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Harmonic morphisms from solvable Lie groups

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Dedicated to the memory of Professor James Eells.

Abstract

In this paper we introduce two new methods for constructing harmonic morphisms from solvable Lie groups. The first method yields global solutions from any simply connected nilpotent Lie group and from any Riemannian symmetric space of non-compact type and rank $r \geqslant 3$. The second method provides us with global solutions from any Damek–Ricci space and many non-compact Riemannian symmetric spaces. We then give a continuous family of 3-dimensional solvable Lie groups not admitting any complex-valued harmonic morphisms, not even locally.

1. *Introduction*

The notion of a minimal submanifold of a given ambient space is of great importance in Riemannian geometry. Harmonic morphisms $\phi\colon (M, g) \to (N, h)$ between Riemannian manifolds are useful tools for the construction of such objects. They are solutions to over-determined non-linear systems of partial differential equations determined by the geometric data of the manifolds involved. For this reason harmonic morphisms are difficult to find and have no general existence theory, not even locally.

If the codomain is a surface the problem is invariant under conformal changes of the metric on N^2. Therefore, at least for local studies, the codomain can be taken to be the complex plane with its standard flat metric. For the general theory of harmonic morphisms between Riemannian manifolds we refer to the self-contained book [**3**] and the regularly updated on-line bibliography [**7**].

In [**2**] Baird and Wood classify harmonic morphisms from the famous 3-dimensional homogeneous geometries to surfaces. They construct a globally defined solution $\phi\colon Nil \to \mathbb{C}$ from the 3-dimensional nilpotent Heisenberg group and prove that the 3-dimensional solvable group *Sol* does not admit any solutions, not even locally.

In this paper we introduce two new methods for constructing complex-valued harmonic morphisms from solvable Lie groups. We prove that any simply connected nilpotent Lie group can be equipped with left-invariant Riemannian metrics admitting globally defined solutions.

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Our methods yield a variety of harmonic morphisms from a large collection of solvable Lie groups for example all Damek–Ricci spaces. We prove existence for most of the solvable Iwasawa groups NA associated with Riemannian symmetric spaces. Combining this with earlier results from [**8**], [**9**] and [**10**] we are able to prove the following conjecture except in the cases when the symmetric space is $G_2/\mathbf{SO}(4)$ or its non-compact dual.

CONJECTURE 1·1 *Let (M^m, g) be an irreducible Riemannian symmetric space of dimension $m \geqslant 2$. For each point $p \in M$ there exists a non-constant complex-valued harmonic morphism $\phi\colon U \to \mathbb{C}$ defined on an open neighbourhood U of p. If the space (M, g) is of non-compact type then the domain U can be chosen to be the whole of M.*

In the last section of the paper, we study the local existence of harmonic morphisms on 3-dimensional solvable Lie groups and prove the following result.

THEOREM 13·6 *Let $\mathfrak{g}$ be a 3-dimensional centerless, solvable Lie algebra and G a connected Lie group with Lie algebra $\mathfrak{g}$ and a left-invariant metric. Let $\mathcal{V}$ be a local conformal foliation by geodesics on G. Then G has constant sectional curvature.*

In contrast to the nilpotent case, we find a continuous family of solvable Lie groups not admitting any complex-valued harmonic morphisms, independent of their left-invariant metric. This family contains the famous example *Sol*.

2. *Harmonic morphisms*

Let M and N be two manifolds of dimensions m and n, respectively. A Riemannian metric g on M gives rise to the notion of a *Laplacian* on (M, g) and real-valued *harmonic functions* $f\colon (M, g) \to \mathbb{R}$. This can be generalized to the concept of *harmonic maps* $\phi\colon (M, g) \to (N, h)$ between Riemannian manifolds, which are solutions to a semi-linear system of partial differential equations, see [**3**].

Definition 2·1. A map $\phi\colon (M, g) \to (N, h)$ between Riemannian manifolds is called a *harmonic morphism* if, for any harmonic function $f\colon U \to \mathbb{R}$ defined on an open subset U of N with $\phi^{-1}(U)$ non-empty, $f \circ \phi\colon \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

The following characterization of harmonic morphisms between Riemannian manifolds is due to Fuglede and Ishihara. For the definition of horizontal (weak) conformality we refer to [**3**].

THEOREM 2·2 [**6**, **12**]. *A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.*

The next result of Baird and Eells gives the theory of harmonic morphisms a strong geometric flavour and shows that the case when $n = 2$ is particularly interesting. The conditions characterizing harmonic morphisms are then independent of conformal changes of the metric on the surface N^2.

THEOREM 2·3 [**1**]. *Let $\phi\colon (M^m, g) \to (N^n, h)$ be a horizontally (weakly) conformal map between Riemannian manifolds. If:*

(i) *$n = 2$, then ϕ is harmonic if and only if ϕ has minimal fibres at regular points*;

(ii) *$n \geqslant 3$, then two of the following conditions imply the other*:

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(a) *ϕ is a harmonic map;*
(b) *ϕ has minimal fibres at regular points;*
(c) *ϕ is horizontally homothetic.*

In this paper we are interested in complex-valued functions $\phi, \psi\colon (M, g) \to \mathbb{C}$ from Riemannian manifolds. In this situation the metric g induces the complex-valued Laplacian $\tau(\phi)$ and the gradient $\mathrm{grad}(\phi)$ with values in the complexified tangent bundle $T^{\mathbb{C}}M$ of M. We extend the metric g to be complex bilinear on $T^{\mathbb{C}}M$ and define the symmetric bilinear operator κ by

$$\kappa(\phi, \psi) = g(\mathrm{grad}(\phi), \mathrm{grad}(\psi)).$$

Two maps $\phi, \psi\colon M \to \mathbb{C}$ are said to be *orthogonal* if

$$\kappa(\phi, \psi) = 0.$$

The harmonicity and horizontal conformality of $\phi\colon (M, g) \to \mathbb{C}$ are expressed by the relations

$$\tau(\phi) = 0 \text{ and } \kappa(\phi, \phi) = 0.$$

Definition 2·4. Let (M, g) be a Riemannian manifold. Then a set

$$\Omega = \{\phi_k : M \to \mathbb{C} \mid k \in I\}$$

of complex-valued functions is said to be an *orthogonal harmonic family* on M if, for all $\phi, \psi \in \Omega$,

$$\tau(\phi) = 0 \text{ and } \kappa(\phi, \psi) = 0.$$

The problem of finding an orthogonal harmonic family on a Riemannian manifold can often be reduced to finding a harmonic morphism with values in $\mathbb{R}^n$.

PROPOSITION 2·5 *Let $\Phi : (M, g) \to \mathbb{R}^n$ be a harmonic morphism from a Riemannian manifold to the standard Euclidean $\mathbb{R}^n$ with $n \geqslant 2$. If V is an isotropic subspace of $\mathbb{C}^n$ then*

$$\Omega_V = \{\phi_v(x) = (\Phi(x), v) \mid v \in V\}$$

is an orthogonal harmonic family of complex-valued functions on (M, g).

Here and elsewhere in this paper $(\cdot, \cdot)$ refers to the standard symmetric bilinear form on the n-dimensional complex linear space $\mathbb{C}^n$.

