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The von Neumann threshold of self-reproducing systems: theory and application

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SUMMARY
This paper is devoted to the study of systems of entities that are capable of generating other entities of the same kind and, possibly, self-reproducing. The main technical issue addressed is to quantify the requirements that such entities must meet to be able to produce a progeny that is not degenerative, i.e., that has the same reproductive capability as the progenitor. A novel theory that allows an explicit quantification of these requirements is presented. The notion of generation rank of an entity is introduced, and it is proved that the generation process, in most cases, is degenerative in that it strictly and irreversibly decreases the generation rank from parent to descendent. It is also proved that there exists a threshold of rank such that this degeneracy can be avoided if and only if the entity has a generation rank that meets that threshold – this is the von Neumann rank threshold. On the basis of this threshold, an information threshold is derived, which quantifies the minimum amount of information that must be provided to specify an entity such that its descendents are not degenerative. Furthermore, a complexity threshold is obtained, which quantifies the minimum length of the description of that entity in a given language. As an application, self-assembly for a 2 Degrees of Freedom planar robot is considered, and simulation results are presented. A robot arm capable of picking up and placing the components of another arm, in the presence of errors, is considered to have successfully reproduced if these are placed within an allowable tolerance. The example shows that, due to the kinematics of the robot, errors can grow from one generation to the next, until the reproduction process fails eventually. However, error correction (via error sensing and feedback control) can then be used to prevent such degeneracy. The von Neumann generation rank and information thresholds are computed for this example, and are consistent with the simulation results in predicting degeneracy in the case without error correction, and predicting successful self-reproduction in the case with error correction.

KEYWORDS: Robotic self-replication; Self-reproduction; Degeneracy; von Neumann threshold; Error correction.

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1. Introduction

This paper is devoted to the study of entities that are capable of generating other entities of the same kind and, possibly, self-reproducing. The main technical issue treated in this paper is to quantify the requirements that such entities must meet to be able to produce a progeny that is not degenerative, i.e., whose reproductive capability is not diminished compared to that of the progenitor. Research on self-reproduction can potentially have significant impact in many areas of societal importance, including evolutionary software, autonomous manufacturing, space colonization, and bioengineering.

While many human cultures have referred to mythical, man-made, self-reproducing entities, the modern scientific study of this subject originated in the work of von Neumann on self-reproducing cellular automata. In a series of lectures in the 1940s,[1] he argued that self-reproducing entities must have four functions. First, they must include a description of themselves. Second, they must have a set of instructions for building a descendent. Third, the first two properties must be coupled, so that, as the entity builds a descendent, it transcribes its description of itself into the latter. Finally, the self-reproduction process must include some unspecified input, so that the genetic identity of the parent entity may be modified in the descendent. Remarkably, these lectures were given about 10 years before J. Watson and F. Crick published their discovery of the DNA structure, which led to clarifying how biology implements the four functionalities stipulated by von Neumann.

In his lectures, von Neumann also argued the existence of a threshold of complexity, below which any attempt at self-reproduction is doomed to degeneracy.[1] However, he did not provide a method for computing it, and optimistically stated that he expected someone would soon do so. An extensive literature survey[2] indicates that in the following 60 years no one published an evaluation of this threshold.

This paper presents a novel theory – the so-called generation theory – that allows an explicit quantification of the fundamental requirement that an entity must meet to produce nondegenerate offspring. This yields the amount of information required to specify an entity that has such a capability, and the minimal complexity of the description of this entity. The theory is general enough to encompass many practical instances of self-reproduction. We present in Section 4 an application to a simple robotic self-assembly system, and show through simulation

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that the theory successfully predicts when degeneracy is guaranteed.[2]

Von Neumann's seminal ideas were the impetus for a vast body of work on self-replicating entities – see the excellent surveys in[3,4] for an overview of this field. This literature includes studies on cellular automata,[5–27] computer programs,[28–40] experimental machines,[41–48] theory of machines,[49–59] molecular machines,[60,61] mathematical models,[62–64] and error correction.[65–68] Ref. [73] identifies an interesting connection between self-replication and the halting problem.[74] As mentioned above, however, no result in the literature directly addresses the threshold alluded to by von Neumann. Perhaps the closest result is in ref. [64], where a probabilistic measure of replicability is computed as the logarithm of the ratio of integrals over time of the probabilities that a system will be present in two environments. Our work is different from ref. [64] in two ways. First, we allow the reproduction process to depend on an external input, thereby complying with the fourth functional requirement of von Neumann. Second and most importantly, our results yield a necessary and sufficient condition for nondegeneracy in self-reproduction.

The original contributions of this paper can be listed as follows:

- For generation systems (to be defined in Section 2) we introduce the notions of rank of a generation system and rank of an entity. We prove that the generation process, in most cases, is degenerative as it strictly and irreversibly decreases the rank from parent to descendent.
- We prove the existence of a threshold of rank such that this degeneracy can be avoided if and only if the entity has a rank that meets that threshold – this is the von Neumann rank threshold.
- On the basis of this threshold, we obtain an information threshold, which quantifies the minimum amount of information that must be provided to specify an entity whose descendents are not necessarily degenerate.
- On the basis of this information threshold, we obtain a complexity threshold, which quantifies the minimum length of the description of that entity in a given language.
- We clarify the relationship between these three thresholds: the von Neumann rank threshold is fundamental. It induces, but is not equivalent to, the information threshold. This latter one, in turn, induces the complexity threshold.
- A planar robotic self-assembly example is presented to confirm, via simulation results and use of the von Neumann rank and the information thresholds, the predictions of degeneracy or sustainability of the self-reproduction process.
- We provide an algorithm that, given a generation system, allows the computation of the von Neumann rank threshold of the system, the information and complexity thresholds, and the generation rank of every entity in the system.

The significance of these original results is that they bring to light fundamental limitations of all self-reproduction processes. These results can potentially be used to achieve a better understanding of existing natural self-reproducing systems and to set requirements in the design of future artificial ones.

The remainder of the paper is as follows. Section 2 presents generation theory and establishes the Principle of Degeneracy. Section 3 presents the von Neumann rank threshold, the information and complexity thresholds, and discusses their relationship. Section 4 presents a robotic self-assembly example, and compares simulation results to predictions regarding degeneracy versus sustainability of the self-reproduction process based on generation theory thresholds. Section 5 presents an algorithm that allows computation of these thresholds. Section 6 states our conclusions and plans for future work. The Appendix contains all the proofs, together with additional mathematical examples that illustrate generation theory.

2. Generation Theory

Below, we refer to the entities that can potentially self-reproduce as "cells," regardless of their physical makeup (e.g., robotic, biological, or software), physical scale (e.g., dimension or mass), or organization (e.g., single individual or population of several individuals). Any theory that studies self-reproduction must at least be able to discuss four items: the cells, the resources that these cells use, the actions that these cells perform, and the outcomes of these actions. Accordingly, we formulate the following:

Definition 1 A *generation system* is a quadruple $\Gamma = (U, M, R, G)$, where

- U is a *universal set* that contains cells, resources and outcomes of attempts at self-reproduction,
- $M \subseteq U$ is a *set of cells* – these are the entities that can potentially self-reproduce,
- $R \subseteq U$ is a *set of resources* that can be used for self-reproduction,
- $G : M \times R \to U$ is a *generation function* that maps a cell and a resource into an outcome.

