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COMBINATORIAL PROOFS FOR TWO-COLOUR PARTITIONS

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Abstract

Andrews and El Bachraoui ['On two-colour partitions with odd smallest part', Preprint, arXiv:2410.14190] explored many integer partitions in two colours, some of which are generated by the mock theta functions of third order of Ramanujan and Watson. They also posed questions regarding combinatorial proofs for these results. In this paper, we establish bijections to provide a combinatorial proof of one of these results and a companion result. We give analytic proofs of further companion results.

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1. Introduction

In 2004, Corteel and Lovejoy [3] introduced $\overline{p}(n)$, the number of overpartitions of n; an overpartition of n is a partition of n where the first occurrence of each part may be overlined. Hirschhorn and Sellers [4] presented the generating function of $\overline{p}_o(n)$ which denotes the number of overpartitions of n into odd parts, namely

$$\sum_{n=0}^{\infty} \overline{p}_o(n)q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \quad \text{for } |q| < 1, \tag{1.1}$$

where the q-shifted factorial [1] is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j) \quad \text{for } |q| < 1.$$

Recently, Andrews and El Bachraoui [2] introduced $E(n)$ as the number of two-colour partitions of n in which parts are distinct and the even parts may occur only in the blue colour. They denoted by $E_0(n)$ and $E_1(n)$ the number of partitions counted by $E(n)$ in which the number of even parts is respectively even or odd, and $E_2(n)$ and $E_3(n)$ as the

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number of partitions counted by $E(n)$ in which the number of parts is respectively even or odd. They gave some identities linking these two-colour partitions and $\overline{p}_o(n)$.

THEOREM 1.1 [2, Theorem 1]. *For any nonnegative integer n,*

$$\begin{aligned}
&(a)\quad E(n) = \overline{p}_o(n),\\
&(b)\quad E_0(n) = \begin{cases} \frac{1}{2}\overline{p}_o(n) + 1 & \text{if } n \text{ is a square},\\ \frac{1}{2}\overline{p}_o(n) & \text{otherwise}, \end{cases}\\
&(c)\quad E_1(n) = \begin{cases} \frac{1}{2}\overline{p}_o(n) - 1 & \text{if } n \text{ is a square},\\ \frac{1}{2}\overline{p}_o(n) & \text{otherwise}, \end{cases}\\
&(d)\quad E_2(n) = \begin{cases} \frac{1}{2}\overline{p}_o(n) + (-1)^n & \text{if } n \text{ is a square},\\ \frac{1}{2}\overline{p}_o(n) & \text{otherwise}, \end{cases}\\
&(e)\quad E_3(n) = \begin{cases} \frac{1}{2}\overline{p}_o(n) - (-1)^n & \text{if } n \text{ is a square},\\ \frac{1}{2}\overline{p}_o(n) & \text{otherwise}. \end{cases}
\end{aligned}$$

They also proved the identity

$$\sum_{n\geq 0}\overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{1}{(q;q^2)_\infty^2(q^2;q^2)_\infty} =: \sum_{n\geq 0}F(n)q^n,$$

where $F(n)$ counts the number of two-colour partitions of n in which the even parts may only occur in the blue colour.

DEFINITION 1.2. Let n be a nonnegative integer and let $F_0(n)$ and $F_1(n)$ be the number of partitions counted by $F(n)$ in which the number of even parts is respectively even or odd. Furthermore, let $F_2(n)$ and $F_3(n)$ be the number of partitions counted by $F(n)$ in which the number of parts is respectively even or odd.

Andrews and El Bachraoui questioned whether there are identities similar to Theorem 1.1 for $F(n)$ and $\overline{p}(n)$. We present such a family of identities.