Proof. Let $v, w \in V$, $\{X_1, \ldots, X_m\}$ be a local orthonormal frame for the tangent bundle TM and $\lambda : M \to \mathbb{R}_0^+$ be the dilation of Φ. Then

$$\tau(\phi_v) = \sum_{k=1}^{n} \tau(\Phi_k) v_k = 0$$

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and

$$\begin{aligned}\kappa(\phi_v, \phi_w) &= \sum_{k=1}^{m} X_k(\phi_v) \cdot X_k(\phi_w) \\ &= \sum_{k=1}^{m} \sum_{i,j=1}^{n} X_k(\Phi_i) v_i \cdot X_k(\Phi_j) w_j \\ &= \sum_{i,j=1}^{n} \kappa(\Phi_i, \Phi_j) v_i w_j \\ &= \lambda^2(v, w) \\ &= 0.\end{aligned}$$

The next result shows that orthogonal harmonic families can be useful for producing a variety of harmonic morphisms.

THEOREM 2·6 [**8**]. *Let (M, g) be a Riemannian manifold and*

$$\Omega = \{\phi_k : M \to \mathbb{C} \mid k = 1, \ldots, n\}$$

be a finite orthogonal harmonic family on (M, g). Let $\Phi : M \to \mathbb{C}^n$ be the map given by $\Phi = (\phi_1, \ldots, \phi_n)$ and U be an open subset of $\mathbb{C}^n$ containing the image $\Phi(M)$ of Φ. If $\mathcal{H} = \{h_i : U \to \mathbb{C} \mid i \in I\}$ is a family of holomorphic functions then the family $\mathcal{F}$ given by

$$\mathcal{F} = \{\psi : M \to \mathbb{C} \mid \psi = h(\phi_1, \ldots, \phi_n),\ h \in \mathcal{H}\}$$

is an orthogonal harmonic family on M. In particular, every element of $\mathcal{F}$ is a harmonic morphism.

3. *The general linear group* $\mathbf{GL}_n(\mathbb{R})$

Let $\mathbf{GL}_n(\mathbb{R})$ be the general linear group equipped with its standard Riemannian metric induced by the Euclidean scalar product on the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ given by

$$\langle X, Y \rangle = \operatorname{trace} XY^t.$$

For $1 \leqslant i, j, k, l \leqslant n$ we shall by E_{ij} denote the elements of $\mathfrak{gl}_n(\mathbb{R})$ satisfying

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl} \quad \text{and} \quad E_{ij}E_{kl} = \delta_{jk}E_{il}.$$

Let G be a subgroup of $\mathbf{GL}_n(\mathbb{R})$ with Lie algebra $\mathfrak{g}$ equipped with the induced Riemannian metric g. If $X \in \mathfrak{g}$ is a left-invariant vector field on G and $\phi, \psi : U \to \mathbb{C}$ are two complex-valued functions defined locally on G then

$$\begin{aligned}X(\phi)(p) &= \frac{d}{ds}[\phi(p \cdot \exp(sX))]\Big|_{s=0}, \\ X^2(\phi)(p) &= \frac{d^2}{ds^2}[\phi(p \cdot \exp(sX))]\Big|_{s=0}.\end{aligned}$$

This means that the operator κ is given by

$$\kappa(\phi, \psi) = \sum_{X \in \mathcal{B}} X(\phi) X(\psi),$$

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where $\mathcal{B}$ is any orthonormal basis for the Lie algebra $\mathfrak{g}$. Employing the Koszul formula for the Levi-Civita connection ∇ on G we see that

$$\begin{aligned} g(\nabla_X X, Y) &= g([Y, X], X) \\ &= \operatorname{trace}(YX - XY)X^t \\ &= \operatorname{trace} Y(XX^t - X^tX)^t \\ &= g([X, X^t], Y). \end{aligned}$$

Let $[X, X^t]_{\mathfrak{g}}$ be the orthogonal projection of the bracket $[X, X^t]$ onto $\mathfrak{g}$ in $\mathfrak{gl}_n(\mathbb{R})$. Then the above calculation shows that

$$\nabla_X X = [X, X^t]_{\mathfrak{g}}$$

so the Laplacian τ satisfies

$$\tau(\phi) = \sum_{X \in \mathcal{B}} X^2(\phi) - [X, X^t]_{\mathfrak{g}}(\phi).$$

4. *The nilpotent Lie group N_n*

The standard example of a nilpotent Lie group is the subgroup N_n of $\mathbf{GL}_n(\mathbb{R})$ consisting of $n \times n$ upper-triangular unipotent matrices

$$N_n = \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & x_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathbf{GL}_n(\mathbb{R}) \mid x_{ij} \in \mathbb{R} \right\}.$$

This inherits a natural left-invariant Riemannian metric from $\mathbf{GL}_n(\mathbb{R})$. The Lie algebra $\mathfrak{n}_n$ of N_n has the canonical orthonormal basis $\mathcal{B} = \{E_{rs} \mid r < s\}$ and its Levi–Civita connection ∇ satisfies

$$\nabla_{E_{rs}} E_{rs} = [E_{rs}, E_{rs}^t]_{\mathfrak{n}_n} = (E_{rr} - E_{ss})_{\mathfrak{n}_n} = 0$$

for all $E_{rs} \in \mathcal{B}$. Hence the Laplacian τ is given by

$$\tau(\phi) = \sum_{r<s} E_{rs}^2(\phi).$$

LEMMA 4·1 *Let $x_{ij} : N_n \to \mathbb{R}$ be the real valued coordinate functions*

$$x_{ij} : x \longmapsto e_i \cdot x \cdot e_j^t$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis for $\mathbb{R}^n$. If $i < j$ then the following relations hold

$$\tau(x_{ij}) = 0 \text{ and } \kappa(x_{ij}, x_{kl}) = \delta_{jl} \cdot \sum_{\max\{i,k\} \leqslant r < l} x_{ir} x_{kr}.$$

Proof. For an element X of the Lie algebra $\mathfrak{n}_n$ we have

$$X(x_{ij}) : x \longmapsto e_i \cdot x \cdot X \cdot e_j^t \text{ and } X^2(x_{ij}) : x \longmapsto e_i \cdot x \cdot X^2 \cdot e_j^t.$$

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This leads to the following

$$\begin{aligned}
\tau(x_{ij}) &= \sum_{r<s} E_{rs}^2(x_{ij}) = \sum_{r<s} e_i \cdot x \cdot E_{rs}^2 \cdot e_j^t = 0,\\
\kappa(x_{ij}, x_{kl}) &= \sum_{r<s} E_{rs}(x_{ij}) E_{rs}(x_{kl})\\
&= \sum_{r<s} e_i \cdot x \cdot E_{rs} \cdot e_j^t \cdot e_l \cdot E_{rs}^t \cdot x^t \cdot e_k^t\\
&= \sum_{r<s} e_i \cdot x \cdot E_{rs} \cdot E_{jl} \cdot E_{rs}^t \cdot x^t \cdot e_k^t\\
&= \sum_{r<s} \delta_{sj}\delta_{sl}\, e_i \cdot x \cdot E_{rr} \cdot x^t \cdot e_k^t\\
&= \delta_{jl} \sum_{r<l} e_i \cdot x \cdot e_r^t \cdot e_r \cdot x^t \cdot e_k^t\\
&= \delta_{jl} \cdot \sum_{\max\{i,k\} \leqslant r<l} x_{ir} x_{kr}.
\end{aligned}$$

THEOREM 4·2 *Let N_n be the nilpotent Lie group of real $n \times n$ upper-triangular unipotent matrices equipped with its standard Riemannian metric. Then the group epimorphism $\Phi : N_n \to \mathbb{R}^{n-1}$ defined by*

$$\Phi(x) = (x_{12}, \ldots, x_{n-1,n})$$

is a harmonic morphism.

Proof. As a direct consequence of Lemma 4·1 we see that the components $\phi_1, \ldots, \phi_{n-1}$ of the epimorphism $\Phi\colon N_n \to \mathbb{R}^{n-1}$ satisfy the following system of partial differential equations

$$\tau(\phi_k) = 0 \text{ and } \kappa(\phi_k, \phi_l) = \delta_{kl}.$$

The result is a direct consequence of these formulae.

