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The chaos game on a general iterated function system

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Abstract. The main theorem of this paper establishes conditions under which the 'chaos game' algorithm almost surely yields the attractor of an iterated function system. The theorem holds in a very general setting, even for non-contractive iterated function systems, and under weaker conditions on the random orbit of the chaos game than obtained previously.

1. *Introduction*

There are two methods for computing pictures of fractals that are attractors of iterated function systems: the deterministic algorithm and the more efficient 'chaos game' algorithm [**2**]. This paper concerns the chaos game on a general iterated function system (IFS) $\mathcal{F}$ defined on a complete metric space $(\mathbb{X}, d)$. The IFS is 'general' in the following sense: the only restriction placed on $\mathbb{X}$ is that it is proper, i.e. closed balls are compact, and the only restriction on the functions in $\mathcal{F}$ is that they are continuous. In particular, they need not be contractions on $\mathbb{X}$ with respect to any metric giving the same topology as the original metric d. A general IFS may possess more than one attractor, one attractor, or no attractor. Examples of iterated function systems that are non-contractive yet possess attractors are given in §4.

The main result, Theorem 1, is new in that it shows that the chaos game algorithm, applied to such a general IFS, almost always yields an attractor. More precisely, if the IFS has an attractor A, then a random orbit starting with a point in the basin of attraction of A converges with probability one to the attractor. We show this under weaker conditions than have heretofore been described. In all other papers on this topic of which we are aware, for example [**3–5, 7–9, 11, 13, 14**], it is required that the IFS be either contractive or contractive on the average. It is also required that the process by which the functions are selected to generate the orbit is stationary and that the selection process depends Hölder continuously on the initial point; see, for example, [**12**]. For our result, none of these conditions are required.

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Section 2 of this paper contains basic definitions, in particular the definition of an attractor of an IFS. Lemma 2 in §2 provides an expression for an attractor, of some independent interest, that will be used to prove the main result. The main result on the chaos game is Theorem 1 in §3. Section 4 contains examples that illustrate the practical value of Theorem 1.

2. *General iterated function systems*

Throughout this paper, $(\mathbb{X}, d_{\mathbb{X}})$ is a complete metric space.

Definition 1. If $f_m : \mathbb{X} \to \mathbb{X}$, $m = 1, 2, \ldots, M$, are continuous mappings, then $\mathcal{F} = (\mathbb{X}; f_1, f_2, \ldots, f_M)$ is called an *iterated function system* (IFS).

With a slight abuse of terminology, we use the same symbol $\mathcal{F}$ for the IFS, the set of functions in the IFS, and the following mapping: letting $2^{\mathbb{X}}$ denote the collection of subsets of $\mathbb{X}$, define $\mathcal{F} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ by

$$\mathcal{F}(B) = \bigcup_{f \in \mathcal{F}} f(B)$$

for all $B \in 2^{\mathbb{X}}$. Let $\mathbb{H} = \mathbb{H}(\mathbb{X})$ be the set of non-empty compact subsets of $\mathbb{X}$. Since $\mathcal{F}(\mathbb{H}) \subset \mathbb{H}$, we can also treat $\mathcal{F}$ as a mapping $\mathcal{F} : \mathbb{H} \to \mathbb{H}$. Let $d_{\mathbb{H}}$ denote the Hausdorff metric on $\mathbb{H}$, defined in terms of $d := d_{\mathbb{X}}$.

A metric space $\mathbb{X}$ is *locally compact* if every point has a compact neighborhood, and it is *proper* if every closed ball $\{y : d(x, y) \leq r\}$ is compact. Proper spaces are locally compact, but the converse is not true in general. The proof of the following result is not difficult. We use the notation

$$S + r = \{y \in \mathbb{X} : d(x, y) \leq r \text{ for some } x \in S\}$$

with $S \subset \mathbb{X}$ and $r > 0$.

LEMMA 1.

(1) *A metric space is proper if and only if $C + r$ is compact whenever $C \subset \mathbb{X}$ is compact and r is a positive real number.*

(2) *If $\mathbb{X}$ is proper, then $\mathcal{F} : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ is continuous.*

For $B \subset \mathbb{X}$, let $\mathcal{F}^k(B)$ denote the k-fold composition of $\mathcal{F}$, the union of $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(B)$ over all finite words $i_1 i_2 \cdots i_k$ of length k. Define $\mathcal{F}^0(B) = B$.

