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Asymptotics for crank of overpartitions

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Abstract. Let $\overline{M}(a,c,n)$ denote the number of overpartitions of n with first residual crank congruent to a modulo c with $c \geq 3$ being odd and $0 \leq a < c$. The central objective of this paper is twofold: firstly, to establish an asymptotic formula for the crank of overpartitions; and secondly, to establish several inequalities concerning $\overline{M}(a,c,n)$ that encompasses crank differences, positivity, and strict log-subadditivity.

1 Introduction

The objective of this paper is to derive an asymptotic formula for the crank of overpartitions, and subsequently utilize this formula to demonstrate several inequalities of crank function. A partition of a positive integer n is understood to be a sequence of non-decreasing positive integers that sum to n. The function $p(n)$ represents the total number of partitions of n. In exploring a combinatorial explanation for the three notable congruences of $p(n)$ discovered by Ramanujan, Dyson [13] proposed the notion of the rank of a partition. This rank is determined by the discrepancy between the difference between the largest part and the number of parts. Dyson conjectured that the rank of a partition could explain the congruences for $p(5n+4)$ and $p(7n+5)$, but not for $p(11n+6)$, which was confirmed by Atkin and Swinnerton–Dyer [3]. Furthermore, Dyson hypothesized the existence of another statistic, specifically the crank, which he proposed could illuminate all of Ramanujan's congruences. This crank concept was subsequently brought to fruition by Andrews and Garvan [2, 14], where the crank of a partition λ, is defined as the largest part of λ if there are no ones in the partition; otherwise, it is defined as the number of parts larger than the number of ones minus the number of ones.

Since their inception, the ranks and cranks of partitions have become subjects of extensive study, with particular focus on their asymptotic behavior and differences in rank or crank. Let $N(a,c,n)$ (resp. $M(a,c,n)$) denote the number of partitions of n with rank (resp. crank) congruent to a modulo c. Bringmann [5] utilized the Hardy–Ramanujan circle method to derive an asymptotic formula for $N(a,c,n)$.

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Expanding on this foundation, Bringmann and Kane [8] proceeded to establish an inequality for $N(a,c,n)$. Moreover, Rolon [18] demonstrated analogous asymptotics and inequalities for $M(a,c,n)$.

As expected, this prompted further study of the rank and crank for other types of restricted partitions. With the definition of overpartitions, the concepts of rank [17] and crank [7] were straightforwardly extended to these. Recall that an overpartition [11] of a nonnegative integer n is a partition of n, but with the addition that the first occurrence of each distinct part may be overlined, and $\overline{p}(n)$ represents the total number of overpartitions of n. Many mathematicians have contributed to understanding the asymptotics [6, 9] and inequalities [10, 12, 15, 16] related to the rank of overpartitions.

Analogous to the crank of an ordinary partition, Bringmann, Lovejoy, and Osburn [7] proposed the concept of the first residual crank for an overpartition. The first residual crank of an overpartition is defined as the crank of the subpartition formed by the non-overlined parts. For instance, consider the overpartition $\lambda = (\bar{9}, 9, 7, \bar{6}, 5, 5, \bar{4}, 4, 3, \bar{1}, 1, 1)$. The subpartition composed of its non-overlined parts is $(9, 7, 5, 5, 4, 3, 1, 1)$. Consequently, the first residual crank of λ is determined to be 4.

Before we state one of main results of this paper, we recall certain facts about Dedekind sums. If $x \in \mathbb{R}$, then let

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{for } x \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

For coprime integers h and k, let $s(h,k)$ be the usual Dedekind sum

$$s(h,k) := \sum_{\mu} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

In terms of these sums, we define $\omega_{h,k}$ by

$$\omega_{h,k} := e^{\pi i s(h,k)}.$$

For coprime integers h and k, with $k > 0$, we let h' denote a choice of an inverse of $-h$ modulo k, that is, h' is a solution to $hh' \equiv -1 \pmod{k}$. Furthermore, consider $0 < a < c$ as coprime integers with $c \geq 3$ being odd. Let $k_1 := \frac{k}{(k,c)}$, $c_1 := \frac{c}{(k,c)}$ and $0 < l < c_1$ be defined by the congruence $l \equiv ak_1 \pmod{c_1}$. Lastly, for $m, n \in \mathbb{Z}$ and $c|k$, we proceed to define the following sum of Kloosterman type

$$\overline{B}_{a,c,k}(n,m) := (-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{h \pmod{k}^*} \frac{\omega_{h,k}^2 \omega_{2h,k}^{-1}}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{\pi i a^2 k h'}{c^2}} e^{\frac{2\pi i}{k}(hn+h'm)},$$

where the sum runs over all primitive residue classes h modulo k. For the case $c \nmid k$, we introduce another Kloosterman-type sum as follows:

$$\overline{D}_{a,c,k}(n,m) := (-1)^{ak+l} \sum_{h \pmod{k}^*} \omega_{h,k}^2 \omega_{2h,k}^{-1} e^{\frac{2\pi i}{k}(hn+h'm)}.$$

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Moreover, we define

$$\delta^{+}_{a,c,k,r} = \frac{l^2}{2c_1^2} + \frac{1}{16} - \left(r + \frac{1}{2}\right)\frac{l}{c_1},$$

$$\delta^{-}_{a,c,k,r} = \frac{l^2}{2c_1^2} - \frac{15}{16} - r\left(1 - \frac{l}{c_1}\right) + \frac{l}{2c_1},$$

and

$$m^{+}_{a,k,c,r} = -\frac{a^2k_1^2}{2c_1^2} + \frac{lak_1}{c_1^2} - \frac{l^2}{2c_1^2} + \frac{l}{2c_1} + \frac{lr}{c_1} - \frac{ak_1}{2c_1} - \frac{ark_1}{c_1},$$

$$m^{-}_{a,k,c,r} = -\frac{a^2k_1^2}{2c_1^2} + \frac{lak_1}{c_1^2} - \frac{l^2}{2c_1^2} + 1 + r - \frac{l}{2c_1} - \frac{lr}{c_1} + \frac{ak_1}{2c_1} + \frac{ark_1}{c_1}.$$

Let $\overline{M}(m,n)$ denote the number of overpartitions of n with the first residual crank equal to m. Bringmann, Lovejoy, and Osburn [7] obtained the generating function of $\overline{M}(m,n)$, that is,

$$\overline{C}(x;q) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{M}(m,n)x^m q^n = \frac{(q,-q;q)_\infty}{(xq,x^{-1}q;q)_\infty}. \tag{1.1}$$

In particular,

$$\overline{C}(1;q) = \sum_{n=0}^{\infty}\overline{p}(n)q^n.$$

If $0 < a < c$ are coprime integers, c is odd and n is a positive integer, we define

$$\overline{C}\left(e^{\frac{2\pi i a}{c}};q\right) = 1 + \sum_{n=1}^{\infty}\overline{A}\left(\frac{a}{c};n\right)q^n.$$

Using the circle method, we obtain the following asymptotic formulas for the coefficient $\overline{A}\left(\frac{a}{c};n\right)$.

Theorem 1.1 *If $0 < a < c$ are coprime integers, $c \geq 3$ is odd and n is a positive integer, then the following holds:*

$$\overline{A}\left(\frac{a}{c};n\right) = in^{-\frac{1}{2}}\sum_{\substack{1\leq k\leq\sqrt{n}\\ c|k,\ 2\nmid k}}\frac{\overline{B}_{a,c,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{n}}{k}\right) + 2n^{-\frac{1}{2}}\sin\left(\frac{\pi a}{c}\right)$$

$$\times\sum_{\substack{1\leq k\leq\sqrt{n}\\ c\nmid k,\ 2\nmid k\\ \delta^{i}_{a,c,k,r}>0\\ r\geq 0,\ i\in\{+,-\}}}\frac{\overline{D}_{a,c,k}(-n,m^{i}_{a,c,k,r})}{\sqrt{k}}\sinh\left(\frac{4\pi}{k}\sqrt{n\delta^{i}_{a,c,k,r}}\right) + \overline{R}_c(n),$$

where

$$|\overline{R}_c(n)| \leq (1700009.4994c + 7358.2224)n^{\frac{3}{8}}.$$

Employing Theorem 1.1, a straightforward analysis leads us to the ensuing corollary.

