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Nonlinear impulse of ocean waves on floating bodies

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A new formulation is presented of the nonlinear loads exerted on floating bodies by steep irregular surface waves. The forces and moments are expressed in terms of the time derivative of the fluid impulse which circumvents the time-consuming computation of the temporal and spatial derivatives in Bernoulli's equation. The nonlinear hydrostatic force on a floating body is shown to point vertically upwards and the nonlinear Froude–Krylov force and moment are derived as the time derivative of an impulse that involves the time derivative of a simple integral of the ambient velocity potential over the time-dependent body wetted surface. The nonlinear radiation and diffraction forces and moments are expressed as time derivatives of two impulses, a body impulse and a free surface impulse that represents higher-order wave loads acting along the body waterline. Numerical results are presented illustrating the accuracy of the new force expressions. Applications discussed include the nonlinear seakeeping of ships and offshore platforms and the extreme wave loads and responses of offshore wind turbines.

Key words: surface gravity waves, wave–structure interactions

1. Introduction

The momentum conservation principle has been widely used in fluid mechanics for the evaluation of the steady and unsteady forces and moments acting on bodies in lifting and non-lifting flows. Examples include the derivation of expressions for the unsteady potential flow forces on a body undergoing an arbitrary motion in an infinite fluid and of the induced drag on a three-dimensional lifting surface in terms of the free vorticity shed downstream on the Trefftz plane. Distinct advantages of such expressions for the forces are that they circumvent the computation of the hydrodynamic pressure over the body boundary from Bernoulli's equation which is computationally demanding, they are exact, easy to implement and offer valuable insights into the fluid flow physics responsible for the force and moments exerted on the body.

The momentum conservation principle has found little use for the derivation of expressions for the nonlinear loads exerted on floating bodies by steep irregular waves. The Bernoulli equation is used instead to obtain the hydrodynamic pressure over the instantaneous body wetted surface which upon integration leads to the force and

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moment acting on the body. A drawback of this approach is that the computation of the partial time derivative and spatial gradients of the velocity potential over the instantaneous body wetted surface is computationally challenging leading to small time steps, fine meshing and large computational efforts. The proper application of the momentum conservation principle for such problems would circumvent these computational shortcomings of the Bernoulli equation and lead to expressions for the time dependent force and moment that do not require the evaluation of the time or space derivatives of the velocity potential on the body boundary. A classical example is the expression for the exact force and moment acting on a body in an infinite potential flow in terms of the time derivative of the flow impulse which is a simple integral of the velocity potential over the body boundary.

The concept of the impulse in fluid flow dates back to Kelvin (1868). It arises from the insight that an inviscid flow field governed by the Euler equations may be started from rest if an impulse is applied instantaneously in the fluid domain in the form of a local push over the body boundary or a twist in the parts of the fluid domain where vorticity is shed. This topic is discussed in Lamb (1932) where the impulsive form of Euler's equations are derived and expressions are obtained for the forces and moments acting on bodies as time derivatives of impulses in potential and vortical flows. In the absence of vorticity the potential flow around a rigid body in a domain of infinite extent may be started from rest by imparting an impulse on its boundary equal to the velocity potential multiplied by the unit normal vector. In lifting flows with shed wakes the impulse also includes an integral over the domain of free vorticity in the fluid domain. Its time derivative provides expressions for the lifting forces and moments which are generalizations of the Kutta–Joukowski theorem (Lighthill 1986). The time derivative of the impulse provides the forces and moments exerted on the rigid body in terms of the time invariant added mass tensor leading to the generalized equations of motion of a rigid body in an ideal fluid (Newman 1977).

The present article derives expressions for the impulses in the nonlinear interaction of steep and irregular surface waves with floating bodies in the time domain. The force and moment acting on the body follow at time derivatives of the impulses which involve simple integrals of the velocity potentials over the body boundary and the nonlinear ambient wave profile assumed known *a priori*. The ambient waves are assumed irregular of large amplitude and modelled by potential flow theory using a perturbation or a nonlinear method. A floating body interacting with the ambient waves is allowed to undergo large-amplitude motions and the radiation and diffraction wave disturbances induced by the body are also governed by potential flow theory. Expressions are obtained of components of the impulse, including the Froude–Krylov (FK) impulse and the impulse due to the radiation–diffraction (RD) wave disturbances. The nonlinear wave force acting on the body is shown to be the sum of a nonlinear hydrostatic force which points vertically upwards at all times and of the time derivative of the FK and the RD impulses which involve simple integrals of the velocity potentials over the instantaneous body boundary. Expressions are also derived for the corresponding moment impulses. The use of Bernoulli's equation is circumvented mitigating the time-consuming computation of time and space derivatives of the velocity potential known to be an impediment of the efficiency of nonlinear wave body computations.

Viscous effects are neglected. In certain applications they are important and may be accounted for as additive forces which are nonlinear functions of the ambient flow kinematics. The application of the new expressions for the nonlinear wave forces is discussed for the evaluation of the large-amplitude motion and nonlinear loads of

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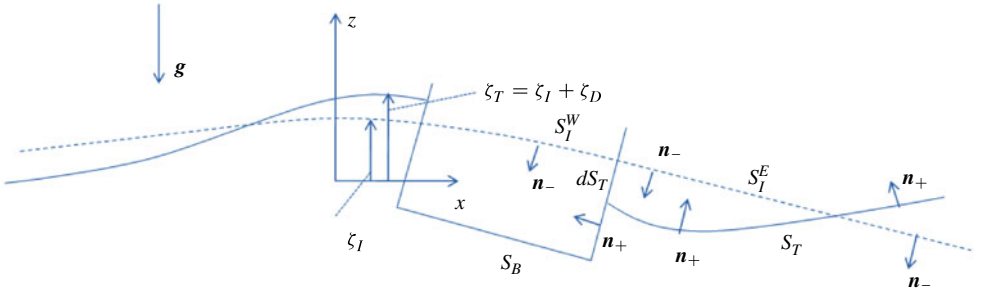


FIGURE 1. (Colour online available at journals.cambridge.org/flm) Plots the exact position of the body boundary and the fully nonlinear free surface (solid curves) and the ambient wave free surface that does not include the radiation and diffraction wave disturbances (dashed curve). The unit normal vector to the fully nonlinear free surface points out of the fluid domain, while the unit normal vector to the ambient wave surface points into the fluid domain. This selection is motivated by the application of Gauss' theorem later in the paper over the differential volume bounded by the solid and dashed free surfaces.

ships, large-volume offshore platforms and small-volume offshore wind turbines either bottom mounted or supported by floaters.

2. Boundary value problem and nonlinear forces

Figure 1 illustrates a body floating on a free surface interacting with a nonlinear ambient wave which is assumed irregular. The reference coordinate system (X, Y, Z) is fixed in space with its origin located on the calm water surface and the positive z-axis pointing upwards. The free surface elevation due to the ambient wave is denoted by the dashed line and the free surface elevation that includes the radiation and diffraction wave disturbances by the solid line.