When a cell $x \in M$ processes a resource $r \in R$ to produce an outcome $y \in U$, we write

$$y = G(x, r). \tag{2.1}$$

We say that cell x *is capable of producing outcome* y if there exists a resource $r \in \mathrm{R}$ such that $y = G(x, r)$. We say that cell x *self-reproduces using resource* r if

$$x = G(x, r). \tag{2.2}$$

We say that cells $x_1, x_2, \ldots, x_n$ form a *generation cycle of order* n if x_1 is capable of producing x_2, x_2 is capable of producing $x_3, \ldots, x_{n-1}$ is capable of producing x_n, and, finally, x_n is capable of producing x_1. Hence, in a generation cycle of order n, every cell is capable of producing an offspring that is identical to itself in n generations, and self-reproduction can be viewed as setting up a generation cycle of order 1.

Note that we allow $M \cap R \neq \varnothing$, in other words, some entities can be both cells and resources. This way, we allow

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"cannibalism" whereby a cell utilizes another cell to produce an outcome. Also note that we allow $M \cup R \neq U$, in other words, the outcome of a generation attempt may be neither a cell nor a resource.

Example 1 The formalism of Definition 1 and Eqs. (2.1) and (2.2) is general enough to encompass many instances of self-reproduction, and in this example we show how the quantities (U, M, R, G) can be interpreted in some of these instances. First, consider the work of von Neumann where functioning automata containing several parts are immersed in an environment that has an inexhaustible supply of parts. As an automaton encounters a part, it can retain or discard it. If it retains it, it appends it to the sub-assembly of an outcome. After enough parts have been retained, the automaton produces an assembled outcome that may or may not itself be a functioning automaton. In this case, the cells are automata, each resource is a sequence of parts that an automaton may encounter, the generation function maps a particular automaton and a particular sequence of parts into an assembled outcome, and the universal set contains functioning automata, and sequences of automaton parts together with assembled outcomes that are not functioning automata.

As a second instance, consider single-cell living organisms that reproduce through mitosis. An organism receives a sequence of molecules of nutrients and photons. After enough of these have been received, the organism produces an offspring. Here, the cells are all the single-cell organisms under consideration, each resource is a sequence of nutrients and photons that an organism could encounter The generation function maps a particular organism and a particular sequence of nutrients and photons into an outcome that may or may not be a living cell itself, and the universal set contains the single-cell organisms under consideration, and sequences of nutrients and photons together with outcomes of unsuccessful attempts at self-reproduction.

Finally, consider a robotic colony containing several robots that are specialized in the tasks that they perform, and that have the capability, as a group, to manufacture a replica of each individual robot. In order to do so, they are given raw material that they process into robot parts, which they assemble. In this case, a cell is a robotic colony, the set of resources is the set of all possible amounts of raw material that could be provided, the generation function maps a particular colony and a particular amount of raw material into an outcome that may or may not be a functioning colony, and the universal set contains functioning colonies, amounts of raw materials together with groups of entities produced by colonies, but that are themselves incapable of reproducing further. Note that in this case, it is the colony of robots that is the self-reproducing entity rather than any individual robot.

Remark 1 The formalism of Definition 1 and Eq. (2.2) encapsulates the four functional requirements of von Neumann for self-reproduction. Indeed, the description of the cell is x. The instructions for building a descendent are the function G. The transcription of the description of x into its descendent is performed through Eq. (2.2). Finally, the external input is provided by r.

Definition 2 In a generation system, we define the *generation sets* as follows:

- $M_0 = M$ is the set of all cells.
- M_1 is the set of all cells that are capable of producing a cell of M_0.
- M_2 is the set of all cells that are capable of producing a cell of M_1.
- ...
- M_{i+1} is the set of all cells that are capable of producing a cell of M_i, and so forth.

The generation sets are nested in the following way:

Proposition 1 The generation sets satisfy $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \supseteq \cdots$

Proof See Appendix.

Based on Proposition 1, we define the innermost generation set, M_∞, as

$$M_\infty = \bigcap_{i=0}^{\infty} M_i. \tag{2.3}$$

This set will be crucial for the study of self-reproduction because, as will be proved later, any self-reproducing cell must be an element of M_∞.

The nesting in Proposition 1 has the following consequence:

Proposition 2 If cell $x \in M_i \backslash M_{i+1}$, where "\" denotes the standard set difference operation, then for all resource $r \in R$, we have $G(x, r) \notin M_i$.

Proof See Appendix.

Propositions 1 and 2 imply that the generation function G always proceeds outwards, from the innermost generation sets to the outermost. We have the following implications:

Corollary 1 If cell $x \in M_i \backslash M_{i+1}$, then any generation sequence starting from x will produce an outcome $y \notin M$ in at most $i + 1$ steps.

Corollary 2 If cell $x \in M_i \backslash M_{i+1}$, then it cannot self-reproduce.

The nesting of the generation sets of Propositions 1 and 2 is illustrated in Fig. 1. Any cell in $M_0 \backslash M_1$, such as, for instance, x_1, can only produce descendents that are outside M_0. Any cell in $M_1 \backslash M_2$, such as, for instance, x_2, can only produce descendents that are either in $M_0 \backslash M_1$ or outside M_0, and so forth.

We can now define the Rank of a Generation System as follows. Since the sequence of generation sets $M_0, M_1, M_2, \ldots$ is nested, one of the two exclusive possibilities will occur: either there exists an integer i such that $M_i = M_{i+1}$, or for all integer i we have $M_i \neq M_{i+1}$.

Definition 3 The *rank of a generation system* $\Gamma = (U, M, R, G)$ with generation sets M_i, $i \geq 0$, is the smallest integer ρ such that $M_\rho = M_{\rho+1}$. If for all integer i we have $M_i \neq M_{i+1}$, we say that the generation system has infinite rank. In either case we use the notation $\rho(\Gamma)$ for the rank of the generation system.

For generation systems of finite rank, we have the following:

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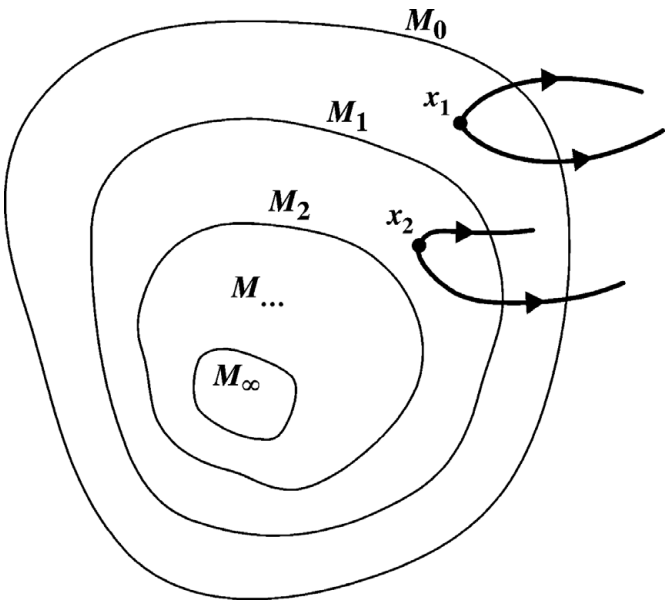


Fig. 1. Nesting of the generation sets in a generation system.

Proposition 3 If in a generation system we have $M_i = M_{i+1}$, then for all $j \geq i$, we have $M_j = M_i$.

Proof See Appendix.

Proposition 3 implies that for a generation system of finite rank ρ, the nesting inclusions of Proposition 1 stop at order ρ, i.e., $M_{\rho-1} \neq M_\rho = M_{\rho+1} = M_{\rho+2} = \cdots = M_\infty$.

When a generation system has a finite number of cells, its rank is necessarily finite. Specifically, we have the following result:

Proposition 4 Assume that for the generation system $\Gamma = (U, M, R, G)$, the number of cells, $|M|$, is finite. Then its rank satisfies

$$\rho(\Gamma) \leq |M| \,. \tag{2.4}$$

Proof See Appendix.

