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Hypercomplex manifolds and hyperholomorphic bundles

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Abstract

Using twistor techniques we shall show that there is a hypercomplex structure in the neighbourhood of the zero section of the tangent bundle TX of any complex manifold X with a real-analytic torsion-free connection compatible with the complex structure whose curvature is of type $(1,1)$. The zero section is totally geodesic and the Obata connection restricts to the given connection on the zero section.

We also prove an analogous result for vector bundles: any vector bundle with real-analytic connection whose curvature is of type $(1,1)$ over X can be extended to a hyperholomorphic bundle over a neighbourhood of the zero section of TX.

1. *Introduction*

Hypercomplex manifolds can be thought of as non-metric analogues of hyperkähler manifolds, i.e. they are differentiable manifolds with three complex structures I,J and K satisfying the relations $I^2 = J^2 = K^2 = IJK = -1$.

Hyperholomorphic vector bundles over hypercomplex manifolds are vector bundles with connection whose curvature is of type $(1,1)$ with respect to all three complex structures I, J and K.

A hypercomplex manifold possesses a unique torsion-free connection, the Obata connection [**8**], which preserves all three complex structures. The tangent bundle with the Obata connection is hyperholomorphic. In the hyperkähler case, the Obata connection is just the Levi–Civita connection of the metric.

In [**5**] we have shown that any real-analytic Kähler manifold can be extended to a hyperkähler manifold in its cotangent bundle. In this paper we shall use a similar construction to obtain the twistor space and show the existence of twistor lines to prove:

Theorem A. *A complex manifold X with a real-analytic torsion-free affine connection preserving the complex structure whose curvature is of type* $(1,1)$ *defines a natural hypercomplex structure on a neighbourhood of the zero section of the tangent bundle TX.*

We point out that X is a complex submanifold of the hypercomplex manifold in TX; however the complex structure is not the canonically induced complex structure on TX.

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In the case where X is a Kähler manifold we can use the metric to identify TX and T^*X; the hypercomplex structure (coming from X equipped with the Levi–Civita connection using Theorem A) and the hyperkähler structure mentioned above correspond under this identification. In this case the complex structure for which X is a complex submanifold corresponds to the canonically induced complex structure of the cotangent bundle.

Using the correspondence between hyperholomorphic bundles over a hypercomplex manifold and certain holomorphic bundles over its twistor space we shall prove:

THEOREM B. *Let E be a vector bundle with real-analytic connection whose curvature is of type $(1,1)$ over the complex manifold X. Then there exists a hyperholomorphic extension of E over a neighbourhood of the zero section of TX.*

Finally we shall show:

THEOREM C. *The zero section X is totally geodesic in the hypercomplex manifold given by Theorem* A *and the Obata connection restricts to the given affine connection on X.*

Recall that a submanifold X of a manifold with connection ∇ is totally geodesic if $\nabla_v w \in TX$ whenever $v, w \in TX$; hence ∇ induces a connection on a totally geodesic submanifold.

Since the Obata connection has curvature of type $(1,1)$ for all complex structures, in particular for the complex structure induced from the base, it is necessary for this result that the connection on X has curvature of type $(1,1)$.

2. *Hypercomplex extensions*

Any complex manifold has an underlying real-analytic structure; if $z_1, \dots, z_n$ are holomorphic coordinates, $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ are real-analytic coordinates. Now let X be a complex manifold with a real-analytic torsion-free affine connection ∇ which preserves the complex structure I. By a real-analytic connection we mean that its Christoffel symbols (or its connection form) are locally given by real-analytic coefficient functions in the real-analytic coordinates $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$.

Remark 1.

(i) A torsion-free connection compatible with the complex structure need not have curvature of type $(1,1)$. However, the $(2,0)$-part of the curvature must vanish on vector fields of type $(0,1)$ and, analogously, the $(0,2)$-part on vector fields of type $(1,0)$.

(ii) There are torsion-free connections compatible with the complex structure whose curvature is of type $(1,1)$ which are not the Levi–Civita connection of a (pseudo-) Kähler metric on X.

Any manifold with a real-analytic structure can be complexified. More precisely, the complexification X^c of X is locally given by holomorphic coordinates $z_1, \dots, z_n$, $\tilde{z}_1, \dots, \tilde{z}_n$ and we can extend the connection for $\tilde{z}_i$ near $\bar{z}_i$, $1 \leqslant i \leqslant n$, because the Christoffel symbols (or the connection form) are given by real-analytic functions in $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$. By abuse of notation we call this neighbourhood X^c again.

On X^c there is a real structure $\sigma\colon X^c \to X^c$ given locally by

$$\sigma(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n) = (\bar{\tilde{z}}_1, \dots, \bar{\tilde{z}}_n, \bar{z}_1, \dots, \bar{z}_n).$$

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We can then identify X with the set of real points, i. e. the fixed points of σ:

$$(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) \in X^c,$$

but under this identification X is of course not a complex submanifold of X^c.

The complexification X^c carries two natural foliations given locally by $z_i = \text{const}$ for $1 \leqslant i \leqslant n$ and $\tilde{z}_i = \text{const}$, respectively. We call these the + foliation (respectively − foliation). Since we assume that the curvature F of ∇ is of type $(1,1)$, i.e. $F = \sum_{ij} F_{ij} dz_i \wedge d\tilde{z}_j$, the connection restricts to a flat torsion-free affine connection along each leaf of either foliation.

The real structure interchanges the foliations, and $X \subset X^c$ intersects each leaf in exactly one point, namely in

$$(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) = (\bar{\tilde{z}}_1, \dots, \bar{\tilde{z}}_n, \tilde{z}_1, \dots, \tilde{z}_n).$$

We can therefore identify the local space of leaves B_+ as a complex manifold with X, and the space of leaves B_- with $\overline{X}$, i.e. X with the conjugate complex structure.

2·1. *Construction*

2·1·1. *The bundle of affine linear functions*

We consider one of the foliations, without loss of generality the + foliation, with space of leaves $B = B_+$.

As each leaf carries a *flat* torsion-free affine connection, we can consider for each $b \in B$ the vector space V_b of affine linear functions on the leaf Λ_b determined by b, i.e. of functions $f : \Lambda_b \to \mathbb{C}$ such that $\nabla(df) = 0$.