Denote the total velocity potential by $\varphi(X, Y, Z, t)$ subject to the Laplace equation in the fluid domain

$$\frac{\partial^2\varphi}{\partial X^2} + \frac{\partial^2\varphi}{\partial Y^2} + \frac{\partial^2\varphi}{\partial Z^2} = 0. \tag{2.1}$$

On the body boundary the normal velocity of the total potential is equal to the normal velocity of the body boundary U_n due to its oscillatory motions

$$\frac{\partial\varphi}{\partial n} = U_n \quad \text{on } S_B(t). \tag{2.2}$$

On the nonlinear free surface $Z = \zeta_T(X, Y, t)$ the total velocity potential φ satisfies the nonlinear free-surface condition

$$\left(\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial Z} + 2\nabla\varphi\cdot\nabla\frac{\partial\varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\left(\nabla\varphi\cdot\nabla\varphi\right)\right)_{Z=\zeta_T(X,Y,t)} = 0, \tag{2.3a}$$

$$\zeta_T(X, Y, t) = -\frac{1}{g}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\right)_{Z=\zeta_T(X,Y,t)}. \tag{2.3b}$$

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The ambient wave potential φ_I also satisfies an analogous free-surface condition on the ambient wave free surface $Z = \zeta_I(X, Y, t)$

$$\left(\frac{\partial^2\varphi_I}{\partial t^2} + g\frac{\partial\varphi_I}{\partial Z} + 2\nabla\varphi_I\cdot\nabla\frac{\partial\varphi_I}{\partial t} + \frac{1}{2}\nabla\varphi_I\cdot\nabla\left(\nabla\varphi_I\cdot\nabla\varphi_I\right)\right)_{Z=\zeta_I(X,Y,t)} = 0, \quad (2.4a)$$

$$\zeta_I(X, Y, t) = -\frac{1}{g}\left(\frac{\partial\varphi_I}{\partial t} + \frac{1}{2}\nabla\varphi_I\cdot\nabla\varphi_I\right)_{Z=\zeta_I(X,Y,t)}. \quad (2.4b)$$

It is hereafter assumed that the nonlinear ambient wave velocity potential and free-surface elevation are known. They may be determined by a perturbation method or by a nonlinear numerical algorithm. In the remainder of the article the ambient waves are assumed to be irregular and of finite amplitude.

The definition of the hydrodynamic force and moment acting on the body follows from the integration of the hydrodynamic pressure obtained form Bernoulli's equation over the instantaneous body wetted surface $S_B(t)$:

$$\boldsymbol{F}(t) = -\rho\int_{S_B(t)}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + gZ\right)\boldsymbol{n}\,\mathrm{d}s \quad (2.5a)$$

$$\boldsymbol{M}(t) = -\rho\int_{S_B(t)}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + gZ\right)[\boldsymbol{X}\times\boldsymbol{n}]\,\mathrm{d}s. \quad (2.5b)$$

At a large distance from the body both the ambient wave and RD wave elevations are assumed to vanish. The decay of the ambient wave disturbance at infinity is physically equivalent to the transition to the calm water region outside a seastate of finite spatial extent.

Denote by $V(t)$ the fluid volume enclosed by the fully nonlinear free surface $Z = \zeta_T(X, Y, t)$ the body wetted surface $S_B(t)$ and a control surface at infinity. By virtue of the transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\int_{V(t)}\nabla\varphi\,\mathrm{d}v = \int\int_{V(t)}\frac{\partial}{\partial t}\nabla\varphi\,\mathrm{d}v + \int_{S(t)}\nabla\varphi(\boldsymbol{U}\cdot\boldsymbol{n})\,\mathrm{d}s. \quad (2.6)$$

The control surface $S(t)$ encloses a potential flow which consists of the ambient wave and the body-induced radiation and diffraction wave disturbances. The asymptotic decay of the RD disturbance at large radial distances is dipole like and leads to the vanishing of all surface integrals over a control surface at infinity. Therefore, reference to that surface is hereafter omitted. In (2.6) $U_n = \boldsymbol{U}\cdot\boldsymbol{n} = \partial\varphi/\partial n$ is the normal velocity of the total flow over the free surface and the body wetted surface and is equal to the normal velocity of the respective geometric surfaces.

Applying Gauss's theorem in the left-hand side and the first term in the right-hand side of (2.6), and exchanging the time derivative with the gradient under the integral sign

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}\varphi\,\boldsymbol{n}\,\mathrm{d}s = \int_{S(t)}\left[\frac{\partial\varphi}{\partial t}\boldsymbol{n} + \nabla\varphi\frac{\partial\varphi}{\partial n}\right]\mathrm{d}s. \quad (2.7)$$

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Adding the quadratic term in Bernoulli's equation on both sides of (2.7) we obtain after a simple rearrangement of terms

$$\int_{S(t)} \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\right] \boldsymbol{n}\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)} \varphi\,\boldsymbol{n}\,\mathrm{d}s - \int_{S(t)} \left[\nabla\varphi\frac{\partial\varphi}{\partial n} - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\,\boldsymbol{n}\right] \mathrm{d}s. \tag{2.8}$$

The following identity holds for any velocity potential over a closed surface $S(t)$

$$\int_{S(t)} \left[\nabla\varphi\frac{\partial\varphi}{\partial n} - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\,\boldsymbol{n}\right] \mathrm{d}s = 0. \tag{2.9}$$

Combining (2.8) and (2.9) and adding the hydrostatic term on both sides of (2.8) we obtain

$$\int_{S_B(t)} \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + gZ\right] \boldsymbol{n}\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_{S_B(t)+S_T(t)} \varphi\,\boldsymbol{n}\,\mathrm{d}s + g\int_{S_B(t)+S_T(t)} Z\,\boldsymbol{n}\,\mathrm{d}s. \tag{2.10}$$

Expression (2.10) is valid for any nonlinear free surface flow around a floating body assuming that all disturbances vanish sufficiently fast at infinity. The surface $S_B(t)$ denotes the body instantaneous wetted surface and $S_T(t)$ the nonlinear free surface exterior to the body over which the hydrodynamic pressure vanishes. The left-hand side of (2.10) multiplied by $-\rho$ is the total nonlinear hydrodynamic force acting on the floating body. The right-hand side involves no spatial or time derivatives under the integral sign and is the starting point for the derivation of the impulses and the new expressions for the forces below.

The corresponding expression for the moment follows from the application of the following identity over a closed surface $S(t)$

$$\int_{S(t)} \boldsymbol{X}\times\left[\nabla\varphi\frac{\partial\varphi}{\partial n} - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\,\boldsymbol{n}\right] \mathrm{d}s = 0. \tag{2.11}$$

The resulting expression for the moment follows a derivation analogous to that used to obtain (2.10)

$$\int_{S_B(t)} \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + gZ\right] (\boldsymbol{X}\times\boldsymbol{n})\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_{S_B(t)+S_T(t)} \varphi(\boldsymbol{X}\times\boldsymbol{n})\,\mathrm{d}s + g\int_{S_B(t)+S_T(t)} Z(\boldsymbol{X}\times\boldsymbol{n})\,\mathrm{d}s. \tag{2.12}$$

The proof of the integral theorems (2.9), (2.11) and (2.12) is outlined in Newman (1977) (equations (89) and (92)).

If the body is allowed to shrink to a point with vanishing displacement the left-hand side of (2.10) reduces to

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I(t)} \varphi_I\boldsymbol{n}\,\mathrm{d}s\right] + g\int_{S_I(t)} Z\boldsymbol{n}\,\mathrm{d}s. \tag{2.13}$$

In (2.13) φ_I is the ambient wave velocity potential and $S_I(t)$ the free surface of the ambient wave in the absence of a floating body. Therefore (2.13) represents an integral form of the nonlinear free surface condition satisfied by an irregular

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ambient wave. It states the balance between the time derivative of the impulse applied over the free surface and the restoring effect of the ambient wave free-surface elevation. Expression (2.13) is an alternative form of the classical nonlinear free-surface condition (2.4) expressing an integral balance between inertial and restoring effects for ambient wave disturbances that vanish at infinity. A new mathematical representation of ocean waves that allows their decay at infinity is presented by Sclavounos (2011).

The second term in the right-hand side of (2.13) is a vector pointing in the vertical direction by virtue of Gauss' theorem applied over the volume bounded by the free surface $S_I(t)$ and the $Z = 0$ plane over which the integrand vanishes. The horizontal components of (2.13) lead to

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I(t)} \varphi_I n_X \,\mathrm{d}s\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I(t)} \varphi_I n_Y \,\mathrm{d}s\right] \tag{2.14a}$$

$$\int_{S_I(t)} \varphi_I n_X \,\mathrm{d}s = C_1, \quad \int_{S_I(t)} \varphi_I n_Y \,\mathrm{d}s = C_2. \tag{2.14b}$$

The conservation law (2.14) states that the horizontal impulses of free ambient wave disturbances are constant and time invariant as they are not counteracted by restoring effects. Any nonlinear representation of ambient waves subject to (2.4) and the Laplace equation also obeys (2.13) and (2.14).

