}$ as a subfamily of $\binom{[wn]}{j}$, and apply Theorem 5.3, which implies $|\mathcal{U}^{(j)}| \leqslant C_j w^{t-1}n^{t-1}s^{j-t+1}$.

Thus, in either case, we have $|\mathcal{U}_{i+1}^{(j)}| \leqslant C_t w^{t-1}n^{t-1}s^t$, since C_t is increasing in t. Substituting this bound into (9.4), we get

$$|\mathcal{T}_i[\mathcal{U}_{i+1}]| \leqslant \alpha^{k-i-1}(2t-1) \cdot C_t n^t w^t s^t \sum_{j=0}^{2t-2} \alpha^{-j} \leqslant \tilde{C}_t w^t \alpha^{k-i} n^t s^t,$$

where we introduced a constant $\tilde{C}_t$ that depends on t only. □

To bound $|\mathcal{F} \setminus \mathcal{F}[\mathcal{T}]|$, note that for each $F \in \mathcal{F} \setminus \mathcal{F}[\mathcal{T}]$ there exists $i \leqslant k - (2t+1)$ such that some subset $F' \subset F$ of size $k-i$ belongs to $\mathcal{T}_i[\mathcal{U}_{i+1}] \cup \mathcal{W}_i$. It yields

$$\begin{aligned} |\mathcal{F} \setminus \mathcal{F}[\mathcal{T}]| &\leqslant \sum_{i=0}^{k-(2t+1)} (|\mathcal{T}_i[\mathcal{U}_{i+1}]| + |\mathcal{W}_i|) \cdot A_{k-i} \\ &\leqslant 2\tilde{C}_t s^t \left(\sum_{i=0}^{k-(2t+1)} \alpha^{k-i} w^t n^t \left(\frac{wn}{k}\right)^{-(k-i-t)} \right) \cdot A_t, \end{aligned}$$

where we used $(wn/k, t)$-spreadness of $\mathcal{A} = \binom{[n]}{k/w}^w$ due to Proposition 4.8. Next, we bound the sum. We have

$$\begin{aligned} \sum_{i=0}^{k-(2t+1)} \alpha^{k-i} w^t n^t \left(\frac{wn}{k}\right)^{-(k-i-t)} &\leqslant n^{2t} w^{2t} \sum_{i=0}^{k-(2t+1)} \left(\frac{\alpha k}{wn}\right)^{k-i} \\ &\leqslant 2n^{2t} w^{2t} \left(\frac{\alpha k}{wn}\right)^{2t+1}. \end{aligned}$$

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Recall that $\alpha = sk$. Since $wn \geqslant (\alpha k)^{4t+2}$ by assumptions of the lemma, the above is at most $2/\sqrt{n}$. Thus, we have

$$|\mathcal{F} \setminus \mathcal{F}[\mathcal{T}]| \leqslant \frac{4\tilde{C}_t s^t}{\sqrt{n}} A_t. \qquad \square$$

Now we are ready to prove Lemma 9.5.

Proof of Lemma 9.5. Take $\mathcal{T}$ as in Lemma 9.6. Consider an arbitrary partition of $\mathcal{F}[\mathcal{T}]$ into $|\mathcal{T}|$ families $\mathcal{F}_T \subset \mathcal{F}(T), T \in \mathcal{T}$, so

$$\mathcal{F}[\mathcal{T}] = \bigsqcup_{T \in \mathcal{T}} \mathcal{F}_T \vee \{T\}.$$

Choose $\tau = n^{1/2t}$, $q = 2t$, and for each $T \in \mathcal{T}$ apply Lemma 5.11 to $\mathcal{F}_T$. We obtain a family $\mathcal{S}_T$, a remainder $\mathcal{R}_T \subset \mathcal{A}[T]$ and a decomposition:

$$\mathcal{F}_T \setminus \mathcal{R}_T = \bigsqcup_{S \in \mathcal{S}_T} \mathcal{F}_{T \sqcup S} \vee \{S\},$$

such that each $\mathcal{F}_{T\sqcup S}$ is τ-homogeneous and $|\mathcal{R}_T| \leqslant n^{-1} A_{|T|}$. Then, we have

$$\sum_{T \in \mathcal{T}} |\mathcal{R}_T| \leqslant n^{-1} \sum_{j=2t}^{2t+1} \sum_{T \in \mathcal{T}^{(j)}} A_j.$$