If we choose an open set $U \subset X^c$ such that $\Lambda_b \cap U$ is simply-connected for all $b \in B$, then $\dim V_b = n + 1$ because there is a basis of covariant constant 1-forms $\theta_1, \dots, \theta_n$ on each leaf (as the connection is flat) which are closed (as the connection is torsion-free), and therefore $\theta_i = df_i$, $1 \leqslant i \leqslant n$, and $f_1, \dots, f_n$ together with $f_0 \equiv 1$ form a basis of V_b. Furthermore as b varies, we obtain a holomorphic vector bundle $V \to B$ of rank $n + 1$.

We consider the dual bundle $V^* \to B$ and define the maps

(i) $\pi : V^* \to \mathbb{C}$ by evaluating $\alpha \in V_b^*$ at the affine linear function $1 \in V_b$, i.e. $\pi(\alpha) = \alpha(1)$.

(ii) $\phi : U \times \mathbb{C} \to V^*, (x, \lambda) \mapsto \lambda \delta_x$ where δ_x denotes the evaluation map at x of an affine linear function.

As $\pi \delta_x = \delta_x(1) = 1$,

$$\left.\begin{aligned} s_x : \mathbb{C} &\longrightarrow V^* \\ \lambda &\longmapsto \phi(x, \lambda) \end{aligned}\right\} \tag{2·1}$$

is a section of π for each $x \in U \subset X^c$.

Remark 2. We describe a certain standard coordinate system $q_1, \dots, q_n, p_1, \dots, p_n$ for X^c for either foliation which we shall use in the proof of Proposition 1.

Assume that the foliation is given by $q_i = \text{const}$ (i.e. $q_i = z_i$ for the + foliation and $q_i = \tilde{z}_i$ for the − foliation, respectively), and $1, p_1, \dots, p_n$ are a basis of affine linear functions constructed as follows:

Let $\theta_1, \dots, \theta_n$ be 1-forms on X^c which form a basis of covariant constant 1-forms on each leaf of the foliation. They can be obtained using parallel transport along the

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leaves after first choosing a transverse section and any basis of cotangent vectors along the section. As the connection is torsion-free, $d\theta_i = 0$ on each leaf. We then solve $dp_i = \theta_i$ on each leaf (as an ordinary differential equation with parameters) to obtain affine linear functions $p_1, \dots, p_n$.

Hence we have found coordinates $q_1, \dots, q_n, p_1, \dots, p_n$ such that the foliation is given by $q_i = \text{const}$ and $1, p_1, \dots, p_n$ are a basis of affine linear functions. In coordinates with respect to the dual basis the map ϕ is given by

$$\phi(q_1, \dots, q_n, p_1, \dots, p_n, \lambda) = (q_1, \dots, q_n, \lambda, \lambda p_1, \dots, \lambda p_n).$$

PROPOSITION 1. *ϕ is a holomorphic diffeomorphism of $U \times \mathbb{C}^*$ onto its image.*

Proof. Using the coordinates introduced in Remark 2 we deduce that ϕ is a local diffeomorphism by the inverse function theorem. As ϕ is also injective, ϕ is a diffeomorphism onto its image.

Remark 3. The exterior derivative maps V into the bundle W of covariant constant 1-forms along the leaves and its kernel is the space of constant functions, hence dualizing we have an exact sequence

$$0 \longrightarrow W^* \longrightarrow V^* \longrightarrow \mathbb{C} \longrightarrow 0,$$

i.e. the covariant constant vector fields along the leaves can be embedded into V^* and, in fact, into $\pi^{-1}(0)$. As the tangent bundle of the manifold can be identified with the space of covariant constant vector fields W^* using parallel transport we can identify the fibre $\pi^{-1}(0)$ over zero with the tangent bundle TX.

2·1·2. *The attaching map and explicit twistor lines*

Applying the above construction to both foliations we obtain V_+^* and V_-^* as well as maps ϕ_+ and ϕ_- which will be used for the gluing. In order to construct a twistor space which is Hausdorff the two halves have to be defined as the following two subspaces of V_+^* and V_-^* respectively. We choose $c \in]0, 1[$ and let

$$Z_+ = V_+^*|_{\pi_+^{-1}(\{\lambda \in \mathbb{C}; |\lambda| < c\})} \cup \phi_+(U \times \{\lambda \in \mathbb{C}; c \leqslant |\lambda| < 1/c\})$$

and

$$Z_- = V_-^*|_{\pi_-^{-1}(\{\tilde{\lambda} \in \mathbb{C}; |\tilde{\lambda}| < c\})} \cup \phi_-(U \times \{\tilde{\lambda} \in \mathbb{C}; c \leqslant |\tilde{\lambda}| < 1/c\}).$$

We shall show that

$$Z = Z_+ \coprod_{\sim} Z_-$$

where $\phi_+(x, \lambda) \sim \phi_-(x, \lambda^{-1})$ for $c < |\lambda| < 1/c$ is the desired twistor space.

The twistor projection $p\colon Z \to \mathbb{C}P^1$ is obtained by using π_+ and π_- on Z_+ and Z_-, respectively. Under the above identification the sections s_x^+ and s_x^- (which are defined by Formula (2·1)) patch together and give a family of explicit twistor lines $s_x\colon \mathbb{C}P^1 \to Z$ parametrised by $x \in U$. By the general theory of the twistor correspondence all other twistor lines will be deformations of these explicit lines.

PROPOSITION 2. *Z is a $(2n + 1)$-dimensional complex manifold, and the projection $p\colon Z \to \mathbb{C}P^1$ is holomorphic.*

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Proof. The only point is Hausdorffness which can be dealt with by proving that the graph of the identification map is closed in the same way as in the metric case, (see [**5**, proposition 3]).

Therefore we have constructed a $(2n+1)$-dimensional complex manifold Z together with a holomorphic projection $p\colon Z \to \mathbb{C}P^1$ and a $2n$-dimensional family of sections.

Remark 4. We note that locally the vertical directions in the tangent bundle are given by $\partial/\partial\tilde{z}_1, \ldots, \partial/\partial\tilde{z}_n$ (as the vertical directions are independent of the connection) which implies the complex structure on the vertical directions is $-I$ (see the discussion of the complexification in Section 2).