3. Nonlinear impulses

Taking the difference between (2.10) and (2.13) and multiplying with both sides with $-\rho$ we obtain the following expression for the nonlinear force acting on the floating body

$$\begin{aligned}
\boldsymbol{F}_B(t) &= -\rho \int_{S_B(t)} \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + gZ\right] \boldsymbol{n}_+ \,\mathrm{d}s \\
&= -\rho g\left[\int_{S_B(t)} Z\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^W(t)} Z\boldsymbol{n}_- \,\mathrm{d}s\right] \\
&\quad - \rho\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_B(t)} \varphi\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^W(t)} \varphi_I\boldsymbol{n}_- \,\mathrm{d}s\right] \\
&\quad - \rho\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_T(t)} \varphi\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} \varphi_I\boldsymbol{n}_- \,\mathrm{d}s\right] \\
&\quad - \rho g\left[\int_{S_T(t)} Z\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} Z\boldsymbol{n}_- \,\mathrm{d}s\right].
\end{aligned} \tag{3.1}$$

In (3.1) the ambient wave free-surface interior to the body is denoted by S_I^W and its surface exterior to the body is denoted by S_I^E (figure 1). Moreover, the notation is adopted that unit normal vectors pointing out of the fluid domain are denoted by $\boldsymbol{n}_+$ and unit normal vectors pointing into the fluid domain by $\boldsymbol{n}_-$. This will facilitate the application of Gauss' theorem in the derivation that follows.

Denote by $S_B^W(t)$ the body wetted surface defined as the intersection of its boundary with the ambient wave profile. The exact body wetted surface is $S_B(t) = S_B^W(t) + \Delta S(t)$, where $\Delta S(t)$ is the differential surface around the waterline bounded by the exact free-surface elevation and the free-surface elevation of the ambient wave. Introducing

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this notation in (3.1) and re-grouping terms we obtain

$$\begin{aligned}\boldsymbol{F}_B(t) &= -\rho \int_{S_B(t)} \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + gZ\right]\boldsymbol{n}_+ \,\mathrm{d}s \\ &= -\rho g\left[\int_{S_B^W(t)} Z\,\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^W(t)} Z\boldsymbol{n}_- \,\mathrm{d}s\right] \\ &\quad -\rho\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_B^W(t)} \varphi\,\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^W(t)} \varphi_I\,\boldsymbol{n}_- \,\mathrm{d}s\right] \\ &\quad -\rho\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_T(t)+\Delta S(t)} \varphi\,\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} \varphi_I\,\boldsymbol{n}_- \,\mathrm{d}s\right] \\ &\quad -\rho g\left[\int_{S_T(t)+\Delta S(t)} Z\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} Z\boldsymbol{n}_- \,\mathrm{d}s\right]. \end{aligned} \tag{3.2}$$

The total velocity potential is now expressed in the form $\varphi = \varphi_I + \varphi_D$ in (3.2) namely as the sum of the ambient wave and disturbance radiation/diffraction disturbance potentials. It is assumed that both the ambient and disturbance potentials accept an analytic continuation around their respective free surfaces permitting the use of a Taylor series expansion used later in the derivation. Substituting in (3.2) and re-grouping terms we obtain

$$\begin{aligned}\boldsymbol{F}_B(t) &= -\rho \int_{S_B(t)} \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + g\,Z\right]\boldsymbol{n}_+ \,\mathrm{d}s \\ &= -\rho g\left[\int_{S_B^W(t)} Z\,\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^W(t)} Z\,\boldsymbol{n}_- \,\mathrm{d}s\right] \\ &\quad -\rho\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_B^W(t)} \varphi_I\,\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^W(t)} \varphi_I\,\boldsymbol{n}_- \,\mathrm{d}s\right] \\ &\quad -\rho\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_B^W(t)} \varphi_D\,\boldsymbol{n}_+ \,\mathrm{d}s\right] \\ &\quad -\rho\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_T(t)+\Delta S(t)} \varphi_D\,\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} \varphi_D\,\boldsymbol{n}_- \,\mathrm{d}s\right] + \rho\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I^E(t)} \varphi_D\,\boldsymbol{n}_- \,\mathrm{d}s\right] \\ &\quad -\rho\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_T(t)+\Delta S(t)} \varphi_I\,\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} \varphi_I\,\boldsymbol{n}_- \,\mathrm{d}s\right] \\ &\quad -\rho g\left[\int_{S_T(t)+\Delta S(t)} Z\,\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} Z\,\boldsymbol{n}_- \,\mathrm{d}s\right]. \end{aligned} \tag{3.3}$$

In the fourth term on the right-hand side of (3.3) the integral of the disturbance potential over the ambient wave surface was added and subtracted. The reason is that the surfaces $S_T(t) + \Delta S(t) + S_I^E(t)$ define a closed differential volume (see figure 1) over which Gauss' theorem will be applied to further reduce the integrals in the last two lines of the right-hand side of (3.3). It is also noted that the unit normal vectors over this closed differential surface have been defined so that they point either always

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outside or inside the enclosed volume leading to a positive or negative sign in the application of Gauss' theorem, respectively.

Expression (3.3) provides an exact expression for the nonlinear force acting on the body under the assumption that the ambient and disturbance potentials accept an analytic continuation about their respective exact free surfaces. In § 4 it will be further assumed that the radiation/diffraction disturbance is a small perturbation to the ambient wave field. Expression (3.3) contains five terms on its right-hand side. Each is discussed separately next.

Nonlinear buoyancy force and moment

The first term in the right-hand side of (3.3) defines the nonlinear buoyancy force. Applying Gauss' theorem and noting that with the unit normal vectors point inside the body boundary (see the discussion following figure 1), we obtain

$$\begin{aligned} \boldsymbol{F}_{B,1}(t) &= -\rho g \left[\int_{S_B^W(t)} Z\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^W(t)} Z\boldsymbol{n}_- \,\mathrm{d}s \right] \\ &= \rho g \forall_W(t) \boldsymbol{k}. \end{aligned} \tag{3.4}$$

In (3.4) $\boldsymbol{k}$ is the unit vector pointing in the positive z-direction and $\forall_W(t)$ is the volume enclosed by the body wetted surface and the ambient wave surface interior to the body. Therefore, the nonlinear hydrostatics force given by (3.4) acts on a closed volume and always points upwards. In the classical definition of the nonlinear body force obtained by integration of the hydrodynamic pressure using Bernoulli's equation in (2.5) the nonlinear hydrostatic force is coupled with the flow dynamics and does not necessarily point upwards.

The nonlinear hydrostatic moment follows from (2.12) and a derivation that parallels that for the force

$$\boldsymbol{M}_{B,1}(t) = -\rho g \left[\int_{S_B^W(t)} Z(\boldsymbol{X} \times \boldsymbol{n}_+) \,\mathrm{d}s + \int_{S_I^W(t)} Z(\boldsymbol{X} \times \boldsymbol{n}_-) \,\mathrm{d}s \right]. \tag{3.5}$$

It follows from (3.4) and (3.5) that the nonlinear buoyancy force points upwards and passes through the geometrical centre of the volume $\forall_W(t)$. These expressions extend the classical Archimedean hydrostatics in calm water via the introduction of the ambient wave free surface as the dynamic waterplane area of the floating body.

Froude–Krylov impulse force and moment

The second term in the right-hand side of (3.3) leads to the definition of the FK impulse and the force defined as its time derivative

$$\boldsymbol{F}_{B,2}(t) = -\rho \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_{S_B^W(t)} \varphi_I \boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^W(t)} \varphi_I \boldsymbol{n}_- \,\mathrm{d}s \right] = \frac{\mathrm{d}\boldsymbol{I}_{F-K}}{\mathrm{d}t} \tag{3.6a}$$

$$\begin{aligned} \boldsymbol{I}_{F-K} &= -\rho \int_{S_B^W(t)} \varphi_I \boldsymbol{n}_+ \,\mathrm{d}s - \rho \int_{S_I^W(t)} \varphi_I \boldsymbol{n}_- \,\mathrm{d}s \\ &= \rho \int \int_{\forall_W(t)} \nabla \varphi_I \,\mathrm{d}v. \end{aligned} \tag{3.6b}$$

The surface integrations in (3.6) are again carried out over the intersection of the body boundary and the ambient wave profile which is assumed known. An additional integration is carried out over the ambient wave free-surface interior to the body. An application of Gauss's theorem provides an alternative definition of the FK impulse as

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the integral of the ambient wave velocity vector over the volume internal to the body surface and its dynamic waterplane area. The evaluation of the new FK forces and moments requires knowledge of just the velocity potential of the ambient wave over the body boundary and not its partial time derivative or its spatial gradients.

