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Exponential Additive Runge-Kutta Methods for Semi-Linear Differential Equations

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Abstract. Exponential additive Runge-Kutta methods for solving semi-linear equations are discussed. Related order conditions and stability properties for both explicit and implicit schemes are developed, according to the dimension of the coefficients in the linear terms. Several examples illustrate our theoretical results.

AMS subject classifications: 65L20

Key words: Semi-linear differential equations, exponential Runge-Kutta methods, order conditions, stability.

1. Introduction

Recently, various semi-linear problems have emerged in many scientific fields such as physics, chemistry, biology, ecology and economics [5, 10, 11, 20]. Semi-linear differential equations are the most basic and important nonlinear evolution equations arising from the spatial discretisation of nonlinear time dependent partial differential equations (PDE) — cf. Refs. [8, 10]. There are already many results on the global existence of solutions for related initial boundary value and Cauchy problems — e.g. see Refs. [11, 24]. The numerical solution of initial value problems with systems of differential equations often but not always invoke discretisation in both space and time, and the exact solution may be approximated iteratively. In this article, we develop a new class of numerical methods for time integration of the following semi-linear problem in the time-dependent variable $u(t)$, an *a priori* partial linearisation of a nonlinear problem where the prime denotes the time derivative, L is a linear differential operator in the space variable, and $f(t,u(t))$ is a known function:

$$u'(t) = Lu(t) + f\big(t,u(t)\big)\,, \quad t \geq t_0\,, \tag{1.1}$$

subject to the initial condition $u(t_0) = u_0$.

Numerical methods for solving differential equations began to develop rapidly over fifty years ago, but more recently the construction of exponential integrators has provided

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exponential Runge-Kutta methods [12,13,18], exponential Rosenbrock methods [4,10,14], exponential general linear methods [19, 22], exponential linear multistep methods [21], etc.. Exponential integration methods deal more efficiently with the linearity in semi-linear equations, and as their name suggests use the exponential function (and related functions) of the Jacobian or an approximation to it inside the related classical numerical method. In passing, we note that Ref. [15] handled the linear portions "exactly" by using integrating factors in formulating Krylov IIF-WENO methods for advection-diffusion-reaction systems, similar to exponential methods.

In many applications, it is convenient to use splitting methods to take advantage of the special structure of the differential operator in the mathematical model. The space discretisation of some time-dependent PDE produces ordinary differential equations that contain additive terms with different stiffness properties. In these situations, the additive Runge-Kutta methods have been used — e.g. see Refs. [1,2,6,17]. In molecular dynamics applications, the different structures may correspond to forces of different stiffness [2]. It is sometimes inappropriate to sample the net force, and one may wish to sample the stiffer parts more frequently than the non-stiff parts. For instance, in a reaction-diffusion partial differential problem, an implicit method for the diffusion may be combined with an explicit method for the reaction component [3, 23]. A general approach is to construct implicit-explicit (IMEX) methods, where the stiff term is handled implicitly, and the non-stiff term is handled explicitly, thereby gaining the benefit of the implicit method for the stiffness but potentially saving computational effort by handling some terms explicitly. IMEX Runge-Kutta methods are also known as additive Runge-Kutta methods [6].

In Section 2, we establish exponential additive Runge-Kutta methods for semi-linear equations. The analysis of the order conditions for these methods is provided in Section 3, and stability properties are discussed in Section 4. We define exponentially algebraic stability and *EB*-stability; and then focus on exponentially additive Runge-Kutta methods dealing with equations with one-dimensional linear term coefficients, before considering the asymptotic stability of explicit exponential additive Runge-Kutta methods for equations with higher dimensional linear term coefficients. Section 5 is devoted to several examples that satisfy the order conditions, with numerical experiments to illustrate our theory. Some concluding remarks are made in Section 6.

2. Exponential Additive Runge-Kutta Methods

We consider the decomposition

$$f(t,u(t)) = \sum_{\nu=1}^{N} f_\nu\big(t,u(t)\big) \tag{2.1}$$

in Eq. (1.1), and exponential methods corresponding to non-additive counterparts. Let u_n denote the numerical approximation to the exact solution $u(t_0+nh)$ after n step sizes h. We obtain the general form of exponential additive Runge-Kutta methods on invoking a

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variation-of-constants formula — i.e.

$$U_n^i = \mathrm{e}^{c_i hL} u_n + h \sum_{\nu=1}^{N} \sum_{j=1}^{m} a_{ij}^{[\nu]}(hL) f_\nu(t_n + c_j h, U_n^j)\,, \quad i = 1, \cdots, m\,, \tag{2.2}$$

$$u_{n+1} = \mathrm{e}^{hL} u_n + h \sum_{\nu=1}^{N} \sum_{i=1}^{m} b_i^{[\nu]}(hL) f_\nu(t_n + c_i h, U_n^i)\,, \tag{2.3}$$

where $h > 0$ denotes a positive time step and u_{n+1} is the numerical approximation to the exact solution at time $t_{n+1} = t_n + h$. The method is built on m internal stages U_n^i that approximate the solution at $t_n^i = t_n + c_i h$, where the real numbers c_i are called nodes of the method. To simplify the order conditions, we always assume the row conditions $c_i = \sum_{j=1}^{m} a_{ij}^{[\nu]}$ for $\nu = 1, \cdots, N$. So, it is possible to find some reasonable high-order schemes. Due to the coupling between different $a_{ij}^{[\nu]}(hL)$, the order conditions and the stability analysis for the exponential additive Runge-Kutta methods are much more complicated than for the standard exponential Runge-Kutta methods.

As usual in exponential integrators [14], the weights $b_i^{[\nu]}(z)$, $a_{ij}^{[\nu]}(z)$ are linear combinations of the entire functions

$$\varphi_k(z) = \int_0^1 \mathrm{e}^{(1-\sigma)z} \frac{\sigma^{k-1}}{(k-1)!}\, \mathrm{d}\sigma\,, \quad k \geq 1\,, \tag{2.4}$$

and the respective related functions $\varphi_k(c_i z)$. We note for later use that $\varphi_0(z) = \mathrm{e}^z$, and the recurrence relation is

$$\varphi_{k+1}(z) = \frac{\varphi_k(z) - \dfrac{1}{k!}}{z}\,, \qquad \varphi_k(0) = \frac{1}{k!}\,, \qquad k \geq 0\,.$$

We consider a special case of exponential additive Runge-Kutta methods, which is of collocation type. Adopting Eq. (2.1) and writing $u(t) = \mathrm{e}^{tL} C(t)$ in the ordinary differential equation (1.1), we obtain

$$L\mathrm{e}^{tL} C(t) + \mathrm{e}^{tL} C'(t) = L\mathrm{e}^{tL} C(t) + \sum_{\nu=1}^{N} f_\nu\big(t, u(t)\big),$$

so that

$$C'(t) = \mathrm{e}^{-tL} \sum_{\nu=1}^{N} f_\nu\big(t, u(t)\big)\,. \tag{2.5}$$

Then integrating from t_n to t_{n+1} produces

$$\mathrm{e}^{-t_{n+1}L} u(t_{n+1}) - \mathrm{e}^{-t_n L} u(t_n) = \int_{t_n}^{t_{n+1}} \mathrm{e}^{-tL} \sum_{\nu=1}^{N} f_\nu\big(t, u(t)\big)\, \mathrm{d}t\,,$$

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and on setting $t = t_n + hs$ (and noting that $t_{n+1} = t_n + h$ corresponds to $s = 1$) we obtain

$$u(t_{n+1}) - \mathrm{e}^{hL}u(t_n) = h\int_0^1 \mathrm{e}^{hL(1-s)}\sum_{\nu=1}^{N} f_\nu\big(t_n + hs, u(t_n + hs)\big)\,\mathrm{d}s\,. \tag{2.6}$$