We note that there is a canonical real structure τ on Z covering the antipodal map on $\mathbb{C}P^1$ which is induced by the real structure σ on X^c by setting $\tau(z) = [\phi_+(\sigma(x), -\bar{\lambda}^{-1})]$ where $z \in Z|_{p^{-1}\{\lambda\in\mathbb{C}:c<|\lambda|<1/c\}}$ is represented by $\phi_+(x,\lambda)$ and $[\alpha]$ denotes the equivalence class of α under the identification $\phi_+(x,\lambda) \sim \phi_-(x,\lambda^{-1})$.

2·2. *Proof of Theorem* A

First we have to recall the inverse twistor theorem in the hypercomplex case.

Theorem D (for example see [**10**]).
Let Z be a complex manifold of dimension $2n+1$ such that
(i) *there is a holomorphic projection $p\colon Z \to \mathbb{C}P^1$ onto the projective line,*
(ii) *there is a family of holomorphic sections of p, each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes p^*\mathcal{O}(1)$,*
(iii) *Z has a real structure covering the antipodal map on $\mathbb{C}P^1$,*
then the parameter space of real sections is a $4n$-dimensional manifold with a natural hypercomplex structure for which Z is the twistor space.

We recall that, using Kodaira–Spencer deformation theory [**6**] each of the above sections is part of a $4n$-dimensional family, and the tangent space T_mM^c to the parameter space M^c at m is given by $T_mM^c = H^0(N)$ where N is the normal bundle of the twistor line.

In order to prove that the data constructed in Section 2·1·2 does indeed give the twistor space of a hypercomplex manifold we have to show that the explicit sections have the correct normal bundle. The hypercomplex manifold thus obtained can be identified with a neighbourhood of the zero section of the tangent bundle using Remark 3. We start with the following:

Lemma 1. *On each leaf $z = z_0 = \text{const}$ there exists a basis $f_0 \equiv 1, f_1, \ldots, f_n$ of affine linear functions such that $f_i = \tilde{z}_i +$ higher order terms.*

Proof. We define covariant constant 1-forms $\theta_1, \ldots, \theta_n$ on $z = z_0$ by the requirement $\theta_i(\tilde{z}_0) = d\tilde{z}_i$ (where $\tilde{z}_0$ is a point in the leaf) using parallel transport. As the connection is torsion-free, these 1-forms are closed, hence locally $\theta_i = df_i$. If

$$f_i = \tilde{z}_i + \sum_{j=1}^{n} a_{ji}(z_0)\tilde{z}_j + g_i(z_0, \tilde{z})$$

where g_i is of order at least 2 in $\tilde{z}_1, \ldots, \tilde{z}_n$ then $df_i(\tilde{z}_0) = \theta_i(\tilde{z}_0) = d\tilde{z}_i$ implies $a_{ji}(z_0) = 0$ and $(\partial g_i/\partial\tilde{z}_j)(z_0, \tilde{z}_0) = 0$ for all i, thus f_i are as claimed.

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The next step is to show:

PROPOSITION 3. *The degree of the normal bundle N of any of the twistor lines $s_x(\mathbb{C}P^1)$ is $2n$.*

Proof. We use the basis $f_0 \equiv 1, f_1, \dots, f_n$ of affine linear functions from Lemma 1. Using this lemma for the leaves of the $-$ foliation we obtain a basis $\tilde{f}_0 \equiv 1, \tilde{f}_1, \dots, \tilde{f}_n$ of affine linear functions on its leaves. We choose coordinates $z_1, \dots, z_n, x_0, \dots, x_n$ (respectively $\tilde{z}_1, \dots, \tilde{z}_n, \tilde{x}_0, \dots, \tilde{x}_n$) with respect to the dual bases. The maps ϕ_+ and ϕ_- are then given by

$$\phi_+(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n, \lambda) = (z_1, \dots, z_n, \lambda, \lambda f_1(z, \tilde{z}), \dots, \lambda f_n(z, \tilde{z}))$$

(where $z = (z_1, \dots, z_n)$ and $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_n)$) and

$$\phi_-(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n, \lambda^{-1}) = (\tilde{z}_1, \dots, \tilde{z}_n, \lambda^{-1}, \lambda^{-1}\tilde{f}_1(z, \tilde{z}), \dots, \lambda^{-1}\tilde{f}_n(z, \tilde{z})).$$

Now consider

$$dz_1 \wedge \cdots \wedge dz_n \wedge dx_1 \wedge \cdots \wedge dx_n \wedge dx_0 = \lambda^n \alpha(z, \tilde{z}) dz_1 \wedge \cdots \wedge dz_n \wedge d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_n \wedge d\lambda.$$

where $\alpha = \det(\partial f_i / \partial \tilde{z}_j)$ and

$$\begin{aligned} &d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_n \wedge d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n \wedge d\tilde{x}_0 \\ &\quad = (-1)^{n^2+1} \lambda^{-(2n+2)} (\lambda^n \tilde{\alpha}(z, \tilde{z}) dz_1 \wedge \cdots \wedge dz_n \wedge d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_n \wedge d\lambda). \end{aligned}$$

where $\tilde{\alpha} = \det(\partial \tilde{f}_i / \partial z_j)$. The above equations define a $(2n+1)$-form on each half of the twistor space which are related by a twist. Thus $dz_1 \wedge \cdots \wedge dz_n \wedge dx_1 \wedge \cdots \wedge dx_n \wedge dx_0$ and $d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_n \wedge d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_n \wedge d\tilde{x}_0$ patch together to a non-vanishing section of $K_Z \otimes p^*\mathcal{O}(2n+2) \otimes L$ where $L \to Z$ is a line bundle given by the transition function $\tilde{\alpha}/\alpha$. As $\alpha(z_0, \tilde{z}_0) = \tilde{\alpha}(z_0, \tilde{z}_0) = 1$ by Lemma 1, L is trivial along the twistor line $s_x(\mathbb{C}P^1)$, $x = (z_0, \tilde{z}_0)$. Hence $\deg K_z|_{s_x(\mathbb{C}P^1)} = -(2n+2)$, thus $\deg N = 2n$.

Remark 5. In the hyperkähler case the line bundle L is trivial for the following reason: in the above proof we showed that $L^* = K_Z \otimes p^*\mathcal{O}(2n+2)$, but using the exact sequence $0 \to T_F \to TZ \to T\mathbb{C}P^1 \to 0$ yields $L^* = \Lambda^{2n} T_F^* \otimes p^*\mathcal{O}(2n)$. Under the twistor correspondence the hyperkähler metric gives rise to a twisted symplectic form along the fibres $\omega \in \Gamma(\Lambda^2 T_F^* \otimes p^*\mathcal{O}(2))$, hence ω^n trivializes L^*.