In the limit of waves of small steepness the integrals over the interior waterplane area in (3.4) and (3.6) cancel out by virtue of the linearized dynamic free-surface condition satisfied by the ambient wave on the $Z = 0$ plane. Yet this cancellation does not occur in the nonlinear case.

The FK moment and its impulse take the form

$$\boldsymbol{M}_{B,2}(t) = \frac{\mathrm{d}\boldsymbol{L}_{F-K}}{\mathrm{d}t} \tag{3.7a}$$

$$\boldsymbol{L}_{F-K} = -\rho \int_{S_B^W(t)} \varphi_I(\boldsymbol{X} \times \boldsymbol{n}_+)\, \mathrm{d}s - \rho \int_{S_I^W(t)} \varphi_I(\boldsymbol{X} \times \boldsymbol{n}_-)\, \mathrm{d}s. \tag{3.7b}$$

Radiation and diffraction body impulse, force and moment

The third term in the right-hand side of (3.3) leads to the body RD disturbance impulse and respective force and moments

$$\boldsymbol{F}_{B,3} = \frac{\mathrm{d}\boldsymbol{I}_{B-D,I}}{\mathrm{d}t} \tag{3.8a}$$

$$\boldsymbol{I}_{B-D,I} = -\rho \int_{S_B^W(t)} \varphi_D\, \boldsymbol{n}_+\, \mathrm{d}s \tag{3.8b}$$

$$\boldsymbol{M}_{B,3} = \frac{\mathrm{d}\boldsymbol{L}_{B-D,I}}{\mathrm{d}t} \tag{3.8c}$$

$$\boldsymbol{L}_{B-D,I} = -\rho \int_{S_B^W(t)} \varphi_D(\boldsymbol{X} \times \boldsymbol{n}_+)\, \mathrm{d}s. \tag{3.8d}$$

The integrations in (3.8) are carried out over the body wetted surface defined by its intersection with the ambient wave profile. Again the evaluation of the forces and moments requires just the radiation and diffraction velocity potential over the body boundary and not their partial time derivative or spatial gradients.

Radiation and diffraction free surface impulse, force and moment

The fourth and fifth terms on the right-hand side of (3.3) have been reduced to the form

$$\begin{aligned}\boldsymbol{F}_{B,5} = &-\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_T(t)+\Delta S(t)} \varphi_D\, \boldsymbol{n}_+\, \mathrm{d}s + \int_{S_I^E} \varphi_D\, \boldsymbol{n}_-\, \mathrm{d}s\right] + \rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I^E} \varphi_D\, \boldsymbol{n}_-\, \mathrm{d}s\right] \\ &-\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_T(t)+\Delta S(t)} \varphi_I\, \boldsymbol{n}_+\, \mathrm{d}s + \int_{S_I^E(t)} \varphi_I\, \boldsymbol{n}_-\, \mathrm{d}s\right] \\ &-\rho g\left[\int_{S_T(t)+\Delta S(t)} Z\, \boldsymbol{n}_+\, \mathrm{d}s + \int_{S_I^E(t)} Z\, \boldsymbol{n}_-\, \mathrm{d}s\right].\end{aligned} \tag{3.9}$$

The integration in the first, third and fourth terms on the right-hand side is over a closed differential surface bounded by the nonlinear free surface, the ambient wave surface and the differential surface around the body waterline. The unit normal vector over this closed surface points outside the enclosed volume when the total wave elevation is larger than the ambient wave elevation and points inside the differential

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volume otherwise. The first, third and fourth terms may be reduced by an application of Gauss' theorem over the volume bounded by the closed differential surface defined above. Denoting by $\zeta_D = \zeta_T - \zeta_I$ the RD disturbance wave elevation assumed to be a signed quantity the unit normal vectors over the differential surface point outside the differential volume when $\zeta_D > 0$ and inside the volume when $\zeta_D < 0$. Grouping terms and applying Gauss' theorem in (3.9) we obtain

$$\begin{aligned}
\boldsymbol{F}_{B,5} &= -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I^E} \varphi_D \boldsymbol{n}_+ \,\mathrm{d}s\right] \\
&\quad -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_T(t)+\Delta S(t)} (\varphi_I + \varphi_D)\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} (\varphi_I + \varphi_D)\boldsymbol{n}_- \,\mathrm{d}s\right] \\
&\quad -\rho g\left[\int_{S_T(t)+\Delta S(t)} Z\boldsymbol{n}_+ \,\mathrm{d}s + \int_{S_I^E(t)} Z\boldsymbol{n}_- \,\mathrm{d}s\right] \\
&= -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I^E(t)} \varphi_D \boldsymbol{n}_+ \,\mathrm{d}s\right] \\
&\quad -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int\int_{\upsilon(t)} (\nabla\varphi_I + \nabla\varphi_D)\,\mathrm{d}v\right] \mathrm{sgn}(\zeta_D) - \rho g \boldsymbol{k}\upsilon(t)\mathrm{sgn}(\zeta_D) \qquad (3.10)
\end{aligned}$$

where $\upsilon(t)$ is the volume bounded by the exact and ambient wave free surfaces. The second and third terms in the right-hand side of the second equality of (3.10) may be expressed as integrals over the ambient wave free-surface exterior to the body by invoking a Taylor series expansion

$$\begin{aligned}
\boldsymbol{F}_{B,5} = \frac{\mathrm{d}\boldsymbol{I}_{B,5}}{\mathrm{d}t} &= -\rho \frac{\mathrm{d}}{\mathrm{d}t}\int_{S_I^E(t)} \varphi_D \boldsymbol{n}_+ \,\mathrm{d}s - \rho g\boldsymbol{k}\int_{S_I^E(t)} \zeta_D \,\mathrm{d}s \\
&\quad -\rho \frac{\mathrm{d}}{\mathrm{d}t}\int_{S_I^E(t)} \left[\zeta_D \nabla(\varphi_I + \varphi_D) + \frac{1}{2}\zeta_D^2 \frac{\partial}{\partial Z}\nabla(\varphi_I + \varphi_D) + \cdots\right] \mathrm{d}s \qquad (3.11a) \\
\boldsymbol{I}_{B,F-S} &= -\rho \int_{S_I^E(t)} \varphi_D \boldsymbol{n}_+ \,\mathrm{d}s - \rho g\boldsymbol{k}\int_t \mathrm{d}\tau \int_{S_I^E(t)} \zeta_D \,\mathrm{d}s \\
&\quad -\rho \int_{S_I^E(t)} \left[\zeta_D \nabla(\varphi_I + \varphi_D) + \frac{1}{2}\zeta_D^2 \frac{\partial}{\partial Z}\nabla(\varphi_I + \varphi_D) + \cdots\right] \mathrm{d}s. \qquad (3.11b)
\end{aligned}$$

Expression (3.11) provides the impulse imparted upon the body by the RD disturbances via the ambient wave free surface.