If we now replace each function $f_\nu(t_n + hs, u(t_n + hs))$ by a Lagrange interpolation formula

$$\sum_{i=1}^{m} l_i(s) f_\nu\big(t_n + c_i h, u(t_n + c_i h)\big)\,,$$

then Eq. (2.6) becomes

$$u(t_{n+1}) = \mathrm{e}^{hL}u(t_n) + h\sum_{\nu=1}^{N}\sum_{i=1}^{m}\left(\int_0^1 \mathrm{e}^{hL(1-s)} l_i(s)\mathrm{d}s\right) f_\nu\big(t_n + c_i h, u(t_n + c_i h)\big)\,.$$

Thus the methods (2.3) and (2.2) evolve into

$$u_{n+1} = \mathrm{e}^{hL}u_n + h\sum_{\nu=1}^{N}\sum_{i=1}^{m} b_i(hL) f_\nu\left(t_n^i, U_n^i\right), \tag{2.7}$$

$$b_i(hL) = \int_0^1 \mathrm{e}^{hL(1-s)} l_i(s)\,\mathrm{d}s\,, \tag{2.8}$$

and

$$U_n^i = \mathrm{e}^{c_i hL}u_n + h\sum_{\nu=1}^{N}\sum_{j=1}^{m} a_{ij}(hL) f_\nu\left(t_n^j, U_n^j\right), \quad i = 1, \cdots, m\,, \tag{2.9}$$

$$a_{ij}(hL) = \int_0^{c_i} \mathrm{e}^{hL(c_i - s)} l_j(s)\,\mathrm{d}s\,. \tag{2.10}$$

The resulting scheme (2.7) and (2.9) is called an exponential additive Runge-Kutta method of collocation type. For $N = 1$ this procedure reduces to the corresponding exponential Runge-Kutta methods, and for $L = 0$ it reduces to the additive Runge-Kutta method. (If $N = 1$ and $L = 0$ both hold, we have the corresponding classical Runge-Kutta method.) Especially for the stages $m = 1$ and $m = 2$, we are now in the position to give the following examples of such collocation exponential Runge-Kutta methods that are useful components in our additive schemes.

Example 2.1. With a specific Runge-Kutta method, the Butcher array for the Radau IIA method of one order is

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}\,.$$

According to Eqs. (2.8) and (2.10), we readily obtain $b_1(hL) = a_{11}(hL) = \varphi_1(hL)$, and substituting these values into the specific positions gives the Butcher array for the exponential Radau IIA method of order one:

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$$\begin{array}{c|c} 1 & \varphi_1(hL) \\ \hline & \varphi_1(hL) \end{array}.$$

Example 2.2. For the Lobatto IIIA method of order two, we have

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

From Eqs. (2.8) and (2.10), we readily obtain $a_{21}(hL) = b_1(hL) = \varphi_1(hL) - \varphi_2(hL)$ and $a_{22}(hL) = b_2(hL) = \varphi_2(hL)$, so the Butcher array for the exponential Lobatto IIIA method of order two correspondingly is

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \varphi_1(hL)-\varphi_2(hL) & \varphi_2(hL) \\ \hline & \varphi_1(hL)-\varphi_2(hL) & \varphi_2(hL) \end{array}.$$

3. Order Conditions of the Exponential Additive Runge-Kutta Methods

We follow Ref. [13] and choose an abstract Banach space framework of sectorial operators and locally Lipschitz continuous non-linearities, to include interesting cases such as reaction-diffusion equations. Our analysis below is based on the abstract formulation of Eq. (1.1) as an evolution equation in a Banach space $(X, \|\cdot\|)$. Let $\mathscr{D}(L)$ denote the domain of L in X.

Assumption 3.1 (cf. Ref. [13])**.** Let $L : \mathscr{D}(L) \to X$ be sectorial — i.e. L is a densely defined and closed linear operator on X satisfying the resolvent condition

$$\left\|(zI - L)^{-1}\right\|_{X \leftarrow X} \leq \frac{M}{|z-a|}$$

on the sector $\{z \in \mathbb{C} : \vartheta \leq |arg(z-a)| \leq \pi, z \neq a\}$ for $M \geq 1$, $a \in \mathbb{R}$, and $0 < \vartheta < \pi/2$.

Under this assumption, the operator L is the infinitesimal generator of an analytic semigroup $\{e^{tL}\}_{t\geq 0}$. We choose $0 \leq \eta < 1$ and define $V = \mathscr{D}(\tilde{L}^\eta) \subset X$, where $\tilde{L}$ denotes the shifted operator $\tilde{L} = L + \omega I$ with $\omega > -a$. The linear space V is a Banach space with norm $\|v\|_V = \|\tilde{L}^\eta v\|$. It is notable that V does not depend on ω, since different choices of ω lead to equivalent norms. Our main hypothesis on the nonlinearity $f_\nu(t, u(t))$ is as follows.

Assumption 3.2. Let $f_\nu(t, u(t)) : [0, T] \times V \to X$, $\nu = 1, \cdots, N$ be locally Lipschitz-continuous in a strip along the exact solution u. Then there exist real numbers $\gamma_\nu(R, T)$ such that

$$\|f_\nu(t, v) - f_\nu(t, w)\| \leq \gamma_\nu \|v - w\|_V ,$$

for all $t \in [0, T]$ and $max\,(\|v\|_V, \|w\|_V) \leq R$.

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Assumption 3.3. The continuous vector-valued functions $f_\nu(t,u(t)) : \mathbb{R}_0^+ \times \mathbb{R}^D \to \mathbb{R}^D$ with components $f_{\nu(i)}, i = 1,\cdots,D$ satisfy the one-sided Lipschitz condition with respect to u on the interval $[t_0,+\infty)$ — i.e. there exist constants $\sigma_\nu \le 0$, such that

$$\big\langle u_1-u_2, f_\nu(t,u_1)-f_\nu(t,u_2)\big\rangle \le \sigma_\nu \|u_1-u_2\|^2 \ \forall\, u_1,u_2 \in \mathbb{R}^D, \quad \nu = 1,\cdots,N\,. \tag{3.1}$$

(The relevant Euclidean inner product is defined as $\langle a,b\rangle = a^T b$, where a,b are vectors in the Euclidean space.)

The following stability bounds are crucial in our analysis.

Lemma 3.1 (cf. Ref. [13]). *Under Assumption* 3.1 *and fixed* $\omega > -a$, *the following bounds hold uniformly on* $0 \le t \le T$*:*

$$\left\|\mathrm{e}^{tL}\right\|_{X\leftarrow X} + \left\|t^\gamma \widetilde{L}^\gamma \mathrm{e}^{tL}\right\|_{X\leftarrow X} \le C\,, \quad \left\| hL\sum_{j=1}^{n-1} \mathrm{e}^{jhL}\right\|_{X\leftarrow X} \le C\,, \quad \gamma \ge 0\,. \tag{3.2}$$

The stability estimate (3.2) enables us to define the bounded operators (2.4) to derive the order conditions of exponential additive Runge-Kutta methods. We integrate Eq. (2.5) from t_n to $t_n+\theta h$ to obtain

$$\mathrm{e}^{-(t_n+\theta h)L}u(t_n+\theta h) - \mathrm{e}^{-t_nL}u(t_n) = \int_{t_n}^{t_n+\theta h} \mathrm{e}^{-tL}\sum_{\nu=1}^{N} f_\nu\big(t,u(t)\big)\,\mathrm{d}t\,,$$

and the change of variable $t = t_n+\tau$ gives

$$u(t_n+\theta h) = \mathrm{e}^{\theta hL}u(t_n) + \int_0^{\theta h} \mathrm{e}^{(\theta h-\tau)L}\sum_{\nu=1}^{N} f_\nu\big(t_n+\tau,u(t_n+\tau)\big)\,\mathrm{d}\tau\,. \tag{3.3}$$