Due to Theorem 5.3, for each layer $j = 2t, 2t+1$, we have

$$|\mathcal{T}^{(j)}| \leqslant C_j s^t \binom{wn}{j-t}.$$

Using $(wn/k, t)$-spreadness of $\mathcal{A} = \binom{[n]}{k/w}^w$, guaranteed by Proposition 4.8, we obtain

$$\begin{aligned}
\sum_{T \in \mathcal{T}} |\mathcal{R}_T| &\leqslant \frac{C_{2t+1} s^t}{n} \sum_{j=2t}^{2+1} \binom{wn}{j-t} A_j \\
&\leqslant \frac{C'_t s^t}{n} \sum_{j=2t}^{2t+1} (wn)^{j-t} \left(\frac{wn}{k}\right)^{-(j-t)} A_t \\
&\leqslant \frac{2C'_t s^t}{n} \cdot k^{t+1} A_t \leqslant \frac{2C_{2t+1} k^{t+1} s^t}{n} \cdot A_t.
\end{aligned}$$

Define

$$\mathcal{S}^* = \{\{T\} \vee \mathcal{S}_T \,:\, T \in \mathcal{T}\}.$$

Then, $\mathcal{S}^*$ is a τ-homogeneous core of $\mathcal{F}[\mathcal{T}] \setminus \bigsqcup_{T\in\mathcal{T}}(\{T\} \vee \mathcal{R}_T)$. Since $k \geqslant 5t \geqslant |T| + q + (t-1)$ and $wn/k \geqslant \sqrt{2^{30}(sk^2)^{4t+2}}/k \cdot \sqrt{n} \geqslant 2^{10}(4k)^t s\sqrt{n}$, we may apply Lemma 9.3 with $\alpha = 1/4k$, $r = wn/k$, $q = 4t+1$ to $\mathcal{S}^*$, and obtain[9] $(\mathcal{S}^*, s, t)$ system $\mathcal{U}$, such that for each $V \in \mathcal{S}^*$

$$|\mathcal{U}_V| \geqslant \frac{1}{2}|\mathcal{F}_V| \text{ and } |\partial_h(\mathcal{U}_V)| \geqslant (\tau/2)^{-h}|\mathcal{A}(V)| \geqslant 2^{-h} n^{-h/2t} |\mathcal{A}(V)|$$

[9]Note that q here is the parameter of Lemma 9.3 instead of $q = 2t$ used in the spread approximation.

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for any $h < k - |V|$. Applying Proposition 9.4 with $\lambda = 2^{-t} n^{-1/2}$ and $r = wn/k$, we infer that there exists a family $\hat{\mathcal{T}}$ such that

$$|\hat{\mathcal{T}}| \leqslant 2^t \sqrt{n} \left\{ 1 + 2\phi(s,t) \ln \sum_{j=0}^{q} \binom{wn}{q} \right\} \leqslant 2^t \sqrt{n} \left\{ 1 + 2\phi(s,t) \ln \sum_{j=0}^{q} (wn)^{2q} \right\}$$
$$\leqslant 2^t \sqrt{n} \cdot 7q\phi(s,t) \ln(wn) = 35t \cdot 2^t \phi(s,t) \sqrt{n} \ln(wn) \leqslant 70t 2^t \phi(s,t) \sqrt{n} \ln n,$$

and

$$\sum_{\mathcal{S}^* \setminus \mathcal{S}^*[\hat{\mathcal{T}}]} |\mathcal{U}_S| \leqslant \frac{8 \cdot 2^t s^2 (5t)^3 k}{w\sqrt{n}} \cdot \phi(s,t) \ln \left(\sum_{j=0}^{q} \binom{wn}{j} \right) \cdot A_t$$
$$\leqslant \frac{2^{14+t} s^2 t^4 k \ln(wn)}{w\sqrt{n}} \cdot \phi(s,t) \cdot A_t \leqslant \frac{2^{15+t} s^2 t^4 k \ln n}{w\sqrt{n}} \cdot \phi(s,t) \cdot A_t$$