Definition 2. A non-empty compact set $A \subset \mathbb{X}$ is said to be an *attractor* of the IFS $\mathcal{F}$ if:

(1) $\mathcal{F}(A) = A$; and

(2) there is an open set $U \subset \mathbb{X}$ such that $A \subset U$ and $\lim_{k \to \infty} \mathcal{F}^k(B) = A$ for all $B \in \mathbb{H}(U)$, where the limit is with respect to the Hausdorff metric.

The largest open set U for which (2) is true is called the *basin of attraction* for the attractor A of the IFS $\mathcal{F}$.

The notation $\overline{S}$ is used to denote the closure of a set S, and, when $U \subset \mathbb{X}$ is non-empty, $\mathbb{H}(U) = \mathbb{H}(\mathbb{X}) \cap 2^U$. The quantity on the right-hand side of the equation below is sometimes called the *topological upper limit* of the sequence $\{\mathcal{F}^k(B)\}_{k=1}^{\infty}$ and is related to other definitions of attractors of generalizations of the notion of an IFS; see, for example,

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McGehee [**6**] and Lesniak [**10**], as well as the references in these two papers. We will use Lemma 2 in the proof of Theorem 1, our main result.

LEMMA 2. *If $\mathcal{F}$ is an IFS that has an attractor A with basin of attraction U, then*

$$A = \lim_{K\to\infty} \overline{\bigcup_{k\geq K} \mathcal{F}^k(B)}$$

for all $B \in \mathbb{H}(U)$.

Proof. We first show that $\overline{\bigcup_{k\geq K} \mathcal{F}^k(B)}$ is compact for K sufficiently large. Let $\{\mathcal{O}_i : i \in \mathcal{I}\}$ be an open cover of $\overline{\bigcup_{k\geq K} \mathcal{F}^k(B)}$. Since

$$\bigcup_{k\geq K} \mathcal{F}^k(\overline{B}) \subset \overline{\bigcup_{k\geq K} \mathcal{F}^k(B)}$$

and $A = \lim_{k\to\infty} \mathcal{F}^k(\overline{B})$, we have that $\{\mathcal{O}_i : i \in \mathcal{I}\}$ is also an open cover of A. Because A is compact, $\{\mathcal{O}_i : i \in \mathcal{I}\}$ contains a finite subcollection, say $\{\mathcal{O}_m : m = 1, 2, \ldots, M\}$, such that $A \subset \mathcal{O} := \bigcup_{m=1}^{M} \mathcal{O}_m$. Because a metric space is normal, there is an open set $\mathcal{O}'$ containing A such that $\overline{\mathcal{O}'} \subset \mathcal{O}$. Again using that fact that $\mathcal{F}^k(\overline{B})$ converges in the Hausdorff metric to A, there is an integer K' such that $\mathcal{F}^k(\overline{B}) \subset \mathcal{O}'$ for all $k \geq K'$. It follows that $\bigcup_{k\geq K'} \mathcal{F}^k(\overline{B}) \subset \mathcal{O}'$ and therefore

$$\overline{\bigcup_{k\geq K} \mathcal{F}^k(B)} \subset \mathcal{O} = \bigcup_{m=1}^{M} \mathcal{O}_m \quad \text{for all } K \geq K'.$$

Thus $\overline{\bigcup_{k\geq K} \mathcal{F}^k(B)}$ is compact if $K \geq K'$.

By the same argument used above to show that $\overline{\bigcup_{k\geq K} \mathcal{F}^k(B)} \subset \mathcal{O}$, it can be shown that there is a K' such that $B_K := \overline{\bigcup_{k\geq K} \mathcal{F}^k(B)} \subset U$ for $K \geq K'$. Since $B_K \in \mathbb{H}(U)$, by the definition of attractor we have

$$\begin{aligned} A &= \lim_{K\to\infty} \mathcal{F}^K(B_{K'}) = \lim_{K\to\infty} \mathcal{F}^K\left(\overline{\bigcup_{k\geq K'} \mathcal{F}^k(B)}\right) \\ &= \lim_{K\to\infty} \overline{\bigcup_{k\geq K'+K} \mathcal{F}^k(B)} = \lim_{K\to\infty} \overline{\bigcup_{k\geq K} \mathcal{F}^k(B)}. \end{aligned}$$

□

3. *The chaos game algorithm*

The following lemma will be used in the proof of Theorem 1.