5. *The first construction*

In this section we generalize the above construction for N_n to a large class of Lie groups. We find an algebraic condition on the Lie algebra which ensures the existence of at least local harmonic morphisms.

PROPOSITION 5·1 *Let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ and non-trivial quotient algebra $\mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of dimension n. Then there exists a natural group epimorphism $\Phi : G \to A \cong \mathbb{R}^n$ and left-invariant Riemannian metrics on G turning Φ into a Riemannian submersion.*

Proof. The two Lie algebras $\mathfrak{a}$ and $\mathbb{R}^n$ are Abelian so there exists an isomorphism $\psi : \mathfrak{a} \to \mathbb{R}^n$ which lifts to a Lie group isomorphism $\Psi\colon A \to \mathbb{R}^n$ from the connected and simply connected Lie group A with Lie algebra $\mathfrak{a}$.

On $\mathbb{R}^n$ we have the standard Euclidean scalar product. Equip the Lie algebra $\mathfrak{a}$ with the unique scalar product turning ψ into an isometry. This induces a left-invariant metric on A and the isomorphism $\Psi\colon A \to \mathbb{R}^n$ is clearly an isometry.

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Then equip the Lie algebra $\mathfrak{g}$ of G with *any* Euclidean scalar product such that the projection map $\pi\colon \mathfrak{g} \to \mathfrak{a}$ is a Riemannian submersion. This gives a Riemannian metric on G and the induced group epimorphism $\Pi\colon G \to A$ is a Riemannian submersion. It follows from the above construction that the composition $\Phi = \Psi \circ \Pi\colon G \to \mathbb{R}^n$ is a group epimorphism and its differential $d\Phi_e$ at e is a Lie algebra homomorphism. Furthermore Φ is a Riemannian submersion.

THEOREM 5·2 *Let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ and non-trivial quotient algebra $\mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of dimension n. Let g be a left-invariant Riemannian metric on G such that the natural group epimorphism $\Phi\colon (G, g) \to A \cong \mathbb{R}^n$ is a Riemannian submersion. For the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ let $\{X_1, \ldots, X_n\}$ be the orthonormal basis of the horizontal space $\mathcal{H}_e$ with $d\Phi_e(X_i) = e_i$ and define the vector $\xi \in \mathbb{C}^n$ by*

$$\xi = (\operatorname{trace} \operatorname{ad}_{X_1}, \ldots, \operatorname{trace} \operatorname{ad}_{X_n}).$$

For a maximal isotropic subspace W of $\mathbb{C}^n$ put

$$V = \{w \in W \mid (w, \xi) = 0\}.$$

If the dimension of the isotropic subspace V is at least 2 then

$$\Omega_V = \{\phi_v(x) = (\Phi(x), v) \mid v \in V\}$$

is an orthogonal family of globally defined harmonic morphisms on (G, g).

Proof. Since Φ is a group homomorphism from G to the Abelian group $\mathbb{R}^n$, we have for any $X \in \mathfrak{g}$,

$$\begin{aligned}\Phi(L_p \exp(tX)) &= \Phi(p) + \Phi(\exp(tX)) \\ &= \Phi(p) + \exp(td\Phi_e(X)) \\ &= \Phi(p) + td\Phi_e(X),\end{aligned}$$

where we have used exp to denote the exponential maps of both G and $\mathbb{R}^n$. Thus,

$$X^k(\Phi)(p) = \frac{d^k}{dt^k}(\Phi(L_p(\exp(tX)))|_{t=0} = 0$$

for $k > 1$.

Fix an orthonormal basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ is an orthonormal basis for $[\mathfrak{g}, \mathfrak{g}]$ and $\mathcal{B}_2$ for the orthogonal complement $[\mathfrak{g}, \mathfrak{g}]^\perp$. The tension field of Φ can now be calculated as follows:

$$\begin{aligned}\tau(\Phi)(p) &= \sum_{X \in \mathcal{B}} X^2(\Phi)(p) - d\Phi_p(\nabla_X X) \\ &= -d\Phi_e\left(\sum_{X \in \mathcal{B}} \nabla_X X\right) \\ &= -d\Phi_e\left(\sum_{X, Z \in \mathcal{B}} \langle \nabla_X X, Z\rangle Z\right) \\ &= \sum_{Z \in \mathcal{B}_2} \sum_{X \in \mathcal{B}} \langle [Z, X], X\rangle d\Phi_e(Z) \\ &= \sum_{Z \in \mathcal{B}_2} (\operatorname{trace} \operatorname{ad}_Z) d\Phi_e(Z).\end{aligned}$$

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6. *Nilpotent Lie groups*

In this section we show that every connected, simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$ can be equipped with natural Riemannian metrics admitting complex-valued harmonic morphisms on G. When the algebra is Abelian the problem is completely trivial, so we shall assume that $\mathfrak{g}$ is not Abelian. In that case we have the following well-known fact.

LEMMA 6·1 *Let $\mathfrak{g}$ be a non-Abelian nilpotent Lie algebra. Then the dimension of the quotient algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is at least* 2.

Proof. Since the algebra $\mathfrak{g}$ is nilpotent we have $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Assume that the quotient algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is of dimension 1 i.e.

$$\mathfrak{g} = \mathbb{R}X \oplus [\mathfrak{g}, \mathfrak{g}]$$

for some $X \in \mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ and since of course $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \subseteq [\mathfrak{g}, \mathfrak{g}]$ we must have $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$. But as $\mathfrak{g}$ is nilpotent, this is only possibible if $[\mathfrak{g}, \mathfrak{g}] = 0$ i.e. if $\mathfrak{g}$ is Abelian.

For non-Abelian nilpotent Lie groups the existence result of Theorem 5·2 simplifies to the following.

THEOREM 6·2 *Let G be a connected, simply connected, non-Abelian and nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then there exist Riemannian metrics g on G such that the natural group epimorphism Φ: $(G, g) \to A \cong \mathbb{R}^n$ is a Riemannian submersion. If V is a maximal isotropic subspace of $\mathbb{C}^n$ then*

$$\Omega_V = \{\phi_v(x) = (\Phi(x), v) \mid v \in V\}$$

is an orthogonal family of globally defined harmonic morphisms on (G, g).

Proof. The Lie algebra $\mathfrak{g}$ is nilpotent, so if $Z \in \mathfrak{g}$ then $\operatorname{trace} \operatorname{ad}_Z = 0$. This means that the vector ξ defined in Theorem 5·2 vanishes. The result is then a direct consequence of Lemma 6·1.

7. *The nilpotent Heisenberg group H_n*

We shall now apply Theorem 6·2 to yield an orthogonal family of complex-valued harmonic morphisms from the well-known $(2n + 1)$-dimensional nilpotent Heisenberg group

$$H_n = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & I_n & y \\ 0 & 0 & 1 \end{pmatrix} \in N_{n+2} \mid x, y^t \in \mathbb{R}^n,\ z \in \mathbb{R} \right\},$$

where I_n is the $n \times n$ identity matrix. A canonical orthonormal basis $\mathcal{B}$ for the nilpotent Lie algebra $\mathfrak{h}_n$ of H_n consists of the following matrices

$$X_k = \begin{pmatrix} 0 & e_k & 0 \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0_n & e_k^t \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis for $\mathbb{R}^n$. The derived algebra $[\mathfrak{h}_n, \mathfrak{h}_n]$ is generated by Z and the quotient $\mathfrak{h}_n/[\mathfrak{h}_n, \mathfrak{h}_n]$ can be identified with the $2n$-dimensional subspace generated

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by $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. The natural group epimorphism Φ: $H_n \to \mathbb{R}^{2n}$ is given by

$$\Phi: \begin{pmatrix} 1 & x & z \\ 0 & I_n & y \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x_1, \ldots, x_n, y_1, \ldots, y_n).$$

THEOREM 7·1 *Let H_n be the $(2n+1)$-dimensional Heisenberg group and Φ: $H_n \to \mathbb{R}^{2n}$ be the natural group epimorphism. If V is a maximal isotropic subspace of $\mathbb{C}^{2n}$ then*

$$\Omega_V = \{\phi_v : x \longmapsto (\Phi(x), v) \mid v \in V\}$$

is an orthogonal family of globally defined harmonic morphisms on H_n.