As rank $N = 2n$, by the Birkhoff–Grothendieck theorem $N = \bigoplus_{i=1}^{2n} \mathcal{O}(m_i)$ where $m_i \in \mathbb{Z}$ and $\sum_{i=1}^{2n} m_i = 2n$. Now the following lemma clearly implies that N is isomorphic to $\mathbb{C}^{2n} \otimes p^*\mathcal{O}(1)$.

LEMMA 2. *$m_i > 0$ for all $i \in \{1, \dots, 2n\}$.*

Proof. Using Proposition 1 and the definition of Z_+ and Z_- in Section 2·1·2 we can identify a neighbourhood U of X in X^c with any fibre $p^{-1}(\lambda) \subset Z$ for $c \leqslant |\lambda| \leqslant 1/c$. We thus obtain two foliations in $p^{-1}(1)$ using the two foliations of X^c and $\phi_+(\cdot, 1)$ and $\phi_-(\cdot, 1)$, respectively.

We shall show that we can find sections $s_1, \dots, s_n, t_1, \dots, t_n$ of N such that $s_1(0) = \cdots = s_n(0) = 0$, $t_1(\infty) = \cdots = t_n(\infty) = 0$, $s_1(1), \dots, s_n(1)$ are linearly

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independent and tangential to the + foliation and $t_1(1), \dots, t_n(1)$ are linearly independent and tangential to the $-$foliation. Then $s_1(1), \dots, s_n(1), t_1(1), \dots, t_n(1)$ form a basis for $N_{s_x(1)}$.

Using Remark 2 there exist coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ of X^c such that the + foliation is given by $q_i = \text{const}$ and

$$\phi_+(p_1, \dots, p_n, q_1, \dots, q_n, \lambda) = (q_1, \dots, q_n, \lambda, \lambda p_1, \dots, \lambda p_n).$$

We consider the 1-parameter family of sections $s_{t,j} : \mathbb{C}P^1 \to Z$ given by

$$s_{t,j}(\lambda) = (q_1, \dots, q_n, \lambda, \lambda p_1, \dots, \lambda(p_j + t), \dots, \lambda p_n).$$

Then $s_j(\lambda) = (d/dt)|_{t=0} s_{t,j}(\lambda)$, $1 \leqslant j \leqslant n$ are the desired sections of the normal bundle. Similarly, we construct $t_1, \dots, t_n$.

Now suppose $m_i < 0$ for some i, then, as $\mathcal{O}(m_i)$ does not then have sections, $s_1(1), \dots, s_n(1)$, $t_1(1), \dots, t_n(1)$ lie in a space of at most $2n-1$ dimensions contradicting linear independence.

Suppose $m_i = 0$ for some i. Let $s_j^i, t_j^i : \mathbb{C}P^1 \to \mathcal{O}(m_i) = \mathcal{O}$ be the corresponding components of s_j, t_j. Then s_j^i and t_j^i are constant and, in fact, $s_j^i = t_j^i = 0$ since $s_j(0) = t_j(\infty) = 0$. Therefore $s_1(1), \dots, s_n(1), t_1(1), \dots, t_n(1)$ again lie in a subspace of dimension at most $2n-1$, giving a contradiction.

2·3. *The S^1-action on the twistor space*

We develop a correspondence between the complexified manifold X^c and certain twistor lines which will be needed in the proof of Proposition 5 and in Section 4.

There is an S^1-action on the hypercomplex manifold $M \subset TX$ given by scalar multiplication in the fibres.

Lemma 3. *The S^1-action on M lifts to a holomorphic action on the twistor space Z such that the twistor projection p is equivariant where S^1 acts on $\mathbb{C}P^1$ keeping 0 and ∞ fixed.*

Proof. We work locally, without loss of generality on Z_+, and choose a coordinate system $z_1, \dots, z_n, x_0, \dots, x_n$ as in the proof of Proposition 3. Then the projection is given by x_0 and $x_1, \dots, x_n$ are coordinates in the fibre of the tangent bundle. Hence the S^1-action is given in these coordinates by $u \cdot z_j = z_j$ for $1 \leqslant j \leqslant n$, and $u \cdot x_j = u x_j$ for $1 \leqslant j \leqslant n$ where ux denotes multiplication by u. It can be extended by letting $u \cdot x_0 = u x_0$, and it is obvious that the action is holomorphic on Z and that the projection p is equivariant.

We note that the fixed point set in Z has two components, namely $X \subset p^{-1}(0)$ and $\overline{X} \subset p^{-1}(\infty)$. The action also extends to a $\mathbb{C}^*$-action for $\lambda \in \mathbb{C}^*$ with $|\lambda|$ near 1.

As $X \subset M$ is the fixed point set for the S^1-action, the complexified manifold X^c corresponds to the parameter space of all $\mathbb{C}^*$-invariant twistor lines, hence all twistor line s such that $s(0) \in X$ and $s(\infty) \in \overline{X}$; for if such a twistor line were not $\mathbb{C}^*$-invariant the action would give rise to a 1-parameter family of twistor lines keeping $s(0)$ and $s(\infty)$ fixed, hence a section of the normal bundle $N \cong \mathbb{C}^{2n}(1)$ vanishing at 0 and ∞ which would be identically zero.

In this picture the leaf Λ_z^+ corresponds to all twistor lines s such that $s(0) = z \in X$ and $s(\infty) \in \overline{X}$, and similarly $\Lambda_{\tilde{z}}^-$ can be identified with the space of all twistor lines s such that $s(0) \in X$ and $s(\infty) = \tilde{z} \in \overline{X}$. Tangent vectors to Λ_z^+ are of the form

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$s \otimes s_0$ under the decomposition $H^0(N) = H^0(N(-1)) \otimes H^0(\mathcal{O}(1))$ where $s_0 \in H^0(\mathcal{O}(1))$ such that $s_0(0) = 0$. Similarly, tangent vectors to $\Lambda_{\tilde{z}}^-$ are of the form $t \otimes s_\infty$ where $s_\infty \in H^0(\mathcal{O}(1))$ such that $s_\infty(\infty) = 0$. Notice that s_0 and s_∞ form a basis of $H^0(\mathcal{O}(1))$.