We now expand the integrand into a Taylor series of order q at the point $(t_n,u(t_n))$ with the remainder in integral form to get

$$\begin{aligned} &f_\nu\big(t_n+\tau,u(t_n+\tau)\big)\\ &=\sum_{j=1}^{q}\frac{\tau^{j-1}}{(j-1)!}f_\nu^{(j-1)}\big(t_n,u(t_n)\big) + \int_0^\tau \frac{(\tau-\sigma)^{q-1}}{(q-1)!}f_\nu^{(q)}\big(t_n+\sigma,u(t_n+\sigma)\big)\,\mathrm{d}\sigma\,, \end{aligned}$$

so with the functions defined in Eq. (2.4) from Eq. (3.3) we have

$$\begin{aligned} u(t_n+\theta h) =& \mathrm{e}^{\theta hL}u(t_n) + \sum_{j=1}^{q}\sum_{\nu=1}^{N}(\theta h)^j\varphi_j(\theta Lh)f_\nu^{(j-1)}\big(t_n,u(t_n)\big)\\ &+\sum_{\nu=1}^{N}\int_0^{\theta h}\mathrm{e}^{(\theta h-\tau)L}\int_0^\tau \frac{(\tau-\sigma)^{q-1}}{(q-1)!}f_\nu^{(q)}\big(t_n+\sigma,u(t_n+\sigma)\big)\mathrm{d}\sigma\,\mathrm{d}\tau\,. \end{aligned} \tag{3.4}$$

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On the other hand, according to the expression (2.2) defining the internal stages of the exponential additive Runge-Kutta methods, on expanding the functions $f_\nu(t_n + c_j h, u(t_n + c_j h))$ into a Taylor series of order q about the point $(t_n, u(t_n))$ with remainder in integral form we have

$$\begin{aligned} &f_\nu\big(t_n + c_j h, u(t_n + c_j h)\big) \\ &= \sum_{k=1}^{q} \frac{(c_j h)^{k-1}}{(k-1)!} f_\nu^{(k-1)}\big(t_n, u(t_n)\big) + \int_0^{c_j h} \frac{(c_j h - \sigma)^{q-1}}{(q-1)!} f_\nu^{(q)}\big(t_n + \sigma, u(t_n + \sigma)\big)\,\mathrm{d}\sigma\,. \end{aligned}$$

After substituting back into (2.2), we therefore obtain

$$\begin{aligned} U_n^i =& \mathrm{e}^{c_i h L} u_n + h \sum_{\nu=1}^{N} \sum_{k=1}^{q} \sum_{j=1}^{m} a_{ij}^{[\nu]}(hL) \frac{(c_j h)^{k-1}}{(k-1)!} f_\nu^{(k-1)}\big(t_n, u(t_n)\big) \\ &+ h \sum_{\nu=1}^{N} \sum_{j=1}^{m} a_{ij}^{[\nu]}(hL) \int_0^{c_j h} \frac{(c_j h - \sigma)^{q-1}}{(q-1)!} f_\nu^{(q)}\big(t_n + \sigma, u(t_n + \sigma)\big)\,\mathrm{d}\sigma\,. \end{aligned} \tag{3.5}$$

Taking $\theta = c_i$, $j = k$ in Eq. (3.4) and comparing the corresponding terms in h with those in Eq. (3.5), we have

$$\sum_{j=1}^{m} a_{ij}^{[\nu]}(hL) \frac{c_j^{k-1}}{(k-1)!} = c_i^k \varphi_k(c_i h L)\,. \tag{3.6}$$

Similarly, according to expression (2.3) that is the last stage of the exponential additive Runge-Kutta methods, on expanding the functions $f_\nu(t_n + c_i h, u(t_n + c_i h))$ into a Taylor series of order q about the point $(t_n, u(t_n))$ with the remainder in integral form we have

$$\begin{aligned} &f_\nu\big(t_n + c_i h, u(t_n + c_i h)\big) \\ &= \sum_{k=1}^{q} \frac{(c_i h)^{k-1}}{(k-1)!} f_\nu^{(k-1)}\big(t_n, u(t_n)\big) + \int_0^{c_i h} \frac{(c_i h - \sigma)^{q-1}}{(q-1)!} f_\nu^{(q)}\big(t_n + \sigma, u(t_n + \sigma)\big)\,\mathrm{d}\sigma\,. \end{aligned}$$

After substituting back into Eq. (2.3), we therefore obtain

$$\begin{aligned} u_{n+1} =& \mathrm{e}^{hL} u_n + h \sum_{\nu=1}^{N} \sum_{k=1}^{q} \sum_{i=1}^{m} b_i^{[\nu]}(hL) \frac{(c_i h)^{k-1}}{(k-1)!} f_\nu^{(k-1)}\big(t_n, u(t_n)\big) \\ &+ h \sum_{\nu=1}^{N} \sum_{i=1}^{m} b_i^{[\nu]}(hL) \int_0^{c_i h} \frac{(c_i h - \sigma)^{q-1}}{(q-1)!} f_\nu^{(q)}\big(t_n + \sigma, u(t_n + \sigma)\big)\,\mathrm{d}\sigma\,, \end{aligned} \tag{3.7}$$

so taking $\theta = 1$, $j = k$ in Eq. (3.4) and comparing the corresponding terms in h with those in Eq. (3.7) we have

$$\sum_{i=1}^{m} b_i^{[\nu]}(hL) \frac{c_i^{k-1}}{(k-1)!} = \varphi_k(hL)\,. \tag{3.8}$$

Eqs. (3.6) and (3.8) are thus the relevant order conditions for the methods.

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Table 1: Stiff order conditions for the explicit exponential additive Runge-Kutta methods.

No.	Order	Order condition
1	1	$\psi_1^{[\nu]}(hL)=0$
2	2	$\psi_2^{[\nu]}(hL)=0$
3	2	$\psi_{1,i}^{[\nu]}(hL)=0$
4	3	$\psi_3^{[\nu]}(hL)=0$
5	3	$\sum_{i=1}^{m} b_i^{[\nu]}(hL)J\psi_{2,i}^{[\nu]}(hL)=0$
6	4	$\psi_4^{[\nu]}(hL)=0$
7	4	$\sum_{i=1}^{m} b_i^{[\nu]}(hL)J\psi_{3,i}^{[\nu]}(hL)=0$
8	4	$\sum_{i=1}^{m} b_i^{[\nu]}(hL)J\sum_{j=2}^{i-1} a_{ij}^{[\nu]}(hL)J\psi_{2,j}^{[\nu]}(hL)=0$
9	4	$\sum_{i=1}^{m} b_i^{[\nu]}(hL)c_iK\psi_{2,i}^{[\nu]}(hL)=0$

Remark 3.1. Example 2.1 and Example 2.2 both satisfy the order conditions (3.6) and (3.8).

Similar to Table 2 in Ref. [12], we combine Eqs. (3.6) and (3.8) to list order conditions for the explicit exponential additive Runge-Kutta methods in Table 1, where J and K denote arbitrary bounded operators on X.

We consider the functions

$$\psi_{k,i}^{[\nu]}(hL)=\varphi_k(c_ihL)c_i^k-\sum_{j=1}^{i-1}a_{ij}^{[\nu]}(hL)\frac{c_j^{k-1}}{(k-1)!},$$

$$\psi_k^{[\nu]}(hL)=\varphi_k(hL)-\sum_{i=1}^{m}b_i^{[\nu]}(hL)\frac{c_i^{k-1}}{(k-1)!}.$$

Under Assumptions 3.1 and 3.2, our convergence result is a straightforward extension of a result in Ref. [13], which we therefore state without proof.