In the case $n = 1$, there is essentially only one choice for V giving the harmonic morphism ϕ: $H_1 \to \mathbb{C}$ with

$$\phi: \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \longmapsto x + iy.$$

Baird and Wood show in [3] that, up to weakly conformal transformations on the codomain, any harmonic morphism from an open subset of $Nil = H_1$ to a surface is the restriction of this map ϕ.

8. *The nilpotent Lie group K_n*

In this section we employ Theorem 6·2 to construct a harmonic morphism on the $(n+1)$-dimensional nilpotent subgroup K_n of $\mathbf{GL}_{n+1}(\mathbb{R})$ given by

$$K_n = \left\{ \left(\begin{array}{ccccccc} 1 & x & p_2(x) & p_3(x) & \cdots & p_{n-1}(x) & y_1 \\ 0 & 1 & x & p_2(x) & \cdots & p_{n-2}(x) & y_2 \\ & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & 0 & 1 & x & p_2(x) & y_{n-2} \\ & & & 0 & 1 & x & y_{n-1} \\ & & & & 0 & 1 & y_n \\ & & & & & 0 & 1 \end{array} \right) \middle| \; x, y_k \in \mathbb{R} \right\},$$

where the polynomials $p_2, \ldots, p_{n-1}$ are given by $p_k(x) = x^k/k!$ For the Lie algebra

$$\mathfrak{k}_n = \left\{ \left(\begin{array}{ccccccc} 0 & \alpha & 0 & & & 0 & \beta_1 \\ & 0 & \alpha & 0 & & 0 & \beta_2 \\ & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & 0 & \alpha & 0 & \beta_{n-2} \\ & & & & 0 & \alpha & \beta_{n-1} \\ & & & & & 0 & \beta_n \\ & & & & & & 0 \end{array} \right) \in \mathfrak{gl}_{n+1}(\mathbb{R}) \mid \alpha, \beta_k \in \mathbb{R} \right\}$$

of K_n we have the orthonormal basis $\mathcal{B}$ consisting of the matrices

$$Y_1 = E_{1,n+1}, \ldots, Y_n = E_{n,n+1} \text{ and } X = \frac{1}{\sqrt{n-1}}(E_{12} + \cdots + E_{n-1,n}).$$

The derived algebra $[\mathfrak{k}_n, \mathfrak{k}_n]$ is generated by the vectors $Y_1, \ldots, Y_{n-1}$ and the quotient algebra $\mathfrak{k}_n/[\mathfrak{k}_n, \mathfrak{k}_n]$ can be identified with the 2-dimensional subspace of $\mathfrak{k}_n$ generated by X and Y_n.

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The natural group epimorphism $\Phi\colon K_n \to \mathbb{C}$ given by

$$\Phi\colon p \longmapsto (x\sqrt{n-1} + iy_n)$$

is a globally defined harmonic morphism on K_n.

9. *Compact nilmanifolds*

A *nilmanifold* is a homogeneous space of a nilpotent Lie group. As is well known, any compact nilmanifold is of the form G/Γ, where G is a nilpotent Lie group and Γ a *uniform* subgroup of G, i.e., a co-compact discrete subgroup. We have proved the global existence of harmonic morphisms on any simply connected nilpotent Lie group with a left-invariant metric. We include here a section where we prove the existence of a globally defined harmonic morphism on any compact nilmanifold G/Γ, for which G has rational structure constants.

To begin with, assume that (M, g) is a compact Riemannian manifold and $\omega_1, \dots, \omega_k$ be a basis for the linear space of harmonic 1-forms on M; thus k is the first Betti number of M. Define the lattice Λ in $\mathbb{R}^k$ as the integer span of the vectors

$$\left(\int_\gamma \omega_1, \dots, \int_\gamma \omega_k\right) \text{ with } \gamma \in H_1(M, \mathbb{Z}).$$

After fixing a point $p \in M$ we can define the correponding *Albanese map*

$$\pi_p\colon M \longrightarrow \mathbb{R}^k/\Lambda,\ \pi_p(q) = \left(\int_p^q \omega_1, \dots, \int_p^q \omega_k\right) \bmod \Lambda.$$

Note that π_p is a harmonic map with respect to the flat metric on the torus $\mathbb{R}^k/\Lambda$.

To continue our construction, we need the following simple lemma, where $\mathcal{L}$ denotes Lie derivative.

LEMMA 9·1 *Assume that ω is a harmonic p-form on a compact Riemannian manifold (M, g), and X a Killing vector field. Then*

$$\mathcal{L}_X \omega = 0.$$

Proof. Let φ_t be the flow of X. As φ_t is an isometry, $\varphi_t^*\omega$ is also harmonic. Hence $\mathcal{L}_X\omega = d\iota_X\omega$ is a harmonic p-form, but as it is also exact, it must vanish.

Let G be a Lie group and Γ be a co-compact, discrete subgroup of G. We equip G with a left-invariant metric and G/Γ with the metric which makes the quotient map $G \to G/\Gamma$ into a Riemannian submersion. Thus G acts by isometries on G/Γ. By the above lemma, any harmonic 1-form on G/Γ is left-invariant by G, and so has constant pointwise norm. Hence, any two harmonic 1-forms which are orthogonal at one point, remain orthogonal everywhere. Thus we can choose $\omega_1, \dots, \omega_k$ in the above construction such that the relation

$$\langle \omega_i, \omega_j \rangle = \delta_{ij}$$

holds everywhere on G/Γ. This makes the Albanese map into a Riemannian harmonic submersion and hence a harmonic morphism.

Malcev proves in [**13**] that a simply connected nilpotent Lie group G contains a uniform subgroup Γ if and only if the Lie algebra $\mathfrak{g}$ of G has rational structure constants with respect to some basis. It is known that this condition is always satisfied if the dimension of $\mathfrak{g}$ is less than 7. In [**15**] Nomizu shows that the first Betti number of the quotient G/Γ equals the

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codimension of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$, which by Lemma 6·1 is at least 2 unless the algebra is Abelian. The above arguments now deliver the following result.

THEOREM 9·2 *Let G be a non-Abelian nilpotent Lie group for which the Lie algebra $\mathfrak{g}$ has rational structure constants in some basis. For any uniform subgroup Γ and any G-invariant Riemannian metric g on G/Γ, there exists a harmonic morphism from $(G/\Gamma, g)$ into a flat torus of dimension at least 2.*

To show that the lift of the Albanese map to a map between the universal coverings is essentially the same as we constructed in the previous section, we show the following result.