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Corollary 1.2 *Let $0 \le a < c$ with c an odd integer. For any positive integer n, we have*

$$\begin{aligned}\overline{M}(a,c,n) = {} & \frac{1}{2c\pi}\sum_{\substack{k=1\\ 2\nmid k}}^{\infty}\sqrt{k}\sum_{\substack{h=0\\ (h,k)=1}}^{k}\frac{\omega_{h,k}^2}{\omega_{2h,k}}e^{-\frac{2\pi inh}{k}}\frac{d}{dn}\left(\frac{\sinh\frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right)+\frac{1}{c}\sum_{j=1}^{c-1}\zeta_c^{-aj}in^{-\frac{1}{2}}\\ & \times\sum_{\substack{1\le k\le\sqrt{n}\\ c'\mid k,\ 2\nmid k}}\frac{\overline{B}_{j',c',k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{n}}{k}\right)+\frac{1}{c}\sum_{j=1}^{c-1}\zeta_c^{-aj}2n^{-\frac{1}{2}}\sin\left(\frac{\pi j}{c}\right)\\ & \times\sum_{\substack{1\le k\le\sqrt{n}\\ c_j\nmid k,\ 2\nmid k\\ \delta^i_{j',c',k,r}>0\\ r\ge 0,\ i\in\{+,-\}}}\frac{\overline{D}_{j',c',k}(-n,m^i_{j',c',k,r})}{\sqrt{k}}\sinh\left(\frac{4\pi}{k}\sqrt{n\delta^i_{j',c',k,r}}\right)+R_c(n),\end{aligned}$$

where $c' := \frac{c}{(c,j)}$, $j' := \frac{j}{(c,j)}$ and

$$|R_c(n)| \le (1700009.4994c + 7358.2224)n^{\frac{3}{8}}. \tag{1.2}$$

Remark While the bounds on the error terms in Theorem 1.1 and Corollary 1.2 appear identical, they fundamentally differ. Consequently, we employ distinct notations to denote them.

Corollary 1.2 offers an asymptotic expression for $\overline{M}(a,c,n)$ along with an error term $O(n^{\varepsilon})$.

Theorem 1.3 *Let $0 \le a < c$ with c an odd integer. Then we have*

$$\frac{\overline{M}(a,c,n)}{\overline{p}(n)} = \frac{1}{c} + \overline{R}(a,c,n), \tag{1.3}$$

where

$$|\overline{R}(a,c,n)| \le \begin{cases} 10^5(464.45c+2.02)n^{\frac{11}{8}}e^{-\frac{2}{3}\pi\sqrt{n}}, & \text{if } c<9,\\ 10^5(464.45c+2.02)n^{\frac{11}{8}}e^{-\left(1-\sqrt{1-\frac{8}{c}+\frac{8}{c^2}}\right)\pi\sqrt{n}}, & \text{if } c\ge 9.\end{cases}$$

Subsequently, we further refine this analysis by establishing an equality relationship between $\overline{M}(a,c,n)$ and $\overline{p}(n)$. In addition, we provide explicit bounds for the associated error term.

As a direct application of Theorem 1.1, we further establish the inequality of crank differences.

Theorem 1.4 *Suppose that $0 \le a < b \le \frac{c-1}{2}$ and that $c \ge 9$ is an odd integer. For $n > \overline{N}_{a,b,c}$, where $\overline{N}_{a,b,c}$ is an explicit constant, we have the inequality:*

$$\overline{M}(a,c,n) > \overline{M}(b,c,n).$$

We employ Theorem 1.3 to demonstrate the positivity of $\overline{M}(a,c,n)$.

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Theorem 1.5 *Let $0 \le a < c$ with c an odd integer. For $c \ge 9$, define the constant*

$$\overline{Q}_c := \frac{c^4\left(2+\sqrt{4+\frac{16\pi}{c}\log\left(2\times 10^5\times\left(464.45c^2+2.02c\right)\right)}\right)^4}{8^4\pi^4}. \tag{1.4}$$

Then we have

$$\overline{M}(a,c,n) > 0,$$

if $c < 9$ and $n \ge 196$, or if $c \ge 9$ and $n > \max\{840c, \overline{Q}_c\}$.

Bessenrodt and Ono [4] were the first to discover the strict log-subadditivity property of the partition function $p(n)$. Subsequently, numerous studies emerged investigating the strict log-subadditivity property of other partition functions. The final result of this paper is to demonstrate the strict log-subadditivity property of $\overline{M}(a,c,n)$.

Theorem 1.6 *Let $0 \le a < c$ with c an odd integer. We have*

$$\overline{M}(a,c,n_1+n_2) < \overline{M}(a,c,n_1)\overline{M}(a,c,n_2), \tag{1.5}$$

if $c < 9$ and $n_1, n_2 \ge 196$, or if $c \ge 9$ and $n_1, n_2 \ge \overline{Q}_c$, where $\overline{Q}_c$ is defined in (1.4).

The paper is structured as follows. In Section 2, we introduce a transformation formula for $\overline{C}(x;q)$. Section 3 is dedicated to breaking down the coefficient $\overline{A}\left(\frac{a}{c};n\right)$ into four terms and providing explicit expressions for the main terms and establishing bounds for the error terms within each component. This allows us to establish Theorem 1.1. Moving on to Section 4, we leverage Theorem 1.1 to prove Corollary 1.2 and Theorem 1.3. Finally, in Section 5, we apply the derived results to demonstrate various inequalities pertaining to the crank function, covering crank differences, positivity, and strict log-subadditivity.

2 A transformation law

In this section, we establish a transformation formula for the function $\overline{C}(x;q)$, which plays an important role in the proof of the asymptotics of crank of overpartitions. First, we introduce the needed quantities to state the transformation rules. The Dedekind η-function is defined as

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}}\prod_{n=0}^{\infty}\left(1-e^{2\pi in\tau}\right), \tag{2.1}$$

and the Jacobi ϑ-function is defined as

$$\vartheta(u;\tau) = \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} e^{\pi i\nu^2\tau+2\pi i\nu\left(u+\frac{1}{2}\right)}.$$

For convenience, we let

$$\tau_1 := \frac{h+iz}{k}, \qquad \tau_2 := \frac{h'+iz^{-1}}{k},$$

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and

$$\omega_1 =: \frac{c}{a}, \qquad \omega_2 := \frac{ah'}{c}, \qquad \omega_3 := -\omega_2 + \frac{l}{c_1}\tau_2,$$

where h, k are coprime integer with h' like previous section. In principle, one can make any choice of h', but must carry that specific choice through all relevant calculations. To allow simplifications in various formulas and calculations, we make some assumptions about our choice of h'. These assumptions deal only with fixed k and h and only appear in the proofs. When $2 \nmid k$, we assume that $h_2' = \frac{h'}{2}$. These are viewed as choices made for a fixed value of k and h, so such a choice is possible. Clearly such choices would be impossible for all k and h simultaneously.

For $z \in \mathbb{C}$ with $Re(z) > 0$, we have

$$\eta(\tau_1) = \sqrt{\frac{i}{z}}\chi(h, h', k)\eta(\tau_2), \tag{2.2}$$

where we take the principal branch of the square root and

$$\chi(h, h', k) = i^{-\frac{1}{2}}\omega_{h,k}^{-1}e^{-\frac{\pi i}{12k}(h'-h)}. \tag{2.3}$$

Moreover, η is a modular form of weight $\frac{1}{2}$ with multiplier system.

Define $x = e^{2\pi i\omega}$ and $q = e^{2\pi i\tau}$, where $\omega \in \mathbb{C}$ and $\tau \in \mathbb{H}$. Then, ϑ satisfies the Jacobi triple product identity

$$\vartheta(\omega; \tau) = -2\sin(\pi\omega)q^{\frac{1}{8}}(q, xq, x^{-1}q; q)_\infty, \tag{2.4}$$

and if $\mathrm{Re}(z) > 0$, then

$$\vartheta(\omega; \tau_1) = \chi^3(h, h', k)\sqrt{\frac{i}{z}}e^{-\frac{\pi k\omega^2}{z}}\vartheta\left(\frac{i\omega}{z}; \tau_2\right). \tag{2.5}$$

Plugging (2.1) and (2.4) in (1.1), we obtain that

$$\overline{C}(x; q) = -2\sin(\pi\omega)\frac{\eta(\tau)\eta(2\tau)}{\vartheta(\omega; \tau)}. \tag{2.6}$$

We are now ready to show the transformation law of $\overline{C}\left(e^{2\pi i\omega_1}; e^{2\pi i\tau_1}\right)$.