Remark 2 As per Proposition 4, for a generation system to have infinite rank, it is required that it has an infinite number of cells. It is indeed possible to construct generation systems with an infinite number of cells and infinite rank. However, it is also possible to construct others with an infinite number of cells and arbitrary finite rank, including zero.

We can now define the Generation Rank of a Cell as follows. In a generation system, for any cell x, one of the two exclusive possibilities must occur: either there exists an integer i such that $x \in M_i \backslash M_{i+1}$, or for all integer i we have $x \in M_i$.

Definition 4 In a generation system (U, M, R, G) with generation sets M_i, $i \geq 0$ and rank ρ, the *generation rank of a cell* x is defined as follows. If $x \in M_i \backslash M_{i+1}$ for some i, we say that cell x has deficient generation rank i. Otherwise, $x \in \bigcap_{i=0}^{\infty} M_i$ and we say that cell x has full generation rank ρ, whether this rank is finite or infinite. In all cases, when there is no risk of confusion, we will use the notation $\rho(x)$ for the generation rank of the cell.

Note that, according to Definition 4, the innermost generation set M_∞ in Eq. (2.3) contains all the cells of full generation rank. In terms of this set, we have the following requirement for nondegeneracy and self-reproduction:

Proposition 5 In a generation system, if cells $x_1, x_2, \ldots, x_n$ form a generation cycle of order n, then $x_i \in M_\infty$, $1 \leq i \leq n$.

Proof See Appendix.

Proposition 5 specifies that any generation cycle of order n must belong to M_∞. As a particular case, since self-reproduction as defined in Eq. (2.2) can be viewed as setting up a generation cycle of order 1, we have the following:

Corollary 3 If in a generation system a cell x is capable of self-reproducing, then $x \in M_\infty$.

The next result considers whether a cell in M_∞ is necessarily capable of producing an offspring that also belongs to M_∞.

Proposition 6 Assume that a generation system has finite rank, and let $x \in M_\infty$. Then there must exist a resource $r \in R$ such that $G(x, r) \in M_\infty$.

Proof See Appendix.

Corollary 4 Assume that a generation system has finite rank, such that its innermost generation set, M_∞, is not empty and has a finite number of cells. Then M_∞ contains at least one generation cycle of order at most $|M_\infty|$.

Proposition 7 Assume that a generation system has positive finite rank ρ and that a cell $y \in M_{\rho-1} \backslash M_\rho$ can be generated. Then the resource set must contain at least two elements.

Proof See Appendix.

Proposition 7 has the following consequences:

Corollary 5 If in a generation system every cell can be generated and the resource set is a singleton, then the rank of the system must be either 0 or ∞.

Propositions 1–7 paint a comprehensive picture of the nest of generation sets within a generation system with finite rank, and highlight the importance of the concept of generation rank. The innermost generation set, M_∞, contains all the cells of full generation rank. Any cell in M_∞ is capable of producing an outcome that is still in M_∞, i.e., an offspring without decrease in generation rank. A cell in M_∞ may also be able to produce an outcome that is no longer in M_∞, i.e., that has deficient rank. However, if that happens, then two fundamental facts are guaranteed. First, this exit from M_∞ is irreversible: no cell outside M_∞ is capable of producing an outcome in M_∞. Moreover, terminal degeneracy is assured: every cell outside M_∞ will always produce an offspring that has strictly lower generation rank than itself and, after a finite number of generations, will produce an outcome that is no longer a cell at all. Hence, we can formulate the following.

Principle of Degeneracy. In a generation system, a cell is capable of producing a nondegenerate offspring if and only if it has full generation rank, i.e., it belongs to the innermost generation set M_∞. In particular, any cell that is capable of self-reproducing must belong to M_∞. Any exit from M_∞ is

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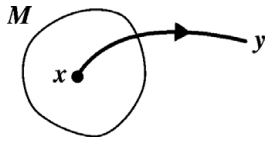


Fig. 2. Venn diagram of the generation system of Example 2(a).

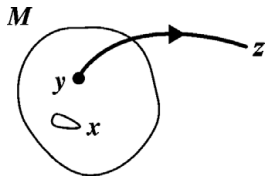


Fig. 3. Venn diagram of the generation system of Example 2(b).

irreversible. Moreover, outside M_∞, the generation process always strictly and irreversibly decreases the generation rank.

Example 2(a) To illustrate the concepts introduced in this Section, consider the generation system whose Venn diagram is given in Fig. 2.

Here, $M = \{x\}$, $R = \{r\}$, $G(x, r) = y$, and $y \notin M$. Then Definition 2 yields the generation sets: $M_0 = \{x\}$, $M_1 = M_2 = \cdots = M_\infty = \varnothing$. Definition 3 implies that the rank of this generation system is $\rho(\Gamma) = 1$. Since $x \in M_0 \backslash M_1$, Definition 4 yields that cell x has deficient rank $\rho(x) = 0$.

(b) Now consider the generation system whose Venn diagram is in Fig. 3.

In this case, $M = \{x, y\}$, $R = \{r\}$, $G(x, r) = x$, $G(y, r) = z$, and $z \notin M$. Applying Definitions 2–4, we obtain the generation sets $M_0 = \{x, y\}$, $M_1 = M_2 = \cdots = M_\infty = \{x\}$. The rank of the system is $\rho(\Gamma) = 1$. Cell x has full rank $\rho(x) = 1$, but cell y has deficient rank $\rho(y) = 0$.

3. The von Neumann Threshold

In this Section, we quantify the requirements that a cell must satisfy to be able to produce nondegenerate offspring. Such a quantification was anticipated by von Neumann in the following words:

> The conclusion one should draw from this is that complication is degenerative below a certain minimum level. This conclusion is quite in harmony with other results in formal logics, to which I have referred a few times earlier during these lectures*. We do not now know what complication is, or how to measure it, but I think that something like this conclusion is true even if one measures complication by the crudest possible standard, the number of elementary parts. There is a minimum number of parts below which complication is degenerative, in the sense that if one automaton makes another the second is less complex than the first, but above which it is possible for an automaton to construct other automata of equal or higher complexity. Where this number lies depends upon how you define the parts. I think that with reasonable definitions of parts, like those I will partially indicate later, which give one or two dozen parts with simple properties, this minimum number is large, in the millions. I don't have a good estimate of it, although I think that one will be produced before terribly long, but to do so will be laborious. [1(pp. 79,80)]

To quantify the requirements, first we evaluate the generation rank that is required for a cell to be able to generate an offspring without decrease of generation rank. This evaluation yields the *von Neumann rank threshold*. Second, we evaluate the amount of information required to specify a cell in M_∞ from among all the cells in M_0. This evaluation yields an *information threshold*. Finally, if the specification of the cell is done in a language, we evaluate how long the specification must be for a cell in M_∞. This evaluation yields a *complexity threshold*.

According to the Principle of Degeneracy, in a generation system $\Gamma = (U, M, R, G)$, one can define the von Neumann rank threshold as

$$\tau_r = \rho(\Gamma), \tag{3.1}$$

and it represents the generation rank required for a cell to be able to generate an offspring that do not degenerate. The set of cells that meet this rank threshold requirement is precisely M_∞.

The idea for evaluating the amount of information required to specify the cells in M_∞ stems from recognizing the connection between the generation theory and the information theory.[69] Indeed, in the process of generation, a message (the identity of the parent cell) is transmitted from a message originator (the parent) to a message recipient (the descendent) through a channel (the process whereby the descendent is made from the parent). To exploit this connection, we now consider that every cell is represented by a message consisting of a sequence of symbols drawn from an alphabet. Examples of such symbolic representations include cell parts together with specifications on how to assemble them into a cell, straight and curved lines together with instructions on how to connect them into a blueprint of the cell, and the nucleotide sequence of a DNA molecule. Then, for generation systems with a finite number of cells, based on the information contents of the set M_∞ we obtain the following information threshold[69]:

$$\begin{aligned} \tau_i &= \log_2 \frac{|M_0|}{|M_\infty|} \quad \text{for} \quad |M_\infty| \neq 0, \\ &= \infty \qquad\quad \text{for} \quad |M_\infty| = 0. \end{aligned} \tag{3.2}$$

The above expression represents the number of bits of information required to specify a cell in M_∞ from among all the cells in M_0. For generation systems where the set of cells is measurable, Eq. (3.2) is easily extended by replacing the cardinalities of the sets by their measures.