3. *Hyperholomorphic extensions*

It is well known that hyperholomorphic bundles over hyperkähler manifolds correspond to holomorphic bundles over its twistor space which are trivial along each real twistor line (see, for example, [**7**, **12**] for the four-dimensional situation; [**2**] for the quaternionic-Kähler case; also [**11**]).

Examining the proof of this correspondence we notice that the same statement is true if the base manifold is hypercomplex instead of hyperkähler.

We briefly recall how to construct the hyperholomorphic bundle E and its connection from such a bundle $W \to Z$ where Z is the twistor space of a hypercomplex manifold M. For $m \in M$ denote by $s_m(\mathbb{C}P^1)$ the twistor line parameterized by m and let

$$E_m = H^0(s_m(\mathbb{C}P^1), W).$$

For $z \in Z$ denote by S_z the submanifold of all twistor lines through z. Then there is a natural trivialization of E on S_z given by evaluation at the point z, i.e.

$$\begin{aligned} E|_{S_z} &\longrightarrow S_z \times W_z \\ (m, s) &\longmapsto (m, s(p(z))) \end{aligned}$$

where $p\colon Z \to \mathbb{C}P^1$ is the twistor projection. Hence we obtain a flat connection on $E|_{S_z}$ for each $z \in Z$ by choosing the standard flat connection on the product.

Locally, a connection on a vector bundle is given by a matrix of 1-forms, thus by a matrix of elements of $H^0(\mathbb{C}P^{4n-1}, \mathcal{O}(1))$ since $T_m^* M^c = H^0(\mathbb{P}(T_m M^c), \mathcal{O}(1))$ and $\dim M^c = 4n$. We can already define covariant differentiation on the cone C of vectors at m tangent to some S_z through m, hence using the tensor product decomposition of the tangent bundle of a hypercomplex manifold $T_m M^c = H^0(N) = H^0(N(-1)) \otimes H^0(\mathcal{O}(1))$, we can differentiate along the decomposable tangent vectors $v \otimes \psi$ with respect to this decomposition.

Consider the map

$$\begin{aligned} f\colon \mathbb{C}P^{2n-1} \times \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^{4n-1} \\ (v, \psi) &\longmapsto v \otimes \psi\,. \end{aligned}$$

The map f is linear in v for fixed ψ and linear in ψ for fixed v, thus of bidegree $(1,1)$, and $\mathcal{O}(1)$ pulls back to $\mathcal{O}(1,1)$. We consider the induced map

$$H^0(\mathbb{C}P^{4n-1}, \mathcal{O}(1)) \longrightarrow H^0(\mathbb{C}P^{2n-1} \times \mathbb{C}P^1, \mathcal{O}(1,1)).$$

It is injective because $\mathbb{C}^{2n} \otimes \mathbb{C}^2$ is spanned by the decomposable elements, hence $v \otimes \psi$ do not lie in any linear subspace. By the Künneth formula,

$$\begin{aligned} \dim H^0(\mathbb{C}P^{2n-1} \times \mathbb{C}P^1, \mathcal{O}(1,1)) &= \dim H^0(\mathbb{C}P^{2n-1}, \mathcal{O}(1)) \cdot \dim H^0(\mathbb{C}P^1, \mathcal{O}(1)) \\ &= 4n, \end{aligned}$$

and as $\dim H^0(\mathbb{C}P^{4n-1}, \mathcal{O}(1)) = 4n$ the induced map is an isomorphism, hence there is a unique connection on E which extends the given flat connection on $E|_{S_z}$.

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3·1. *Construction*

3·1·1. *A holomorphic bundle over each half of the twistor space*

We start with a vector bundle E of rank k over a complex manifold X with a real-analytic connection whose curvature is of type $(1,1)$.

That means, locally in a holomorphic coordinate system $\underline{z} = (z_1, \dots, z_n)$ of X and in a local trivialization

$$E|_U \longrightarrow U \times \mathbb{C}^k$$
$$(\underline{z}, v) \longmapsto (\underline{z}, \phi(\underline{z})v)$$

where U are open sets in X and $\phi\colon U \to \mathbb{C}^{k\times k}$ is holomorphic the connection form a of E is locally given by

$$a = \sum_{i=1}^{n} a_i(\underline{z}, \underline{\bar{z}})dz_i + \sum_{i=1}^{n} b_i(\underline{z}, \underline{\bar{z}})d\bar{z}_i$$

where $a_1, \dots, a_n, b_1, \dots, b_n$ are real-analytic. Using the following lemma we can assume that the connection form is of type $(1,0)$.

Lemma 4. *There exists a real-analytic gauge in which the connection form a is of type* $(1,0)$.

Proof. The proof is essentially the same as the proof given by Donaldson and Kronheimer of the integrability theorem for holomorphic structures (see [**3**, pp. 50–53]).

If we let

$$\alpha = \sum_{i=1}^{n} \alpha_i(z)d\bar{z}_i$$

be the $(0,1)$ part of the connection form, we have to find locally a real-analytic gauge transformation g such that

$$g\alpha g^{-1} - (\bar{\partial}g)g^{-1} = 0.$$

The gauge transformation can be found inductively by solving successively the one-dimensional equations

$$\frac{\partial g}{\partial \bar{z}_j} = g\alpha_j$$

for $j = 1, \dots, n$ where we regard $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$ as parameters. Since α_j is real-analytic, so is g by elliptic regularity. As α_j depends real-analytically on the parameters, so does g which gives the desired gauge transformation.

We complexify to $E^c \xrightarrow{q} X^c$ where X^c is the complexified complex manifold of our twistor construction introduced in Section 2: locally X^c has coordinates $(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n) = (\underline{z}, \underline{\tilde{z}})$.

The complexified bundle can be given in terms of the local trivialization

$$E^c|_{U^c} \longrightarrow U^c \times \mathbb{C}^k$$
$$((\underline{z}, \underline{\tilde{z}}), v) \longmapsto ((\underline{z}, \underline{\tilde{z}}), \phi(\underline{z})v),$$

where $U^c = \{(\underline{z}, \underline{\tilde{z}}) \in X^c; \underline{z} \in U\}$.