Proposition 2.1 *Adopting the previously mentioned notation and further, let $w \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Define $q := e^{2\pi i\tau_1} = e^{\frac{2\pi i}{k}(h+iz)}$ and $q_1 := e^{2\pi i\tau_2} = e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}$.*

(1) If $c|k$ and k is even, we have

$$\overline{C}\left(e^{2\pi i\omega_1}; q\right) = \frac{i\sin(\pi\omega_1)}{z^{\frac{1}{2}}\sin(\pi\omega_2)}(-1)^{ak+1}\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1}e^{-\frac{\pi i a^2k_1h'}{c}}\overline{C}\left(e^{2\pi i\omega_2}; q_1\right). \tag{2.7}$$

(2) If $c|k$ and k is odd, we have

$$\overline{C}\left(e^{2\pi i\omega_1}; q\right) = \frac{\sqrt{2}i\sin(\pi\omega_1)}{2z^{\frac{1}{2}}\sin(\pi\omega_2)}(-1)^{ak+1}\omega_{h,k}^2\omega_{2h,k}^{-1}e^{\frac{\pi ih'}{8k}-\frac{\pi i a^2k_1h'}{c}} \cdot \frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta(2\tau_2)}\overline{C}\left(e^{2\pi i\omega_2}; q_1\right). \tag{2.8}$$

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(3) If $c \nmid k$ and k is even, we have

$$\overline{C}\left(e^{2\pi i\omega_1};q\right) = 4i\sin\left(\pi\omega_1\right) z^{-\frac{1}{2}}(-1)^{ak+l+1}\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1} \cdot e^{\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2k_1h'}{cc_1}} q_1^{-\frac{l^2}{2c_1^2}}\overline{C}\left(ah', \frac{lc}{c_1}, c; q_1\right). \tag{2.9}$$

(4) If $c \nmid k$ and k is odd, we have

$$\overline{C}\left(e^{2\pi i\omega_1};q\right) = 2\sqrt{2}i\sin\left(\pi\omega_1\right) z^{-\frac{1}{2}}(-1)^{ak+l+1}\omega_{h,k}^2\omega_{2h,k}^{-1}e^{\frac{\pi ih'}{8k}} \cdot q_1^{-\frac{l^2}{2c_1^2}} e^{\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2k_1h'}{cc_1}} \frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta\left(2\tau_2\right)}\overline{C}\left(ah', \frac{lc}{c_1}, c; q_1\right), \tag{2.10}$$

where

$$\overline{C}(a,b,c;q) = \frac{i}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{-\frac{\pi ia}{c}} q^{\binom{m+1}{2}+\frac{b}{2c}}}{1-e^{-\frac{2\pi ia}{c}}q^{m+\frac{b}{c}}}.$$

Proof Substituting ω and τ by ω_1 and τ_1 in (2.6), respectively, we find that

$$\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right) = -2\sin\left(\pi\omega_1\right)\eta\left(\tau_1\right)\eta\left(2\tau_1\right)\vartheta^{-1}\left(\omega_1;\tau_1\right). \tag{2.11}$$

By plugging (2.5) and (2.3) into (2.11), we infer that

$$\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right) = -2iz^{\frac{1}{2}}\sin\left(\pi\omega_1\right)\omega_{h,k}^3 e^{\frac{\pi i}{4k}(h'-h)+\frac{\pi ka^2}{zc^2}}\eta\left(\tau_1\right)\eta\left(2\tau_1\right)\vartheta^{-1}\left(\frac{i\omega_1}{z};\tau_2\right). \tag{2.12}$$

For k is even, since $(h,k)=1$, we have $\left(h,\frac{k}{2}\right)=1$. From (2.2) it follows that

$$\eta\left(\tau_1\right)\eta\left(2\tau_1\right) = \sqrt{\frac{i}{z}}\chi(h,h',k)\eta\left(\tau_2\right)\sqrt{\frac{i}{z}}\chi\left(h,h',\frac{k}{2}\right)\eta\left(2\tau_2\right). \tag{2.13}$$

Inserting (2.3) into (2.13), we arrive at

$$\eta\left(\tau_1\right)\eta\left(2\tau_1\right) = z^{-1}\omega_{h,k}^{-1}\omega_{h,\frac{k}{2}}^{-1}e^{-\frac{\pi i}{4k}(h'-h)}\eta\left(\tau_2\right)\eta\left(2\tau_2\right).$$

Thus for k is even, (2.12) equals to

$$\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right) = -2iz^{-\frac{1}{2}}\sin\left(\pi\omega_1\right)\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1}e^{\frac{\pi ka^2}{zc^2}}\frac{\eta\left(\tau_2\right)\eta\left(2\tau_2\right)}{\vartheta\left(\frac{i\omega_1}{z};\tau_2\right)}. \tag{2.14}$$

We now assume that $c|k$. It is obvious that $\frac{ak}{c}\in\mathbb{Z}$. Rewriting $\eta\left(\tau_2\right)\eta\left(2\tau_2\right)$ by (2.6), we obtain

$$\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right) = iz^{-\frac{1}{2}}\frac{\sin\left(\pi\omega_1\right)}{\sin\left(\pi\omega_2\right)}\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1}e^{\frac{\pi ka^2}{zc^2}}\overline{C}\left(e^{2\pi i\omega_2};e^{2\pi i\tau_2}\right)\frac{\vartheta\left(\omega_2;\tau_2\right)}{\vartheta\left(\frac{i\omega_1}{z};\tau_2\right)}. \tag{2.15}$$

Recall the following identity in Rolon [18]:

$$\frac{\vartheta\left(\omega_2;\tau_2\right)}{\vartheta\left(\frac{i\omega_1}{z};\tau_2\right)} = (-1)^{ak+1}e^{-\frac{\pi ia^2kh'}{c^2}-\frac{\pi a^2k}{zc^2}}. \tag{2.16}$$

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Then (2.15) becomes

$$\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right)=(-1)^{ak+1}iz^{-\frac{1}{2}}\frac{\sin(\pi\omega_1)}{\sin(\pi\omega_2)}\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1}e^{-\frac{\pi ia^2kh'}{c^2}}\overline{C}\left(e^{2\pi i\omega_2};e^{2\pi i\tau_2}\right),$$

which coincides with (2.7).

The case $c\nmid k$ presents more complexity, as typically $\frac{ak}{c}\notin\mathbb{Z}$. Given that l was determined as the solution to the congruence $l\equiv ak_1(\mathrm{mod}\ c_1)$ by definition, it follows that $B=\frac{l-ak_1}{c_1}\in\mathbb{Z}$. This allows for the adjustment of the theta function in the elliptic variable by a factor of B times the modular variable, as defined previously.

Recall the following identity in Rolon [18]:

$$\vartheta\left(\frac{i\omega_1}{z};\tau_2\right)=(-1)^{ak+l}e^{\frac{\pi i(l-ak_1)^2\tau_2}{c_1^2}+2\pi i\left(\frac{l-ak_1}{c_1}\right)\cdot\frac{i\omega_1}{z}}\vartheta(\omega_3;\tau_2). \tag{2.17}$$

Combining with (2.14), we get

$$\begin{aligned}\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right)&=2(-1)^{ak+l+1}iz^{-\frac{1}{2}}\sin(\pi\omega_1)\,\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1}\\&\quad\times q_1^{-\frac{l^2}{2c_1}}e^{\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2h'k_1}{cc_1}}\frac{\eta(\tau_2)\,\eta(2\tau_2)}{\vartheta(\omega_3;\tau_2)}\\&=(-1)^{ak+l}iz^{-\frac{1}{2}}\sin(\pi\omega_1)\,\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1}\\&\quad\times q_1^{-\frac{l^2}{2c_1}}e^{\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2h'k_1}{cc_1}}\frac{\overline{C}\left(e^{2\pi i\omega_3};q_1\right)}{\sin(\pi\omega_3)}.\end{aligned} \tag{2.18}$$

We define $x=e^{2\pi i\omega_3}$, and use the exponential representation of the sine to deduce

$$\begin{aligned}\frac{\overline{C}\left(e^{2\pi i\omega_3};q_1\right)}{\sin(\pi\omega_3)}&=\frac{2i(1-x)}{x^{\frac{1}{2}}-x^{-\frac{1}{2}}}\cdot\frac{(-q_1;q_1)_\infty}{(q_1;q_1)_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^mq_1^{\binom{m+1}{2}}}{1-xq_1^m}\\&=-\frac{2ix^{\frac{1}{2}}(-q_1;q_1)_\infty}{(q_1;q_1)_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^mq_1^{\binom{m+1}{2}}}{1-xq_1^m}\\&=-4\overline{C}\left(ah',\frac{lc}{c_1},c;q_1\right).\end{aligned} \tag{2.19}$$

Thus, (2.9) can be given by inserting this expression into (2.18).