We can use Eq. (3.2) to quantify the complexity required for the specification of a cell in M_∞. Since the cells are represented by sequences of symbols, it is possible to obtain a measure of the expected information carried by a symbol in a message in such a language. This measure is the *entropy*[69]** and is quantified in bits per symbol. For instance, if the

* Here, von Neumann was alluding to type theory, where it is shown that one "can perform within the logical type that's involved everything that's feasible, but the question of whether something is feasible in a type belongs to a higher logical type." [1(p. 51)]

** Note that entropy is also used in ref. [77] to assess the difficulty of self-replication. However, here entropy pertains to cells in a generation system rather than the environment.

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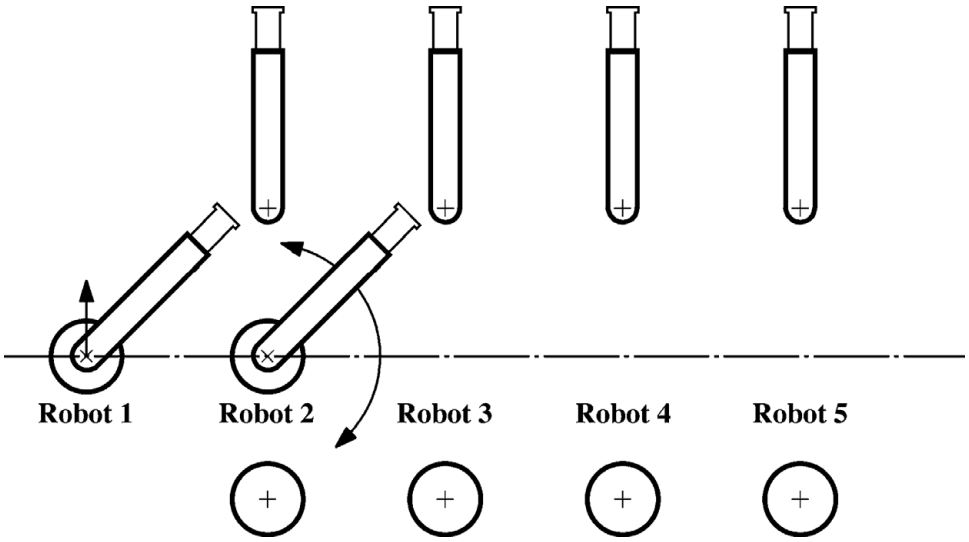


Fig. 4. Configuration for robotic self-assembly.

language uses n symbols $s_1, \ldots, s_n$ that occur independently with probabilities $p_1, \ldots, p_n$, respectively, the entropy is

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i. \tag{3.3}$$

Knowing τ_i, the information threshold for a generation system, and H, the entropy of the symbolic representation used to specify a cell, we obtain the complexity threshold as

$$\tau_c = \frac{\tau_i}{H}, \tag{3.4}$$

which is quantified in number of symbols, and represents the required length for the description of a cell in M_∞.

Note that the three thresholds, i.e., Eqs. (3.1), (3.2), and (3.4), have different units: τ_r has units of generations, τ_i has units of bits, and τ_c has units of symbols. Among these three thresholds, the most fundamental one is the von Neumann rank threshold, i.e., Eq. (3.1), because it provides a necessary and sufficient condition for a cell to be able to generate another cell of equal generation rank. The rank threshold (Eq. (3.1)) induces, but is not equivalent to, the information threshold, i.e., Eq. (3.2). In other words, specifying a cell in M_∞ requires an amount of information given by Eq. (3.2); however, specifying an amount of information (Eq. (3.2)) about a cell does not guarantee that this cell belongs to M_∞. The information threshold (Eq. (3.2)), in turn, induces the complexity threshold, i.e., Eq. (3.4) as the latter depends on the former but, in addition, requires the choice of a particular language. Hence, the von Neumann rank threshold quantifies the requirement that a cell must be capable of self-reproduction.

Example 3 To illustrate the thresholds introduced above, consider again the generation system in Fig. 2. Equation (3.1) yields a von Neumann rank threshold, $\tau_r = 1$ and no cell meets that rank requirement. Equation (3.2) yields an information threshold, $\tau_i = \infty$, and hence, for any representation with finite nonzero entropy the complexity threshold is $\tau_c = \infty$.

Now consider the generation system in Fig. 3. Here also we have $\tau_r = 1$. However, since one of the two cells meets the rank threshold, the information threshold is $\tau_i = 1$ bit. Assume that we specify the cells using a string of symbols, from an alphabet of two symbols, e.g., $\{0, 1\}$, occurring with equal probability. Then, Eq. (3.3) implies that the entropy of the language is $H = 1$ bit per symbol, and Eq. (3.4) implies that the complexity threshold is $\tau_c = 1$ symbol. Clearly, we can specify cell x with the symbol 1 and cell y with the symbol 0. Note, however, that even though cell y meets both information and complexity thresholds, it does not meet the rank threshold and does not belong to M_∞.

4. A Robotic Self-Assembly Application

Consider a simple self-reproducing cell, which consists of a 2-Degree of Freedom (DOF) planar robot arm capable of picking up and placing the components of another arm, as shown in Fig. 4.[2] If the robot places the components within an allowable tolerance, then the original arm has successfully reproduced. An assembly line is constructed so that a self-reproduction process can proceed along a track. If this process fails eventually, because one robot is not capable of assembling another, then the system is said to be degenerative. Otherwise, the self-assembly is sustainable. The propagation of errors in assembly, and other manufacturing operations, is a well-studied subject.[71,78] Recently, the propagation of variation in multistage manufacturing processes has been represented using state space models, where the independent variable index denotes not time but a particular stage of the manufacturing (or assembly) process.[75,76] Here we utilize such an approach to model, for the first time, a process of self-assembly as depicted in Fig. 4.

Consider the planar 2-DOF robot shown in Fig. 5, and the definition of errors as illustrated in Fig. 6. The mapping of errors from one generation of robot to the next is governed by robot kinematics, describing the transfer of the part from the

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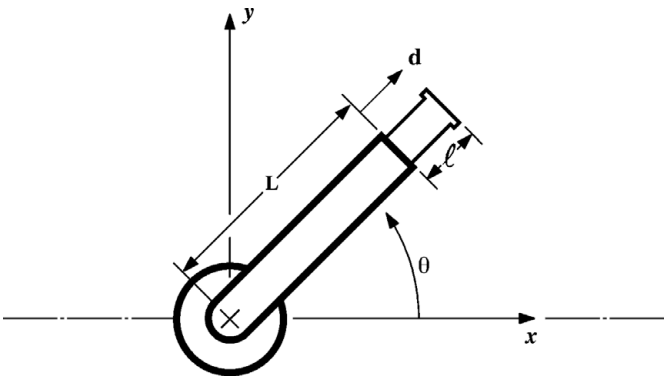


Fig. 5. Robot arm configuration, and degrees of freedom (DOF) θ and d.