Theorem 3.1. *Let the initial value problem defined by Eqs.* (1.1) *and* (2.1) *satisfy Assumptions* 3.1 *and* 3.2*, and consider an m-stage exponential additive Runge-Kutta method of collocation-type given by Eqs.* (2.2) *and* (2.3)*. Then for sufficiently small h there exists a unique numerical solution* u_n $(0\le nh\le T)$*, where the error satisfies*

$$\|u_n-u(t_n)\|_V\le C\cdot h^m\cdot\sup_{\substack{0\le\tau\le T\\ \nu=1,\cdots,N}}\left\|f_\nu^{(m)}(\tau,u(\tau))\right\|_X$$

provided that the right-hand side is bounded. The constant C depends on T, but is independent of n and h.

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4. Stability Analysis of the Exponential Additive Runge-Kutta Methods

Let L in Eq. (1.1) be a matrix that satisfies $\mu(L) \leq 0$, and consider the logarithmic matrix norm. Originally defined for matrices, the logarithmic norm can be extended to bounded linear operators, but extensions to nonlinear maps and unbounded operators have required a functional and analytic redefinition of the concept. For an integer D, let us introduce a norm $\|\cdot\|$ on $\mathbb{R}^D$ which induces a norm on $\mathbb{R}^{D\times D}$ denoted by the same symbol. Moreover, let $\mu(L)$ denote the logarithmic norm of L relevant to the given norm on $\mathbb{R}^D$. We recall the following two properties of the logarithmic norm, which will be widely used later:

(i) $\|e^M\| \leq e^{\mu(M)}$ for $M \in \mathbb{R}^{D\times D}$,

(ii) $\mu(\alpha M) = \alpha\mu(M)$ for $M \in \mathbb{R}^{D\times D}$ and $\alpha \geq 0$.

4.1. Stability of the one-dimensional case

We first consider on the coefficients of the linear terms in Eqs. (1.1) and (2.1) for one dimension, when L is a real number, $\lambda \leq 0$ say:

$$\begin{cases} u'(t) = \lambda u(t) + \sum_{\nu=1}^{N} f_\nu(t, u(t)), \quad t \geq t_0, \\ u(t_0) = u_0. \end{cases} \tag{4.1}$$

Let $v(t)$ be a solution to Eq. (4.1) with a different initial value $v(t_0) = v_0$. The condition $\lambda + \sum_{\nu=1}^{N} \sigma_\nu \leq 0$ guarantees the stability of the exact solution of the equations with different initial values — i.e. $\|u(t) - v(t)\| \leq \|u_0 - v_0\|$. Applying the exponential additive Runge-Kutta methods to Eq. (4.1) gives

$$\begin{cases} U_n^i = e^{c_i h\lambda} u_n + h \sum_{\nu=1}^{N} \sum_{j=1}^{m} a_{ij}^{[\nu]}(h\lambda) f_\nu\left(t_n^j, U_n^j\right), \quad i = 1, \cdots, m, \\ u_{n+1} = e^{h\lambda} u_n + h \sum_{\nu=1}^{N} \sum_{i=1}^{m} b_i^{[\nu]}(h\lambda) f_\nu\left(t_n^i, U_n^i\right), \end{cases} \tag{4.2}$$

where the collocation nodes $c_i \in [0, 1]$, weight coefficients $b_i^{[\nu]}(h\lambda)$ and $a_{ij}^{[\nu]}(h\lambda)$ are analytic functions about $h\lambda$. We discuss the stability of the established methods below, called exponential B-stability (i.e. EB-stability), in order to distinguish it from the B-stability in the non-exponential case.

Definition 4.1. An exponential additive Runge-Kutta method is EB-stable if, for any Eq. (4.1) satisfying (3.1), $\|u_{n+1} - v_{n+1}\| \leq \|u_n - v_n\|$ holds for $\lambda \leq 0$ and $h > 0$.

We denote $b_i^{[\nu]}(h\lambda) = b_i^{[\nu]}$, $a_{ij}^{[\nu]}(h\lambda) = a_{ij}^{[\nu]}$ and in this stability discussion introduce the

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following notation:

$$\widetilde{B}^{[\nu]} = diag\left(b_1^{[\nu]}(h\lambda)\mathrm{e}^{(1-c_1)h\lambda}, \cdots, b_m^{[\nu]}(h\lambda)\mathrm{e}^{(1-c_m)h\lambda}\right), \qquad b^{[\nu]} = \begin{pmatrix} b_1^{[\nu]}(h\lambda) \\ \vdots \\ b_m^{[\nu]}(h\lambda) \end{pmatrix},$$

$$A^{[\nu]} = \begin{pmatrix} a_{11}^{[\nu]}(h\lambda) & \cdots & a_{1m}^{[\nu]}(h\lambda) \\ \vdots & & \vdots \\ a_{m1}^{[\nu]}(h\lambda) & \cdots & a_{mm}^{[\nu]}(h\lambda) \end{pmatrix}, \qquad Q^{[\nu\kappa]} = \begin{pmatrix} q_{11}^{[\nu\kappa]} & \cdots & q_{1m}^{[\nu\kappa]} \\ \vdots & & \vdots \\ q_{m1}^{[\nu\kappa]} & \cdots & q_{mm}^{[\nu\kappa]} \end{pmatrix}.$$

Definition 4.2. An exponential additive Runge-Kutta method of type (4.2) is exponentially algebraically stable if the matrices $\widetilde{B}^{[\nu]}$ and $Q^{[\nu\kappa]} = \widetilde{B}^{[\nu]}A^{[\kappa]} + A^{[\nu]T}\widetilde{B}^{[\kappa]} - b^{[\nu]}b^{[\kappa]T}$ are non-negative definite for $\nu, \kappa = 1, \cdots, N$.

Theorem 4.1. *A sufficient condition for an exponential additive Runge-Kutta method* (4.2) *to be EB-stable is that it is exponentially algebraically stable.*

Proof. Consider the exponential additive Runge-Kutta method applied to Eq. (4.1) with a different initial value;

$$\begin{cases} V_n^i = \mathrm{e}^{c_i h\lambda} v_n + h\sum_{\nu=1}^{N}\sum_{j=1}^{m} a_{ij}^{[\nu]}(h\lambda) f_\nu\left(t_n^j, V_n^j\right), \quad i = 1, \cdots, m, \\ v_{n+1} = \mathrm{e}^{h\lambda} v_n + h\sum_{\nu=1}^{N}\sum_{i=1}^{m} b_i^{[\nu]}(h\lambda) f_\nu\left(t_n^i, V_n^i\right). \end{cases}$$

Let $\delta_n = u_n - v_n$, $\triangle_n^i = U_n^i - V_n^i$ and $\triangle f_\nu^i = h[f_\nu(t_n^i, U_n^i) - f_\nu(t_n^i, V_n^i)]$. Then we have

$$U_n^i - V_n^i = \triangle_n^i = \mathrm{e}^{c_i h\lambda}\delta_n + \sum_{\nu=1}^{N}\sum_{j=1}^{m} a_{ij}^{[\nu]}\triangle f_\nu^j, \quad i = 1, \cdots, m,$$

$$u_{n+1} - v_{n+1} = \delta_{n+1} = \mathrm{e}^{h\lambda}\delta_n + \sum_{\nu=1}^{N}\sum_{i=1}^{m} b_i^{[\nu]}\triangle f_\nu^i,$$

and hence

$$\|\delta_{n+1}\|^2 = \left\|\mathrm{e}^{h\lambda}\delta_n\right\|^2 + 2\sum_{\nu=1}^{N}\sum_{i=1}^{m}\left\langle \mathrm{e}^{h\lambda}\delta_n, b_i^{[\nu]}\triangle f_\nu^i\right\rangle + \left\|\sum_{\nu=1}^{N}\sum_{k=1}^{m} b_k^{[\nu]}\triangle f_\nu^k\right\|^2.$$