For k is odd, then we have $(2h,k)=1$, by (2.2),

$$\eta(\tau_1)\,\eta(2\tau_1)=\sqrt{\frac{i}{z}}\chi(h,h',k)\eta(\tau_2)\sqrt{\frac{i}{2z}}\chi\left(2h,\frac{h'}{2},k\right)\eta\left(\frac{\tau_2}{2}\right).$$

Using (2.3), we infer that

$$\eta(\tau_1)\,\eta(2\tau_1)=\frac{\sqrt{2}}{2}z^{-1}\omega_{h,k}^{-1}\omega_{2h,k}^{-1}e^{-\frac{\pi i}{4k}\left(\frac{h'}{2}-h\right)}\eta(\tau_2)\,\eta\left(\frac{\tau_2}{2}\right).$$

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Thus for k is odd, (2.12) equals to

$$\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right)=-\sqrt{2}iz^{-\frac{1}{2}}\sin\left(\pi\omega_1\right)\omega_{h,k}^2\omega_{2h,k}^{-1}e^{\frac{\pi ih'}{8k}+\frac{\pi ka^2}{zc^2}}\frac{\eta\left(\tau_2\right)\eta\left(\frac{\tau_2}{2}\right)}{\vartheta\left(\frac{i\omega_1}{z};\tau_2\right)}. \tag{2.20}$$

Similarly, we consider the cases $c|k$ and $c\nmid k$. For $c|k$, we have $\frac{ak}{c}$ is an integer, and we obtain

$$\begin{aligned}\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right)&=i(2z)^{-\frac{1}{2}}\frac{\sin\left(\pi\omega_1\right)}{\sin\left(\pi\omega_2\right)}\omega_{h,k}^2\omega_{2h,k}^{-1}e^{\frac{\pi ih'}{8k}+\frac{\pi ka^2}{zc^2}}\frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta\left(2\tau_2\right)}\\&\quad\cdot\frac{\vartheta\left(\omega_2;\tau_2\right)}{\vartheta\left(\frac{i\omega_1}{z};\tau_2\right)}\overline{C}\left(e^{2\pi i\omega_2};q_1\right).\end{aligned} \tag{2.21}$$

Utilizing (2.16), (2.21) can be rewritten as

$$\begin{aligned}\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right)&=(-1)^{ak+1}i(2z)^{-\frac{1}{2}}\frac{\sin\left(\pi\omega_1\right)}{\sin\left(\pi\omega_2\right)}\omega_{h,k}^2\omega_{2h,k}^{-1}e^{\frac{\pi ih'}{8k}-\frac{\pi ia^2k_1h'}{c}}\\&\quad\cdot\frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta\left(2\tau_2\right)}\overline{C}\left(e^{2\pi i\omega_2};q_1\right),\end{aligned}$$

which yields (2.8).

It remains to prove (2.10). Combining (2.17) and (2.19) with (2.20), we see that

$$\begin{aligned}\overline{C}\left(e^{2\pi i\omega_1};e^{2\pi i\tau_1}\right)&=\sqrt{2}(-1)^{ak+l+1}iz^{-\frac{1}{2}}\sin\left(\pi\omega_1\right)\omega_{h,k}^2\omega_{2h,k}^{-1}q_1^{-\frac{l^2}{2c_1}}e^{\frac{\pi ih'}{8k}+\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2h'k_1}{cc_1}}\\&\quad\cdot\frac{\eta\left(\tau_2\right)\eta\left(\frac{\tau_2}{2}\right)}{\vartheta\left(\omega_3;\tau_2\right)}\\&=(-1)^{ak+l}i(2z)^{-\frac{1}{2}}\sin\left(\pi\omega_1\right)\omega_{h,k}^2\omega_{2h,k}^{-1}q_1^{-\frac{l^2}{2c_1}}e^{\frac{\pi ih'}{8k}+\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2h'k_1}{cc_1}}\\&\quad\cdot\frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta\left(2\tau_2\right)}\frac{\overline{C}\left(e^{2\pi i\omega_3};q_1\right)}{\sin\left(\pi\omega_3\right)}\\&=4(-1)^{ak+l+1}i(2z)^{-\frac{1}{2}}\sin\left(\pi\omega_1\right)\omega_{h,k}^2\omega_{2h,k}^{-1}q_1^{-\frac{l^2}{2c_1}}e^{\frac{\pi ih'}{8k}+\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2h'k_1}{cc_1}}\\&\quad\cdot\frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta\left(2\tau_2\right)}\overline{C}\left(ah',\frac{lc}{c_1},c;q_1\right).\end{aligned}$$

This completes the proof. ∎

3 Asymptotics for $\overline{A}\left(\frac{a}{c};n\right)$

In this section, we establish the asymptotic formula for $\overline{A}\left(\frac{a}{c};n\right)$ by decomposing it into the sum of Σ_1, Σ_2, Σ_3, and Σ_4, and then assessing the error and principal terms for each component. Key to our analysis are the following lemmas.

Lemma 3.1 *Let $n,m,k,D\in\mathbb{Z}$ with $(D,k)=1$, $0\le\sigma_1<\sigma_2\le k$. Then there are constants $\tilde{C}_1$ and $\tilde{C}_2$, which are independent of a and k, such that:*

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(1) *We have*

$$\left|\sum_{\substack{h(\bmod k)^*\\ \sigma_1\le Dh'\le\sigma_2}} \omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1} e^{\frac{2\pi i}{k}(hn+h'm)}\right| \le \tilde{C}_1\cdot\gcd(24n+1,k)^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon}. \tag{3.1}$$

(2) *We have*

$$\left|\frac{\sin\frac{\pi a}{c}}{(-1)^{ak+1}}\sum_{\substack{h(\bmod k)^*\\ \sigma_1\le Dh'\le\sigma_2}} \frac{\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1}}{\sin\left(\frac{\pi ah'}{c}\right)} e^{-\frac{\pi i a^2k_1h'}{c}} e^{\frac{2\pi i}{k}(hn+h'm)}\right| \le \tilde{C}_2\cdot\gcd(24n+1,k)^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon}. \tag{3.2}$$

Proof The proof of equation 3.1 is essentially established in [1], with only a slight modification required. Part two is a direct consequence of part one and the proof of Lemma 3.2 in [5]. ∎

Lemma 3.2 *Define the integral $I_{k,t}$ as follows:*

$$I_{k,t} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} e^{\frac{2\pi}{k}\left(zn+\frac{t}{z}\right)}\mathrm{d}\Phi. \tag{3.3}$$

We then evaluate this integral to obtain:

$$I_{k,t} = \sqrt{\frac{2}{kn}}\sinh\left(\frac{4\pi\sqrt{nt}}{k}\right) + I_{k,t}^{err}, \tag{3.4}$$

where the error term $I_{k,t}^{err}$ is given by

$$I_{k,t}^{err} \le \frac{2}{k}e^{2\pi+2\pi t}\left(\frac{4}{3}+2^{\frac{5}{4}}\right)n^{-\frac{1}{8}}. \tag{3.5}$$

Proof Substituting $z = \frac{k}{n} - ik\Phi$ into (3.3) gives

$$I_{k,t} = \frac{1}{ki}\int_{\frac{k}{n}-\frac{i}{N}}^{\frac{k}{n}+\frac{i}{N}} z^{-\frac{1}{2}} e^{\frac{2\pi}{k}\left(zn+\frac{t}{z}\right)}\mathrm{d}z. \tag{3.6}$$