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The functions $a_1, \ldots, a_n$ can be extended to holomorphic functions in $\underline{z}, \underline{\tilde{z}}$ for $\underline{\tilde{z}}$ near $\underline{\bar{z}}$. This gives rise to a connection form a^c on E^c over an open subset of X^c which we – by abuse of notation – again call X^c.

We recall that X^c is equipped with two foliations, namely $z_i = \text{const}$ and $\tilde{z}_i = \text{const}$ respectively (see Section 2). The connection on E^c is flat along each leaf of either of the two foliations as E has curvature of type $(1,1)$ and so the $(0,2)$ and $(2,0)$ components vanish.

For any $b \in B_+ = X$ we consider a simply-connected subspace Λ_b^+ of the leaf of the + foliation parameterized by b. We define $(W_+)_b$ to be the space of covariant constant sections of E^c on Λ_b^+, i.e.

$$(W_+)_b = \{\sigma_+ \in \Gamma(\Lambda_b^+, E^c); \nabla\sigma_+ = 0\}.$$

As the connection is flat along each leaf and Λ_b^+ is simply-connected, $\dim(W_+)_b = k$ for all $b \in B_+$, and we obtain a holomorphic vector bundle $W_+ \to B_+$. Similarly, we construct a holomorphic vector bundle $W_- \to B_-$ by setting

$$(W_-)_b = \{\sigma_- \in \Gamma(\Lambda_b^-, E^c); \nabla\sigma_- = 0\},$$

where Λ_b^- is a simply-connected subspace of the leaf of the $-$ foliation parametrized by $b \in B_- = \overline{X}$.

Each half of our twistor space is part of the total space of V_+^* and V_-^*, respectively where $q_+ \colon V_+^* \to B_+$ (respectively $q_- \colon V_-^* \to B_-$) is the dual bundle to the bundle of affine-linear functions on each leaf of the + foliation (respectively $-$ foliation) (cf. Section 2·1·1). By pulling back W_+ and W_- to V_+^* and V_-^* respectively we obtain holomorphic vector bundles on each half of the twistor space.

3·1·2. *The gluing procedure*

We identify points in $q_+^* W_+^*$ and $q_-^* W_-^*$ using parallel transport.

More precisely, if $x = (z, \tilde{z}) \in X^c$ is then $\Lambda_z^+ \bigcap \Lambda_{\tilde{z}}^- = \{x\}$, and

$$q_+\phi_+(x,\lambda) = z,\ q_-\phi_-(x,\tilde{\lambda}) = \tilde{z}$$

for all $\lambda, \tilde{\lambda} \in \mathbb{C}$.

We identify $\alpha_+ \in (q_+^* W_+)_{\phi_+(x,\lambda)}$ and $\alpha_- \in (q_-^* W_-)_{\phi_-(x,\lambda^{-1})}$ if and only if $\sigma_+(x) = \sigma_-(x) \in E_x^c$ where $\sigma_+ = q_{+*}\alpha_+ \in (W_+)_z$ and $\sigma_- = q_{-*}\alpha_- \in (W_-)_{\tilde{z}}$.

This procedure respects the gluing of V_+^* and V_-^* in the construction of the twistor space (see Section 2·1·2) and thus yields a holomorphic vector bundle $W \to Z$ over the twistor space Z of rank k.

3·2. *Proof of Theorem* B

In order to prove that the bundle $W \to Z$ indeed encodes a hyperholomorphic bundle E' over the hypercomplex manifold in the tangent bundle, we have to show:

PROPOSITION 4. *The vector bundle $W \to Z$ is trivial along each real twistor line.*

Proof. We show that W is trivial along each of the explicit twistor lines s_x, $x \in X^c$, defined in Section 2·1·2. The inverse twistor construction for hypercomplex manifolds uses Kodaira's relative deformation theory and the complexified hypercomplex manifold can be identified with the parameter space of all twistor lines which are deformations of the given set of explicit lines $\mathbb{P}_x = s_x(\mathbb{C}P^1)$. If $W|_{\mathbb{P}_x}$ is trivial for each

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explicit line, then $W|_{\mathbb{P}_y}$ is also trivial for each nearby line. This follows from the semi-continuity of $|m_i - m_j|$ where m_i are the coefficients in the Birkhoff–Grothendieck decomposition $W|_{\mathbb{P}_x} = \sum_{i=1}^{2n} \mathcal{O}(m_i)$ (see, for example, [**1**, chapter 4]).

Now let $x = (z, \tilde{z}) \in X^c$ and pick a basis $v_1, \dots, v_k$ of E^c_x. We define $\sigma^i_+ \in (W_+)_z$ and $\sigma^i_- \in (W_-)_{\tilde{z}}$ for $i = 1, \dots, k$, by $\sigma^i_+(x) = \sigma^i_-(x) = v_i$ and parallel transport on the leaves Λ^+_z and $\Lambda^-_{\tilde{z}}$ respectively. Then the maps

$$\begin{aligned} U_0 = \mathbb{C}P^1 \setminus \{0\} &\longrightarrow (W_+)_z \times \mathbb{C} \quad \cong (q_+^* W_+)_{s^+_x(\mathbb{C})} \\ \lambda &\longmapsto \sigma^i_+ \times \lambda \end{aligned}$$

and

$$\begin{aligned} U_1 = \mathbb{C}P^1 \setminus \{\infty\} &\longrightarrow (W_-)_{\tilde{z}} \times \mathbb{C} \cong (q_-^* W_-)_{s^-_x(\mathbb{C})} \\ \tilde{\lambda} &\longmapsto \sigma^i_- \times \tilde{\lambda} \end{aligned}$$

give a basis of sections of $W|_{s_x(\mathbb{C}P^1)} \to \mathbb{C}P^1$, hence $W|_{s_x(\mathbb{C}P^1)}$ is trivial.

The bundle $W \to Z$ gives rise to a hyperholomorphic bundle $q' \colon E' \to M^c$ over the complexified hypercomplex manifold M^c where $E'_m = H^0(s_m(\mathbb{C}P^1), W)$ for any $m \in M^c$ where s_m is the twistor line parametrized by m.

To complete the proof of Theorem B we have to show:

Proposition 5. *The hyperholomorphic bundle E' restricted to X is isomorphic to the original bundle $E \to X$ as bundle with connection.*

The proposition follows from two lemmas of which the first shows the natural equivalence as vector bundles and the second the equivalence of the connections in the complexified situation.