Furthermore, $\|\mathrm{e}^{h\lambda}\delta_n\| \le \mathrm{e}^{h\lambda}\|\delta_n\| \le \|\delta_n\|$ holds because $h > 0$ and $\lambda \le 0$, so the last two terms are

$$\begin{aligned} &2\sum_{\nu=1}^{N}\sum_{i=1}^{m}\left(\mathrm{e}^{h\lambda}\delta_n\right)^T b_i^{[\nu]}\triangle f_\nu^i + \left(\sum_{\kappa=1}^{N}\sum_{j=1}^{m} b_j^{[\kappa]}\triangle f_\kappa^j\right)^T\left(\sum_{\nu=1}^{N}\sum_{i=1}^{m} b_i^{[\nu]}\triangle f_\nu^i\right) \\ =&2\sum_{\nu=1}^{N}\sum_{i=1}^{m} b_i^{[\nu]}\mathrm{e}^{(1-c_i)h\lambda}\triangle_n^{iT}\triangle f_\nu^i - \sum_{\nu,\kappa=1}^{N}\sum_{i,j=1}^{m} q_{ij}^{[\nu\kappa]}\triangle f_\kappa^{jT}\triangle f_\nu^i, \end{aligned}$$

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which is established according to $q_{ij}^{[\nu\kappa]} = b_i^{[\nu]} e^{(1-c_i)h\lambda} a_{ij}^{[\kappa]} + b_j^{[\kappa]} e^{(1-c_j)h\lambda} a_{ji}^{[\nu]} - b_i^{[\nu]} b_j^{[\kappa]}$. Due to inequality (3.1), we obtain

$$\triangle_n^{iT} \triangle f_\nu^i = h \left\langle U_n^i - V_n^i, f_\nu(t_n^i, U_n^i) - f_\nu(t_n^i, V_n^i) \right\rangle \le h\sigma_\nu \left\| U_n^i - V_n^i \right\|^2 \le 0 .$$

Noting that the $\widetilde{B}^{[\nu]}$ are non-negative definite (i.e. $b_i^{[\nu]} e^{(1-c_i)h\lambda} \ge 0$ for $i = 1, \cdots, m$ and $\nu = 1, \cdots, N$), we obtain $2\sum_{\nu=1}^{N} \sum_{i=1}^{m} b_i^{[\nu]} e^{(1-c_i)h\lambda} \triangle_n^{iT} \triangle f_\nu^i \le 0$. The positive definite property of $Q^{[\nu\kappa]}$ implies $\sum_{\nu,\kappa=1}^{N} \sum_{i,j=1}^{m} q_{ij}^{[\nu\kappa]} \xi_i^T \xi_j \ge 0$, where $\xi_1, \cdots, \xi_m$ are any vectors in $\mathbb{R}^D$, therefore

$$- \sum_{\nu,\kappa=1}^{N} \sum_{i,j=1}^{m} q_{ij}^{[\nu\kappa]} \triangle f_\kappa^{jT} \triangle f_\nu^i \le 0.$$

In summary, we obtain $\|\delta_{n+1}\| \le \|\delta_n\|$ — i.e. $\|u_{n+1} - v_{n+1}\| \le \|u_n - v_n\|$, which satisfies the definition of EB-stability so that this exponential additive Runge-Kutta method is EB-stable. □

Remark 4.1. The exponential Radau IIA method of order one in Example 2.1 is exponentially algebraically stable — i.e.

$$b_1 a_{11} + a_{11} b_1 - b_1^2 = b_1^2 \ge 0 .$$

According to Theorem 4.1, the exponential Radau IIA method is EB-stable.

Remark 4.2. The exponential Lobatto IIIA method of order 2 in Example 2.2 is not exponentially algebraically stable since

$$\begin{pmatrix} b_1 e^{h\lambda} & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} + \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} b_1 e^{h\lambda} & 0 \\ 0 & b_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \end{pmatrix} = \begin{pmatrix} -b_1^2 & 0 \\ 0 & b_2^2 \end{pmatrix},$$

so the method is not EB-stable.

Example 4.1. When $N = 2$, it is necessary to modify Example 2.2 into the exponential additive Lobatto IIIA method

$$\begin{array}{c|cc|cc} 0 & \dfrac{2(h\lambda-1)e^{h\lambda}+2}{(h\lambda)^2 e^{h\lambda}} & 0 & \dfrac{(h\lambda-1)e^{h\lambda}+1}{(h\lambda)^2 e^{h\lambda}} & 0 \\ 1 & \dfrac{e^{h\lambda}-1}{h\lambda} & \dfrac{e^{h\lambda}-h\lambda-1}{(h\lambda)^2} & \dfrac{(h\lambda-1)e^{h\lambda}+1}{(h\lambda)^2} & \dfrac{e^{h\lambda}-h\lambda-1}{(h\lambda)^2} \\ \hline & \dfrac{(h\lambda-1)e^{h\lambda}+1}{(h\lambda)^2} & \dfrac{e^{h\lambda}-h\lambda-1}{(h\lambda)^2} & \dfrac{(h\lambda-1)e^{h\lambda}+1}{(h\lambda)^2} & \dfrac{e^{h\lambda}-h\lambda-1}{(h\lambda)^2} \end{array} .$$

According to Definition 4.2 we have

$$Q^{[11]} = \begin{pmatrix} 3b_1^2 & b_2^2 \\ b_2^2 & b_2^2 \end{pmatrix}, Q^{[12]} = \begin{pmatrix} 2b_1^2 & b_2^2 \\ 0 & b_2^2 \end{pmatrix}, Q^{[21]} = \begin{pmatrix} 2b_1^2 & 0 \\ b_2^2 & b_2^2 \end{pmatrix}, Q^{[22]} = \begin{pmatrix} b_1^2 & 0 \\ 0 & b_2^2 \end{pmatrix},$$

which are non-negative definite and imply the exponential additive Lobatto IIIA method is exponentially algebraically stable and EB-stable.

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4.2. Stability of the exponential additive Runge-Kutta methods when applied to general semi-linear equations

Let us now consider the general problem

$$v'(t) = Lv(t) + \sum_{\nu=1}^{N} f_\nu(t, v(t)), \quad t \geq t_0, \quad v(t_0) = v_0\,, \tag{4.3}$$

where L is a matrix. If

$$\mu(L) + \sum_{\nu=1}^{N} \gamma_\nu \leq 0\,, \tag{4.4}$$

then the stability of the solution is guaranteed. Let $\delta(t) = u(t) - v(t)$, $\delta_0 = \delta(t_0) = u_0 - v_0$. Stability means $\|\delta(t)\| \leq \|\delta_0\|$, $t \geq t_0$, holds for every u_0, v_0. Moreover, when the symbol $\leq$ in (4.4) is replaced by $<$, the system (1.1) with (2.1) is asymptotically stable — i.e. $\|\delta(t)\| \to 0$, $t \to +\infty$, holds for every u_0 and v_0.