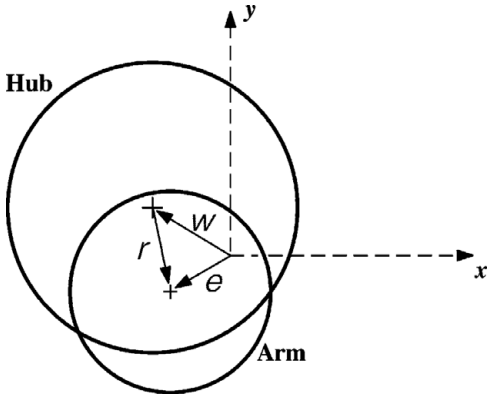


Fig. 6. Definition of errors $\boldsymbol{e}$, $\boldsymbol{w}$, and $\boldsymbol{r}$.

original location to the placement location. One can derive this mapping by using standard robotics notation. First, derive the transformation of coordinates from the world origin to the robot tool tip. This transformation includes a pure translation from world origin to a hub frame, a pure rotation to the frame rotating with the arm, a pure translation from the frame in the hub rotating with the arm to a frame in the arm rotating with the arm, a pure translation to the end-effector on the arm, and finally a pure translation into a frame attached to the tool tip and moving with the prismatic joint. The overall transformation is

$$\mathbf{T} = \begin{bmatrix} c\theta & -s\theta & 0 & c\theta[d+(L+l)] + [c\theta \cdot r_x - s\theta \cdot r_y + w_x] \\ s\theta & c\theta & 0 & s\theta[d+(L+l)] + [s\theta \cdot r_x + c\theta \cdot r_y + w_y] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.1}$$

where $c\theta$ and $s\theta$ denote $\cos\theta$ and $\sin\theta$, respectively.

Using the transformation in Eq. (4.1), one can derive how the errors in hub and arm placement in the local world frame are transformed into errors in the new local world frame, as well as how the errors in the hub supply, $\mathbf{v}$, add to the errors in the new local world frame.[2] The error in hub placement away from this origin is $\mathbf{w}$. The error in arm placement away from this origin is $\mathbf{e}$. The relative distance between these two points, or the error between the hub and the arm, is $\mathbf{r}$ (See Fig. 6). If $\mathbf{r}$ exceeds the tolerance, c_r, then the construction of the robot fails. Since this is a planar model, the acceptable values of $\mathbf{r}$ form a disk, $\|\mathbf{r}\| \leq c_r$. We can use this $\mathbf{r}$ value

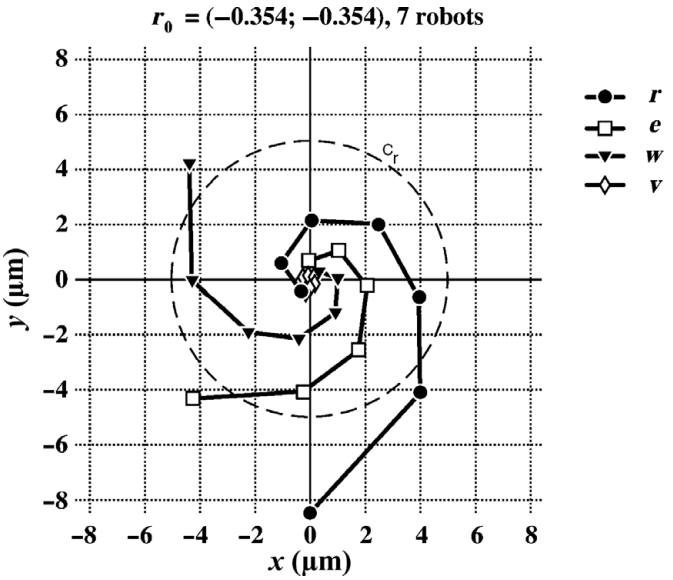


Fig. 7. Increasing errors with each robot generation (for $\mathbf{v} = N([^0_0], [^{1\ 0}_{0\ 1}]0.175\,\mu\text{m})$).

to define the identity of individual robots. Thus, the bound c_r defines the set of all possible robots. As we will see later, this also defines the set of all possible cells, M_0, according to the generation theory. Using the index, k, to denote each generation, the composite error state at generation k is defined as $\mathbf{s}_k = [\,\mathbf{w}_k^T\ \mathbf{e}_k^T\,]^T$. The new relative error for generation $k+1$ can then be calculated as $\mathbf{r}_{k+1} = \mathbf{e}_{k+1} - \mathbf{w}_{k+1}$, and the linearized state and output equations, which describe the generational change in errors, can be derived as[2]:

$$\mathbf{s}_{k+1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \mathbf{s}_k + \begin{bmatrix} 0 & 1 \\ (L+l) & 0 \\ 0 & 1 \\ (L+l) & 0 \end{bmatrix} \cdot \mathbf{u}_k + \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \mathbf{v}_{k+1} = \mathbf{A}\mathbf{s}_k + \mathbf{B}\mathbf{u}_k + \mathbf{E}\mathbf{v}_{k+1}, \tag{4.2}$$

$$\mathbf{r}_k = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdot \mathbf{s}_k = \mathbf{C}\mathbf{s}_k. \tag{4.3}$$

Various simulation results are reported in ref. [2], for parameters $L = 50$ cm and $l = 5$ cm, and the system exhibits exponential growth in component placement errors (e.g., see Fig. 7) when there is no error correction (i.e., $\mathbf{u}_k = \mathbf{0}$ in Eq. (4.2)). The tolerance on the relative error is specified as $c_r = 5\,\mu$m. The eigenvalues of $\mathbf{A}$ are 0, 0, $1 \pm 1j$, thus, the magnitude of the relative error increases by a multiplicative factor of $\sqrt{2}$ at each generation, and the direction of the error in the plane is rotated by 45° at each generation. Thus, this self-reproduction system is degenerative.

However, if this system is augmented to provide error-correction during the assembly process, the self-reproduction process can become sustainable (see Fig. 8), and the natural tendency of construction errors to grow from generation

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to generation can be counteracted by this error-correction. This error-correction was modeled as an asymptotically stabilizing feedback control law, $\mathbf{u}_k = -\mathbf{K}\mathbf{s}_k$, with $\mathbf{K}$ selected to assign the eigen values of $(\mathbf{A}-\mathbf{BK})$ at 0.1, 0.1, 0.25, and 0.25. The error-corrected system becomes

$$\mathbf{s}_{k+1} = (\mathbf{A} - \mathbf{BK}) \cdot \mathbf{s}_k + \mathbf{E} \cdot \mathbf{v}_{k+1}, \tag{4.4}$$

$$\mathbf{K} = \begin{bmatrix} -1.795 \times 10^{-6} & 1.182 \times 10^{-6} & 2.273 \times 10^{-8} & 0 \\ .6500 & .9875 & 0 & .0125 \end{bmatrix}. \tag{4.5}$$

Equations (4.3)–(4.5) now represent a new self-reproduction system, which exhibits better transference of robot construction errors from generation to generation. Although not reported here, the minimal amount of error-correction required to achieve sustainable self-reproduction was also investigated in the presence of sensor quantization, and it was shown that the amount of fidelity in the error-correction signal determines the success of the self-reproduction process.[2]

This robotic self-reproduction system was also analyzed in the context of the generation theory. The theory is able to predict the observed simulation results regarding degeneracy or sustainability. An element of the cell set, $\mathbf{r} \in M$, is defined as the relative error coordinates between the positioning of the hub and the arm subassemblies for a given robot. An element of the resource set, $\mathbf{v} \in R$, is defined as the absolute error coordinates of the hub supply position. The generation function, $G(\boldsymbol{r}, \boldsymbol{v})$, is defined as the kinematic operator that maps the relative errors from one robot to its progeny. So the set of cells is $M = \{\mathbf{r}_i | \, \|\mathbf{r}_i\| \leq c_r\}$, the set of resources is $R = \{\mathbf{v}_i | \, \|\mathbf{v}_i\| \leq c_r\}$, and the generation function maps a robot and a resource onto an outcome through Eqs. (4.2) and (4.3). A robot successfully produces another robot if $\|\mathbf{r}_k\| \leq c_r$, $G(\mathbf{r}_k, \mathbf{v}_k) = \mathbf{r}_{k+1}$, and also $\|\mathbf{r}_{k+1}\| \leq c_r$. Although any relative error coordinates within the tolerance c_r represent a viable cell, the magnitude of the relative error can be thought of as a measure of the cell quality.