Employing the bound from Claim 4.5, we conclude that there is some constant C_t' depending on t only such that the following holds:

$$|\mathcal{F} \setminus \mathcal{F}[\hat{\mathcal{T}}]| \leqslant |\mathcal{F} \setminus \mathcal{F}[\mathcal{T}]| + \sum_{T \in \mathcal{T}} |\mathcal{R}_T| + \sum_{V \in \mathcal{S}^* \setminus \mathcal{S}^*[\hat{\mathcal{T}}]} |\mathcal{F}_V|$$
$$\leqslant |\mathcal{F} \setminus \mathcal{F}[\mathcal{T}]| + \sum_{T \in \mathcal{T}} |\mathcal{R}_T| + 2 \sum_{V \in \mathcal{S}^* \setminus \mathcal{S}^*[\hat{\mathcal{T}}]} |\mathcal{U}_V|$$
$$\leqslant \frac{C_t' s^t}{n} \cdot A_t + \frac{C_t' s^t k^{t+1}}{n} \cdot A_t + \frac{C_t' s^t k \ln n}{\sqrt{n}} \cdot A_t$$
$$\leqslant \frac{\hat{C}_t s^t k \ln n}{\sqrt{n}} \cdot A_t,$$

where $\hat{C}_t$ is some constant depending on t only. Setting $\mathcal{C}^* = \hat{\mathcal{T}}$, we obtain the lemma. □

9.3. *Proof of Theorem 6.2*

Proof. We have $n \geqslant k^{20t} \cdot s^{10t} \sqrt{g_0(t)}$, so conditions of Lemma 9.5 are satisfied, provided $g_0(t)$ is a large enough function of t. Due to Lemma 9.5, we have a family $\mathcal{C}^* \in \binom{n}{t}$ of size at most $70t2^t \phi(s,t)\sqrt{n} \ln n$, such that

$$|\mathcal{F} \setminus \mathcal{F}[\mathcal{C}^*]| \leqslant \frac{C_t s^t k \ln n}{\sqrt{n}} \cdot A_t. \tag{9.5}$$

Consider an arbitrary partition of $\mathcal{F}[\mathcal{C}^*]$ into $|\mathcal{C}^*|$ families $\mathcal{F}_C$, $C \in \mathcal{C}^*$, such that $\mathcal{F}_C \subset \mathcal{F}(C)$ and

$$\mathcal{F}[\mathcal{C}^*] = \bigsqcup_{C \in \mathcal{C}^*} \mathcal{F}_C \vee \{C\}.$$

For each $\mathcal{F}_C$, we construct a family $\mathcal{F}_C^*$ as follows:

$$\mathcal{F}_C^* = \left\{ F \in \mathcal{F}_C \,:\, \forall X \in \binom{F}{\leqslant t-1} \text{ the covering number of } \mathcal{F}_C(X) \text{ is at least } sk \right\}.$$

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Then, using $(wn/k, t)$-spreadness of $\mathcal{A}$ from Proposition 4.8, we obtain

$$\begin{aligned}|\mathcal{F}_C \setminus \mathcal{F}_C^*| &\leqslant \sum_{j=0}^{t-1} \sum_{X \in \partial_j \mathcal{A} \colon \tau(\mathcal{F}_C(X)) < sk} \tau(\mathcal{F}_C(X)) \cdot A_{|C|+j+1} \\ &\leqslant sk \sum_{j=0}^{t-1} \binom{wn}{j} \left(\frac{wn}{k}\right)^{-(j+1)} A_t \\ &= \frac{sk}{wn} \sum_{j=0}^{t-1} k^j \cdot A_t = \frac{sk}{wn} \cdot \frac{k^t - 1}{k-1} \overset{(k \geqslant 5)}{\leqslant} \frac{2sk^t}{n} \cdot A_t, \end{aligned} \tag{9.6}$$

where $\tau(\mathcal{F}_C(X))$ stands for the covering number of $\mathcal{F}_C(X)$. Finally, we consider the decomposition

$$\sum_{C \in \mathcal{C}^*} |\mathcal{F}_C^*| = \sum_{F \in \partial_{k-t}\mathcal{A}} |\{C \in \mathcal{C}^* \,:\, F \in \mathcal{F}_C^*\}|.$$

Given $F \in \partial_{k-t}\mathcal{A}$, we denote $\mathcal{C}_F = \{C \in \mathcal{C}^* \,:\, F \in \mathcal{F}_C^*\}$. We claim that $\mathcal{C}_F$ does not contain a sunflower with s petals and the core of arbitrary size. Indeed, let $C_1, \dots, C_s \in \mathcal{C}_F$ be such a sunflower. Take any $X \subset F$ of size $t - 1 - \left|\cap_{\ell=1}^s C_\ell\right|$, and note that X can be empty. Then, for each $\ell = 1, \dots, s$, $\mathcal{F}_{C_\ell}(X)$ has covering number at least sk. Take the maximum number of pairwise disjoint sets $F_1 \in \mathcal{F}_{C_1}(X), \dots, F_m \in \mathcal{F}_{C_m}(X)$, that do not intersect $\cup_{\ell=1}^m C_\ell$. To avoid contradiction, we should assume that $m < s$, and so consider $\mathcal{F}_{C_{m+1}}(X)$. But since the covering number of $\mathcal{F}_{C_{m+1}}(X)$ is at least sk, there is a set F_{m+1} that does not intersect $\cup_{\ell=1}^m C_\ell \cup F_\ell$, contradicting the maximality of m. Hence, we have