An explicit exponential additive Runge-Kutta method applied to Eqs. (1.1), (2.1) and (4.3) gives

$$\begin{cases} U_n^i = \mathrm{e}^{c_i hL} u_n + h \sum\limits_{\nu=1}^{N} \sum\limits_{j=1}^{i-1} a_{ij}^{[\nu]}(hL) f_\nu\left(t_n^j, U_n^j\right), \quad i = 1, \cdots, m\,, \\ u_{n+1} = \mathrm{e}^{hL} u_n + h \sum\limits_{\nu=1}^{N} \sum\limits_{i=1}^{m} b_i^{[\nu]}(hL) f_\nu\left(t_n^i, U_n^i\right), \end{cases} \tag{4.5}$$

$$\begin{cases} V_n^i = \mathrm{e}^{c_i hL} v_n + h \sum\limits_{\nu=1}^{N} \sum\limits_{j=1}^{i-1} a_{ij}^{[\nu]}(hL) f_\nu\left(t_n^j, V_n^j\right), \quad i = 1, \cdots, m\,, \\ v_{n+1} = \mathrm{e}^{hL} v_n + h \sum\limits_{\nu=1}^{N} \sum\limits_{i=1}^{m} b_i^{[\nu]}(hL) f_\nu\left(t_n^i, V_n^i\right). \end{cases}$$

We again denote the approximate grids by u_{n+1} and v_{n+1}, and internal stages as U_n^i and V_n^i. In notation similar to the previous section, on subtracting these schemes we obtain

$$\begin{aligned} \triangle_n^i &= \mathrm{e}^{c_i hL} \delta_n + h \sum_{\nu=1}^{N} \sum_{j=1}^{i-1} a_{ij}^{[\nu]}(hL) \left[f_\nu\left(t_n^j, U_n^j\right) - f_\nu\left(t_n^j, V_n^j\right) \right], \\ \delta_{n+1} &= \mathrm{e}^{hL} \delta_n + h \sum_{\nu=1}^{N} \sum_{i=1}^{m} b_i^{[\nu]}(hL) \left[f_\nu\left(t_n^i, U_n^i\right) - f_\nu\left(t_n^i, V_n^i\right) \right]; \end{aligned}$$

and then taking the norm of both sides we have

$$\left\|\triangle_n^i\right\| \leq \mathrm{e}^{c_i \mu(hL)} \|\delta_n\| + h \sum_{\nu=1}^{N} \gamma_\nu \sum_{j=1}^{i-1} \left\| a_{ij}^{[\nu]}(hL) \right\| \left\|\triangle_n^j\right\|, \quad i = 1, \cdots, m\,, \tag{4.6}$$

$$\|\delta_{n+1}\| \leq \mathrm{e}^{\mu(hL)} \|\delta_n\| + h \sum_{\nu=1}^{N} \gamma_\nu \sum_{i=1}^{m} \left\| b_i^{[\nu]}(hL) \right\| \left\|\triangle_n^i\right\|, \tag{4.7}$$

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on invoking the triangle inequality and properties of the logarithmic norm.

We now introduce some definitions concerning stability properties of exponential additive Runge-Kutta methods, by considering a family $\mathscr{F}$ of semi-linear systems of the type (1.1) satisfying the inequality (4.4).

Definition 4.3. An exponential additive Runge-Kutta method (4.5) is called unconditionally stable on the family $\mathscr{F}$ with respect to the norm $\|\cdot\|$ if $\|\delta_1\| \le \|\delta_0\|$ holds for all u_0, v_0 and all stepsizes h when applied to any system in $\mathscr{F}$. Furthermore, it is called unconditionally asymptotically stable if $\delta_n \to 0$ as $n \to \infty$ whenever $\mu(L) + \sum_{\nu=1}^{N} \gamma_\nu < 0$.

Let $\mathbb{M}$ be a class of matrices (even of different dimensions) that is closed with respect to multiplication by positive scalars, so that $lM \in \mathbb{M}$ for any $l > 0$, if $M \in \mathbb{M}$. We investigate the stability properties of the family $\mathscr{F}(\mathbb{M})$ of equations (1.1) with (2.1) and $L \in \mathbb{M}$ with respect to a given norm $\|\cdot\|$. To do so, for $\alpha \le 0$ we introduce the $(1 \times m)$-vector $\bar{b}(\alpha)$ with components

$$\bar{b}_i(\alpha) = \sup_{\substack{M \in \mathbb{M},\ \mu(M) \le \alpha \\ \nu = 1, \cdots, N}} \left\| b_i^{[\nu]}(M) \right\|, \quad i = 1, \cdots, m,$$

and the $(m \times m)$-matrix $\bar{A}(\alpha)$ with the elements

$$\bar{a}_{ij}(\alpha) = \sup_{\substack{M \in \mathbb{M},\ \mu(M) \le \alpha \\ \nu = 1, \cdots, N}} \left\| a_{ij}^{[\nu]}(M) \right\|, \quad i, j = 1, \cdots, m.$$

Since $\mathbb{M}$ contains all scalars, we have $\sup_{x \le \alpha} |b_i^{[\nu]}(x)| \le \bar{b}_i(\alpha)$ and $\sup_{x \le \alpha} |a_{ij}^{[\nu]}(x)| \le \bar{a}_{ij}(\alpha)$, $\alpha \le 0$, $i, j = 1, \cdots, m$, $\nu = 1, \cdots, N$. Moreover, we remark that the functions $\bar{b}_i$ and $\bar{a}_{ij}$, $i, j = 1, \cdots, m$ are non-decreasing. Now we give a bound for $\|\delta_{n+1}\|$ in terms of $\|\delta_n\|$. From inequalities (4.6) and (4.7) we obtain

$$\begin{aligned} \|\delta_{n+1}\| &\le \mathrm{e}^{h\mu(L)} \|\delta_n\| + h \sum_{\nu=1}^{N} \gamma_\nu \bar{b}(h\mu(L)) \bar{\triangle}_n, \\ \left\| \triangle_n^i \right\| &\le \mathrm{e}^{c_i h\mu(L)} \|\delta_n\| + h \sum_{\nu=1}^{N} \gamma_\nu \sum_{j=1}^{i-1} \bar{a}_{ij}(h\mu(L)) \left\| \triangle_n^j \right\|, \end{aligned} \tag{4.8}$$

so we have

$$\begin{aligned} &\left\| \triangle_n^1 \right\| \le \mathrm{e}^{c_1 h\mu(L)} \|\delta_n\|, \\ &\left\| \triangle_n^2 \right\| \le \mathrm{e}^{c_2 h\mu(L)} \|\delta_n\| + h \sum_{\nu=1}^{N} \gamma_\nu \bar{a}_{21}(h\mu(L)) \left\| \triangle_n^1 \right\|, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\left\| \triangle_n^m \right\| \le \mathrm{e}^{c_m h\mu(L)} \|\delta_n\| + h \sum_{\nu=1}^{N} \gamma_\nu \left(\bar{a}_{m1}(h\mu(L)) \left\| \triangle_n^1 \right\| + \cdots + \bar{a}_{mm-1}(h\mu(L)) \left\| \triangle_n^{m-1} \right\| \right). \end{aligned}$$

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Substituting the above successively gives

$$\bar{\triangle}_n \leq \mathrm{e}^{ch\mu(L)} 1_m \|\delta_n\| + h \sum_{\nu=1}^{N} \gamma_\nu \bar{A}(h\mu(L)) \bar{\triangle}_n \,,$$

where $\bar{\triangle}_n$ is the $(m \times 1)$-vector of components $\left\|\triangle_n^i\right\|$, $i = 1, \cdots, m$, c is the $(m \times 1)$-vector of components c_i, $i = 1, ..., m$, 1_m is the $(m \times 1)$-vector with all components equal to 1, the notation $x \leq y$, $x, y \in \mathbb{R}^m$ stands for $x_i \leq y_i$, $i = 1, ..., m$, and the notation diag e^x, $x \in \mathbb{R}^m$ for the diagonal $(m \times m)$-matrix diag $(\mathrm{e}^{x_1}, \cdots, \mathrm{e}^{x_m})$.

