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On the Maximum Cut of Line Graphs

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For an undirected graph $G = (V, E)$, let $\sigma(L(G))$ be the size of the maximum cut of the line graph $L(G)$. Let d_v denote the degree of the vertex v in G. Then we have the following results.

(A) If $G = (V, E)$ is Eulerian, then

$$\sigma(L(G)) = \begin{cases} \frac{1}{4}\sum_{v\in V} d_v^2 & \text{if } |E| \text{ is even,} \\ \frac{1}{4}\sum_{v\in V} d_v^2 - 1 & \text{if } |E| \text{ is odd.} \end{cases}$$

(B) If G is not Eulerian, then

$$\sigma(L(G)) = \frac{1}{4}\left(\sum_{v\in V} d_v^2 - l\right), \text{ where } l \text{ is the number of vertices of odd degree.}$$

Let $V = V_0 \cup V_1$ be a partition of the vertex set of $G = (V, E)$. A *cut* of G is the set (V_0, V_1) of edges from V_0 to V_1. A *uniform cut* of G is a cut such that $|V_0| = |V_1|$ or $|V_0| = |V_1| + 1$. A cut (V_0, V_1) of G is *maximum* if $|(V_0, V_1)| \geqslant |(V_0', V_1')|$ for any other cut (V_0', V_1') of G. The *line graph* of G, denoted $L(G)$, is defined as follows. The vertices of $L(G)$ are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent. An *Euler tour* in a graph is a circuit that traverses every edge exactly once. A graph is called *Eulerian* if it has an Euler tour. A graph is Eulerian if and only if the degree of every vertex is even.

In general, the size of the maximum cut of a graph is difficult to estimate well and the maximum cut problem is NP-complete [2].

In the present paper, we derive the size of the maximum cut of the line graph exactly by the application of algorithmic reasoning. We give formulae for the size of the maximum

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cut in terms of the degrees of each vertex and the number of edges. The linear algorithm used in the Theorem is in NC [1, 3].

$L(G)$ is the edge disjoint union of complete graphs corresponding to vertices of G. The best we can do with the graph corresponding to v is $(1/4)d_v^2$ if d_v is even and $(1/4)d_v^2 - 1$ if d_v is odd. This gives the bounds in (A) and (B) from above, except when G is Eulerian and $|E|$ is odd, when we must fail at one vertex. The bounds are then obtained by the algorithm given in the proof.

The main result is as follows.

Theorem. *For an undirected graph $G = (V, E)$, let $\sigma(L(G))$ be the size of the maximum cut of the line graph $L(G)$. Let d_v denote the degree of the vertex v in G. Then we have the following results.*

(A) *If $G = (V, E)$ is Eulerian, then*

$$\sigma(L(G)) = \begin{cases} \dfrac{1}{4}\displaystyle\sum_{v \in V} d_v^2 & \text{if } |E| \text{ is even,} \\ \dfrac{1}{4}\displaystyle\sum_{v \in V} d_v^2 - 1 & \text{if } |E| \text{ is odd.} \end{cases}$$

(B) *If G is not Eulerian, then*

$$\sigma(L(G)) = \frac{1}{4}\left(\sum_{v \in V} d_v^2 - l\right),$$

where l is the number of vertices of odd degree.

Proof. We remark that a cut of $L(G)$ corresponds to a 2-colouring of the edges of G. We consider the following maximum uniform cut algorithm using the Euler tour technique.

MAXIMUM UNIFORM CUT ALGORITHM

INPUT: A connected graph $G = (V, E)$.
OUTPUT: A maximum uniform cut (V_0, V_1) of the line graph $L(G)$.

(1) Compute the degree of each vertex in G.
(2) If G is not Eulerian, make G a graph of even degrees G^* by connecting all odd degree vertices to a dummy vertex v^*.
(3) Find an Euler circuit in G^*.
(4) Label the edges of G^* from v^* by 1 and 0 alternately following the route determined by its Euler circuit.
(5) Let $V_0 = \{e \mid \text{edge } e \text{ labelled 0 in } G\}$ and $V_1 = \{e \mid \text{edge } e \text{ labelled 1 in } G\}$.

We remark that, if G is Eulerian, we start at any vertex v when labelling.

Let $E(v, label) = \{vx | vx \text{ in } G \text{ and } vx \text{ has } label\ (= 0 \text{ or } 1)\}$. Then the following two facts guarantee that step (5) of the algorithm correctly generates a maximum uniform cut of $L(G)$.

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Fact 1. *If v is a start vertex, then $|E(v,0)| = |E(v,1)|$ or $|E(v,0)| + 2 = |E(v,1)|$, else $|E(v,0)| = |E(v,1)|$ or $|E(v,0)| = |E(v,1)| \pm 1$.*

Fact 2. *$|V_0| = |V_1|$ or $|V_0| + 1 = |V_1|$.*

These two facts are obtained by $1 - 0$ labelling of step (4). $E(v,0) + 2 = E(v,1)$ holds when G is an Euler circuit with an odd number of edges.

Each vertex of G corresponds to a complete subgraph of $L(G)$, so that by Fact 1 (except for $|E(v,0)| + 2 = |E(v,1)|$) each such subgraph is partitioned uniformly, and $L(G)$ itself is also partitioned uniformly by Fact 2. Each such complete subgraph is edge disjoint; therefore the maximum uniform cut is guaranteed. If $|E(v,0)| + 2 = |E(v,1)|$ holds, the corresponding complete subgraph is not partitioned uniformly. However, the maximum uniform cut holds. This can be easily shown.

By the definition of the line graph, each edge in $E(v,0)$ joins all edges in $E(v,1)$, so that the number of the joins about the vertex v is $|E(v,0)| \cdot |E(v,1)|$. Therefore the number $|(V_0, V_1)|$ of the maximum uniform cut of $L(G)$ is $\sum_{v \in V} |E(v,0)| \cdot |E(v,1)|$.

If $G = (V, E)$ is Eulerian with an even number of edges, then, for any vertex v in G, $|E(v,0)| = |E(v,1)| = d_v/2$ and therefore $|(V_0, V_1)| = (1/4) \sum_{v \in V} d_v^2$. If $|E|$ is odd, then it is the same as the even number case except for exactly one vertex v. Since $|E(v,0)| + 2 = |E(v,1)|$ and $|E(v,0)| + |E(v,1)| = d_v$ hold, we get $|(V_0, V_1)| = (1/4) \sum_{v \in V} d_v^2 - 1$.

If G is not Eulerian and v is a vertex of odd degree, then $|E(v,0)| \cdot |E(v,1)| = (d_v - 1)/2 \cdot (d_v + 1)/2$ holds. Therefore we get $|(V_0, V_1)| = (1/4)(\sum_{v \in V} d_v^2 - l)$, where l is the number of vertices of odd degree. □

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