$$\sum_{C \in \mathcal{C}^*} |\mathcal{F}_C^*| = \sum_{F \in \partial_{k-t}\mathcal{A}} |\mathcal{C}_F| \leqslant \phi(s,t)|\partial_{k-t}\mathcal{A}|.$$

Combining the above, (9.6) and (9.5), we obtain

$$|\mathcal{F}| \leqslant \phi(s,t)|\partial_{k-t}\mathcal{A}| + \frac{2sk^t|\mathcal{C}^*|}{n} \cdot A_t + \frac{C_t s^t k \ln n}{\sqrt{n}} \cdot A_t.$$

Since $|\mathcal{C}^*| \leqslant 14t \cdot 2^t \phi(s,t)\sqrt{n}\ln n$, using Claim 4.5, we get that $|\mathcal{C}^*| \leqslant C_t' s^t \sqrt{n} \ln n$ for some constant depending on t only. Therefore, we have

$$|\mathcal{F}| \leqslant \phi(s,t) \cdot |\partial_{k-t}\mathcal{A}| + \frac{\tilde{C}_t s^t k^t \ln n}{\sqrt{n}} \cdot A_t.$$

for some constant $\tilde{C}_t$ depending on t only. By the conditions of the theorem, $n \geqslant k^{20t}$, so $k^t \leqslant n^{1/20}$. Similarly, we have that $s^t \leqslant n^{1/40}$ and $C_t' \ln n \leqslant n^{1/40}$, provided $g_0(t)$ is large enough. Therefore, we have

$$C_t' s^t k^t \ln n \leqslant n^{1/10},$$

and so

$$|\mathcal{F}| \leqslant \phi(s,t) \cdot |\partial_{k-t}\mathcal{A}| + n^{-1/3} \cdot A_t.$$

□

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10. Proof sketch of Theorem 7.6

We start by applying Lemma 9.5 to $\mathcal{F}$. Then, we obtain a family $\mathcal{C}^* \subset \partial_t \mathcal{A}$, $|\mathcal{C}^*| \leqslant 70t \cdot 2^t \phi(s,t)\sqrt{n}\ln n$, such that

$$|\mathcal{F} \setminus \mathcal{F}[\mathcal{C}^*]| \leqslant \frac{C_t s^t k \ln n}{\sqrt{n}} \cdot A_t.$$

Consider an arbitrary partition of $\mathcal{F}[\mathcal{C}^*]$ into $|\mathcal{C}^*|$ families $\mathcal{F}_C, C \in \mathcal{C}^*$, such that $\mathcal{F}_C \subset \mathcal{F}(C)$ and

$$\mathcal{F}[\mathcal{C}^*] = \bigsqcup_{C \in \mathcal{C}^*} \mathcal{F}_C \vee \{C\}.$$

As in the proof of Theorem 6.2, we define

$$\mathcal{F}_C^* = \left\{ F \in \mathcal{F}_C \,:\, \forall X \in \binom{F}{\leqslant t-1} \text{ the covering number of } \mathcal{F}_C(X) \text{ is at least } sk \right\}.$$

Due to (9.6), we have

$$|\mathcal{F}_C \setminus \mathcal{F}_C^*| \leqslant \frac{2sk^t}{n} \cdot A_t. \tag{10.1}$$

The idea of the proof is to apply a Kruskal–Katona–type result to show that the number of $C \in \mathcal{C}^*$, such that the shadow of $\mathcal{F}_C$ is large, is at most $\left(1 + e^{-O(k/t)}\right)\phi(s,t)$, and if the shadow of $\mathcal{F}_C$ is small, then the impact of $\mathcal{F}_C$ on $|\mathcal{F}|$ is negligible. Although the exact counterpart of the Kruskal–Katona theorem for $[n]^k$ is known [32], we would like to highlight that a similar but less accurate result can be deduced from Assumptions 3,4.