The corresponding expression for the moment follows in an analogous manner

$$\begin{aligned}
\boldsymbol{M}_{B,5} &= -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I^E} \varphi_D(\boldsymbol{X} \times \boldsymbol{n}_+)\,\mathrm{d}s\right] \\
&\quad -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_T(t)+\Delta S(t)} (\varphi_I + \varphi_D)(\boldsymbol{X} \times \boldsymbol{n}_+)\,\mathrm{d}s + \int_{S_I^E(t)} (\varphi_I + \varphi_D)(\boldsymbol{X} \times \boldsymbol{n}_-)\,\mathrm{d}s\right] \\
&\quad -\rho g\left[\int_{S_T(t)+\Delta S(t)} Z(\boldsymbol{X} \times \boldsymbol{n}_+)\,\mathrm{d}s + \int_{S_I^E(t)} Z(\boldsymbol{X} \times \boldsymbol{n}_+)\,\mathrm{d}s\right]. \qquad (3.12)
\end{aligned}$$

Applying again Gauss' theorem the second and third terms in the right-hand side of (3.12) may be converted into an integral over the differential volume between the

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exact and the ambient wave surfaces

$$\boldsymbol{M}_{B,5} = -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{S_I^E(t)} \varphi_D(\boldsymbol{X}\times \boldsymbol{n}_+)\,\mathrm{d}s\right] - \rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\int\int_{\upsilon(t)} \boldsymbol{X}\times(\nabla\varphi_I + \nabla\varphi_D)\,\mathrm{d}\upsilon\right]\mathrm{sgn}(\zeta_D) - \rho g \int\int_{\upsilon(t)} \boldsymbol{X}\times\boldsymbol{k}\,\mathrm{d}\upsilon\,\mathrm{sgn}(\zeta_D). \tag{3.13}$$

The further reduction of (3.13) into a form analogous to (3.11) is omitted for brevity. Summarizing, the nonlinear fluid force and moment acting on a body floating in an ambient irregular wave of large amplitude has been derived as the sum of a nonlinear buoyancy force pointing upwards and the time derivative of a sequence of impulses. The FK nonlinear impulse involves an integral of the ambient wave velocity potential over the instantaneous body wetted surface and the interior waterplane area defined by the ambient wave elevation. The body RD nonlinear impulse involves an integral of the RD velocity potentials over the body wetted surface. The free-surface RD nonlinear impulse involves integrals of the RD disturbances over the ambient wave free surface. It is shown in the next section that the free-surface nonlinear RD impulse is small relative to the nonlinear FK and body RD forces and may be neglected in a number of applications.

4. Free-surface impulse

All expressions for the nonlinear impulses and forces derived above involve integrals of the velocity potentials over the instantaneous body wetted surface defined as its intersection with the ambient wave profile. For fixed bodies this wetted surface is known in terms of the ambient wave profile which is known *a priori*. For bodies free to oscillate it follows as the geometric intersection of the instantaneous body position as obtained from the time marching of its equations of motion and the ambient wave profile. This geometric intersection may be simply carried out during each time step of the simulation of the body motions using a standard computer aided design program.

Nonlinear ambient waves are assumed to have an amplitude $A = O(1)$ comparable to the body dimension d, either the diameter of a leg of an offshore platform or the draft of a section of a ship where in both cases $d \sim 10$–20 m. The ambient wave steepness $kA = 2\pi A/\lambda$ is assumed to be small and of $O(\delta)$ where $\delta \sim 0.1$. For fixed offshore structures the magnitude of the body wave disturbance $(\varphi_D, \zeta_D) = O(kd) = O(2\pi d/\lambda) = O(\varepsilon)$. The case of the interaction of a breaking wave with a structure requires a fully nonlinear treatment which is beyond the scope of the present article. Steep waves encountered in design seastates typically have a length larger than the ship length therefore they are long compared with the ship draft and beam. Therefore, the body wave disturbance is again of the order of $(\varphi_D, \zeta_D) = O(kd) = O(2\pi d/\lambda) = O(\varepsilon)$, where d is the characteristic transverse dimension of the ship. Typically $\varepsilon < 0.1$ in design seastates. These order of magnitude estimates justify the linearization of the body wave disturbance about the ambient wave profile which is assumed to have a large amplitude.

Invoking the fully nonlinear free-surface condition (2.3) and linearizing the body disturbance about the ambient wave profile we obtain the dynamic surface condition

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for (φ_D, ζ_D)

$$\left[\frac{\partial \varphi_D}{\partial t} + \nabla\varphi_D \cdot \nabla\varphi_I\right] + \zeta_D\left\{g + \frac{\partial}{\partial Z}\left[\frac{\partial \varphi_I}{\partial t} + \frac{1}{2}\nabla\varphi_I \cdot \nabla\varphi_I\right]\right\} = 0, Z = \zeta_I(X, Y, t) \tag{4.1a}$$

$$\frac{\partial \varphi_D}{\partial t} + g\zeta_D = -\left\{\nabla\varphi_D \cdot \nabla\varphi_I - \zeta_D\frac{\partial}{\partial Z}\left[\frac{\partial \varphi_I}{\partial t} + \frac{1}{2}\nabla\varphi_I \cdot \nabla\varphi_I\right]\right\} = O(\varepsilon\delta), Z = \zeta_I(X, Y, t). \tag{4.1b}$$

The kinematic free-surface condition takes the form

$$\frac{\partial \zeta_D}{\partial t} - \frac{\partial \varphi_D}{\partial Z} + \nabla\varphi_I \cdot \nabla\zeta_D + \nabla\zeta_I \cdot \nabla\varphi_D - \zeta_D\left[\frac{\partial^2 \varphi_I}{\partial Z^2} - \nabla\zeta_I \cdot \nabla\left(\frac{\partial \varphi_I}{\partial Z}\right)\right] = 0,$$
$$Z = \zeta_I(X, Y, t). \tag{4.2}$$

Equations (4.1) and (4.2) assume that the component of the fully nonlinear free-surface elevation due to the body-induced radiation and diffraction disturbances is small and of $O(\varepsilon)$ relative to the ambient wave elevation which is assumed to be of $O(1)$. The same applies to the radiation/diffraction disturbance velocity potential φ_D which is also assumed small relative to the incident wave potential φ_I, or $\varphi_D \ll \varphi_I$. These assumptions justify the leading order linearization of the dynamic and kinematic free-surface conditions (4.1) and (4.2), respectively, around the ambient wave free-surface elevation. Equations (4.1) and (4.2) are hereafter referred to as the weak-scatterer free-surface conditions.

The disturbance potential satisfies the Laplace equation in the fluid domain and on the body wetted surface is subject to the condition

$$\frac{\partial \varphi_D}{\partial n} = U_n - \frac{\partial \varphi_I}{\partial n} \quad \text{on } S_B(t). \tag{4.3}$$

For a fixed body $U_n = 0$ and the disturbance potential reduces to the diffraction potential.

Vertical free-surface impulse

It follows from (3.11) that the vertical component of the force and free-surface impulse takes the form

$$F_{Z-B,5} = -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_I^E(t)} \varphi_D n_Z \,\mathrm{d}s - \rho g\int_{S_I^E(t)} \zeta_D \,\mathrm{d}s$$
$$-\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_I^E(t)}\left[\zeta_D\frac{\partial}{\partial Z}(\varphi_I + \varphi_D) + \frac{1}{2}\zeta_D^2\frac{\partial^2}{\partial Z^2}\nabla(\varphi_I + \varphi_D) + \cdots\right]\mathrm{d}s. \tag{4.4}$$

The unit normal vector on the ambient wave free surface is given by the expression

$$\boldsymbol{n} = \frac{\nabla(Z - \zeta_I(X, Y, t))}{|\nabla(Z - \zeta_I(X, Y, t))|} = \frac{-\boldsymbol{i}\zeta_{IX} - \boldsymbol{j}\zeta_{IY} + \boldsymbol{k}}{\sqrt{1 + \zeta_{IX}^2 + \zeta_{IY}^2}} = (-\boldsymbol{i}\zeta_{IX} - \boldsymbol{j}\zeta_{IY} + \boldsymbol{k})[1 + O(\delta^2)]. \tag{4.5}$$

Substituting its vertical component in (4.4) and with errors of $O(\delta^2, \varepsilon^2)$ we obtain

$$F_{Z-B,5} = -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_I^E(t)}\left(\varphi_D + \zeta_D\frac{\partial \varphi_I}{\partial Z}\right)\mathrm{d}s - \rho g\int_{S_I^E(t)} \zeta_D \,\mathrm{d}s. \tag{4.6}$$