After transposition, we obtain

$$\left(I - h \sum_{\nu=1}^{N} \gamma_\nu \bar{A}(h\mu(L)) \right) \bar{\triangle}_n \leq \mathrm{e}^{ch\mu(L)} 1_m \|\delta_n\| \,. \tag{4.9}$$

Since the matrix $\bar{A}(h\mu(L))$ is strictly lower triangular, $I - h \sum_{\nu=1}^{N} \gamma_\nu \bar{A}(h\mu(L))$ is invertible, so from inequality (4.9) we obtain

$$\bar{\triangle}_n \leq \sum_{k=0}^{m-1} \left(h \sum_{\nu=1}^{N} \gamma_\nu \right)^k \bar{A}(h\mu(L))^k \mathrm{e}^{ch\mu(L)} 1_m \|\delta_n\| \,.$$

From this inequality and inequality (4.8) we have

$$\|\delta_{n+1}\| \leq \bar{H} \left(h\mu(L), h \sum_{\nu=1}^{N} \gamma_\nu \right) \|\delta_n\| \,,$$

where

$$\bar{H}(\alpha, \beta) = \mathrm{e}^\alpha + \sum_{k=0}^{m-1} \beta^{k+1} \bar{b}(\alpha) \bar{A}(\alpha)^k \mathrm{e}^{c\alpha} 1_m,$$

with $\alpha \in \mathbb{R}$ and $\beta \geq 0$.

Now we present a sufficient condition for the unconditional stability properties on the family $\mathscr{F}(\mathbb{M})$, relevant to a class $\mathbb{M}$ of matrices, and with respect to a norm $\|\cdot\|$.

Theorem 4.2. *If an explicit exponential additive Runge-Kutta method* (4.5) *satisfies*

$$\bar{H}(\alpha, \beta) \leq 1 \tag{4.10}$$

with $\bar{H}$ defined as above and $\alpha + \beta \leq 0$, then it is unconditionally stable on the family $\mathscr{F}(\mathbb{M})$ with respect to the norm $\|\cdot\|$. Furthermore, it is unconditionally asymptotically stable if $\bar{H}(\alpha, \beta) < 1$.

Proof. This follows at once from Definition 4.3. □

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Example 4.2. We consider the explicit exponential additive Runge-Kutta method with three internal stages. Denoting $\varphi_{i,j} = \varphi_{i,j}(hL) = \varphi_i\left(c_j hL\right)$, $1 \le j \le m$, we have the Butcher tableau:

$$\begin{array}{c|ccc|ccc}
0 & & & & & & \\
\frac{1}{3} & \frac{1}{3}\varphi_{1,2} & & & \frac{1}{3}\varphi_{1,2} & & \\
\frac{2}{3} & \frac{2}{3}\varphi_{1,3}-\frac{4}{3}\varphi_{2,3} & \frac{4}{3}\varphi_{2,3} & & \frac{2}{3}\varphi_{1,3}-\frac{4}{3}\varphi_{2,3} & \frac{4}{3}\varphi_{2,3} & \\
\hline
 & \varphi_1-\frac{3}{2}\varphi_2 & 0 & \frac{3}{2}\varphi_2 & \varphi_1-\frac{9}{2}\varphi_2+9\varphi_3 & 6\varphi_2-18\varphi_3 & -\frac{3}{2}\varphi_2+9\varphi_3
\end{array}.$$

Since this explicit exponential additive Runge-Kutta method satisfies the conditions of Theorem 4.2, it is unconditionally asymptotically stable.

For the problem

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \sin u_1 - \sin\left(\mathrm{e}^t\right) \\ \sin u_2 - \sin t \end{bmatrix} + \begin{bmatrix} |u_1| + 5\mathrm{e}^t - t \\ |u_2| + \mathrm{e}^t + 3t + 1 \end{bmatrix},$$

the exact solution is $u_1(t) = \mathrm{e}^t, u_2(t) = t$. Let us consider the maximum norm and the matrix $L = \left(l_{ij}\right)$. The property of logarithmic norm according to $\|\cdot\|_\infty$ shows that [7]

$$\mu_\infty(L) = \max_i\left(l_{ii} + \sum_{j\neq i}\left|l_{ij}\right|\right).$$

Since $\mu_\infty(L) = -3$, this leads to

$$\bar{b}\left(h\mu_\infty(L)\right) = \begin{pmatrix} 0.2284 \\ 0 \\ 0.7270 \end{pmatrix}, \qquad \bar{A}\left(h\mu_\infty(L)\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0.3231 & 0 & 0 \\ -0.0131 & 0.6397 & 0 \end{pmatrix}$$

for stepsize $h = 0.0625$. One can observe that f_1 and f_2 satisfy the classical Lipschitz condition with Lipschitz constants $\gamma_i = 1$, $i = 1, 2$. Obviously, inequality (4.4) holds for $\mu_\infty(L) + \gamma_1 + \gamma_2 = -1$ and $c = \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \end{pmatrix}$. We are now in the position to compute $\bar{H}$:

$$\begin{aligned}
&\bar{H}\left(h\mu_\infty(L), h\left(\gamma_1+\gamma_2\right)\right) \\
=&\mathrm{e}^{h\mu_\infty(L)} + \sum_{k=0}^{2}\left(h\left(\gamma_1+\gamma_2\right)\right)^{k+1}\bar{b}\left(h\mu_\infty(L)\right)\bar{A}\left(h\mu_\infty(L)\right)^k \mathrm{e}^{ch\mu_\infty(L)}1_m = 0.9447,
\end{aligned}$$

which implies the asymptotic stability of Example 4.2.

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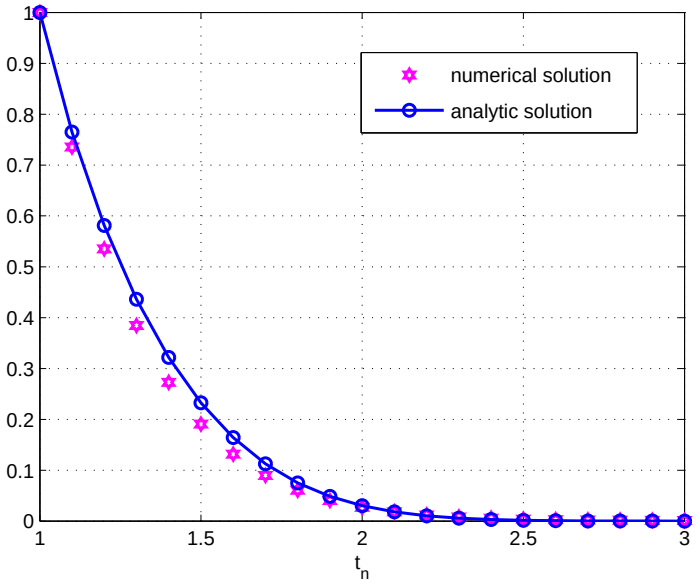


Figure 1: Numerical behaviour of the exponential Radau IIA method for Example 5.1 with stepsize $h = 1/10$.

5. Numerical Experiments

We provide numerical results obtained for the application of the exponential additive methods to semi-linear problems.

Example 5.1. Let us first consider the one-dimensional semi-linear parabolic initial-boundary value problem

$$u'(t) = -\frac{2}{3}u(t) - \frac{u(t)}{t^2} - t^2 u(t), \quad t \geq 1\,, \tag{5.1}$$

where $u(1) = 1$, $f_1(t,u) = -(1/t^2)u(t)$, $f_2(t,u) = -t^2 u(t)$ both satisfy the condition (3.1), and $\lambda = -2/3 \leq 0$. The analytic solution is $u(t) = \mathrm{e}^{-\frac{2}{3}t + \frac{1}{t} - \frac{1}{3}t^3}$, which can be verified to be continuous in its domain.