Proposition 10.1. *Suppose that a k-uniform $\mathcal{A}$ satisfies Assumptions 3,4 with $q = \gamma k$ for some $\gamma < \min\{1, \mu\}$. Let $\mathcal{F} \subset \mathcal{A}$ be any family such that $\mu(\partial_h \mathcal{F}) \leqslant x$. Then, we have*

$$\mu(\mathcal{F}) \leqslant \left(1 - \frac{\gamma}{\mu}\right)^{-\gamma k} x^{\gamma k/h}.$$

Proof. We assume that $x^{1/h} < \left(1 - \frac{\gamma}{\mu}\right)$, since otherwise there is nothing to prove. Consider an arbitrary τ such that $1 < \tau < x^{-1/h}(1-\gamma/\mu)$. Consider the largest R such that $\mu(\mathcal{F}(R)) \geqslant \tau^{|R|}\mu(\mathcal{F})$. In particular, $\mathcal{F}(R) \subset \mathcal{A}(R)$ is τ-homogeneous and $\mu(\mathcal{F}) \leqslant \tau^{-|R|}$.

If $|R| \leqslant \gamma k = q$, then by Proposition 5.8, we have

$$\mu(\partial_h(\mathcal{F}(R))) \geqslant \tau^{-h}.$$

Since $|\partial_h(\mathcal{F}(R))| \leqslant |\partial_h \mathcal{F}|$, the measure $\mu(\partial_h(\mathcal{F}(R)))$ is at most

$$\mu(\partial_h(\mathcal{F}(R))) = \frac{|\partial_h(\mathcal{F}(R))|}{|\partial_h(\mathcal{A}(R))|} \leqslant \frac{|\partial_h \mathcal{F}|}{|\partial_h \mathcal{A}|} \cdot \frac{|\partial_h \mathcal{A}|}{|\partial_h(\mathcal{A}(R))|} = \mu(\partial_h \mathcal{F}) \cdot \frac{|\partial_h \mathcal{A}|}{|\partial_h(\mathcal{A}(R))|}.$$

We bound the above using Assumption 4, and obtain

$$\tau^{-h} \leqslant x \cdot \left(1 - \frac{\gamma}{\mu}\right)^{-h}, \quad \text{which implies} \quad \tau \geqslant x^{-1/h}\left(1 - \frac{\gamma}{\mu}\right),$$

contradicting the definition of τ. Thus, $|R| > \gamma k$, and we have

$$\mu(\mathcal{F}) \leqslant \tau^{-|R|} \leqslant \tau^{-\gamma k}.$$

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Since τ is an arbitrary number less than $x^{-1/h}(1-\gamma/\mu)$, the displayed chain of inequalities above implies the proposition. □

Next, we show that Proposition 10.1 can be applied to $\mathcal{A}(C)$, $C \in \mathcal{C}^*$.

Claim 10.2. *Suppose that $\mathcal{A} = \binom{[n]}{k/w}^w$ for some $w|k$ such that $k \geqslant 4wt$. Then, for any $C \in \partial_t \mathcal{A}$ and $\mathcal{F} \subset \mathcal{A}(C)$ such that $\mu(\partial_h \mathcal{F}) \leqslant x$, we have*

$$\mu(\mathcal{F}) \leqslant \left(1 - \frac{3k}{2wn}\right)^{-k/20} x^{(k-t)/2wh}.$$

Proof. We are going to show that Assumptions 3,4 are satisfied for $\mathcal{A}(C)$ with $q = (k-t)/2w$. Assumption 3 is satisfied due to Claim 5.5. It remains to show that for any R of size at most $q = (k-t)/2w$, we have

$$\frac{|\partial_h(\mathcal{A}(C \cup R))|}{|\partial_h(\mathcal{A}(C))|} \geqslant \left(1 - \frac{q}{\mu'(k-t)}\right)^h = \left(1 - \frac{1}{2w\mu'}\right)^h, \quad \text{where } \mu' = 2\mu/3,$$

so the condition of Proposition 10.1 holds with $\gamma = 1/2w$ and $\mu = \mu'$, $k-t$ in place of k. Note that

$$|R \sqcup C| \leqslant \frac{k-t}{2w} + t \leqslant \frac{k+2wt}{2w} \leqslant \frac{3k}{4w},$$

where we used $k \geqslant 4wt$, so due to Proposition 5.4 applied with $q = 3k/4w$, we have

$$\frac{|\partial_h(\mathcal{A}(C \cup R))|}{|\partial_h(\mathcal{A}(C))|} \geqslant \frac{|\partial_h(\mathcal{A}(C \cup R))|}{|\partial_h \mathcal{A}|} \geqslant \left(1 - \frac{3k}{4wn}\right)^h = \left(1 - \frac{1}{2w\mu'}\right)^h.$$