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The integrals in (4.6) may be projected onto the $Z = 0$ plane using $\mathrm{d}s = \mathrm{d}X\,\mathrm{d}Y\,[1 + O(\delta^2)]$

$$\begin{aligned} F_{Z-B,5} &= -\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_I^E} \left(\varphi_D + \zeta_D \frac{\partial \varphi_I}{\partial Z} \right) (X, Y, \zeta_I, t)\, \mathrm{d}s - \rho g \int_{S_I^E} \zeta_D(X, Y, t)\, \mathrm{d}s \\ &= -\rho \int_{Z=\zeta_I} \left[\frac{\partial}{\partial t} + \frac{\partial \zeta_I}{\partial t} \frac{\partial}{\partial Z} \right] \left(\varphi_D + \zeta_D \frac{\partial \varphi_I}{\partial Z} \right) \mathrm{d}s - \rho g \int_{Z=\zeta_I} \zeta_D(X, Y, t)\, \mathrm{d}s \\ &= -\rho \int_{Z=\zeta_I} \left[\frac{\partial \varphi_D}{\partial t} + g\zeta_D + \frac{\partial \zeta_I}{\partial t} \frac{\partial \varphi_D}{\partial Z} + \zeta_D \frac{\partial^2 \varphi_I}{\partial Z \partial t} + \zeta_D \frac{\partial \zeta_I}{\partial t} \frac{\partial^2 \varphi_D}{\partial Z^2} \right] \mathrm{d}s \\ &= O(\varepsilon\delta). \end{aligned} \tag{4.7}$$

The terms in the last equation of (4.7) are seen to be of $O(\varepsilon\delta)$ by virtue of the second equation in (4.1) and the order of magnitudes of the body wave disturbance and the ambient wave gradients.

By comparison, the nonlinear buoyancy force (3.4) for a surface piercing body is of the order of the wave amplitude hence of $O(1)$ and the same applies for the vertical component of nonlinear FK force (3.6). The body impulse force (3.11) is of the order of the disturbance potential hence of $O(\varepsilon)$. Consequently the vertical component of the free-surface impulse being of order $\varepsilon\delta$ may be neglected. Therefore, the contribution of the vertical impulse and corresponding force may be neglected in most applications.

Horizontal free-surface impulse

Invoking (4.5) in (3.11) the free-surface impulse and force in the x-direction, assumed to coincide with the direction of propagation of the ambient wave, takes the form

$$\begin{aligned} F_{Z-B,5} &= \rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_I^E(t)} \left[\varphi_D \frac{\partial \zeta_I}{\partial X} - \zeta_D \frac{\partial}{\partial X} (\varphi_I + \varphi_D) \right] \mathrm{d}s \\ &\quad - \frac{1}{2} \rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_I^E(t)} \zeta_D^2 \frac{\partial}{\partial Z} \frac{\partial}{\partial X} (\varphi_I + \varphi_D)\, \mathrm{d}s + O(\delta^2) \\ &= \rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_I^E(t)} \left[\varphi_D \frac{\partial \zeta_I}{\partial X} - \zeta_D \frac{\partial}{\partial X} (\varphi_I + \varphi_D) \right] \mathrm{d}s + O(\delta^2, \delta\varepsilon^2). \end{aligned} \tag{4.8}$$

The first two terms in the right-hand side of (4.8) are seen to be of $O(\varepsilon\delta)$ and the last term of $O(\varepsilon^2)$. The nonlinear hydrostatic force in the horizontal direction is in the present formulation identically zero. The horizontal component of the nonlinear FK force (3.6) is of $O(\delta)$ and the force due to the body impulse given by (3.8) is of $O(\varepsilon)$. This order of magnitude analysis suggests that the contribution of the horizontal free-surface impulse (4.8) may be neglected to leading order. The accuracy of the new expressions of the forces derived above as the time derivatives of impulses over the body boundary is illustrated below for a truncated vertical circular cylinder. Applications are discussed in § 6.

5. Numerical results

Computations were carried out to illustrate the performance of the new expressions derived in the present article for the nonlinear wave forces acting on floating bodies. A truncated vertical circular cylinder was considered with diameter $d = 4$ m and draft $T = 10$ m and its wetted surface was discretized with the panel mesh shown in figure 2.

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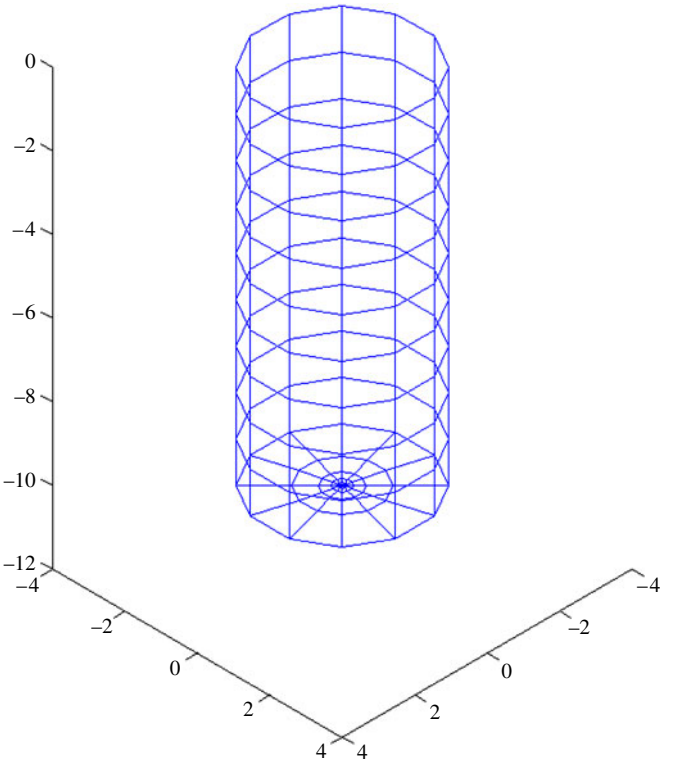


FIGURE 2. (Colour online) Panel mesh on a truncated circular cylinder with $N = 129$ panels.

The nonlinear potential flow diffraction problem was solved subject to the weak-scatterer free-surface conditions (4.1) and (4.2) using a panel method and Green's theorem. Monochromatic ambient waves of finite amplitude interact with the cylinder and the nonlinear heave and surge exciting forces are computed. In the results shown next the forces obtained from the direct application of Bernoulli's equation (2.5) are compared with the body impulse forces derived in the present article and given by (3.4), (3.6) and (3.8).

Figure 3 plots the nonlinear heave exciting force on the truncated circular cylinder in a monochromatic ambient wave with amplitude A and wavenumber k such that $kA = 0.25$, $kd = 0.4$ and $A/d = 0.625$. The heave force due to an ambient wave with steepness $kA = 0.55$, the same wavelength and $A/d = 1.375$ is illustrated in figure 4. Figure 5 plots the heave exciting force in a wave with steepness $kA = 0.55$, shorter wavelength such that $kd = 0.9$ and $A/d = 0.61$. In all cases the nonlinear body impulse force expressions derived in the present article provide a very good approximation of the force obtained via the direct application of Bernoulli's equation.

This finding is consistent with the order of magnitude analysis presented earlier in § 4. It is noted that the heave exciting force presented in figures 3–5 is the sum of the nonlinear hydrostatic, FK impulse and body impulse components. The non-sinusoidal form of the heave exciting force is in part due to the time-dependent wetted surface of the cylinder which gives rise to higher harmonics.