LEMMA 3. *Let $\mathbb{X}$ be a proper complete metric space and $\mathcal{F} = (\mathbb{X}; f_1, f_2, \ldots, f_N)$ an IFS which has an attractor A with basin of attraction U. For any $\varepsilon > 0$, there is an integer $M = M(\varepsilon)$ such that for each $x \in (A + \varepsilon) \cap U$ there is an integer $m = m(x, \varepsilon) < M$ such that*

$$d_{\mathbb{H}}(A, \mathcal{F}^m(\{x\})) < \varepsilon/2.$$

Proof. Because $\mathbb{X}$ is proper and A is compact, $A + \varepsilon$ is also compact by statement (1) of Lemma 1. There is no loss of generality in assuming ε to be small enough that $A + \varepsilon \subset U$. If $x \in A + \varepsilon \subset U$, then there is an integer $m(x, \varepsilon) \geq 0$ such that

$$d_{\mathbb{H}}(A, \mathcal{F}^{m(x,\varepsilon)}(\{x\})) < \varepsilon/4. \tag{3.1}$$

This is because $\lim_{k\to\infty} \mathcal{F}^k(\{x\}) = A$.

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Since $\mathbb{X}$ is proper, it follows from statement (2) of Lemma 1 that $\mathcal{F}:\mathbb{H}\to\mathbb{H}$ is continuous, whence $\mathcal{F}^{m(x,\varepsilon)}:\mathbb{H}\to\mathbb{H}$ is continuous. Since $\mathcal{F}^{m(x,\varepsilon)}:\mathbb{H}\to\mathbb{H}$ is continuous, there is an open ball $B(\{x\}, r_x)$ (in $\mathbb{H}$) of radius $r_x>0$ centered at $\{x\}$ such that $d_{\mathbb{H}}(\mathcal{F}^{m(x,\varepsilon)}\{x\}, \mathcal{F}^{m(x,\varepsilon)}(Y))<\varepsilon/4$ for all $Y\in B(\{x\}, r_x)$. It follows, in particular, that there is a ball $B(x, r_x)$ (in $\mathbb{X}$) centered at x such that $d_{\mathbb{H}}(\mathcal{F}^{m(x,\varepsilon)}(\{x\}), \mathcal{F}^{m(x,\varepsilon)}(\{y\}))<\varepsilon/4$ for all $y\in B(x, r_x)$. Combining this with equation (3.1) above gives $d_{\mathbb{H}}(A, \mathcal{F}^{m(x,\varepsilon)}(\{y\}))<\varepsilon/2$ for all $y\in B(x, r_x)$.

The set of balls $\{B(x, r_x): x\in A+\varepsilon\}$ is an open covering of A_ε. Since $A+\varepsilon$ is compact, there is a finite subcovering $\{B(a_i, r_{a_i}) : i=1, 2, \ldots, q\}$ such that $A+\varepsilon\subset\bigcup_{i=1}^{q} B(a_i, r_{a_i})$. If $M=\max_i m(a_i, \varepsilon)$, then for any $x\in A+\varepsilon$ there is an i such that $x\in B(a_i, r_{a_i})$, in which case $d_{\mathbb{H}}(A, \mathcal{F}^m(\{x\}))<\varepsilon/2$ for $m=m(x, \varepsilon):=m(a_i, \varepsilon)<M$. □

Definition 3. Let $\mathcal{F}=(\mathbb{X}; f_1, f_2, \ldots, f_N)$ be an IFS and let $p\in(0, 1/N]$ be fixed. A sequence $\{x_k\}_{k=0}^{\infty}$ of points in $\mathbb{X}$ is called a *random orbit* of $x_0\in\mathbb{X}$ if $x_k=f_{\sigma_k}(x_{k-1})$ for $k=1, 2, \ldots$ where σ_k is selected randomly from $\{1, 2, \ldots, N\}$ so that the probability that $\sigma_k=n$ is greater than or equal to p, regardless of the preceding outcomes, for all $n\in\{1, 2, \ldots, N\}$ and all k. More formally, in terms of conditional probability,

$$P(\sigma_k=n \mid x_0, \sigma_1, \sigma_2, \ldots, \sigma_{k-1})\geq p.$$

THEOREM 1. *Let $\mathbb{X}$ be a proper complete metric space and let $\mathcal{F}=(\mathbb{X}; f_1, f_2, \ldots, f_N)$ be an IFS with attractor A and basin of attraction U. If $\{x_k\}_{k=0}^{\infty}$ is a random orbit of $x_0\in U$ under $\mathcal{F}$, then, with probability one,*

$$A=\lim_{K\to\infty}\{x_k\}_{k=K}^{\infty}$$

where the limit is with respect to the Hausdorff metric.