PROPOSITION 9·3 *Assume that G is a connected and simply connected Lie group, and Γ a uniform subgroup of G, and equip G and G/Γ with G-invariant metrics for which the covering map $G \to G/\Gamma$ is a local isometry. Then any harmonic map $\phi\colon G \to \mathbb{R}^n$ which is right-invariant under the action of Γ is a homomorphism followed by a translation in $\mathbb{R}^n$.*

Proof. By translating, we may assume that $\phi(e) = 0$. Consider the 1-form $\omega = d\phi$ on G. This is the lift of a harmonic 1-form on G/Γ, and so, by Lemma 9·1, is left-invariant on G. Fix a $g \in G$ and define $\Phi(h) = \phi(g) + \phi(h)$ and $\Psi(h) = \phi(gh)$. As $d\phi$ is left-invariant, it follows easily that $d\Phi = d\Psi$, and since $\Phi(e) = \Psi(e)$, we get $\Phi = \Psi$. Hence ϕ is a homomorphism.

Thus, the lift of the Albanese map is a homomorphism and a Riemannian submersion $\phi\colon G \to \mathbb{R}^n$, where $n = \dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Since the kernel of $d\phi$ necessarily contains $[\mathfrak{g}, \mathfrak{g}]$, these spaces must coincide. This shows that ϕ, up to a translation in $\mathbb{R}^n$, is precisely the map constructed in Theorem 6·2.

10. *The solvable Lie group S_n*

The standard example of a solvable Lie group is the subgroup S_n of $\mathbf{GL}_n(\mathbb{R})$ of $n \times n$ upper-triangular matrices. This inherits a natural left-invariant Riemannian metric from $\mathbf{GL}_n(\mathbb{R})$. The connected component G of S_n containing the neutral element e is given by

$$G = \left\{ \begin{pmatrix} e^{t_1} & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\ 0 & e^{t_2} & \cdots & x_{2,n-1} & x_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & e^{t_{n-1}} & x_{n-1,n} \\ 0 & \cdots & 0 & 0 & e^{t_n} \end{pmatrix} \in \mathbf{GL}_n(\mathbb{R}) \mid x_{ij}, t_i \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}$ of G consist of all upper-triangular matrices and has the orthogonal decomposition $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{n}_n$ where $\mathfrak{n}_n$ is the Lie algebra for N_n and $\mathfrak{d}$ is generated by the diagonal elements $D_1 = E_{11}, \ldots, D_n = E_{nn} \in \mathfrak{g}$. The derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is $\mathfrak{n}_n$ and the quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ can be identified with $\mathfrak{d}$. The natural group epimorphism $\Phi\colon G \to \mathbb{R}^n$ is the Riemannian submersion given by

$$\Phi\colon g \longrightarrow (t_1, \ldots, t_n).$$

THEOREM 10·1 *Let G be the connected component of the solvable Lie group S_n containing the neutral element e and $\Phi\colon G \to \mathbb{R}^n$ be the natural group epimorphism. Let the vector $\xi \in \mathbb{C}^n$ be given by*

$$\xi = ((n+1) - 2, (n+1) - 4, \ldots, (n+1) - 2n).$$

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If $n \geqslant 3$, W is a maximal isotropic subspace of $\mathbb{C}^n$ and

$$V = \{w \in W \mid (w, \xi) = 0\},$$

then

$$\Omega_V = \{\phi_v(x) = (\Phi(x), v) \mid v \in V\}$$

is an orthogonal family of globally defined harmonic morphisms on G.

Proof. The subspace $\mathfrak{d}$ of $\mathfrak{g}$ is the horizontal space $\mathcal{H}_e$ of Φ: $G \to \mathbb{R}^n$ and has the orthonormal basis $\{D_1, \ldots, D_n\}$. For the diagonal elements D_t we have

$$\operatorname{trace} \operatorname{ad}_{D_t} = (n+1) - 2t$$

which gives

$$\begin{aligned} \xi &= (\operatorname{trace} \operatorname{ad}_{X_1}, \ldots, \operatorname{trace} \operatorname{ad}_{X_n}) \\ &= ((n+1) - 2, (n+1) - 4, \ldots, (n+1) - 2n). \end{aligned}$$

This means that for any maximal isotropic subspace W of $\mathbb{C}^n$ the space $V = \{w \in W \mid (w, \xi) = 0\}$ is at least two-dimensional. The result is then a consequence of Theorem 5·2.

11. *Symmetric spaces of rank $r \geqslant 3$*

In this section we employ Theorem 5·2 to construct complex-valued harmonic morphisms from Riemannian symmetric spaces of rank at least 3. For the details of their structure theory we refer to [**11**]. In the next section we will construct harmonic morphisms on symmetric spaces of rank one through a different method.

An irreducible Riemannian symmetric space of non-compact type may be written as G/K where G is a simple, connected and simply connected Lie group and K is a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively, and by

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

the corresponding Cartan decomposition of $\mathfrak{g}$. According to the Iwasawa decomposition of $\mathfrak{g}$ we have

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k},$$

where $\mathfrak{a}$ is a maximal Abelian subalgebra of $\mathfrak{p}$ and $\mathfrak{n}$ a nilpotent subalgebra of $\mathfrak{g}$. Furthermore, the subalgebra

$$\mathfrak{s} = \mathfrak{n} + \mathfrak{a}$$

is a solvable subalgebra of $\mathfrak{g}$. On the group level we have similar decompositions

$$G = NAK \quad \text{and} \quad S = NA$$

where N is a normal subgroup of S.

As S acts simply transitively on G/K we thus obtain a diffeomorphism

$$G/K \cong S$$

and, as the action of S on G/K is isometric, the induced metric on S is left-invariant.

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Since $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{n}$, we see that the codimension of the derived algebra of $\mathfrak{s}$ is the dimension of $\mathfrak{a}$ i.e. the rank of G/K. Hence, when the rank is at least 3, the statement of Theorem 5·2 shows that there exist globally defined complex-valued harmonic morphisms on G/K.

Example 11·1. For the real special linear group $\mathbf{SL}_n(\mathbb{R})$ we have the Iwasawa decomposition NAK where $N = N_n$, $K = \mathbf{SO}(n)$ and A is the subgroup of diagonal matrices with positive diagonal elements. The solvable group $S = N_n A \cong \mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n)$ is given by

$$S = \left\{ \begin{pmatrix} e^{t_1} & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\ 0 & e^{t_2} & \cdots & x_{2,n-1} & x_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & e^{t_{n-1}} & x_{n-1,n} \\ 0 & \cdots & 0 & 0 & e^{t_n} \end{pmatrix} \middle| \; x_{ij}, t_i \in \mathbb{R},\ t_1 + \cdots + t_n = 0 \right\}.$$

The natural epimorphism $\Phi\colon S \to \mathbb{R}^{n-1}$ is given by $\Phi\colon g \mapsto (t_1, \ldots, t_n)$ where

$$\mathbb{R}^{n-1} \cong \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}.$$

We shall now describe a duality principle introduced by the authors in [**8**]. This shows how a locally defined complex-valued harmonic morphism from a Riemannian symmetric space G/K of non-compact type gives rise to a dual locally defined harmonic morphism from its compact dual U/K, and vice versa. Recall that any harmonic morphism between real analytic Riemannian manifolds is real analytic, see [**3**].

Let W be an open subset of G/K and $\phi\colon W \to \mathbb{C}$ a real analytic map. By composing ϕ with the projection $G \to G/K$ we obtain a real analytic K-invariant map $\hat{\phi}\colon \hat{W} \to \mathbb{C}$ from some open subset $\hat{W}$ of G. Let $G^{\mathbb{C}}$ denote the complexification of the Lie group G. Then $\hat{\phi}$ extends uniquely to a K-invariant holomorphic map $\phi^{\mathbb{C}}\colon W^{\mathbb{C}} \to \mathbb{C}$ from some open subset $W^{\mathbb{C}}$ of $G^{\mathbb{C}}$. By restricting this map to $U \cap W^{\mathbb{C}}$ and factoring through the projection $U \to U/K$, we obtain a real analytic map $\phi^*\colon W^* \to \mathbb{C}$ from some open subset W^* of U/K. The following result is a direct consequence of [**8**, Theorem 7.1]

LEMMA 11·2 *Let $\phi\colon W \to \mathbb{C}$ be a map as described above and $\phi^*\colon W^* \to \mathbb{C}$ be the dual map. Then ϕ^* is a harmonic morphism if and only if ϕ is a harmonic morphism.*

This duality principle gives the following interesting result.