Note that the degeneracy of the system without error correction is driven by the instability in the dynamics of Eq. (4.2) rather than the randomness in hub placement error, $\mathbf{v}$. Moreover, setting $\boldsymbol{v} = 0$ does not change the character of the results in Figs. 7 and 8.[2] Hence, we set $\mathbf{v} = 0$ to simplify the following analysis. The generation sets, M_i, are disks in the relative error coordinate plane, centered at the origin and with radii, $d_i = (\sqrt{2})^{-i} d_0$ (See Fig. 9). An increase in i corresponds to a decrease in d_i, and a robot in $M_i \backslash M_{i+1}$ will produce a robot outside M_i. So a cell has to start closer to the origin to be able to successfully reproduce for a larger number of generations. The set of all cells, M_0, is disk-centered at the origin with radius c_r as defined previously. All other generation sets, M_i, are nested subsets of M_0. For the original self-reproduction system in Eqs. (4.2) and (4.3), with $\mathbf{u}_k = \mathbf{0}$ and $\mathbf{v}_k = \mathbf{0}$, it is obvious that $\forall i$, $M_i \neq M_{i+1}$. The innermost generation set, M_∞, is the singleton $\{0\}$. So this generation system has infinite generation ranks. Therefore, the system will always exhibit degeneracy; within a finite number of generations the self-reproduction process will fail.

In the second system, Eq. (4.4), when the control input is set to apply the full-state error-correction, $\mathbf{u}_k = -\mathbf{K} \cdot \mathbf{s}_k$ and

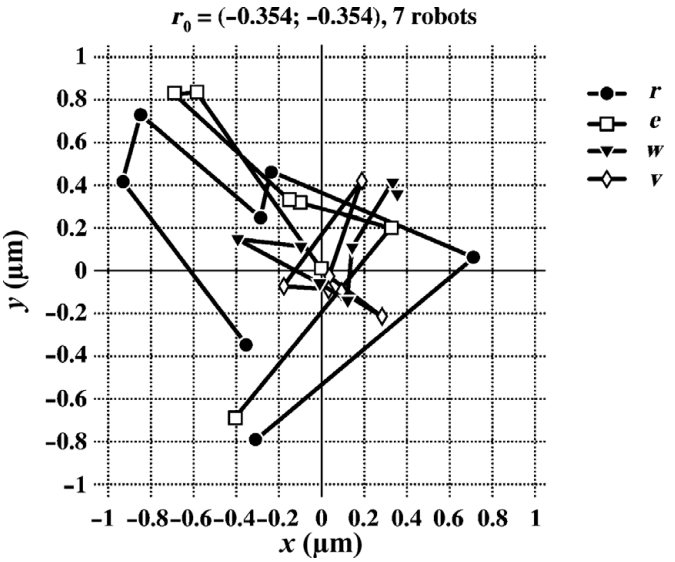


Fig. 8. Bounded errors with each robot generation for the error-corrected system (for $\mathbf{v} = N([\begin{smallmatrix}0\\0\end{smallmatrix}], [\begin{smallmatrix}1&0\\0&1\end{smallmatrix}]\ 0.175\ \mu\text{m})$).

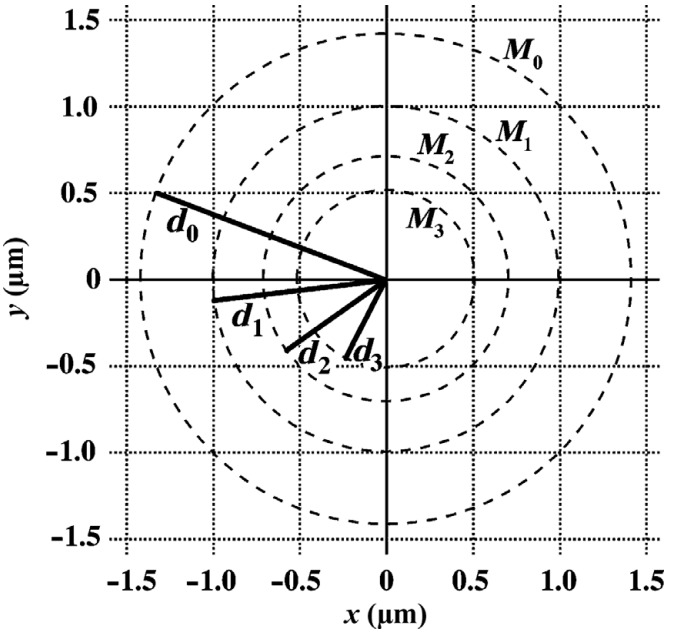


Fig. 9. The nested generation sets (circles) $M_0, M_1, M_2, \ldots M_k$ for $k = 0, 1, 2, 3, \ldots$ contain all cells of increasing accuracy $\|\mathbf{r}_k\|$ as k increases.

$\mathbf{v}_k = \mathbf{0}$, all cells are capable of sustained reproduction. So for this generation system, the generation rank is zero, $M_0 = M_\infty$. Thus, the self-reproduction system is either completely sustainable or completely unsustainable, and in the former case the generation rank is zero and in the latter case the generation rank is infinite.

For these self-reproducing robot systems, the threshold of information, τ_i, can be readily calculated. Remember that the set, M_i, is a disk, centered at the origin and with radius, d_i. Following Eq. (3.2), the information threshold is the logarithm of the ratio of measures of M_0 and M_∞, i.e.,

$$\tau_i = \log_2 \frac{\|M_0\|}{\|M_\infty\|}. \tag{4.6}$$

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For the original system without error correction, $\|M_\infty\|$ is zero; therefore, the threshold of information is infinity. In the generation system with full error-correction, $M_0 = M_\infty$, and therefore $\tau_i = 0$. With $\tau_i = \infty$ in the original system, the requirement for creating a sustainable self-reproducing robot is impossibly high: the initial relative error must be perfectly zero. However, in the error-corrected system, with $\tau_i = 0$, any robot will be able to successfully produce other robots indefinitely. So we can lower the generation rank of a self-reproducing system by using error-correction in the reproduction process. Note that these results predicted by the generation theory are indeed consistent with the simulation results presented in Figs. 7 and 8.

5. A Generation Analysis Algorithm

In Section 4, the computations of the von Neumann rank threshold and information threshold were straightforward. However, this may not always be the case. Hence, we now consider the following basic analysis questions for a given generation system $\Gamma = (U, M, R, G)$ with, unlike Section 4, a finite number of cells:

(a) Compute the von Neumann threshold of the system, τ_r.
(b) Compute the information threshold, τ_i.
(c) Compute the complexity threshold, τ_c.
(d) Compute the generation rank, $\rho(x)$, of a given cell x.
(e) Obtain all the cells of a given generation rank.

The purpose of this section is to present an algorithm that provides answers to these five questions. Specifically, given a generation system of unknown rank ρ, the algorithm returns the sequence of sets $(M_0\backslash M_1), (M_1\backslash M_2), \ldots, (M_{\rho-1}\backslash M_\rho)$, $M_\rho = M_\infty$, and the integer ρ, which is the von Neumann rank threshold. With these results produced by the algorithm, the information threshold is given by Eq. (3.2), the complexity threshold by Eq. (3.4), the generation rank for each cell follows from Definition 4, and the set of all cells of given generation rank k is precisely $M_k\backslash M_{k+1}$ for $0 \le k < \rho$ and M_∞ for $k = \rho$.