Proof. We first claim that given any $\varepsilon>0$, there is an integer $K>0$ such that

$$x_k\in A+\varepsilon \tag{3.2}$$

for all $k\geq K$. By Lemma 2 we have $A=\lim_{L\to\infty}\overline{\bigcup_{j\geq L}\mathcal{F}^j(\{x_0\})}$. It follows that for any $\varepsilon>0$, we can choose K so that $x_k\in\overline{\bigcup_{j\geq K}\mathcal{F}^j(\{x_0\})}\subset A+\varepsilon$ for all $k\geq K$, as claimed.

We next show that for any $\varepsilon>0$, there is an integer $K>0$ such that

$$d_{\mathbb{H}}(A, \{x_k\}_{k=L}^{\infty})<\varepsilon$$

with probability one, for all $L\geq K$. This is equivalent to the assertion of the theorem. To prove this, let $\varepsilon>0$. If K is as specified in the paragraph above, then by (3.2) we have $x_L\in A+\varepsilon$ for $L\geq K$. The attractor A, being compact, is totally bounded. Let $\{a_q : i=1, 2, \ldots, Q\}$ be a set of points such that $A\subset\bigcup_{q=1}^{Q} B(a_i, \varepsilon/2)$, where $B(a_q, \varepsilon/2)$ is the ball of radius $\varepsilon/2$ centered at a_q. Note that each a_q and the integer Q depend on ε. By Lemma 3, there is an integer M such that for each $x\in A+\varepsilon$ there is an $m<M$ with $d_{\mathbb{H}}(A, \mathcal{F}^m(\{x\}))<\varepsilon/2$. Hence

$$d_{\mathbb{H}}(A, \mathcal{F}^m(\{x_L\}))<\varepsilon/2$$

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for some integer $m < M$. Therefore there is a sequence of symbols $\sigma_{L+1}\sigma_{L+2}\cdots\sigma_{L+m}$ such that $f_{\sigma_{L+m}} \circ f_{\sigma_{L+m-2}} \circ \cdots \circ f_{\sigma_{L+1}}(x_L) \in B(a_1, \varepsilon/2)$. (We adopt the convention that the composition on the left equals x_L if $m = 0$.) It follows that

$$B(a_1, \varepsilon/2) \cap \{x_k\}_{k=L}^{L+M-1} \neq \emptyset,$$

or

$$B(a_1, \varepsilon/2) \subset \{x_k\}_{k=L}^{L+M-1} + \varepsilon.$$

The probability that this event occurs, i.e. that the particular sequence $\sigma_{L+1}\sigma_{L+2}\cdots\sigma_{L+m}$ is chosen, is greater than p^M. By repeating this argument, we deduce that the probability that

$$B(a_q, \varepsilon/2) \subset \{x_k\}_{k=L+(q-1)M}^{L+qM-1} + \varepsilon$$

is greater than $p^M > 0$, for each $q \in \{1, 2, \ldots, Q\}$, regardless of whether or not the preceding events occur (which is not to say that the events are independent). It follows that the probability that all of these events occur is greater than p^{QM}. Consider the event E_1 defined by

$$\bigcup_{q=1}^{Q} B(a_q, \varepsilon/2) \not\subseteq \{x_k\}_{k=L}^{L+QM-1} + \varepsilon.$$

The probability of E_1 is less than $1 - p^{QM}$. By a similar argument, the probability of the event E_r for $r \geq 1$, defined by

$$\bigcup_{q=1}^{Q} B(a_q, \varepsilon/2) \not\subseteq \{x_k\}_{k=L+(r-1)QM}^{L+rQM-1} + \varepsilon,$$

is less than $1 - p^{QM}$, regardless of whether or not the previous events $E_1, E_2, \ldots, E_{r-1}$ (for $r = 2, 3, \ldots$) occurred. It follows that the probability of the event $E_1 \cap E_2 \cap \cdots \cap E_r$ is less than $(1 - p^{QM})^r$, for all $r = 1, 2, \ldots$. This inequality holds regardless of the fact that the E_r are not independent. To simplify notation, let $s = 1 - p^{QM} < 1$ so that

$$\mathrm{pr}(E_1 \cap E_2 \cap \cdots) \leq \mathrm{pr}(E_1 \cap E_2 \cap \cdots \cap E_r) = s^r$$

for all r. This implies that $\mathrm{pr}(\bigcap_{r=1}^{\infty} E_r) = 0$. Hence, with probability one, there is an R such that

$$\bigcup_{q=1}^{Q} B(a_q, \varepsilon/2) \subset \{x_k\}_{k=L+(R-1)QM}^{L+RQM-1} + \varepsilon.$$