For convenience, we only take methods of collocation type into consideration — i.e. each function $f_\nu(t,u(t))$ is treated with a similar Lagrange interpolation. We consider the exponential additive Runge-Kutta method with $m = 1$ — cf. Example 2.1 in Section 2.

In Fig. 1, we demonstrate the stability by representing the exact solution and the numerical solution obtained by the exponential Radau IIA method using Matlab software. The several values and errors listed in Table 2 are according to the number of iterations, and $e_n = u(t_n) - u_n$ denotes the global error for Example 2.1. The successive reduction of the error can be seen from the data presented.

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Table 2: Numerical errors of one-order exponential Radau IIA method for the problem (5.1).

n	u_n	$u(t_n)$	$\lvert e_n\rvert$	$\lvert u_{n+1}-u_n\rvert$
11	0.01798741	0.01811872	0.00013131	0.00900571
12	0.01204387	0.01044685	0.00159702	0.00594353
13	0.00814904	0.00577500	0.00237404	0.00389484
14	0.00560680	0.00305391	0.00255289	0.00254224
15	0.00394937	0.00154150	0.00240787	0.00165743
16	0.00286827	0.00074109	0.00212718	0.00108109
17	0.00216339	0.00033862	0.00182477	0.00070488
18	0.00170684	0.00014674	0.00156011	0.00045655
19	0.00141848	0.00006018	0.00135830	0.00028836
20	0.00124993	0.00002331	0.00122663	0.00016855

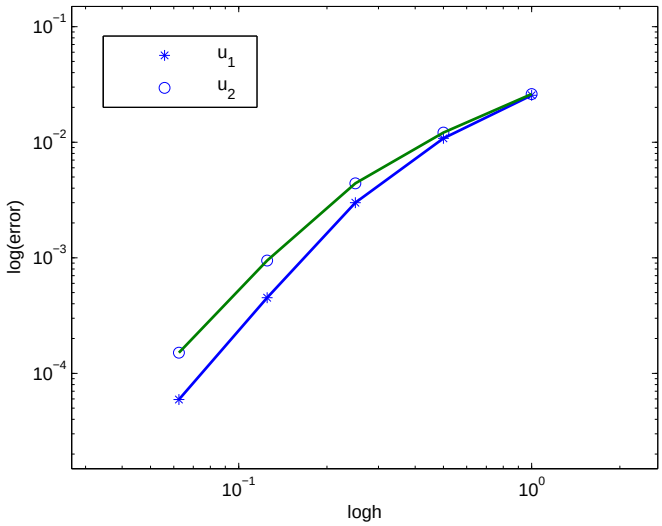


Figure 2: Convergence of Example 4.2 in Example 5.2, showing error versus stepsize.

Example 5.2. We also considered the following well known stiff non-linear problem — cf. Refs. [9, 25]:

$$\begin{cases} u_1' = 0.01-(1+(u_1+1000)(u_1+1))(0.01+u_1+u_2), \\ u_2' = 0.01-\left(1+u_2^2\right)(0.01+u_1+u_2), \\ u_1(0)=u_2(0)=0, \quad 0\le t\le 100, \end{cases} \tag{5.2}$$

The reference solution at $t=100$ is $u_1=-0.99164207$ and $u_2=0.98333636$.

We have compared the numerical solutions obtained with the timestep halved. In Fig. 2, we plot the maximal absolute end point error versus $h=1, 0.5, 0.25, 0.125, 0.0625$ to show

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Table 3: Estimated order for Example 4.2.

	h	1	0.5	0.25	0.125	0.0625
u_1	error	0.0255	0.0108	0.0030	4.5161e-004	5.9410e-005
	order	-	1.2451	1.8480	2.7318	2.9263
u_2	error	0.0261	0.0121	0.0044	9.4678e-004	1.5064e-004
	order	-	1.1102	1.4537	2.2164	2.6519

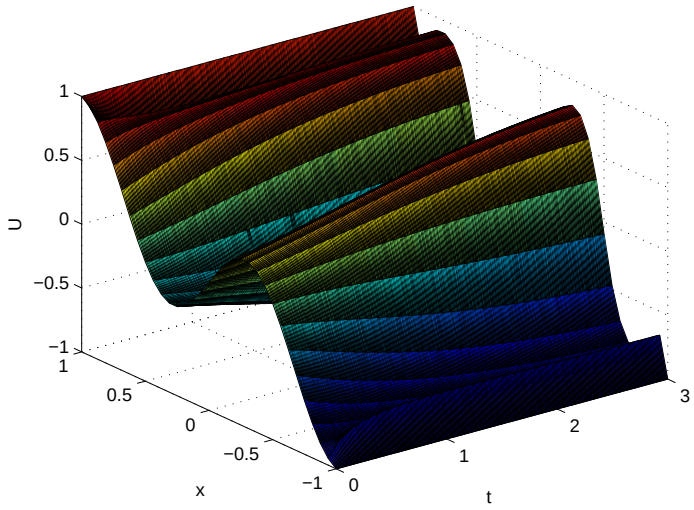


Figure 3: Numerical evolution for the Allen-Cahn problem obtained by the exponential additive Lobatto IIIA method with stepsize $h = 0.01$.

the efficiency of the proposed methods. The results of these numerical experiments shown in Table 3 suggest a convergence order $O(h^3)$ when h is sufficiently small. When the time-step decreases by the above factor 2, we note that the numerically observed convergence orders for the explicit exponential additive Runge-Kutta methods in Example 4.2 of Section 4 agree with the values expected from the theoretical results given in Theorem 3.1.

Example 5.3. Finally, we considered the following problem involving the Allen-Cahn equation [4, 16, 19]:

$$u_t = \epsilon u_{xx} + u - u^3, \ x \in [-1, 1]$$

subject to the homogeneous Dirichlet boundary conditions $u(-1, t) = -1$, $u(1, t) = 1$ and the initial condition $u(x, 0) = 0.53x + 0.47\sin(-1.5\pi x)$.

We chose $\epsilon = 0.01$, and discretised this problem in space with a standard finite difference

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method that renders the stiff ordinary differential system of size $N = 32$:

$$U'(t) = \epsilon \left(\frac{N}{2}\right)^2 \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \end{pmatrix} U(t) + f_1\big(t, U(t)\big) + f_2\big(t, U(t)\big), \qquad (5.3)$$

where

$$\begin{aligned} &u_i(t) = u(x_i, t), \qquad i = 0, 1, \cdots, N, \\ &U(t) = \big(u_1(t), u_2(t), \cdots, u_{N-1}(t)\big)^T, \\ &f_1\big(t, U(t)\big) = U(t), \quad f_2\big(t, U(t)\big) = -\big(U(t)\big)^3, \\ &f_i\big(t, U(t)\big) = \big(f_i(t, u_1(t)), f_i(t, u_2(t)), \cdots, f_i(t, u_{N-1}(t))\big)^T, \quad i = 1, 2. \end{aligned}$$

The ordinary differential equation was integrated in time up to $T = 3$ by the exponential additive Lobatto IIIA method of Example 4.1 with stepsize $h = 0.01$ — producing the solution illustrated in Fig. 3 in the interval [0,3], with a performance similar to the approach of Ref. [16] in obtaining the solution shown in their Fig. 8.

6. Conclusions

We have constructed exponential additive Runge-Kutta methods for semi-linear equations, and introduced the concept of *EB*-stability. The main feature of this class of methods is their high accuracy of dealing with the non-linear part that is obtained using exponential functions. We obtained some results on their stability properties, and several numerical examples have been provided to illustrate our theoretical results.

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