The idea of the algorithm is as follows. For a generation system $\Gamma = (U, M, R, G)$, define the *outer layer* as the set $M_0\backslash M_1$. This is the set of cells such that, no matter what resource they use, they produce an offspring that is no longer a cell, i.e.,

$$\{x \in M : \forall r \in R, G(x, r) \notin M\}. \tag{5.1}$$

Now, for $0 \le i \le \rho - 1$, consider the generation systems, $\Gamma_i = (U, M_i, R, G|_{M_i})$. From the above definition, the outer layer of Γ_i is the set

$$\{x \in M_i : \forall r \in R, G(x, r) \notin M_i\}, \tag{5.2}$$

and from Definition 2, this outer layer is exactly $M_i\backslash M_{i+1}$. Moreover, the outer layer of the generation system, $\Gamma_\rho = (U, M_\rho, R, G|_{M_\rho})$, is the empty set. Hence, we can construct the sequence of sets $(M_0\backslash M_1), (M_1\backslash M_2), \ldots, (M_{\rho-1}\backslash M_\rho)$, by identifying the outer layer of each of the generation systems $\Gamma_i, 0 \le i \le \rho$.

Identifying the outer layer of a generation system $\Gamma = (U, M, R, G)$ can be done as follows. Let $M = \{x_1, x_2, \ldots, x_n\}$ and consider the *descendency matrix*, defined as the $n \times n$ matrix of integers D such that

$$\begin{aligned} D_{ij} &= 1 \quad \text{if} \quad \exists r \in R : x_j = G(x_i, r), \\ &= 0 \ \text{otherwise}, \end{aligned} \tag{5.3}$$

i.e., $D_{ij} = 1$ if cell x_i is capable of generating cell x_j and $D_{ij} = 0$ otherwise. Then, the outer layer of Γ consists of those cells such that the corresponding row in matrix D is completely zero, i.e., cells that are not capable of generating another cell.

Finally, if we know D^k, the descendency matrix of Γ_k, then the descendency matrix of Γ_{k+1} is obtained from D^k simply by removing the rows and columns corresponding to the outer layer of Γ_k, i.e., by removing all the rows that are completely zero and the corresponding columns. This is because the entities in the outer layer of Γ_k are cells in Γ_k, but are no longer cells in Γ_{k+1}. Hence, we obtain the following.

6. Generation Analysis Algorithm

Inputs: A generation system, $\Gamma = (U, M, R, G)$, where $M = \{x_1, x_2, \ldots, x_n\}$
Outputs: The sets $(M_0\backslash M_1), (M_1\backslash M_2), \ldots, (M_{\rho-1}\backslash M_\rho)$, $M_\rho = M_\infty$, and the von Neumann rank threshold $\tau_r = \rho(\Gamma)$
Step 1: Compute the $n \times n$ matrix D in Eq. (5.3).
Step 2: Initialize $i = 0$.
Step 3:
While M is not empty and D has at least one row of zeros, repeat.
Return $M_i\backslash M_{i+1}$ = set of cells corresponding to zero rows of D.
Update M by removing the cells corresponding to zero rows of D.
Update D by removing the zero rows and the corresponding columns.
Increment $i \leftarrow i + 1$.
Step 4: Return $M_\infty = M$.
Step 5: Return $\tau_r = i$.
Stop

Note that in Step 3 of the above algorithm, during the kth iteration of the "while" loop, the algorithm removes from M a number of cells equal to $|M_{k-1}\backslash M_k|$. Hence, the algorithm stops after ρ iterations of that loop. Roughly speaking, the algorithm "peels away" outer layer after outer layer until it reaches a core, from which no outer layer can be peeled – that core is M_∞.

Example 4 To illustrate the use of the generation analysis algorithm, consider the generation system in Fig. 2. Applying the algorithm yields $D = [0]p$, $M_0\backslash M_1 = \{x\}$, $M_\infty = \varnothing$, and $\tau_r = 1$. Similarly, applying the algorithm to the generation system of Fig. 3 yields

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{6.1}$$

$$M_0\backslash M_1 = \{y\},$$

$M_\infty = \{x\}$, and $\tau_r = 1$.

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Example 5 To further illustrate the use of the algorithm, consider a slightly less trivial generation system with set of cells $M = \{x_1, x_2, x_3, x_4, x_5\}$, set of resources $R = \{r_1, r_2\}$, and a generation function specified by

$$\begin{aligned} G(x_1, r_1) &= x_2 \quad G(x_1, r_2) = x_4, \\ G(x_2, r_1) &= x_3 \quad G(x_2, r_2) = x_5, \\ G(x_3, r_1) &= x_5 \quad G(x_3, r_2) = x_1, \\ G(x_4, r_1) &= y_1 \quad G(x_4, r_2) = x_5, \\ G(x_5, r_1) &= y_2 \quad G(x_5, r_2) = y_3, \end{aligned} \tag{6.2}$$

where y_1, y_2, and y_3 are not cells. For this generation system, the descendency matrix is

$$D = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.3}$$

Applying the algorithm, we obtain $M_0 \backslash M_1 = \{x_5\}$, $M_1 \backslash M_2 = \{x_4\}$, $M_\infty = \{x_1, x_2, x_3\}$, and the von Neumann rank threshold is $\tau_r = 2$. Note that, in this generation system no cell is capable of self-reproducing in the sense of Eq. (2.2). However, within M_∞, x_1, x_2, and x_3 form a generation cycle of order 3. Hence, each of them is capable of producing an offspring that is identical to itself in three generations.

Other examples of generation analysis for robotic self-assembly are given in ref. [70], where the generation analysis algorithm is extended to identify a seed for a generation system.

To illustrate the generality of the generation theory, two additional mathematical examples are given in the Appendix.

7. Conclusions and Future Work

This paper has presented the fundamental requirements for all self-reproduction processes. Two major themes seem to emerge from the results. First is the critical importance of the concept of generation rank: the capability of producing a nondegenerate offspring is strictly equivalent to the requirement of meeting a threshold of generation rank – the von Neumann threshold. Whereas one might have postulated, as von Neumann originally did, that the basic requirement for self-reproduction is "complication," it turns out that the fundamental requirement is one of generation rank. The requirements on information and complexity are induced by, are secondary to, and are not equivalent to this primary requirement on generation rank. The second major theme is the inherent irreversibility, with the exception of generation cycles, of all generation events: any exit from the innermost generation set is irreversible and, outside that set, the generation process always strictly and irreversibly decreases the generation rank. Such a prevalence of irreversibility is very reminiscent of the second principle of thermodynamics in mechanics. A third, a minor theme, is the simplicity and generality of the results: by using the abstract mathematical framework of the set theory, we have been able to avoid detailed discussions of cell components and their functionalities. Note, however, that given specific cell components to be used as resources, deriving the corresponding generation system is a rather substantial modeling task.

A simple robot arm, capable of assembling a replica of self, was modeled to include assembly errors and arm kinematics, and simulated to study the sustainability of the self-replication process. The model exhibited a gradual degeneration of the cell over the progression of generations, eventually failing by yielding a terminal generation not capable of self-reproducing. The cell degeneration was caused by the errors accumulation in the self-reproduction process over several generations. It was then shown that this self-reproduction process could be made sustainable by the application of error-correction. The application of the generation theory to the two systems predicted that the original process would be degenerative, while the fully error-corrected process would be sustainable. The von Neumann rank threshold and the information threshold were infinity for the first process and zero for the second one.