Since $A \subset \bigcup_{q=1}^{Q} B(a_q, \varepsilon/2)$, it follows that, with probability one, there is an R such that

$$A \subset \{x_k\}_{k=L+(R-1)QM}^{L+RQM-1} + \varepsilon.$$

Because L is an arbitrary integer greater than or equal to K, we have $A \subset \{x_k\}_{L}^{\infty} + \varepsilon$ for any $L \geq K$. But, by (3.2), we also have $\{x_k\}_{k=L}^{\infty} \subset A + \varepsilon$ for any $L \geq K$. Hence, with probability one, it is the case that $d_{\mathbb{H}}(A, \{x_k\}_{k=L}^{\infty}) < \varepsilon$ for any $L \geq K$. □

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It follows from Theorem 1 that

$$\bigcap_{K\geq 1} \overline{\{x_k\}_{k=K}^{\infty}} = \bigcap_{K\geq 1} \overline{\bigcup_{k\geq K} \mathcal{F}^k(B)}$$

almost surely, for $x_0 \in U$ and $B \in \mathbb{H}(U)$, e.g. for $B = \{x_0\}$. We draw attention to this equality because it seems surprising when $\mathcal{F}$ contains more than one function; the set $\{x_k\}_K^{\infty}$ seems sparse in comparison to $\bigcup_{k\geq K} \mathcal{F}^k(B)$.

4. *Examples*

Example 1. The IFS $\mathcal{F}$ in this example has a unique attractor, yet each $f \in \mathcal{F}$ fails to be a contraction with respect to any metric giving the same topology as the original metric. With probability one, the chaos game applied to this example 'draws a picture' of the attractor of the IFS.

If $\mathbb{X} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and d is the Euclidean metric, then $(\mathbb{X}, d)$ is a compact metric space. Let $\mathcal{F} = (\mathbb{X}; f_1, f_2)$ where $f_1(x, y) = (x, y)$ and $f_2(x, y) = (x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)$, with α/π irrational. The map f_1 is the identity map, and f_2 is a rotation through angle α anticlockwise about the origin. Since neither f_1 nor f_2 has a unique fixed point, it follows that there exists no metric $\tilde{d}$ on $\mathbb{X}$ such that $(\mathbb{X}, \tilde{d})$ is complete and either f_1 or f_2 is a contraction. On the other hand, $\mathcal{F}$ has a unique attractor $A = \mathbb{X}$. To see this, first note that $\mathcal{F}(\mathbb{X}) = \mathbb{X}$. Also, if $(x_0, y_0) \in \mathbb{X}$, then

$$\mathcal{F}^k(x_0, y_0) = \{f_2^j(x_0, y_0)\}_{j=0}^k$$

for all k. The right-hand side is well known to converge in the Hausdorff metric to $\mathbb{X}$ as k tends to infinity. It follows that $\mathcal{F}^k(B)$ converges to $\mathbb{X}$ for all $B \in \mathbb{H}(\mathbb{X})$. By Definition 2, the IFS $\mathcal{F}$ has a unique attractor, namely $\mathbb{X}$. By Theorem 1, with probability one, the chaos game applied to this example 'draws a picture' of the unit circle, the attractor of the IFS.

Example 2. In the same spirit as Example 1, this IFS on the real projective plane possesses an attractor, but there is no equivalent metric with respect to which the maps of the IFS are contractive. Again, with probability one, the chaos game 'draws a picture' of the attractor of the IFS.

This example appears in [**1**]. The metric space is $(\mathbb{R}P^2, d)$, where $\mathbb{R}P^2$ denotes real projective two-dimensional space and d denotes the round metric; see [**1**]. Let $\mathcal{F} = (\mathbb{R}P^2; f_1, f_2)$, where the projective transformations f_1 and f_2 (acting on homogeneous coordinates) are represented, respectively, by the pair of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2\cos\alpha & -2\sin\alpha \\ 0 & 2\sin\alpha & 2\cos\alpha \end{pmatrix},$$

with α/π irrational. In terms of homogeneous coordinates (x, y, z), the attractor of $\mathcal{F}$ is the line $x = 0$. This can be proved by using an argument similar to the one in Example 1.

Such non-contractive IFSs occur often in real projective IFS theory. Theorem 1 tells us that the chaos game algorithm can always be applied to compute approximate pictures of attractors of real projective iterated function systems.

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