We constructed h' from f and g with the help of w. Now suppose that f and g map homeomorphically. It is easy to check that h maps homeomorphically, under these conditions, and consequently so does h'. Therefore

$$D(m+n+1) \leqslant k+p+q+1.$$

We may take $$k = C(m,n), \quad p = D(m), \quad q = D(n).$$

Hence (6) follows at once.

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A NOTE CONCERNING INVARIANT SUBSPACES OF A BOUNDED LINEAR OPERATOR ON A BANACH SPACE

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Let T be a bounded linear operator on a Banach space $\mathfrak{B}$ and σ, $\overline{\sigma}$ isolated complementary parts of its spectrum. By the well-known results of F. Riesz ((1), p. 417) there corresponds to σ a projection P_σ such that the subspaces (closed linear manifolds) $\mathfrak{M}_\sigma = P_\sigma \mathfrak{B}$ and $\mathfrak{N}_\sigma = (I - P_\sigma)\mathfrak{B}$ are invariant with respect to T. Regarded as an operator in the subspaces $\mathfrak{M}_\sigma$ and $\mathfrak{N}_\sigma$, T has as spectrum σ and $\overline{\sigma}$ respectively.

It has been shown in (2) that $\mathfrak{M}_\sigma$ contains every element $x \in \mathfrak{B}$ which for some point $\lambda \in \sigma$ satisfies

$$\|(T-\lambda I)^n x\|^{1/n} \to 0 \quad (n \to \infty). \tag{1}$$

In this note we prove the

Theorem. *Every element of $\mathfrak{M}_\sigma$ satisfies* (1) *for some λ only if σ consists of a single point.*

We shall say that an element $x \in \mathfrak{B}$ belongs to the point λ of the complex plane if it satisfies (1), and we remark that it does so if and only if

$$p^n(T-\lambda I)^n x \to 0 \quad (n \to \infty)$$

for all $p > 0$.

We prove three lemmas.

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LEMMA 1. *An element $x \in \mathfrak{B}$ can belong to at most one point.*

For every λ and $\mu \neq \lambda$ there exist α and β such that the operator

$$S = T - (1 + \|T\|) I$$

can be expressed in the form

$$S = \alpha(T - \mu I) + \beta(T - \lambda I). \tag{2}$$

The operator S has the property that $\|Sy\| \geqslant \|y\|$ for all $y \in \mathfrak{B}$. If, however, the element x belonged to both λ and μ we would obtain, by considering the binomial expansion of the expression for S^n given by (2) (cf. (1), p. 422 and the proof of Lemma 2)

$$S^n x = [\alpha(T - \mu I) + \beta(T - \lambda I)]^n x \to 0,$$

which contradicts $\|S^n x\| \geqslant \|x\|$.

COROLLARY. *If x belongs to λ then, for every $\mu \neq \lambda$, there exists a $q > 0$ such that $\|q^n (T - \mu I)^n x\|$ does not tend to zero.*

The second lemma generalizes a result which is obvious for the special case of an eigen vector.

LEMMA 2. *If the element x belongs to λ then*

$$p^n (T - \mu I)^n x \to 0 \quad (n \to \infty)$$

whenever

$$p\,|\lambda - \mu| < 1. \tag{3}$$

We have

$$\begin{aligned} p^n \|(T - \mu I)^n x\| &= p^n \|[(T - \lambda I) + (\lambda - \mu) I]^n x\| \\ &= p^n \left\| \sum_{r=0}^{m} + \sum_{r=m+1}^{n} {}^nC_r (\lambda - \mu)^{n-r} (T - \lambda I)^r x \right\| \\ &\leqslant p^n \{K(m)\, |\lambda - \mu|^n n^m + (|\lambda - \mu| + \delta_m)^n\}, \end{aligned}$$

where $\delta_m = \sup_{n > m} \|(T - \lambda I)^n x\|^{1/n}$ and $K(m)$ is a constant independent of n. Now $\delta_m \to 0$, so, by taking m sufficiently large, we can ensure, by (3), that $p^n(|\lambda - \mu| + \delta_m)^n \to 0$, which gives the result.

LEMMA 3. *If the elements x and y in $\mathfrak{B}$ belong to μ and ν respectively, $\mu \neq \nu$, the set τ of numbers α ($\alpha \neq 0$) which are such that the element $x + \alpha y$ belongs to some λ is at most countable.*

The point $\lambda(\alpha)$ to which $x + \alpha y$ belongs is, by Lemma 1, unique, and as the set of elements in $\mathfrak{B}$ which belong to any λ is a linear space, and x and y do not belong to the same point, $\lambda(\alpha)$ will be different for different α.

For any $\beta \in \tau$ we can write

$$x + \alpha y = (x + \beta y) + (\alpha - \beta) y,$$

and therefore

$$\begin{aligned} &\|p^n (T - \lambda(\alpha) I)^n (x + \alpha y)\| \\ &\qquad \geqslant |\|p^n (T - \lambda(\alpha) I)^n (x + \beta y)\| - |\alpha - \beta| \|p^n (T - \lambda(\alpha) I)^n y\||. \end{aligned} \tag{4}$$

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For all p the first term of the inequality tends to zero; $\lambda(\alpha) \neq \nu$ and so, by the corollary to Lemma 1, there is a $q > 0$ such that

$$q^n(T-\lambda(\alpha)I)^n y \not\to 0.$$

Therefore, from (4) and Lemma 2, we have that, for all $\beta \in \tau$, $\beta \neq \alpha$,

$$q\,|\lambda(\alpha)-\lambda(\beta)| \geqslant 1.$$

Thus $\lambda(\alpha)$ is an isolated point of the set $\{\lambda(\alpha) : \alpha \in \tau\}$ which is in one-one correspondence with the set τ; both sets are therefore countable.

The main result now follows immediately. By the last lemma, if every element of the subspace $\mathfrak{M}_\sigma$ belongs to some point, they must all belong to the same point, and there is thus a λ such that for every element $x \in \mathfrak{M}_\sigma$

$$\|(T-\lambda I)^n x\|^{1/n} \to 0 \quad (n \to \infty).$$

It follows from this that $(T-\lambda I)$ regarded as an operator in $\mathfrak{M}_\sigma$ is quasi-nilpotent and σ contains the point λ only. Conversely, as noted in (2), if σ consists of the single point λ the operator $(T-\lambda I)$ is quasi-nilpotent in $\mathfrak{M}_\sigma$ which therefore consists of all those elements $x \in \mathfrak{B}$ which belong to the point λ.

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DISCONTINUOUS LINEAR FUNCTIONALS

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All norms for a finite-dimensional real vector space give the same topology; in particular, every linear functional on such a space is continuous with respect to any norm. The purpose of this note is to prove the following theorem, giving a characterization of finite-dimensional spaces:

Let f be a non-zero linear functional on a real vector space X, and suppose that f is not discontinuous with respect to any norm. Then X is finite-dimensional.

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