THEOREM 11·3 *Let (M, g) be an irreducible Riemannian symmetric space of rank $r \geqslant 3$. Then for each point $p \in M$ there exists a non-constant complex-valued harmonic morphism $\phi\colon U \to \mathbb{C}$ defined on an open neighbourhood U of p. If the space (M, g) is of non-compact type then the domain U can be chosen to be the whole of M.*

Theorem 11·3 gives the first known examples of complex-valued harmonic morphisms from the non-compact classical Riemannian symmetric spaces

$$\mathbf{SO}(p, p)/\mathbf{SO}(p) \times \mathbf{SO}(p),$$
$$\mathbf{SO}(p, p+1)/\mathbf{SO}(p) \times \mathbf{SO}(p+1),$$
$$\mathbf{Sp}(p, p)/\mathbf{Sp}(p) \times \mathbf{Sp}(p)$$

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with $p \geqslant 3$ and their compact duals. As for the exceptional symmetric spaces, the result gives the first known examples from the non-compact

$$E_6^2/\mathbf{SU}(6) \times \mathbf{SU}(2),$$
$$E_7^{-5}/\mathbf{SO}(12) \times \mathbf{SU}(2),$$
$$E_8^{-24}/E_7 \times \mathbf{SU}(2),$$
$$F_4^4/\mathbf{Sp}(3) \times \mathbf{SU}(2),$$
$$E_6^6/\mathbf{Sp}(4),$$
$$E_8^8/\mathbf{SO}(16)$$

and their compact duals.

12. *The second construction*

In this section we introduce a new method for constructing complex-valued harmonic morphisms from a certain class of solvable Lie groups. This provides solutions on any Damek–Ricci space and many Riemannian symmetric spaces.

Let $\mathfrak{s}$ be a Lie algebra which can be decomposed as the sum of two subalgebras

$$\mathfrak{s} = \mathfrak{n} + \mathfrak{a}, \quad \mathfrak{n} \cap \mathfrak{a} = \{0\},$$

where $\mathfrak{n}$ is a nilpotent ideal and $\mathfrak{a}$ is Abelian. Furthermore let $\mathfrak{s}$ be equipped with an inner product for which $\mathfrak{n}$ and $\mathfrak{a}$ are orthogonal and the adjoint action of $\mathfrak{a}$ on $\mathfrak{n}$ is self-adjoint with respect to this inner product. Thus, we have a subset of "roots" Σ in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and an orthogonal decomposition

$$\mathfrak{n} = \sum_{\alpha \in \Sigma} \mathfrak{n}_\alpha, \quad \text{where } [V, X] = \alpha(V)X$$

for any $V \in \mathfrak{a}$ and $X \in \mathfrak{n}_\alpha$. Finally, we assume that there is at least one root $\beta \in \Sigma$ with the property that

$$\mathfrak{n}_\beta \perp [\mathfrak{n}, \mathfrak{n}].$$

We write $\Sigma^* = \Sigma \setminus \{\beta\}$ and put

$$\mathfrak{m} = \sum_{\alpha \in \Sigma^*} \mathfrak{n}_\alpha.$$

Let S be a simply connected Lie group with Lie algebra $\mathfrak{s}$ and A, N and M be the Lie subgroups of S with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{m}$, respectively. Then $S = N \cdot A$ and M is a closed normal subgroup of N with N/M Abelian and $A \cdot M = M \cdot A$. Furthermore, A, M and N are simply connected, hence so is the quotient N/M.

We equip S with the left-invariant Riemannian metric induced by the inner product on $\mathfrak{s}$ and give N/M the unique left-invariant metric for which the homogeneous projection $N \to N/M$ is a Riemannian submersion.

THEOREM 12·1 *For the above situation the map*

$$\Psi\colon S = N \cdot A \longrightarrow N/M, \quad \Psi(n \cdot a) = n \cdot M$$

is a harmonic morphism.

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Note that, since N/M is Abelian and simply connected, there is an isometric group isomorphism $N/M \cong \mathbb{R}^k$ where $k = \dim \mathfrak{n}_\beta$.

Proof. Denote by $o = eM$ the identity coset of N/M. The fibre of Ψ over a point $n \cdot o$ is the set $n \cdot M \cdot A = n \cdot A \cdot M$. Thus, the vertical space of Ψ at the point $n \cdot a$ is the left translate of $\mathfrak{a} + \mathfrak{m}$, i.e.

$$\mathcal{V}_{n\cdot a} = d(L_{n\cdot a})_e(\mathfrak{a} + \mathfrak{m});$$

the horizontal space is therefore the left translate of $\mathfrak{n}_\beta$:

$$\mathcal{H}_{n\cdot a} = d(L_{n\cdot a})_e(\mathfrak{n}_\beta).$$

This means that for $X \in \mathfrak{n}_\beta$ we have

$$\begin{aligned} d\Psi_{n\cdot a}(d(L_{n\cdot a})_e(X)) &= \frac{d}{dt}\Psi(n \cdot a \cdot \exp(tX))\Big|_{t=0} \\ &= \frac{d}{dt} n \cdot \exp(t \operatorname{Ad}_a(X)) \cdot o\Big|_{t=0} \\ &= d(L_n)_o(\operatorname{Ad}_a(X)). \end{aligned}$$

Now, there is a unique element $V \in \mathfrak{a}$ such that $a = \exp(V)$. Thus,

$$\|d\Psi_{n\cdot a}(d(L_{n\cdot a})_e(X))\|^2 = \|\operatorname{Ad}_a(X)\|^2 = \|e^{\operatorname{ad}(V)}(X)\|^2 = e^{2\beta(V)}\|X\|^2.$$

This implies that Ψ is horizontally conformal with dilation

$$\lambda^2(n \cdot a) = e^{2\beta(V)}$$

so in particular, Ψ is horizontally homothetic. This means that for proving that Ψ is a harmonic morphism it is sufficient to show that it has minimal fibres. In our special situation it is enough to show that the fibre over $o \in N/M$ is minimal. Since $\Psi^{-1}(o) = M \cdot A$ is a subgroup of S, it even sufficies to check that it is minimal at the identity element $e \in S$.

Let $\{f_i\}$ be an orthonormal basis of $\mathfrak{a}$ and for any $\alpha \in \Sigma^*$ let $\{e_k^\alpha\}$ be an orthonormal basis for $\mathfrak{n}_\alpha$. Then $\{f_i, e_k^\alpha\}$ is an orthonormal basis for the sum $\mathfrak{a}+\mathfrak{m}$. Let H be the mean curvature vector field along the fibre $\Psi^{-1}(o)$ and $X \in \mathfrak{n}_\beta$ as above, then

$$\begin{aligned} \langle H, X\rangle &= \sum_i \langle \nabla_{f_i} f_i, X\rangle + \sum_{\alpha\in\Sigma^*}\sum_k \langle \nabla_{e_k^\alpha} e_k^\alpha, X\rangle \\ &= \sum_i \langle [X, f_i], f_i\rangle + \sum_{\alpha\in\Sigma^*}\sum_k \langle [X, e_k^\alpha], e_k^\alpha\rangle \\ &= 0, \end{aligned}$$

since $[X, f_i] \in \mathfrak{n}_\beta \perp \mathfrak{a}$ and $[X, e_k^\alpha] \in \mathfrak{n}_{\beta+\alpha} \perp \mathfrak{n}_\alpha$.