We recall from Section 2·3 that the complexified complex manifold X^c corresponds to all twistor lines s such that $s(0) \in X$ and $s(\infty) \in \overline{X}$.

Lemma 5. *There is a natural isomorphism of $E' \to X^c$ and $E^c \to X^c$ as vector bundles.*

Proof. Let $(x, s) \in E'$, i.e. $s \in H^0(s_x(\mathbb{C}P^1), W)$ and $x = (z, \tilde{z}) \in X^c$. The maps

$$\begin{aligned} E'_x = H^0(s_x(\mathbb{C}P^1), W) &\longrightarrow (W_+)_z = \{\sigma_+ \in \Gamma(\Lambda^+_z, E^c); \nabla(\sigma_+) = 0\} \\ s &\longmapsto s(0) \end{aligned}$$

and

$$\begin{aligned} (W_+)_z = \{\sigma_+ \in \Gamma(\Lambda^+_z, E^c); \nabla(\sigma_+) = 0\} &\longrightarrow E^c_x \\ \sigma_+ &\longmapsto \sigma_+(x) \end{aligned}$$

are canonical isomorphisms (the inverses are obtained using parallel transport), hence the bundles are naturally isomorphic.

Lemma 6. *The connections on $E' \to X^c$ and $E^c \to X^c$ correspond under the isomorphism of Lemma 5.*

Proof. Using the twistor correspondence there is a unique connection on $E' \to M^c$ such that for any $z \in Z$ the fibre of W over z can be identified as

$$W_z = \{\sigma \in \Gamma(S_z, E'); \nabla(\sigma) = 0\},$$

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where $S_z = \{m \in M^c; z \in s_m(\mathbb{C}P^1)\}$ is the set of all twistor lines passing through the point $z \in Z$. Now let $z \in X \subset p^{-1}(0)$. Then (see Section 2·3)

$$\Lambda_z^+ = \{x \in X^c; s_x(0) = z, s_x(\infty) \in \overline{X}\} \subset S_z$$

and by our construction

$$W_z = (W_+)_z = \{\sigma_+ \in \Gamma(\Lambda_z^+, E^c); \nabla(\sigma_+) = 0\}.$$

Taking Lemma 5 into account, we conclude that E' and E^c have the same covariant constant sections on Λ_z^+ and hence the connection forms are the same on Λ_z^+. Similarly, taking $\tilde{z} \in \overline{X} \subset p^{-1}(\infty)$ and using $W_{\tilde{z}} = (W_-)_{\tilde{z}}$, the connections are equivalent on $\Lambda_{\tilde{z}}^-$.

Tangent vectors to Λ_z^+ are of the form $v \otimes s_0$ and tangent vectors to $\Lambda_{\tilde{z}}^-$ of the form $v \otimes s_\infty$ (under the decomposition $H^0(N) = H^0(N(-1)) \otimes H^0(\mathbb{C}P^1, \mathcal{O}(1))$ (cf. Section 2·3), hence the connections are the same for all directions of the form $v \otimes \psi$ where $\psi = \alpha s_0 + \beta s_\infty$.

As a tensor product is spanned by its decomposable elements $v \otimes \psi$, a connection is uniquely determined by its values on vectors $v \otimes \psi$, hence the connections must correspond under the isomorphism of E' and E.

We recall from Section 2·1·2 that the real structure on the twistor space Z is induced by the real structure on X^c. The construction of $W \to Z$ respects the real structure (as we can identify the relevant covariant constant sections using parallel transport), and thus the hyperholomorphic bundle $E' \to M^c$ (and also its restriction $E' \to X^c$) is compatible with the real structure. Therefore, $E \to X$ and $E' \to X$ are isomorphic as vector bundles with connection which concludes the proof of Theorem B.

4. *Proof of Theorem* C

As for the complexified hypercomplex manifold M^c the tangent space at any $m \in M^c$ can be decomposed into $T_m M^c = H^0(N(-1)) \otimes H^0(\mathcal{O}(1))$ (since $N(-1) = N \otimes p^*\mathcal{O}(-1)$ is trivial), we obtain a tensor product decomposition of the tangent bundle as $TM^c = E \otimes H$ where H is a trivial rank 2 bundle over M^c with fibre $H^0(\mathcal{O}(1))$. As for each twistor line, i.e. for each section of p, the tangent bundle along the fibres T_F restricted to the line equals the normal bundle N, the bundle E is hyperholomorphic as it corresponds to the bundle $T_F(-1) = T_F \otimes p^*\mathcal{O}(-1)$ which is equal to $N(-1)$ and thus trivial along each twistor line.

Denote by ∇^E the hyperholomorphic connection on E and by ∇^H the standard flat connection on H. Then we define a connection on the complexified tangent bundle by $\nabla = \nabla^E \otimes \nabla^H$.

It is obvious from the construction that ∇ preserves the complex structures of M. In order to prove that it is in fact the Obata connection of the hypercomplex manifold M we have to show it is torsion-free.

PROPOSITION 6. *The affine connection $\nabla = \nabla^E \otimes \nabla^H$ is torsion-free.*

Proof. The proof is modelled on the proof of the corresponding theorem in the quaternionic case given in [**9**].

The torsion τ is an element of $\Lambda^2 T^* \otimes T$. Splitting $\Lambda^2 T^*$ into $\Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$

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with respect to the complex structure I, we shall show that τ vanishes on each of the summands.

In order to calculate the $(2,0)$-component of τ we choose two $(1,0)$ vector fields v, w on M. Then $v = s \otimes s_0$ and $w = t \otimes s_0$ where $s_0 \in H^0(\mathcal{O}(1))$ as in the proof of Lemma 6 and s, t are sections of E.

As the connection preserves the complex structure I, $[v, w]$ is again a vector field of type $(1,0)$, and locally at $m \in M$

$$[v, w](m) = v(t)(m) \otimes s_0 - w(s)(m) \otimes s_0,$$

where $v(t)$ and $w(s)$ are the directional derivatives calculated in the trivialization

$$E|S_{s_m(0)} \longrightarrow S_{s_m(0)} \times (T_F(-1))_{s_m(0)},$$

where s_m is the twistor line parameterized by m and S_z is the parameter space of all lines through a given $z \in Z$.