We now define Γ as the circle passing through $\frac{k}{n}\pm\frac{i}{N}$ and tangent to the imaginary axis at 0. For $z = x+iy$, Γ is characterized by $x^2+y^2 = \alpha x$, where $\alpha = \frac{k}{n}+\frac{n}{N^2k}$. Considering that $2>\alpha>\frac{1}{k}$, $\mathrm{Re}(z)\le\frac{k}{n}$ and $\mathrm{Re}\left(\frac{1}{z}\right)<k$ on the smaller arc, similar to the argument of Rolon [18, Eq. (4.6)], we get that the integral over the smaller arc is:

$$I_{k,t}^{err} \le \frac{2}{k}e^{2\pi+2\pi t}\left(\frac{4}{3}+2^{\frac{5}{4}}\right)n^{-\frac{1}{8}}.$$

Furthermore, Cauchy's theorem allows us to modify the path of integration in (3.6) to follow the larger arc of Γ. Transforming the circle to a straight line by $s = \frac{2\pi t}{kz}$ gives:

$$\frac{1}{ki}\int_\Gamma z^{-\frac{1}{2}} e^{\frac{2\pi}{k}\left(zn+\frac{t}{z}\right)}\mathrm{d}z = \frac{2\pi}{k}\left(\frac{2\pi t}{k}\right)^{\frac{1}{2}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} s^{-\frac{3}{2}}e^{s+\frac{\beta}{s}}\mathrm{d}s,$$

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where $\gamma \in \mathbb{R}$ and $\beta = \frac{4\pi^2 tn}{k^2}$. Using the Hankel integral formula in [19], we compute

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} t^{-\frac{3}{2}} \cdot e^{t+\frac{\alpha}{t}}\,\mathrm{d}t = \frac{1}{\sqrt{\pi\alpha}} \cdot \sinh(2\sqrt{\alpha}).$$

We thus get (3.4). This completes the proof. ∎

Lemma 3.3 *If $0 < a < c$ are coprime integers, $c \geq 3$ is odd and n is a positive integer, then*

$$\begin{aligned}
\overline{A}\left(\frac{a}{c};n\right) &= i\sin\left(\frac{\pi a}{c}\right)\sum_h \sum_{\substack{c|k\\ 2|k}} \frac{(-1)^{ak+1}}{\sin\left(\frac{\pi a h'}{c}\right)}\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1} e^{-\frac{2\pi i h n}{k}-\frac{\pi i a^2 k_1 h'}{c}} \\
&\quad\cdot\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}}\overline{C}\left(e^{\frac{2\pi i a h'}{c}};q_1\right)\mathrm{d}\Phi \\
&\quad+\frac{\sqrt{2}}{2}i\sin\left(\frac{\pi a}{c}\right)\sum_h \sum_{\substack{c|k\\ 2\nmid k}} \frac{(-1)^{ak+1}}{\sin\left(\frac{\pi a h'}{c}\right)}\omega_{h,k}^2\omega_{2h,k}^{-1} e^{-\frac{2\pi i h n}{k}+\frac{\pi i h'}{8k}-\frac{\pi i a^2 k_1 h'}{c}} \\
&\quad\cdot\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}}\frac{\eta\left(\frac{h'+iz^{-1}}{2k}\right)}{\eta\left(\frac{2(h'+iz^{-1})}{k}\right)}\overline{C}\left(e^{\frac{2\pi i a h'}{c}};q_1\right)\mathrm{d}\Phi \\
&\quad+4i\sin\left(\frac{\pi a}{c}\right)\sum_h \sum_{\substack{c\nmid k\\ 2|k}}(-1)^{ak+l+1}\omega_{h,k}^2\omega_{h,\frac{k}{2}}^{-1} e^{-\frac{2\pi i h n}{k}+\frac{2\pi i a l h'}{cc_1}-\frac{\pi i a^2 k_1 h'}{cc_1}} \\
&\quad\cdot\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} q_1^{-\frac{l^2}{2c_1^2}}\overline{C}\left(ah',\frac{lc}{c_1},c;q_1\right)\mathrm{d}\Phi \\
&\quad+2\sqrt{2}i\sin\left(\frac{\pi a}{c}\right)\sum_h \sum_{\substack{c\nmid k\\ 2\nmid k}}(-1)^{ak+l+1}\omega_{h,k}^2\omega_{2h,k}^{-1} e^{-\frac{2\pi i h n}{k}+\frac{\pi i h'}{8k}+\frac{2\pi i a l h'}{cc_1}-\frac{\pi i a^2 k_1 h'}{cc_1}} \\
&\quad\cdot\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} q_1^{-\frac{l^2}{2c_1^2}}\frac{\eta\left(\frac{h'+iz^{-1}}{2k}\right)}{\eta\left(\frac{2(h'+iz^{-1})}{k}\right)}\overline{C}\left(ah',\frac{lc}{c_1},c;q_1\right)\mathrm{d}\Phi
\end{aligned}$$

$$:= \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4. \tag{3.7}$$

Proof By Cauchy's theorem we have for $n > 0$

$$\overline{A}\left(\frac{a}{c};n\right) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\overline{C}\left(e^{\frac{2\pi i a}{c}};q\right)}{q^{n+1}}\mathrm{d}q,$$

where $\mathcal{C}$ denotes any path within the unit circle that encircles the origin in a counterclockwise direction. Opting for a circular path of radius $e^{-\frac{2\pi}{n}}$ and parameterizing it as

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$q = e^{-\frac{2\pi}{n}+2\pi it}$ for $0 \le t \le 1$ yields

$$\overline{A}\left(\frac{a}{c};n\right) = \int_0^1 \overline{C}\left(e^{\frac{2\pi ia}{c}};e^{-\frac{2\pi}{n}+2\pi it}\right) e^{2\pi-2\pi int}\mathrm{d}t.$$

We define

$$\vartheta'_{h,k} = \frac{1}{k\left(\widetilde{k_1}+k\right)}, \quad \vartheta''_{h,k} = \frac{1}{k\left(\widetilde{k_2}+k\right)},$$

where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order $N = \lfloor n^{1/2} \rfloor$. We know that

$$\frac{1}{k+\widetilde{k_j}} \le \frac{1}{N+1} \quad (j=1,2).$$

Now we decompose the path of integration along Farey arcs $-\vartheta'_{h,k} \le \Phi \le \vartheta''_{h,k}$, where $\Phi = t - \frac{h}{k}$ and $0 \le h < k \le N$ with $(h,k) = 1$. From this decomposition of the path we can rewrite the integral along these arcs:

$$\overline{A}\left(\frac{a}{c};n\right) = \sum_{h,k} e^{-\frac{2\pi ihn}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \overline{C}\left(e^{\frac{2\pi ia}{c}};e^{\frac{2\pi i}{k}(h+iz)}\right) e^{\frac{2\pi nz}{k}}\mathrm{d}\Phi,$$

where $z = \frac{k}{n} - k\Phi i$. We incorporate our transformation formula into the integral and conclude the proof. ∎

By utilizing the estimates for the sums Σ_1, Σ_2, Σ_3, and Σ_4, we substantiate the proof of Theorem 1.1, wherein Σ_1 and Σ_3 correspond to the error terms, while Σ_2 and Σ_4 pertain to the main terms. Subsequently, we will provide an in-depth analysis, presenting detailed calculations for the main terms as well as establishing bounds for the error terms.

For convenience, we define

$$\varepsilon_1 := \sum_{r=1}^{\infty} \overline{p}(r) e^{-\pi r}, \quad \varepsilon_2 := \sum_{m\ge 0} \frac{e^{-\pi m(m+1)/2}}{1-e^{-\pi m-\frac{\pi}{c}}}, \quad \varepsilon_3 := \sum_{m\ge 1} \frac{e^{-\pi m(m+1)/2}}{1-e^{-\pi m+\pi\left(1-\frac{1}{c}\right)}}, \tag{3.8}$$

where c is an odd integer.