Then, Proposition 10.1 implies the desired bound on $\mu(\mathcal{F})$. □

Given a real number $\delta \in (0,1)$, define a family $\mathcal{C}^*_\delta$ as follows:

$$\mathcal{C}^*_\delta = \left\{C \in \mathcal{C}^* \,:\, \mu(\partial_{t-1}\mathcal{F}^*_C) \geqslant \delta\right\}.$$

Let us show that $|\mathcal{C}^*_\delta| \leqslant \phi(s,t)/\delta$. First, by double-counting, there exists $X \in \partial_{t-1}\mathcal{A}$ and a family $\mathcal{C}' \subset \mathcal{C}^*_\delta$ of size at least $\delta|\mathcal{C}^*_\delta|$ such that $X \in \partial_{t-1}\mathcal{F}^*_C$ for any $C \in \mathcal{C}'$. We claim that $\mathcal{C}'$ does not contain a sunflower with s petals. Arguing indirectly, let $C_1, \dots, C_s$ be a sunflower in $\mathcal{C}'$. Choose $Y \subset X$ of size $t - 1 - |\cap_{j\in[s]} C_j|$. Then, $C_1 \cup Y, \dots, C_s \cup Y$ form a sunflower with s petals and the core of size $t-1$. Since the covering number of $\mathcal{F}_{C_i}(Y)$ is at least sk for each $i = 1, \dots, s$, we can find a sunflower with the core of size $t-1$ in $\mathcal{F}$, a contradiction. Thus, $|\mathcal{C}'| \leqslant \phi(s,t)$, and $|\mathcal{C}^*_\delta| \leqslant \phi(s,t)/\delta$.

Fix $\delta = \left(1 - \sqrt{\frac{t}{k}}\right)$. Then, we decompose the sum $\sum_{C\in\mathcal{C}^*} |\mathcal{F}_C|$ as follows:

$$\sum_{C\in\mathcal{C}^*} |\mathcal{F}^*_C| = \sum_{C\in\mathcal{C}^*_\delta} |\mathcal{F}^*_C| + \sum_{j=1}^{\infty} \sum_{C\in\mathcal{C}^*_{\delta^{j+1}}\setminus\mathcal{C}^*_{\delta^j}} |\mathcal{F}^*_C|. \tag{10.2}$$

Consider two cases.

Case 1. We have $\mathcal{A} = \binom{[n]}{k/w}^w$ for $k \geqslant 2^6 w^6 t$. For each j and $C \in \mathcal{C}^*_{\delta^{j+1}} \setminus \mathcal{C}_{\delta^j}$, we have

$$|\mathcal{C}^*_{\delta^{j+1}} \setminus \mathcal{C}^*_{\delta^j}| \leqslant |\mathcal{C}^*_{\delta^{j+1}}| \leqslant \delta^{-j} \cdot \phi(s,t) \quad \text{and} \quad |\mathcal{F}^*_C| \leqslant A_t \cdot \left(1 - \frac{3k}{2wn}\right)^{-k/20} \delta^{\frac{j(k-t)}{2w(t-1)}},$$

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where the second fact follows from Claim 10.2. We have

$$\sum_{C\in\mathcal{C}^*}|\mathcal{F}_C^*| \leqslant \frac{\phi(s,t)A_t}{1-\sqrt{\frac{t}{k}}} + \left(1-\frac{3k}{wn}\right)^{-k/20}\phi(s,t)A_t\cdot\sum_{j=1}^{\infty}\left(1-\sqrt{\frac{t}{k}}\right)^{j\cdot\left(\frac{k-t}{2w(t-1)}-1\right)}$$
$$\leqslant \frac{\phi(s,t)}{1-\sqrt{\frac{t}{k}}} + \left(1-\frac{3k}{wn}\right)^{-k/20}\phi(s,t)A_t\cdot\sum_{j=1}^{\infty}\exp\left(-j\cdot\left(\frac{k-t}{2w(t-1)}-1\right)\cdot\sqrt{\frac{t}{k}}\right). \tag{10.3}$$