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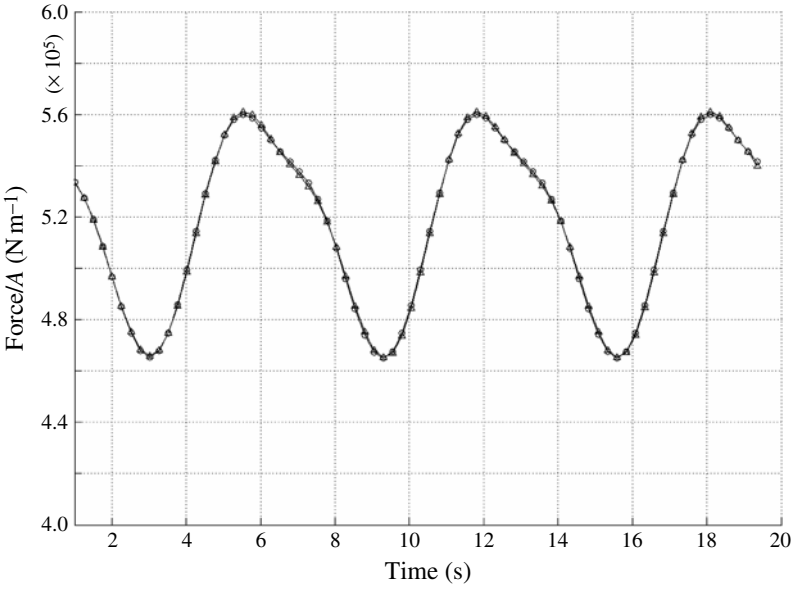


FIGURE 3. Nonlinear heave exciting force in an ambient wave with $kA = 0.25$ and $kd = 0.4$. Triangles: Bernoulli force; circles: body impulse force.

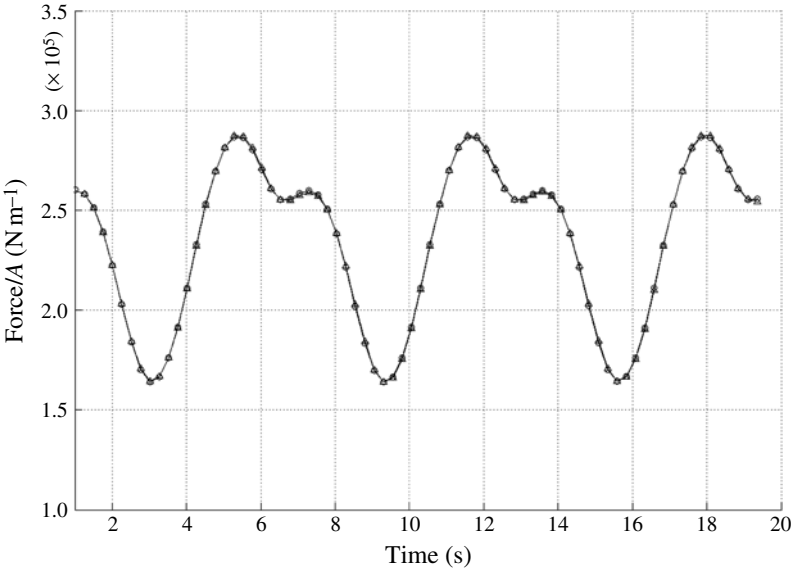


FIGURE 4. Nonlinear heave exciting force in an ambient wave with $kA = 0.55$ and $kd = 0.4$. Triangles: Bernoulli force; circles: body impulse force.

The corresponding nonlinear surge exciting force is plotted in figures 6–8 for the same wave amplitudes and wave lengths as for the heave exciting force. The performance of the nonlinear body impulse surge force expressions is also seen to be very satisfactory even in steep ambient waves but not as accurate as for the heave

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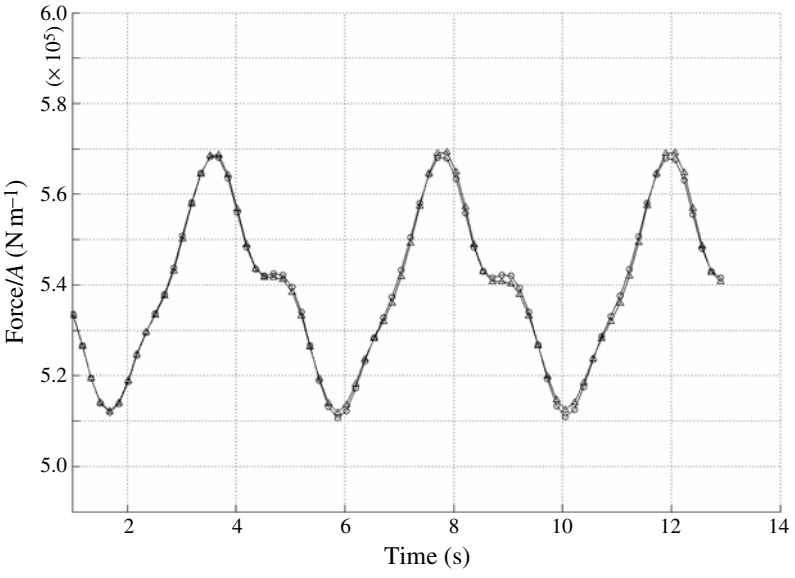


FIGURE 5. Nonlinear heave exciting force in an ambient wave with $kA = 0.55$ and $kd = 0.9$. Triangles: Bernoulli force; circles: body impulse force.

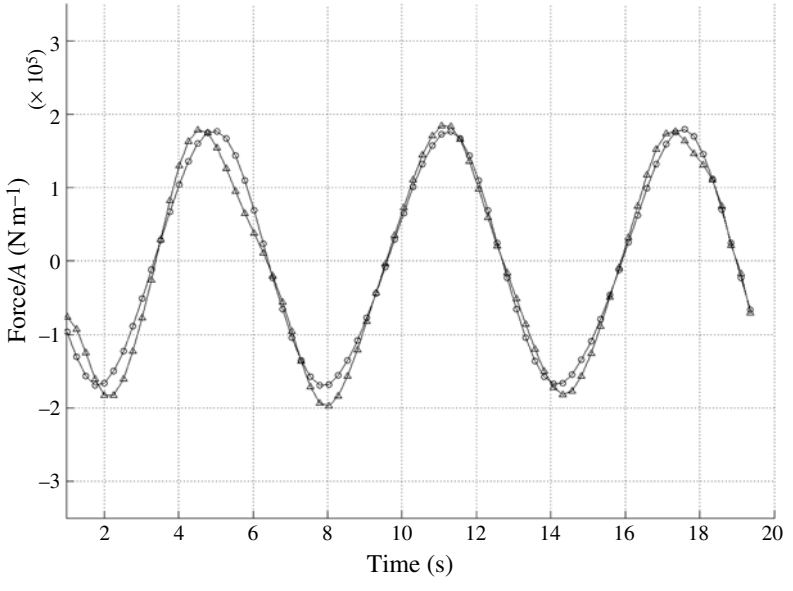


FIGURE 6. Nonlinear surge exciting force in an ambient wave with $kA = 0.25$ and $kd = 0.4$. Triangles: Bernoulli force; circles: body impulse force.

force. As pointed out in § 4 the reason is that the surge hydrostatic force in the new body impulse expressions vanishes identically and the FK and diffraction components are of the same order of magnitude. Therefore, unlike heave, there is no dominant force component.

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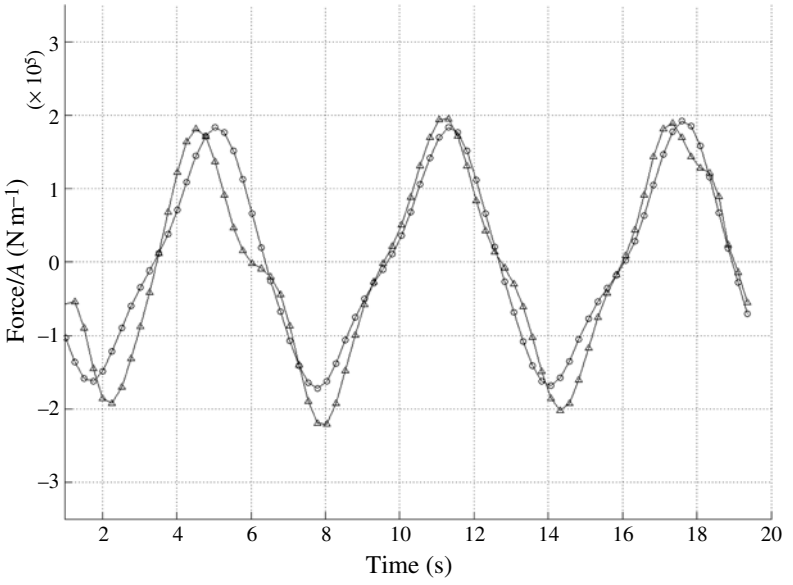


FIGURE 7. Nonlinear surge exciting force in an ambient wave with $kA = 0.55$ and $kd = 0.4$. Triangles: Bernoulli force; circles: body impulse force.