Lemma 3.4 *Σ_1 contributes to an error term, with its definition provided in (3.7). Moreover, the bound for Σ_1 is as follows:*

$$|\Sigma_1| \le \frac{8\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi}\left(1+\varepsilon_1\right)\left(1+\log\left(\frac{c-1}{2}\right)\right) n^{\frac{1}{4}}}{c\pi\left(1-\pi^2/24\right)}.$$

Proof We note that

$$\begin{aligned}\overline{C}\left(e^{\frac{2\pi iah'}{c}};q_1\right) &= 1 + \sum_{r=1}^{\infty}\sum_{m=-\infty}^{\infty} \overline{M}(m,r) e^{\frac{2\pi iah'm}{c}} q_1^r \\ &= 1 + \sum_{r=1}^{\infty} \sum_{s \pmod c} \sum_{t=-\infty}^{\infty} \overline{M}(s+tc,r) e^{\frac{2\pi iah's}{c}} q_1^r.\end{aligned}$$

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Let

$$\overline{a}(r,s) = \sum_{t=-\infty}^{\infty} \overline{M}(s+tc,r),$$

then we have

$$\overline{C}\left(e^{\frac{2\pi i a h'}{c}}; q_1\right) = 1 + \sum_{r=1}^{\infty} \sum_{s \pmod c} \overline{a}(r,s) e^{\frac{2\pi i h'}{c} m_{r,s}} q_1^r, \tag{3.9}$$

where $m_{r,s}$ takes values in $\mathbb{Z}$. Consequently, Σ_1 can be expressed as follows:

$$\Sigma_1 = \widehat{S}_1 + \widehat{S}_2,$$

where

$$\widehat{S}_1 = i \sin\left(\frac{\pi a}{c}\right) \sum_h \sum_{\substack{c|k \\ 2|k}} \frac{(-1)^{ak+1}}{\sin\left(\frac{\pi a h'}{c}\right)} \omega_{h,k}^2 \omega_{h,\frac{k}{2}}^{-1} e^{-\frac{2\pi i h n}{k} - \frac{\pi i a^2 k_1 h'}{c}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathrm{d}\Phi, \tag{3.10}$$

$$\begin{aligned} \widehat{S}_2 = {} & i \sin\left(\frac{\pi a}{c}\right) \sum_h \sum_{\substack{c|k \\ 2|k}} \frac{(-1)^{ak+1}}{\sin\left(\frac{\pi a h'}{c}\right)} \omega_{h,k}^2 \omega_{h,\frac{k}{2}}^{-1} e^{-\frac{2\pi i h n}{k} - \frac{\pi i a^2 k_1 h'}{c}} \\ & \cdot \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \sum_{r=1}^{\infty} \sum_{s \pmod c} \overline{a}(r,s) e^{\frac{2\pi i h'}{c} m_{r,s}} q_1^r \mathrm{d}\Phi, \end{aligned} \tag{3.11}$$

We first consider $\widehat{S}_1$. Splitting the integration path into

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\frac{1}{k(\widetilde{k}_1+k)}} - \int_{\frac{1}{k(\widetilde{k}_2+k)}}^{\frac{1}{kN}}, \tag{3.12}$$

and denoting the associated sums by $\widehat{S}_{11}$, $\widehat{S}_{12}$, and $\widehat{S}_{13}$, then one can demonstrate that $\widehat{S}_{1i}$ $(i = 1, 2, 3)$ contributes to the error term. Specifically, for $\widehat{S}_{11}$, we find

$$\left|\widehat{S}_{11}\right| \le \sum_{\substack{c|k \\ 2|k}} \int_{-\frac{1}{kN}}^{\frac{1}{kN}} \left| z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \right| d\Phi \cdot \left| \frac{\sin\left(\frac{\pi a}{c}\right)}{(-1)^{ak+1}} \sum_{h \atop \ell \le \widetilde{k}_1 + k - 1 \le N + k - 1} \frac{\omega_{h,k}^2 \omega_{h,\frac{k}{2}}^{-1}}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{2\pi i h n}{k} - \frac{\pi i a^2 k_1 h'}{c}} \right|.$$

According to (3.2) and the facts that

$$\mathrm{Re}(z) = \frac{k}{n}, \qquad |z|^{-\frac{1}{2}} \le k^{-\frac{1}{2}} n^{\frac{1}{2}}, \tag{3.13}$$

we deduce that

$$\begin{aligned} \left|\widehat{S}_{11}\right| & \le C \sum_{\substack{c|k \\ 2|k}} \gcd(24n+1, k)^{\frac{1}{2}} k^{\frac{1}{2}+\varepsilon} k^{-\frac{1}{2}} n^{\frac{1}{2}} e^{2\pi} \int_{-\frac{1}{kN}}^{\frac{1}{kN}} \mathrm{d}\Phi \\ & \le C_1 \sum_k \gcd(24n+1, k)^{\frac{1}{2}} k^{-1+\varepsilon} \le C_2 \sum_{k \le N} k^{-1+\varepsilon} \sum_{\substack{d|k \\ d|24n+1}} d^{\frac{1}{2}} \end{aligned}$$

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$$\le C_2 \sum_{\substack{d|24n+1\\ d\le N}} d^{\frac{1}{2}} \sum_{k\le N/d} (kd)^{-1+\varepsilon} \le C_2 \sum_{\substack{d|24n+1\\ d\le N}} d^{-\frac{1}{2}} \sum_{k\le N/d} k^{-1} N^{\varepsilon}$$

$$\le C_3 n^{\varepsilon} \sum_{\substack{d|24n+1\\ d\le N}} d^{-\frac{1}{2}} \le C_4 n^{\varepsilon},$$

where C_1, C_2, C_3 and C_4 are constants.

Turning our attention to $\widehat{S}_{12}$, the integral within this term can be reformulated as:

$$\int_{-\frac{1}{kN}}^{-\frac{1}{k(\widetilde{k}_1+k)}} = \sum_{\ell=N}^{\widetilde{k}_1+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}}.$$

Plugging into $\widehat{S}_{12}$ gives:

$$\widehat{S}_{12} = \sum_{\substack{c|k\\ 2|k}} \sum_{\ell=N}^{\widetilde{k}_1+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathrm{d}\Phi \frac{i \sin\left(\frac{\pi a}{c}\right)}{(-1)^{ak+1}} \sum_{h} \frac{\omega_{h,k}^2 \omega_{h,\frac{k}{2}}^{-1}}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{2\pi i h n}{k} - \frac{\pi i a^2 k_1 h'}{c}}.$$

Given the constraint $\widetilde{k}_1 \le N$, it follows that $\ell \le \widetilde{k}_1 + k - 1 \le N + k - 1$, thereby limiting the range of summation over h. Consequently, we can estimate $\widehat{S}_{12}$ by summing over more integrals:

$$\left|\widehat{S}_{12}\right| \le \sum_{\substack{c|k\\ 2|k}} \sum_{\ell=N}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} \left| z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \right| \mathrm{d}\Phi$$

$$\cdot \left| \frac{\sin\left(\frac{\pi a}{c}\right)}{(-1)^{ak+1}} \sum_{\ell \le \widetilde{k}_1+k-1 \le N+k-1} \sum_{h} \frac{\omega_{h,k}^2 \omega_{h,\frac{k}{2}}^{-1}}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{2\pi i h n}{k} - \frac{\pi i a^2 k_1 h'}{c}} \right|.$$

Now by the theory of Farey fractions we have

$$\widetilde{k}_1 \equiv -h' (\mathrm{mod}\ k), \quad \widetilde{k}_2 \equiv h' (\mathrm{mod}\ k), \quad N - k < \widetilde{k}_i \le N,$$

for $i = 1, 2$. This means the restriction of h' to one or two intervals in the range $0 \le h' < k$. Similar to $\widehat{S}_{11}$, using (3.2) and (3.13), we get $\widehat{S}_{12} = O(n^{\varepsilon})$. The shape of $\widehat{S}_{13}$ is the same as that of $\widehat{S}_{12}$. Hence, representing the expression as

$$\int_{\frac{1}{k(\widetilde{k}_2+k)}}^{\frac{1}{kN}} = \sum_{\ell=k+\widetilde{k}_2-1}^{N} \int_{\frac{1}{k(\ell+1)}}^{\frac{1}{k\ell}}$$

yields $\widehat{S}_{13} = O(n^{\varepsilon})$ by the same argument.