THEOREM 1.3. *For any nonnegative integer n,*

$$\begin{aligned}
&(a)\quad F(n) = \overline{p}(n),\\
&(b)\quad F_0(n) = \tfrac{1}{2}(\overline{p}(n) + \overline{p}_o(n)),\\
&(c)\quad F_1(n) = \tfrac{1}{2}(\overline{p}(n) - \overline{p}_o(n)),\\
&(d)\quad F_2(n) = \tfrac{1}{2}(\overline{p}(n) + (-1)^n\overline{p}_o(n)),\\
&(e)\quad F_3(n) = \tfrac{1}{2}(\overline{p}(n) - (-1)^n\overline{p}_o(n)).
\end{aligned}$$

We use λ_b and λ_g to denote a part λ occurring in blue and green, respectively. In Section 2, we establish a bijection to provide a combinatorial proof for Theorem 1.1(a).

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In Section 3, we give the proof of Theorem 1.3 and we also construct a bijection to prove Theorem 1.3(a). Section 4 is dedicated to remarks inspired by this work.

2. Combinatorial proof of Theorem 1.1(a)

We denote by $\pi_{\overline{p}_o(n)}$ the set of overpartitions of n into odd parts and by $\pi_{E(n)}$ the set of two-colour partitions of n in which parts are distinct and the even parts may occur only in the blue colour. To complete the proof of Theorem 1.1(a), we construct a map $\phi : \pi_{\overline{p}_o(n)} \to \pi_{E(n)}$ as follows.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \in \pi_{\overline{p}_o(n)}$. We define $\phi : \pi_{\overline{p}_o(n)} \to \pi_{E(n)}$ such that $\phi(\lambda) = \mu$, where μ is constructed as follows.

(1) Map the overlined parts in λ to the green parts in μ and keep the part value unchanged.
(2) Map the nonoverlined parts in λ that appear only once to the blue parts in μ and keep the part value unchanged.
(3) For the parts that appear at least twice:
    (i) if the part occurs an odd number of times, the map retains one of the parts and adds the remaining parts two by two;
    (ii) if the part occurs an even number of times, the map adds the parts two by two.

    Repeat these operations on the nonoverlined parts until there is no duplicate part and the colour is blue.

For example, $(\overline{1}, 1, 1, 1, 1)$ is a partition in $\pi_{\overline{p}_o(n)}$ and $\phi((\overline{1}, 1, 1, 1, 1)) = (1_g, 4_b)$.

LEMMA 2.1. *Assume that $\lambda \in \pi_{\overline{p}_o(n)}$ and denote $\phi(\lambda) = \mu$. Then, $\mu \in \pi_{E(n)}$.*

PROOF. To prove this lemma, we need to show that all parts of μ are distinct and that the even parts of μ are exclusively coloured blue.

It is obvious that even parts are all blue. Now we just need to show that all odd parts are different in colour or number. The blue odd parts of μ are mapped from the nonoverlined odd parts that appear only once, and the green odd parts of μ are mapped from the overlined odd parts that occur exactly once. Therefore, the claim is established. □

LEMMA 2.2. *The map ϕ is a bijection between $\pi_{\overline{p}_o(n)}$ and $\pi_{E(n)}$.*

PROOF. By Lemma 2.1, it suffices to construct a map ψ defined on $\pi_{E(n)}$ such that for all $\lambda \in \pi_{\overline{p}_o(n)}$, we have $\psi(\phi(\lambda)) = \lambda$ and for all $\mu \in \pi_{E(n)}$, we have $\phi(\psi(\mu)) = \mu$.

Consider a two-colour partition μ in $\pi_{E(n)}$. The map ψ maps its green parts to the overlined parts in $\pi_{\overline{p}_o(n)}$, maps its odd blue parts to the nonoverlined parts in $\pi_{\overline{p}_o(n)}$ and halves the even parts until they are all odd. Similar to the proof of Lemma 2.1, we deduce that $\psi(\mu)$ is a partition in $\pi_{\overline{p}_o(n)}$.