The results of this paper suggest several possible avenues for further research. First, generalizations and extensions of the work reported here for robotic self-assembly are needed. These could include a study of different levels and types of error correction, relationship to concepts of process capability (e.g., C_p and C_{pk} [71]) in manufacturing, and experimental verification. Moreover, the generation theory appears to be intimately linked with the information theory.[69] Clarifying and exploiting all the connections between these two theories and applying them to robotic self-reproducing systems are parts of our current ongoing work.

The generation theory makes predictions that are very reminiscent of biological processes.[72] Indeed, it is well known that, in living multi-cell organisms, undifferentiated stem cells have the capability of producing both undifferentiated stem cells and terminally differentiated stem cells, which can themselves produce specialized cells. It is widely believed that the processes of differentiation and specialization are irreversible: a terminally differentiated stem cell can no longer produce an undifferentiated stem cell, and a specialized cell can no longer produce a differentiated stem cell. This situation is clearly similar to what happens in a generation system, as summarized in the principle of degeneracy. Modeling of biological systems from the standpoint of the generation theory will also be a part of our future work.

On a final note, based on all instances where self-reproduction has been observed, it can be anticipated that it requires three basic resources: matter, energy, and capability. Of these three, the most crucial seems to be capability: in nature, many microorganisms consist of very small amounts of matter, use very little energy, but have enormous capability in terms of meeting their von Neumann rank threshold. This paper has focused on such capability requirements for self-reproduction. Addressing the matter and energy requirements, and studying how these requirements can be traded off will be a part of our future work too.

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Appendix

Proof of Proposition 1

The proof is by induction. To initialize the induction, note that if $x \in M_1$, then, by definition of M_1, x is a cell, hence, $x \in M_0$. Therefore, $M_1 \subseteq M_0$.

Furthermore, if $x \in M_2$, then x is capable of producing some cell $y \in M_1$. Since $M_1 \subseteq M_0$, we also have $y \in M_0$. As x is capable of producing a cell $y \in M_0$, we have $x \in M_1$. Therefore, $M_2 \subseteq M_1$.

To prove the inductive step, assume that $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_i$, and let $x \in M_{i+1}$. By definition of M_{i+1}, x is capable of producing some cell $y \in M_i$. Since $M_i \subseteq M_{i-1}$, x is capable of producing a cell $y \in M_{i-1}$, which implies that $x \in M_i$. Therefore, $M_{i+1} \subseteq M_i$, which completes the proof of the inductive step.

Proof of Proposition 2

The proof is by contradiction. Assume that there exists a resource $r \in R$ such that $G(x, r) \in M_i$. Then by the definition of M_{i+1} we have $x \in M_{i+1}$, which contradicts the assumption $x \in M_i \backslash M_{i+1}$. This contradiction completes the proof.

Proof of Proposition 3

The proof is by induction. To initialize, we note that by hypothesis the proposition holds for $j = i + 1$.

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To prove the inductive step, assume that $M_i = M_{i+1} = \cdots = M_j$, and let $x \in M_j$. By the definition of M_j, x is capable of producing a cell $y \in M_{j-1}$. Now, by the induction hypothesis, $M_{j-1} = M_j$. Hence, x is capable of producing a cell $y \in M_j$, which implies that $x \in M_{j+1}$. Therefore, $M_j \subseteq M_{j+1}$. Now, by Proposition 1 we know that $M_{j+1} \subseteq M_j$. So we conclude that $M_{j+1} = M_j = M_i$, which completes the proof of the inductive step.

Proof of Proposition 4

First, we note that the rank must be finite. Indeed, if the rank is infinite, then the sets $(M_0 \backslash M_1)$, $(M_1 \backslash M_2), \ldots, (M_{|M|} \backslash M_{|M|+1})$ are mutually disjoint and contain at least one element. Since

$$M \supseteq (M_0 \backslash M_1) \cup (M_1 \backslash M_2) \cup \cdots \cup (M_{|M|} \backslash M_{|M|+1}), \quad \text{(A1)}$$

lower bounding the cardinality of each of the sets on the right-hand side by 1, we obtain $|M| \geq |M| + 1$, which is a contradiction. Hence, the rank must be finite.

Accordingly, let ρ be the finite rank of the generation system. Then M can be decomposed as the union of mutually disjoint sets

$$M = (M_0 \backslash M_1) \cup (M_1 \backslash M_2) \cup \cdots \cup (M_{\rho-1} \backslash M_\rho) \cup M_\rho, \quad \text{(A2)}$$

where, on the right-hand side of Eq. (A2), the sets $(M_0 \backslash M_1)$, $(M_1 \backslash M_2), \ldots, (M_{\rho-1} \backslash M_\rho)$ contain at least one element. Again, lower bounding the cardinality of each of these sets by 1 and lower bounding the cardinality of M_ρ by 0, Eq. (A2) yields Eq. (2.4), which completes the proof.

Proof of Proposition 5

The proof is by contradiction. Assume that any one of the n cells has deficient generation rank and, without loss of generality, let that be x_1. Then Proposition 2 implies the inequalities $\rho(x_2) < \rho(x_1)$, $\rho(x_3) < \rho(x_2), \ldots, \rho(x_n) < \rho(x_{n-1})$, and $\rho(x_1) < \rho(x_n)$. This yields the contradiction $\rho(x_1) < \rho(x_1)$ and completes the proof.

Proof of Proposition 6

Let ρ be the finite rank of the system, so that $M_\rho = M_{\rho+1} = M_\infty$. Since $x \in M_{\rho+1}$, there must exist a cell $y \in M_\rho$ and a resource $r \in R$ such that $y = G(x, r)$. Hence, $G(x, r) \in M_\infty$, which completes the proof.

Proof of Proposition 7

By assumption, there exists a cell $x \in M_\rho$ and a resource $r_1 \in R$ such that $y = G(x, r_1)$. For that particular x, according to Proposition 6, there must exist a resource $r_2 \in R$ such that $G(x, r_2) \in M_\rho$. Since $G(x, r_1) \notin M_\rho$ but $G(x, r_2) \in M_\rho$, we conclude that $r_1 \neq r_2$, which completes the proof.

Mathematical Examples

Here we present two additional mathematical examples that illustrate the generality of generation theory. First, consider the generation system (U, M, R, G) where the set of cells M is the set of all nonzero analytic functions of a single variable t, the resource set is the singleton

$$R = \left\{ \frac{d}{dt} \right\}, \quad \text{(A3)}$$

and the generation function maps an analytic function into its derivative, i.e.,

$$G\left(x, \frac{d}{dt}\right) = \frac{dx}{dt}. \quad \text{(A4)}$$

In this generation system, self-reproduction in the sense of Eq. (2.2) means

$$x(t) = \frac{dx(t)}{dt} 0, \quad x(t) \neq 0. \quad \text{(A5)}$$

Here, the generation set M_k is the set of analytic functions whose kth derivative is nonzero, and the set $M_k \backslash M_{k+1}$ is the set of polynomials of degree k. Since the only functions satisfying Eq. (A5) are exponential and not polynomial, the generation rank and the von Neumann rank threshold of this system are infinite.

Now consider another generation system (U, M, R, G) where M is the set of all nonzero causal functions of the variable t, the resource set is the singleton containing the delta function

$$R = \{\delta(t)\}, \quad \text{(A6)}$$

and the generation function maps each causal signal into the convolution of this signal with the resource:

$$G(x, \delta) = x(t) * \delta(t). \quad \text{(A7)}$$

In this case, all nonzero causal signals self-reproduce in the sense of Eq. (2.2). Hence, the generation rank and the von Neumann rank threshold are zero.

Remark 3 In both examples in this Appendix, every cell can be generated and the resource set is a singleton. It is then not surprising that the generation rank should be either zero or infinity, according to Corollary 5.

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