Next, we bound

$$\left(\frac{k-t}{2w(t-1)}-1\right)\cdot\sqrt{\frac{t}{k}} = \frac{k-t-2w(t-1)}{2w(t-1)}\cdot\sqrt{\frac{t}{k}} \geqslant \frac{k}{4wt}\cdot\sqrt{\frac{t}{k}} = \frac{1}{4w}\cdot\sqrt{\frac{k}{t}},$$

where we used $k\geqslant 2^{12}w^6t\geqslant 2(t+2w(t-1))$. We have $k\geqslant 2^{12}w^6t\geqslant 2^{18}t\geqslant e^5t$, so $\left(\frac{k}{t}\right)^{1/3}\geqslant\frac{1}{2}\ln\frac{k}{t}$, and

$$\sqrt{\frac{k}{t}} \geqslant \left(\frac{k}{t}\right)^{1/6}\ln\sqrt{\frac{k}{t}}.$$

Since $k\geqslant 2^{12}w^6t$, we have

$$\left(\frac{k-t}{2w(t-1)}-1\right)\sqrt{\frac{k}{t}} \geqslant \frac{1}{4w}\left(\frac{k}{t}\right)^{1/6}\ln\sqrt{\frac{k}{t}} \geqslant \ln\sqrt{\frac{k}{t}}.$$

Using the displayed inequality, we may continue (10.3) as follows.

$$\sum_{C\in\mathcal{C}^*}|\mathcal{F}_C^*| \leqslant \frac{\phi(s,t)A_t}{1-\sqrt{\frac{t}{k}}} + \left(1-\frac{3k}{wn}\right)^{-k/20}\phi(s,t)A_t\sum_{j=1}^{\infty}\left(\frac{t}{k}\right)^{j/2}$$
$$= \frac{1+\left(1-\frac{3k}{wn}\right)^{-k/20}\sqrt{\frac{t}{k}}}{1-\sqrt{\frac{t}{k}}}\cdot\phi(s,t)A_t. \tag{10.4}$$

Using $k\geqslant 2^{12}w^6t\geqslant 4t$, we bound

$$\left(1-\sqrt{\frac{t}{k}}\right)^{-1} = 1+\left(1-\sqrt{\frac{t}{k}}\right)^{-1}\sqrt{\frac{t}{k}} \leqslant 1+2\sqrt{\frac{t}{k}}.$$

Similarly, since $wn\geqslant k^{20t}\geqslant 5^{18}k^2t$, we bound

$$\left(1-\frac{3k}{wn}\right)^{-k/20} \leqslant \left(1+\frac{9k}{wn}\right)^{k/20} \leqslant 3.$$

Combining the above and (10.4), we obtain

$$\sum_{C\in\mathcal{C}^*}|\mathcal{F}_C^*| \leqslant \left(1+3\sqrt{\frac{t}{k}}\right)\left(1+2\sqrt{\frac{t}{k}}\right)\phi(s,t)A_t \leqslant \left(1+6\sqrt{\frac{t}{k}}\right)\phi(s,t)A_t. \tag{10.5}$$

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Case 2. We have $\mathcal{A} = [n]^k$. In this case, instead of Claim 10.2, we will use the following theorem due to Frankl, Füredi and Kalai.

Theorem 10.3 (Theorem 5.1 of [32]). *Let $\mathcal{F} \subset [n]^k$, such that $|\mathcal{F}| = y^k$ for some real y. Then, for any $h \in [k]$, we have*

$$|\partial_h \mathcal{F}| \geqslant \binom{k}{h} y^h.$$

Since $|\partial_h [n]^k| = \binom{k}{h} n^h$, this theorem implies the following corollary:

Corollary 10.3.1. *Consider a family of sets $\mathcal{F} \subset [n]^k$, and suppose that $\mu(\partial_h \mathcal{F}) \leqslant x$. Then, we have $\mu(\mathcal{F}) \leqslant x^{k/h}$.*

Hence, for each $j \geqslant 1$ and $C \in \mathcal{C}^*_{\delta^{j+1}} \setminus \mathcal{C}^*_{\delta^j}$, we have

$$|\mathcal{C}^*_{\delta^{j+1}} \setminus \mathcal{C}^*_{\delta^j}| \leqslant \delta^{-j} \cdot \phi(s,t) \quad \text{and} \quad |\mathcal{F}^*_C| \leqslant \delta^{\frac{j(k-t)}{t-1}} A_t.$$