The proof shows that the fibres of Ψ are in fact totally geodesic.

Example 12·2. Consider an irreducible Riemannian symmetric space G/K of non-compact type. We recall again the Iwasawa decomposition

$$G = N \cdot A \cdot K, \quad \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$$

of the group G and its Lie algebra $\mathfrak{g}$. Choose a root space $\mathfrak{n}_\beta \subset \mathfrak{n}$ associated to a simple, restricted root β. The above construction now yields a harmonic morphism

$$G/K \cong N \cdot A \longrightarrow N/M \cong \mathbb{R}^k,$$

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where $k = \dim \mathfrak{n}_\beta$. Thus, whenever β can be chosen with $k \geqslant 2$, there are globally defined, complex-valued harmonic morphisms on G/K, regardless of the rank. When $k = 1$ we get real-valued harmonic morphisms, i.e. harmonic functions, globally defined on G/K.

Besides giving new examples on some of the spaces considered in the previous section, we also find the first known complex-valued harmonic morphisms on the spaces

$$\mathbf{Sp}(2,2)/\mathbf{Sp}(2) \times \mathbf{Sp}(2), \quad E_6^{-26}/F_4, \quad F_4^{-20}/\mathbf{Spin}(9),$$

and their compact duals.

Example 12·3. Recall that a *Damek–Ricci space* is a simply connected solvable Lie group S with a left-invariant metric, such that its Lie algebra is an orthogonal direct sum

$$\mathfrak{s} = \mathfrak{v} + \mathfrak{z} + \mathfrak{a},$$

where $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$ is a generalized Heisenberg algebra with center $\mathfrak{z}$, $\mathfrak{a}$ is 1-dimensional with a fixed, non-zero vector $A \in \mathfrak{a}$, and

$$[A, V] = \frac{1}{2}V, \quad [A, Y] = Y \qquad (A \in \mathfrak{a},\ V \in \mathfrak{v},\ Y \in \mathfrak{z}).$$

Clearly, these spaces generalize non-compact Riemannian symmetric spaces of rank one. They are irreducible Riemannian harmonic manifolds, and were introduced by Damek and Ricci to provide counterexamples to the conjecture by Lichnerowicz, stating that any Riemannian harmonic manifold is 2-point symmetric [**5**]. For more information about Damek-Ricci spaces and generalized Heisenberg algebras, we refer to [**4**].

Let A, N, Z be the subgroups of S with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{z}$, respectively. The above construction gives us a harmonic morphism

$$\varphi : S = N \cdot A \longrightarrow N/Z \cong \mathbb{R}^k,$$

where $k = \dim \mathfrak{v}$.

13. 3-*dimensional solvable Lie groups*

In this section we study the structure of conformal foliations by geodesics on 3-dimensional solvable Lie groups. We start by reviewing some general terminology.

Assume that $\mathcal{V}$ is an involutive distribution on a Riemannian manifold (M, g) and denote by $\mathcal{H}$ its orthogonal complement. As customary, we also use $\mathcal{V}$ and $\mathcal{H}$ to denote the orthogonal projections onto the corresponding subbundles of TM and we identify $\mathcal{V}$ with the corresponding foliation tangent to $\mathcal{V}$. The second fundamental form for $\mathcal{V}$ is given by

$$B^{\mathcal{V}}(U, V) = \frac{1}{2}\mathcal{H}(\nabla_U V + \nabla_V U) \qquad (U, V \in \mathcal{V}),$$

while the second fundamental form for $\mathcal{H}$ is given by

$$B^{\mathcal{H}}(X, Y) = \frac{1}{2}\mathcal{V}(\nabla_X Y + \nabla_Y X) \qquad (X, Y \in \mathcal{H}).$$

Recall that $\mathcal{V}$ is said to be *conformal* if there is a vector field V, tangent to $\mathcal{V}$, such that

$$B^{\mathcal{H}} = g \otimes V,$$

and $\mathcal{V}$ is said to be *Riemannian* if $V = 0$. Furthermore, $\mathcal{V}$ is said to be *totally geodesic* if $B^{\mathcal{V}} = 0$. This is equivalent to the leaves of $\mathcal{V}$ being totally geodesic submanifolds of M.

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It is easy to see that the fibres of a horizontally conformal map (resp. Riemannian submersion) give rise to a conformal foliation (resp. Riemannian foliation). Conversely, any conformal foliation (resp. Riemannian foliation) is locally the fibres of a horizontally conformal map (resp. Riemannian submersion), see [**3**]. When the codimension of the foliation is 2, the map is harmonic if and only if the leaves of the foliation are minimal submanifolds.

Before restricting ourselves to the 3-dimensional situation, we note the following result.

PROPOSITION 13·1 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and a left-invariant metric. Then the left-translation of any subspace of the centre of $\mathfrak{g}$ generates a totally geodesic Riemannian foliation of G.*

Proof Let $\mathcal{V}$ be the foliation thus obtained and $\mathcal{H}$ its orthogonal complement. If U and V are left-invariant vector field in $\mathcal{V}$ and X and Y left-invariant in $\mathcal{H}$ then clearly

$$\langle B^{\mathcal{V}}(U, V), X\rangle = \frac{1}{2}(\langle [U, V], X\rangle + \langle [X, U], V\rangle - \langle [V, X], U\rangle) = 0$$

and

$$\begin{aligned}\langle B^{\mathcal{H}}(X, Y), U\rangle &= \frac{1}{2}(\langle \nabla_X Y, U\rangle + \langle \nabla_Y X, U\rangle)\\ &= \frac{1}{4}(\langle [X, Y], U\rangle + \langle [U, X], Y\rangle - \langle [Y, U], X\rangle\\ &\quad + \langle [Y, X], U\rangle + \langle [U, Y], X\rangle - \langle [X, U], Y\rangle)\\ &= 0.\end{aligned}$$

Thus $\mathcal{V}$ is Riemannian and totally geodesic.

Example 13·2. The solvable Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} \,\middle|\, t_1, t_2, x \in \mathbb{R} \right\}$$

has a 1-dimensional center. Thus the corresponding simply connected Lie group S_2 admits a Riemannian foliation by geodesics, regardless of which left-invariant metric we equip it with.

The following result shows that the 3-dimensional situation is very special with respect to conformal foliations by geodesics.

THEOREM 13·3 [**3**]. *Let M be a 3-dimensional Riemannian manifold with non-constant sectional curvature. Then there are at most two distinct conformal foliations by geodesics of M. If there is an open subset on which the Ricci tensor has precisely two distinct eigenvalues, then there is at most one conformal foliation by geodesics of M.*

This result can now be applied to the situation of 3-dimensional Lie groups with left-invariant metrics.

PROPOSITION 13·4 *Let G be a connected 3-dimensional Lie group with a left-invariant metric of non-constant sectional curvature. Then any local conformal foliation by geodesics of a connected open subset of G can be extended to a global conformal foliation by geodesics of G. This is given by the left-translation of a 1-parameter subgroup of G.*

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This result is clearly false if we omit the assumption on the sectional curvature. See, e.g., [**3**, Chapter 1] for examples of locally defined harmonic morphisms $U \subset \mathbb{R}^3 \to \mathbb{C}$ where the induced foliations do not arise in this way.

Proof. Assume that $\mathcal{V}$ is a conformal foliation by geodesics of some connected neighbourhood U of the identity element e of G and denote by $\mathcal{H}$ the orthogonal complement of $\mathcal{V}$. Let $U' \subset U$ be a connected neighbourhood of e such that $gh \in U$ for all $g, h \in U'$, and let $U'' \subset U'$ be a connected neigbourhood of e for which $g^{-1} \in U'$ for all $g \in U''$.