Following this, our analysis shifts to $\widehat{S}_2$. Analogously, the integral within $\widehat{S}_2$ can be expressed as

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} + \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(\widetilde{k}_2+k)}} + \int_{-\frac{1}{k(\widetilde{k}_1+k)}}^{-\frac{1}{k(N+k)}}, \tag{3.14}$$

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and we denote the associated sums by $\widehat{S}_{21}$, $\widehat{S}_{22}$ and $\widehat{S}_{23}$. Then taking the absolute value of this it is possible to bound the term. Before doing that we define

$$\overline{a}(r) = \sum_{s(\mathrm{mod}\ c)} |\overline{a}(r,s)|,$$

where $\overline{a}(r)$ are exactly $\overline{p}(n)$ except from some constant term ambiguity. Using (3.2) and (3.13) again, we have

$$\begin{aligned}\left|\widehat{S}_{21}\right| &\le \sum_{r=1}^{\infty}\sum_{\substack{c|k\\2|k}}\overline{a}(r)\left|(-1)^{ak+1}\sin\left(\frac{\pi a}{c}\right)\sum_{h}\frac{\omega_{h,k}^{2}\omega_{h,\frac{k}{2}}^{-1}}{\sin\left(\frac{\pi ah'}{c}\right)}e^{-\frac{2\pi ihn}{k}-\frac{\pi ia^{2}k_{1}h'}{c}+\frac{2\pi ih'}{c}}m_{r,s}\right| \\ &\quad\cdot k^{-\frac{1}{2}}n^{\frac{1}{2}}e^{2\pi-\pi r}\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}}\mathrm{d}\Phi \\ &= O(n^{\varepsilon}).\end{aligned}$$

Since $\widehat{S}_{22}$ and $\widehat{S}_{23}$ are treated in exactly the same way as $\widehat{S}_{12}$, by writting

$$\int_{\frac{1}{k(\widetilde{k}_2+k)}}^{\frac{1}{k(N+k)}} = \sum_{\ell=N+k-1}^{\widetilde{k}_2+k}\int_{\frac{1}{k(\ell+1)}}^{\frac{1}{k\ell}}, \qquad \int_{-\frac{1}{k(N+k)}}^{-\frac{1}{k(\widetilde{k}_1+k)}} = \sum_{\ell=k+\widetilde{k}_1}^{N+k-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}},$$

and repeating the steps of the estimation of $\widehat{S}_{12}$, we derive that both $\widehat{S}_{22}$ and $\widehat{S}_{23}$ are of order $O(n^{\varepsilon})$. In conclusion, we have

$$\Sigma_1 = O(n^{\varepsilon}). \tag{3.15}$$

For the bounds of Σ_1, using (3.13) and $\vartheta'_{h,k}+\vartheta''_{h,k}\le\frac{2}{k\sqrt{n}}$, we have

$$|\Sigma_1| \le 2\left|\sin\left(\frac{\pi a}{c}\right)\right|e^{2\pi}\sum_{\substack{k\le N\\c|k}}k^{-\frac{3}{2}}\sum_{\substack{h=1\\(h,k)=1}}^{k}\frac{1}{\left|\sin\left(\frac{\pi h}{c}\right)\right|}\max_{z}\left|\overline{C}\left(e^{\frac{2\pi ih}{c}};q_1\right)\right|.$$

Here we use a change of variables $ah'\to h$. Applying (3.9) and $\mathrm{Re}\left(\frac{1}{z}\right)>\frac{k}{2}$, we have

$$\left|\overline{C}\left(e^{\frac{2\pi ih}{c}};q_1\right)\right| \le 1+\sum_{r=1}^{\infty}\overline{p}(r)e^{-\pi r}.$$

Recall the estimate in [10, Section 4.2]:

$$\sum_{\substack{h=1\\(h,k)=1}}^{k}\frac{1}{\left|\sin\left(\frac{\pi h}{c}\right)\right|} \le \frac{2k\left(1+\log\left(\frac{c-1}{2}\right)\right)}{\pi(1-\pi^{2}/24)}. \tag{3.16}$$

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Using these inequalities above and setting $\varepsilon_1 = \sum_{r=1}^{\infty} \overline{p}(r) e^{-\pi r}$, we can derive that

$$\begin{aligned}|\Sigma_1| &\le 2\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi}(1+\varepsilon_1) \sum_{\substack{k\le N\\ c|k}} k^{-\frac{3}{2}} \sum_{\substack{h=1\\(h,k)=1}}^{k} \frac{1}{\left|\sin\left(\frac{\pi h}{c}\right)\right|}\\ &\le \frac{4\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi}(1+\varepsilon_1)\left(1+\log\left(\frac{c-1}{2}\right)\right)}{\pi\left(1-\pi^2/24\right)} \sum_{\substack{k\le N\\ c|k}} k^{-\frac{1}{2}}\\ &\le \frac{8\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi}(1+\varepsilon_1)\left(1+\log\left(\frac{c-1}{2}\right)\right) n^{\frac{1}{4}}}{c\pi\left(1-\pi^2/24\right)}.\end{aligned}$$

This completes the proof. ∎

In the above proof of Σ_1, we detailed how to identify the error terms and compute their bounds. In the subsequent proof of Σ_4, we will focus on determining the main term and calculating the bounds of the error terms, while omitting the process of identifying these error terms.

Lemma 3.5 *Σ_4 contributes a main term, defined as in (3.7), which can be specifically expressed as follows:*

(3.17)
$$\Sigma_4 = 2n^{-\frac{1}{2}} \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{1\le k\le\sqrt{n}\\ c\nmid k,\ 2\nmid k\\ \delta^i_{a,c,k,r}>0\\ r\ge 0,\ i\in\{+,-\}}} \frac{\overline{D}_{a,c,k}\left(-n, m^i_{a,c,k,r}\right)}{\sqrt{k}} \sinh\left(\frac{4\pi}{k}\sqrt{n\delta^i_{a,c,k,r}}\right) + O(n^{\varepsilon}).$$

Moreover, the error terms coming from Σ_4 can be bounded by

$$8\sqrt{2}e^{2\pi}n^{\frac{1}{4}}f(c) + 16\sqrt{2}e^{2\pi}\frac{e^{2\pi\delta_0}}{1-e^{-\frac{2\pi}{c}}}n^{\frac{1}{4}} + 4\sqrt{2}\left(\frac{4}{3}+2^{\frac{5}{4}}\right)\frac{e^{2\pi\delta_0+2\pi}}{1-e^{-\frac{2\pi}{c}}}n^{\frac{3}{8}},$$

where

$$f(c) := \frac{1+\varepsilon_1 e^{\pi\delta_0}}{\left(1-e^{-\frac{\pi}{c}}\right)} + e^{\pi\delta_0}(1+\varepsilon_1)\varepsilon_2 + e^{\pi\delta_0+\pi}(1+\varepsilon_1)\varepsilon_3 \quad \textit{with} \quad \delta_0 := \frac{1}{2c^2} - \frac{1}{2c} + \frac{1}{16}.$$

Proof Note that

$$\begin{aligned}\overline{C}\left(ah', \frac{lc}{c_1}, c; q_1\right) &= \frac{i}{2}\frac{(-q_1;q_1)_\infty}{(q_1;q_1)_\infty}\\ &\quad\times\left(\sum_{m=0}^{\infty}\frac{(-1)^m e^{-\frac{\pi i a h'}{c}} q_1^{\binom{m+1}{2}+\frac{l}{2c_1}}}{1-e^{-\frac{2\pi i a h'}{c}} q_1^{m+\frac{l}{c_1}}} - \sum_{m=1}^{\infty}\frac{(-1)^m e^{\frac{\pi i a h'}{c}} q_1^{\binom{m+1}{2}-\frac{l}{2c_1}}}{1-e^{\frac{2\pi i a h'}{c}} q_1^{m-\frac{l}{c_1}}}\right)\end{aligned}$$

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$$= \frac{i}{2}\frac{(-q_1;q_1)_\infty}{(q_1;q_1)_\infty}\left(\sum_{m=0}^{\infty}(-1)^m\sum_{r=0}^{\infty}e^{-\frac{\pi iah'}{c}-\frac{2\pi irah'}{c}}q_1^{\binom{m+1}{2}+\frac{l}{2c_1}+rm+\frac{rl}{c_1}}\right.$$

$$\left.-\sum_{m=1}^{\infty}(-1)^m\sum_{r=0}^{\infty}e^{\frac{\pi iah'}{c}+\frac{2\pi irah'}{c}}q_1^{\binom{m+1}{2}-\frac{l}{2c_1}+rm-\frac{rl}{c_1}}\right). \tag{3.18}$$