Substituting the above inequalities into (10.2), we obtain

$$\begin{aligned}\sum_{C \in \mathcal{C}^*} |\mathcal{F}^*_C| &\leqslant \frac{\phi(s,t) \cdot A_t}{1 - \sqrt{\frac{t}{k}}} + \phi(s,t) A_t \cdot \sum_{j=1}^{\infty} \left(1 - \sqrt{\frac{t}{k}}\right)^{j \cdot \left(\frac{k-t}{t-1} - 1\right)} \\ &\leqslant \frac{\phi(s,t) A_t}{1 - \sqrt{\frac{t}{k}}} + \phi(s,t) A_t \cdot \sum_{j=1}^{\infty} \exp\left(-j \sqrt{\frac{t}{k}} \left(\frac{k-t}{t-1} - 1\right)\right).\end{aligned}$$

Since $k \geqslant 5t$ and $\sqrt{y} \geqslant \ln y$ for any positive y, we can bound

$$\sqrt{\frac{t}{k}} \left(\frac{k-t}{t-1} - 1\right) \geqslant \sqrt{\frac{t}{k}} \cdot \frac{k-2t}{t} \geqslant \frac{1}{2} \cdot \sqrt{\frac{k}{t}} \geqslant \frac{1}{2} \ln\left(\frac{k}{t}\right),$$

and obtain

$$\sum_{C \in \mathcal{C}^*} |\mathcal{F}^*_C| \leqslant \frac{1 + \sqrt{\frac{t}{k}}}{1 - \sqrt{\frac{t}{k}}} \cdot \phi(s,t) \cdot A_t \leqslant \left(1 + 6\sqrt{\frac{t}{k}}\right) \phi(s,t) \cdot A_t.$$

Thus, the above bound holds for both $\binom{n}{k/w}^w$, $k \geqslant 2^{12} w^6 t$, and $[n]^k$. Using (10.1), we get

$$\sum_{C \in \mathcal{C}^*} |\mathcal{F}_C| \leqslant \left(1 + 6\sqrt{\frac{t}{k}}\right) \phi(s,t) A_t + |\mathcal{C}^*| \cdot \frac{2sk^t}{n} \cdot A_t \leqslant \left(1 + 9\sqrt{\frac{t}{k}} + \frac{70st2^t k^t \ln n}{\sqrt{n}}\right) \phi(s,t) \cdot A_t,$$

where we use the bound on $|\mathcal{C}^*|$ from Lemma 9.5. Using the bound on $|\mathcal{F} \setminus \mathcal{F}[\mathcal{C}^*]|$ provided by this lemma, we get

$$|\mathcal{F}| \leqslant \left(1 + 6\sqrt{\frac{t}{k}} + \frac{70st2^t k^t \ln n}{\sqrt{n}}\right) \phi(s,t) A_t + \frac{C_t s^t \ln n}{\sqrt{n}} A_t.$$

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Since $n \geqslant k^{20t}$ and $n \geqslant s^{40t} \cdot g_0(t)$, for large enough $g_0(t)$, we have

$$|\mathcal{F}| \leqslant \left(1 + 7\sqrt{\frac{t}{k}}\right)\phi(s,t)A_t.$$

11. Conclusion

In the present paper, we provide new bounds for the problem of Erdős–Duke on forbidden sunflowers with prescribed core size. In particular, we extend the result of Frankl and Füredi [26] to the case when n is allowed to depend linearly on k with the factor polynomial in s, t. To obtain this result, we combine the hypercontractivity technique of Keller, Lifshitz, Keevash et al. [38, 41], the spread approximation technique of Kupavskii and Zakharov [49], the result on forbidden sunflowers of Bradač et al. [8] and some ideas from the original proof of Frankl and Füredi [26]. As a byproduct, we answer the question of Kalai[10] and Bradač et al. [8] about the maximal size of families without sunflowers with the core of size at most $t-1$ in the regime when $s \leqslant (t\ln(n/k))^{-3}\sqrt{n/k}$ and $n \geqslant 2^{11}sk\log_2 k$.

It turns out that our methods are quite general and can be applied to obtain similar results for subfamilies of permutations and $\binom{[n]}{k/w}^w$ for certain ranges of values w.

When looking through the proof of Theorem 3.1, one can ask the following question: could we employ modern techniques to improve dependencies in the Delta-system method? In our problem, $k \geqslant 5t$ is enough to significantly decrease the lower bound on n. Could we go further and avoid double-exponential dependence on t? Could we improve similar lower bounds for other Turán-type problems? After we published the first version of our paper as a preprint, it was discovered that the answer is positive and double-exponential dependencies in most applications of the delta-system method could be avoided [37]. However, we hope that dependencies between parameters in hypergraph Turán-type problems will be further improved in the near future.

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[10]See footnote 3.

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