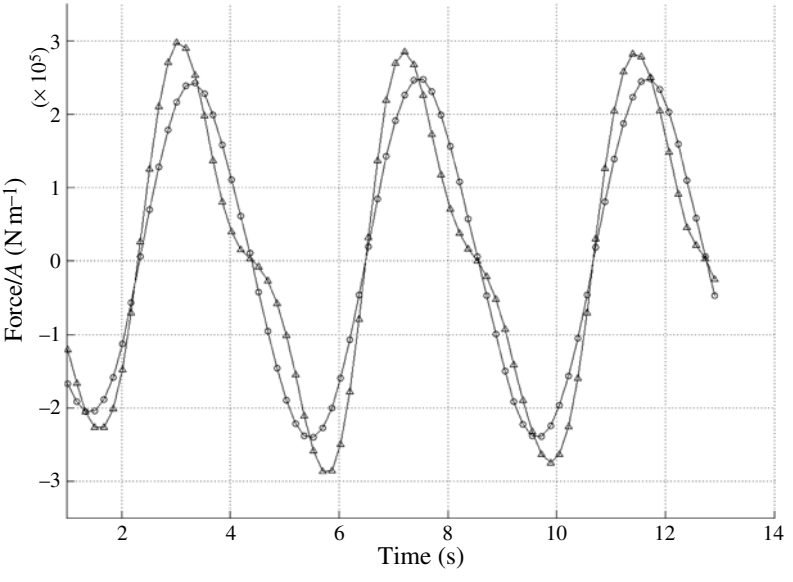


FIGURE 8. Nonlinear surge exciting force in an ambient wave with $kA = 0.55$ and $kd = 0.9$. Triangles: Bernoulli force; circles: body impulse force.

The computations presented above confirm the accuracy of the new nonlinear body impulse force expressions. By circumventing the evaluation of the partial time and spatial derivatives of the velocity potential in Bernoulli's equation the evaluation of the new force expressions has been found to be ten times faster relative to the evaluation

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of the nonlinear forces obtained from the direct application of Bernoulli's equation. The latter require small time steps and dense panel discretizations of the instantaneous body wetted surface necessary for the time-space resolution of the local potential flow around the vessel. This gains in efficiency is expected to be larger for flared bodies.

6. Discussion and applications

Expressions have been derived for the ocean wave impulse and the nonlinear loads on floating bodies which find application in a wide range of problems in naval hydrodynamics and ocean engineering. They are discussed next.

'Impulse dispersion relation' for nonlinear irregular ocean waves

Expression (2.13) states an integral dynamic equilibrium between the inertia and restoring effects of an irregular nonlinear wave train. Its derivation used both the nonlinear dynamic free-surface condition of zero pressure and the kinematic free-surface condition that the normal flow velocity equals the normal velocity of the free surface. It may therefore be regarded as a nonlinear impulse dispersion relation that constrains the irregular wave profiles that are admissible representations of ocean waves. Assuming a space and time evolution of a large-amplitude free-surface elevation, a potential flow problem follows in the fluid domain the solution of which leads to a velocity potential on the presumed free surface which must satisfy (2.13). Therefore, not all presumed ambient wave free-surface elevations are admissible under the nonlinear free-surface conditions. Equation (2.13) may be viewed as a generalized dispersion relation by virtue of the fact that it represents an integral balance between inertia effects, given by the time derivative of the impulse, and restoring effects, given by the integral of the free-surface elevation.

Nonlinear buoyancy and impulse Froude–Krylov force

A principal result of the present paper is the derivation of the coupled nonlinear buoyancy and FK force and moment in nonlinear irregular ambient waves given by expressions (3.4)–(3.8). It was shown that by introducing the ambient wave free-surface elevation as a dynamic waterplane area for surface piercing bodies, the nonlinear buoyancy force always points upwards. The direct use of Bernoulli's equation which is often used in applications keeps the hydrostatic effects strongly coupled with the dynamic effects. Moreover, the expressions of the FK component have the simple form of the time derivative of integrals of just the ambient wave velocity potential over the body wetted surface and the dynamic waterplane area. Gradients of the ambient wave velocity potential are absent and the force takes the form of the time derivative of an impulse which may also be expressed as an integral of the ambient wave velocity inside the body.

The computation of the FK force thus reduces to a simple task if the ambient wave velocity potential is known. Moreover, in a number of seakeeping problems for ships and other surface piercing offshore structures the vertical component of the FK force and moment are known to dominate the force components due to the radiation and diffraction components. Therefore the accurate computation of the nonlinear FK force is essential in wave–body interactions.

Nonlinear seakeeping of ships

The expressions derived in the present article for the wave loads on general floating bodies may be readily applied to ship-like geometries. Slender body approximations

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may be invoked to simplify three-dimensional integrals of the ambient wave and body velocity potentials over sections of the ship hull along its length. The section wetted surface remains time dependent and all dynamic forces follow as time derivatives of sectional impulses relative to the reference frame. By virtue of its simplicity the nonlinear buoyancy and FK force may be evaluated in their exact three-dimensional form in order to account as accurately as possible for three-dimensional effects near the ship ends.

6.1. *Horizontal drift forces*

The horizontal free surface impulse accounts for the entire horizontal drift force acting on a floating body. This follows from the observation that the FK and body disturbance forces have been expressed as time derivatives of the corresponding impulses which are stationary time-dependent quantities in irregular waves. The time average of the time derivative of any stationary signal of sufficiently long duration vanishes. This follows easily by an integration by parts. Therefore, the only remaining force component that may account for the horizontal drift force is the horizontal impulse given in its exact form by (3.11). If the ambient wave interaction starts from rest, the free-surface area over which the integrations in (3.11) are carried out grows in time and this must be accounted for properly when computing the mean value of the force signal. Otherwise it follows from the analysis of § 3 that the free-surface impulse and force are quadratic functions of the ambient wave and body disturbances and therefore their time average is non-zero.

Ringing loads on offshore structures and wind turbines

Offshore structures and wind turbines fixed on the ocean floor or supported by floaters and constrained by mooring lines or vertical tethers as in the case of tension leg platforms have resonant flexural frequencies of the order of a few seconds which may be excited by ambient waves. This subject has been studied extensively by the oil industry and 'ringing' events have been measured when a steep wave impinges upon the offshore structure. 'Ringing' may also occur in offshore wind turbines and is a topic of particular interest since the lowest natural frequency of the tower is around 2 rad s^{-1} and higher tower natural frequencies overlap with the natural frequencies of the blades. Excessive ringing may therefore lead to fatigue and structural failure (Oil Industry JIPs and Wind Industry, R. Jefferys private communication).

The flexural natural frequencies of offshore structures and wind turbines fall well to the right of the modal frequency of design seastate spectra which is $\sim$0.4 rad s^{-1}. Therefore the excitation of ringing events must arise primarily from nonlinearities in the wave loads. Such nonlinearities originate from the ambient wave kinematics, the nonlinear form of the buoyancy, FK and body impulse force. The quadratic wave force and moment due to the horizontal free-surface impulse, approximated in (4.8) for ambient waves of large amplitude and small steepness, contributes a nonlinear wave load which must be compared with the other dominant nonlinear loads with significant power spectral density at the high flexural frequencies of the structure. The power spectral density of these nonlinear force components needs to be determined in design seastates. The force (4.8) may be evaluated numerically or estimated analytically and its power spectral density at the structural natural frequencies compared with that of the other force components. It should be pointed out that the accuracy in the estimation of the force spectral density at high frequencies must be balanced with an equally accurate estimate of the damping of the structure at resonance which is often of viscous origin and hard to estimate.

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