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By the constructions of ϕ and ψ, it is straightforward to check that $\psi(\phi(\lambda)) = \lambda$ for all $\lambda \in \pi_{\overline{p}_o(n)}$ and $\phi(\psi(\mu)) = \mu$ for all $\mu \in \pi_{E(n)}$. This completes the proof. □

3. Proof of Theorem 1.3

Proof of part (a). We denote by $\pi_{F(n)}$ and $\pi_{\overline{p}(n)}$ the sets of partitions in $F(n)$ and $\overline{p}(n)$, respectively.

We first define a map σ on $\pi_{F(n)}$. For $\alpha \in \pi_{F(n)}$, we define $\beta = \sigma(\alpha)$ as follows.

(1) Map the green parts in α to the nonoverlined part in β and keep the part value unchanged.
(2) Map the blue parts in α that appear only once to the overlined parts in β and keep the part value unchanged.
(3) For the blue parts that appear at least twice:
   (i) if the part occurs an odd number of times, the map retains one of the parts which is overlined and adds the remaining parts two by two and makes them nonoverlined;
   (ii) if the part occurs an even number of times, the map adds the parts two by two and makes them nonoverlined.

For example, $\sigma(1_b, 1_b, 1_b, 1_g, 1_g) = (2, \overline{1}, 1, 1)$.

LEMMA 3.1. *σ is a bijection from $\pi_{F(n)}$ to $\pi_{\overline{p}(n)}$.*

PROOF. First, we assert that $\sigma(\alpha) = \beta \in \pi_{\overline{p}(n)}$ for all $\alpha \in \pi_{F(n)}$. We have to prove the uniqueness of the overlined parts of β and this is obvious from the definition of σ.

Next, we need to prove there is an inverse of σ, which we denote by δ, such that for all $\alpha \in \pi_{F(n)}$, we have $\delta(\sigma(\alpha)) = \alpha$ and for all $\beta \in \pi_{\overline{p}(n)}$, we have $\sigma(\delta(\beta)) = \beta$.

For $\beta \in \pi_{\overline{p}(n)}$, the map δ assigns the overlined parts of β to the blue parts of α, maps the nonoverlined odd parts of β to the green parts of α and divides the nonoverlined even parts of β to the same two blue parts of α. From this, we deduce $\delta(\beta)$ is a partition in $\pi_{F(n)}$.

By examining the constructions of σ and δ, it is easy to verify that $\delta(\sigma(\alpha)) = \alpha$ for all $\alpha \in \pi_{F(n)}$ and $\sigma(\delta(\beta)) = \beta$ for all $\beta \in \pi_{\overline{p}(n)}$. This completes the proof. □

Thus, we obtain $F(n) = \overline{p}(n)$ proving Theorem 1.3(a).

Proof of parts (b) and (c). By Definition 1.2,

$$\sum_{n\geq 0}(F_0(n) - F_1(n))q^n = \frac{1}{(-q^2;q^2)_\infty (q;q^2)^2} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n\geq 0}\overline{p_0}(n)q^n$$

and

$$\sum_{n\geq 0}(F_0(n) + F_1(n))q^n = \sum_{n\geq 0}F(n)q^n.$$

Combining these two formulae with part (a) yields the identities for $F_0(n)$ and $F_1(n)$.

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Proof of parts (d) and (e). By Definition 1.2,

$$\sum_{n\geq 0}(F_2(n)-F_3(n))q^n=\frac{1}{(-q^2;q^2)_\infty(-q;q^2)^2}=\frac{(q;q^2)_\infty}{(-q;q^2)_\infty}=\sum_{n\geq 0}\overline{p_0}(n)(-q)^n$$

and

$$\sum_{n\geq 0}(F_2(n)+F_3(n))q^n=\sum_{n\geq 0}F(n)q^n.$$

Combining these two formulae with part (a) yields the identities for $F_2(n)$ and $F_3(n)$.

4. Concluding remarks

We have used q-series identities and generating functions to give the proofs of Theorem 1.3(b)–(e) and given combinatorial proofs for part (a) of Theorems 1.1 and 1.3. It is natural to ask if there are combinatorial proofs for parts (b)–(e) of Theorems 1.1 and 1.3.

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