On the other hand,

$$\begin{aligned}(\nabla_v w - \nabla_w v)(m) &= (\nabla^E_v t)(m) \otimes s_0 - (\nabla^E_w s)(m) \otimes s_0 \\ &= v(t)(m) \otimes s_0 - w(s)(m) \otimes s_0\end{aligned}$$

as ∇^E is defined using the above trivializations and the trivial flat connection on the product. Thus $\tau(m)(v, w) = 0$. Similarly, the $(0,2)$-component of τ vanishes.

Finally, let $v = s \otimes s_0$ be a $(1,0)$ vector field and $w = t \otimes s_\infty$ a $(0,1)$ vector field where $s_\infty \in H^0(\mathcal{O}(1))$ as in the proof of Lemma 6 and s, t are sections of E. Obviously, $[v, w] = 0$. Now

$$(\nabla_v w - \nabla_w v)(m) = (\nabla^E_v t)(m) \otimes s_\infty - (\nabla^E_w s)(m) \otimes s_0$$

hence at $0 \in \mathbb{C}P^1$ (using the trivialization of $E|_{S_{s_m(0)}}$)

$$(\nabla_v w - \nabla_w v)(m)(0) = (\nabla^E_v t)(m)(0) \otimes s_\infty(0) = v(t)(m)(0) = 0$$

since $v(0) = s(0) \otimes s_0(0) = 0$. Similarly, at $\infty \in \mathbb{C}P^1$, using the trivialization of $E|_{S_{s_m(\infty)}}$ we derive $(\nabla_v w - \nabla_w v)(m)(\infty) = 0$.

Hence $\tau(m)(v, w)(0) = \tau(m)(v, w)(\infty) = 0$, but $\tau(m)(v, w)$ is a section of $N \cong \mathbb{C}^{2n}(1)$ hence $\tau(m)$ vanishes identically.

We recall that S^1 acts on M and this action lifts to Z (see Section 2·3). We now consider the induced action on TM.

Proposition 7. *The induced S^1-action on TM preserves the Obata connection.*

Proof. Using Lemma 3 we note that S^1 acts on the tangent bundle along the fibres T_F (as the projection p is equivariant) and hence on $T_F(-1)$. It thus acts on E (which is defined at the beginning of this section) and preserves the connection ∇^E which is defined using trivializations $E|_{S_z} \to S_z \times T_F(-1)_z$ (see Section 3).

Now S^1 also acts on H: we take a basis of sections S_0 and S_∞ of H defined by $S_0(m) = s_0$ and $S_\infty(m) = s_\infty$ for all $m \in M^c$ (where s_0 and s_∞ are defined in Section 2·3) and S^1 acts by multiplying S_0 by u and S_∞ by u^{-1}. Therefore this action also preserves the trivial flat connection ∇^H.

As the Obata connection is given by $\nabla^E \otimes \nabla^H$ (see Proposition 6), the action preserves the Obata connection.

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As we can identify $TX \subset TM|_X$ with the fixed point set of the induced S^1-action, Proposition 7 implies that $\nabla_v w$ is a section of TX whenever v and w are, hence the zero section is totally geodesic and the Obata connection restricts to a connection on TX.

Now we are ready to prove Theorem C.

Proof. Using the general theory of the twistor correspondence for hyperholomorphic bundles as outlined in Section 3 we can identify the covariant constant sections of E on S_z, the parameter space of all twistor lines through $z \in Z$, with $(T_F(-1))_z$. The bundle H is trivial and we can think of it as spanned by S_0 and S_∞ (which are defined in the proof of Proposition 7). We also recall that for $z \in X$, $\Lambda^+_z \subset S_z$ and for $\tilde{z} \in \overline{X}$, $\Lambda^-_{\tilde{z}} \subset S_{\tilde{z}}$.

Next we construct a natural map from the space of covariant constant vector fields along Λ^+_z to $E \otimes S_0 \subset E \otimes H \cong TM^c$.

First we choose a basis of covariant constant vector fields $X_1, \dots, X_n$ on Λ^+_z and denote by $\theta_1, \dots, \theta_n$ the dual basis of covariant constant 1-forms. Next we choose a basis $f_0 \equiv 1, f_1, \dots, f_n$ of affine linear functions on Λ^+_z such that $df_j = \theta_j$ for $1 \leqslant j \leqslant n$. Now we can choose a coordinate system $z_1, \dots, z_n, x_0, \dots, x_n$ of V^*_+ with respect to the dual basis. In these coordinates any twistor line s parameterized by a point in Λ^+_z is given by $s(\lambda) = (z_1, \dots, z_n, \lambda, \lambda f_1(z, \tilde{z}), \dots, \lambda f_n(z, \tilde{z}))$ (see the proof of Proposition 3), hence the tangent direction to $s(\mathbb{C}P^1)$ is given by $\partial/\partial x_0 + \sum_{j=1}^n f_j \partial/\partial x_j$. Therefore, the forms $dx_i - f_i dx_0$ lie in the co-normal bundle, and $dz_1, \dots, dz_n, dx_1 - f_1 dx_0, \dots, dx_n - f_n dx_0$ form a local basis for the co-normal bundle of the twistor line parameterized by $(z, \tilde{z}) \in \Lambda^+_z \subset X^c$. We note that $dx_i - f_i dx_0 = x_0 df_i = x_0 \theta_i$.

We denote by $\partial/\partial z_1, \dots, \partial/\partial z_n, Y_1, \dots, Y_n$ the dual basis of sections of N. Then the above relation yields $x_0 Y_i = X_i$, hence each X_i gives rise to a section s_i of the normal bundle of any twistor line in Λ^+_z which vanishes at $x_0 = 0$, i.e. s_i is of the form $s_i = \sigma_i \otimes s_0$. Thus we have produced an element σ_i of $E_{(z,\tilde{z})}$. The map determined by $X_i \mapsto \sigma_i$ gives the desired identification.

We recall that $T_F(-1)$ is trivial on each twistor line and thus on S_z we can identify $E|_{S_z} = H^0(T_F(-1))|_{S_z}$ with $(T_F(-1))_z$ using evaluation at z, hence any covariant constant vector field along Λ^+_z with respect to the original connection is also covariant constant with respect to the Obata connection.

Similarly, any covariant constant section of TX^c on $\Lambda^-_{\tilde{z}}$ with respect to the original connection is also covariant constant with respect to the Obata connection. Therefore we conclude as in Lemma 6 that the Obata connection restricts to the given connection on TX.

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