For any $g \in G$, we denote by $L_g \colon G \to G$ left translation by G. Take $g \in U''$ and consider the distribution $dL_g\mathcal{V}\big|_{U'}$, obtained by restricting $\mathcal{V}$ to U' and translating with g. As L_g is an isometry, this is also a conformal foliation by geodesics of L_gU', which is a connected neigbourhood of e. It is clear from Theorem 13·3 and by continuity, that this distribution must coincide with $\mathcal{V}$ restricted to L_gU'. It follows that $d(L_g)_h(\mathcal{V}_h) = \mathcal{V}_{gh}$ for all $g, h \in U''$. In particular we have

$$d(L_g)_e(\mathcal{V}_e) = \mathcal{V}_g \qquad (g \in U'').$$

Define a 1-dimensional distribution $\tilde{\mathcal{V}}$ on G by

$$\tilde{\mathcal{V}}_g = (dL_g)_e(\mathcal{V}_e) \qquad (g \in G).$$

Its horizontal distribution $\tilde{\mathcal{H}}$ is clearly given by left translation of $\mathcal{H}_e$. From the above we see that

$$\tilde{\mathcal{V}}\big|_{U''} = \mathcal{V}\big|_{U''}.$$

It follows that

$$B^{\tilde{\mathcal{V}}}\big|_{U''} = B^{\mathcal{V}}\big|_{U''} = 0,$$

and since $\tilde{\mathcal{V}}$ is left-invariant, it follows that $B^{\tilde{\mathcal{V}}} = 0$ everywhere, i.e., $\tilde{\mathcal{V}}$ is totally geodesic. In the same way we see that $\tilde{\mathcal{V}}$ is a conformal distribution and, by Theorem 13·3, we see that

$$\tilde{\mathcal{V}}\big|_{U} = \mathcal{V}.$$

This shows that $\mathcal{V}$ extends to a global conformal, totally geodesic distribution $\tilde{\mathcal{V}}$, which is left-invariant. By picking any unit vector $V \in \mathcal{V}_e$, we see that the corresponding foliation is given by left translation of the 1-parameter subgroup generated by V.

Example 13·5. For two non-zero real numbers $\alpha, \beta \in \mathbb{R}$ let the Lie algebra $\mathfrak{g}$ be the semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$ given by the the commutator relations

$$[Z, X] = \alpha X + \beta Y,\ [Z, Y] = -\beta X + \alpha Y,\ [X, Y] = 0,$$

where $\{X, Y\}$ is the standard basis for $\mathbb{R}^2$, and Z generates $\mathbb{R}$. Then $\mathfrak{g}$ is solvable and centerless and the corresponding simply connected Lie group has the following matrix representation

$$G = \left\{ \begin{pmatrix} e^{z\alpha}\cos(z\beta) & -e^{z\alpha}\sin(z\beta) & x \\ e^{z\alpha}\sin(z\beta) & e^{z\alpha}\cos(z\beta) & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; z, x, y \in \mathbb{R} \right\}.$$

Equip G with the left-invariant metric for which $\{X, Y, Z\}$ is an orthonormal basis for $\mathfrak{g}$. By identifying G with $\mathbb{R}^3$ in the obvious way, we see that this metric is given by

$$e^{-2z\alpha}dx^2 + e^{-2z\alpha}dy^2 + dz^2.$$

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A simple calculation shows that this metric has constant sectional curvature $-\alpha^2$. It is also easy to see that left translation of Z generates a conformal foliation by geodesics on G.

THEOREM 13·6 *Let $\mathfrak{g}$ be a 3-dimensional centerless, solvable Lie algebra and G a connected Lie group with Lie algebra $\mathfrak{g}$ and a left-invariant metric. Let $\mathcal{V}$ be a local conformal foliation by geodesics on G. Then G has constant sectional curvature.*

Proof. According to Proposition 13·4 the foliation $\mathcal{V}$ can be extended to a global foliation on G and is tangent to a left-invariant vector field $V \in \mathfrak{g}$. Let $\mathcal{H}$ be the left-invariant distribution orthogonal to $\mathcal{V}$ and X, Y be a left-invariant orthonormal basis for $\mathcal{H}$. As $\mathcal{V}$ is totally geodesic, we obtain

$$0 = \langle B^{\mathcal{V}}(V, V), X\rangle = \langle [X, V], V\rangle$$

and similarly for Y. We thus get

$$[V, X] = \alpha X + \beta Y,$$

$$[V, Y] = \gamma X + \delta Y.$$

As $\mathcal{V}$ is conformal we have

$$\begin{aligned} 0 &= \langle B^{\mathcal{H}}(X, X), V\rangle - \langle B^{\mathcal{H}}(Y, Y), V\rangle \\ &= \langle [V, X], X\rangle - \langle [V, Y], Y\rangle \\ &= \alpha - \delta \end{aligned}$$

and

$$\begin{aligned} 0 &= \langle B^{\mathcal{H}}(X, Y), V\rangle \\ &= \frac{1}{2}\big\{\langle [V, X], Y\rangle + \langle [X, Y], V\rangle - \langle [Y, V], X\rangle \\ &\qquad + \langle [V, Y], X\rangle + \langle [Y, X], V\rangle - \langle [X, V], Y\rangle\big\} \\ &= \langle [V, X], Y\rangle + \langle [V, Y], X\rangle \\ &= \beta + \gamma. \end{aligned}$$

Thus

$$\begin{aligned} [V, X] &= \ \alpha X + \beta Y \\ [V, Y] &= -\beta X + \alpha Y. \end{aligned} \tag{13·1}$$

The Lie algebra $\mathfrak{g}$ is centerless so we must have $\alpha^2 + \beta^2 \neq 0$, and since

$$\begin{aligned} [V, \beta X + \alpha Y] &= \ (\alpha^2 + \beta^2)Y \\ [V, \alpha X - \beta Y] &= (\alpha^2 + \beta^2)X, \end{aligned}$$

it follows that $X, Y \in [\mathfrak{g}, \mathfrak{g}]$. As $\mathfrak{g}$ is solvable, we must have $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, so $[\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{X, Y\}$. Since $[\mathfrak{g}, \mathfrak{g}]$ is a 2-dimensional, nilpotent Lie algebra, it must be Abelian; hence $[X, Y] = 0$.

By comparing with Example 13·5, it now follows that G must have constant sectional curvature $-\alpha^2$.

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Example 13·7. Consider the 3-dimensional solvable Lie algebra $\mathfrak{g}_\alpha$ spanned by the matrices

$$e_1 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

where α is some fixed real number. Here $[\mathfrak{g}, \mathfrak{g}]$ is spanned by e_2 and e_3, and the operator ad_{e_1} acts on this space with eigenvalues α and -1. The simply connected Lie group G_α with Lie algebra $\mathfrak{g}_\alpha$ is given by

$$G_\alpha = \left\{ \begin{pmatrix} e^{\alpha x} & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}.$$

When $\alpha = 0$, e_2 spans the center of $\mathfrak{g}_\alpha$, and is thus tangent to a Riemannian foliation of G_α by geodesics. For $\alpha \neq 0$, $\mathfrak{g}_\alpha$ is centerless so the only metrics on G_α for $\alpha \neq 0$ which admit a conformal foliation by geodesics are those of constant sectional curvature. For $\alpha > 0$ there are no left-invaraint metric on G_α with constant sectional curvature, see [**14**]. This implies that for $\alpha > 0$ the Lie group G_α does not admit any local conformal foliation by geodesics, regardless of which left-invariant metric we equip it with.

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