Then the product in Σ_4 can be written as

$$e^{\frac{\pi ih'}{8k}+\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2k_1h'}{cc_1}}q_1^{-\frac{l^2}{2c_1^2}}\frac{\eta\left(\frac{h'+iz^{-1}}{2k}\right)}{\eta\left(\frac{2(h'+iz^{-1})}{k}\right)}\overline{C}\left(ah',\frac{lc}{c_1},c;q_1\right)$$

$$=\sum_{r=r_0'}^{\infty}\sum_{s(\mathrm{mod}\ c)}\overline{d}(r,s)e^{\frac{2\pi im'_{r,s}h'}{k}}q_1^r. \tag{3.19}$$

We aim to demonstrate that $m'_{r,s}\in\mathbb{Z}$ and r_0' may be negative. The segments involving negative r contribute to the component. Considering $\tau_2=\frac{h'+iz^{-1}}{k}$, $q_1=e^{2\pi i\tau_2}$, we further elaborate (3.19) by employing

$$\frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta(2\tau_2)}\frac{(-q_1;q_1)_\infty}{(q_1;q_1)_\infty}=q_1^{-\frac{1}{16}}\frac{(q_1^{\frac12};q_1^{\frac12})_\infty}{(q_1;q_1)_\infty^2}=q_1^{-\frac{1}{16}}\left(1+O\left(q_1^{\frac12}\right)\right) \tag{3.20}$$

inside of $\eta\left(\frac{h'+iz^{-1}}{2k}\right)\eta^{-1}\left(\frac{2(h'+iz^{-1})}{k}\right)\overline{C}\left(ah',\frac{lc}{c_1},c;q_1\right)$. So, the main contribution of

$$e^{\frac{\pi ih'}{8k}+\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2k_1h'}{cc_1}}q_1^{-\frac{l^2}{2c_1^2}}\frac{\eta\left(\frac{h'+i(z)^{-1}}{2k}\right)}{\eta\left(\frac{2(h'+iz^{-1})}{k}\right)}\overline{C}\left(ah',\frac{lc}{c_1},c;q_1\right)$$

comes from the following expression:

$$\pm\frac{i}{2}e^{\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2k_1h'}{cc_1}+\frac{\pi ih'}{8k}}q_1^{-\frac{l^2}{2c_1^2}-\frac{1}{16}}(-1)^m q_1^{\frac{m}{2}(m+1)\pm\frac{l}{2c_1}+rm\pm\frac{rl}{c_1}}e^{\mp\frac{\pi iah'}{c}\mp\frac{2\pi iah'r}{c}}. \tag{3.21}$$

From this is possible to split the expression into the roots of unity and to the part that depends on the variable z. The roots of unity look like:

$$\exp\left(\frac{2\pi ih'}{k}\left(-\frac{a^2k_1k}{2cc_1}+\frac{lak}{cc_1}-\frac{l^2}{2c_1^2}+rm\pm\frac{rl}{c_1}\mp\frac{kra}{c}+\frac{m(m+1)}{2}\pm\frac{l}{2c_1}\mp\frac{ak}{2c}\right)\right).$$

Rewriting the expression in the second bracket, using the congruence condition $l\equiv ak_1(\mathrm{mod}\ c_1)$, l^2+l is always even and rearranging the sum it is possible to show that the contribution of the roots of unity looks like $\exp\left(\frac{2\pi ih'm'_{r,s}}{k}\right)$ where $m'_{r,s}$ is a sequence in $\mathbb{Z}$. The interesting part happens for $\exp\left(\frac{\pi}{kz}H_2\right)$, where H_2 is defined in the following way:

$$H_2=\frac{l^2}{c_1^2}+\frac18-2rm\mp2r\frac{l}{c_1}-m(m+1)\mp\frac{l}{c_1}.$$

This part contributes to the circle method exactly if $H_2>0$. A straightforward, but rather tedious computation shows that there are two contributions coming from each

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of the two terms of

$$\eta\left(\frac{h'+iz^{-1}}{2k}\right)\eta^{-1}\left(\frac{2(h'+iz^{-1})}{k}\right)\overline{C}\left(ah',\frac{lc}{c_1},c;q_1\right).$$

The first one comes from the first sum, if $m=0$, and this contributes with

$$\frac{i}{2}e^{\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2k_1h'}{cc_1}-\frac{\pi iah'}{c}+\frac{\pi ih'}{8k}}q_1^{-\frac{l^2}{2c_1^2}-\frac{1}{16}+\frac{l}{2c_1}}\sum_{\substack{r\geq 0\\ \delta^+_{a,c,k,r}>0}}q_1^{\frac{rl}{c_1}}e^{-\frac{2\pi iah'r}{c}},$$

where $\delta^+_{a,c,k,r}=\frac{l^2}{2c_1^2}+\frac{1}{16}-\left(r+\frac{1}{2}\right)\frac{l}{c_1}$. The second contribution comes from the second sum, if $m=1$, and this contributes with

$$\frac{i}{2}e^{\frac{2\pi ialh'}{cc_1}-\frac{\pi ia^2k_1h'}{cc_1}+\frac{\pi ih'}{8k}}q_1^{-\frac{l^2}{2c_1^2}-\frac{1}{16}}q_1^{1-\frac{l}{2c_1}+r-\frac{rl}{c_1}}e^{\frac{\pi iah'}{c}+\frac{2\pi iah'r}{c}},$$

where $\delta^-_{a,c,k,r}=\frac{l^2}{2c_1^2}-\frac{15}{16}-r\left(1-\frac{l}{c_1}\right)+\frac{l}{2c_1}$. Thus, Σ_4 can be rewritten as

$$\Sigma_4=\sqrt{2}\sin\left(\frac{\pi a}{c}\right)\sum_{\substack{c\nmid k,\ 2\nmid k\\ r\geq 0,\ i\in\{-,+\}}}(-1)^{ak+l}\sum_h\omega_{h,k}^2\omega_{2h,k}^{-1}e^{\frac{2\pi i}{k}\left(-nh+m^i_{a,c,k,r}h'\right)}$$

$$\cdot\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}z^{-\frac{1}{2}}e^{\frac{2\pi nz}{k}+\frac{2\pi}{kz}\delta^i_{a,c,k,r}}\mathrm{d}\Phi. \tag{3.22}$$

Now it is possible to rewrite the sum over k into the sum where the k's have the same values for c_1 and l and thus the $\delta^i_{a,c,k,r}$ are constant in each class and the condition $\delta^i_{a,c,k,r}>0$ is independent of k in each class. Moreover it is clear that there are only finitely many solution to the inequality $\delta^i_{a,c,k,r}>0$. This means it is possible to split the sum over r into positive and negative $\delta^i_{a,c,k,r}$, where the part with positive $\delta^i_{a,c,k,r}$ contributes to the main term. Therefore, using (3.1) it is possible to evaluate Σ_4 by the same argument as for Σ_1:

$$\Sigma_4=\sqrt{2}\sin\left(\frac{\pi a}{c}\right)\sum_{r\geq 0}\sum_{\substack{2\nmid k\\ c\nmid k}}\sum_{\substack{\delta^i_{a,c,k,r}>0\\ i\in\{+,-\}}}(-1)^{ak+l}\sum_{h(\mathrm{mod}\ k)^*}\omega_{h,k}^2\omega_{2h,k}^{-1}e^{\frac{2\pi i}{k}\left(-nh+m^i_{a,c,k,r}h'\right)}$$

$$\cdot\int_{-\frac{1}{kN}}^{\frac{1}{kN}}z^{-\frac{1}{2}}e^{\frac{2\pi zn}{k}+\frac{2\pi}{kz}\delta^i_{a,c,k,r}}\mathrm{d}\Phi+O(n^{\varepsilon}). \tag{3.23}$$

Then substituting (3.4) into (3.23), we obtain (3.17).

Now we give a specific bound for the error coming from Σ_4, which is composed of three parts. Firstly, we consider the error corresponding to the terms where $\delta^i_{a,c,k,r}$ is not positive, denoting it by S_4^{err}. Define $\overline{M}(jh',l,c;q_1)$ to be the terms with positive exponents in the q_1-expansion of

$$q_1^{-\frac{l^2}{2c_1^2}}\frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta(2\tau_2)}\overline{C}\left(jh',\frac{lc}{c_1},c;q_1\right),$$

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where $\tau_2 = \frac{h'+iz^{-1}}{k}$. Then using the usual change of variable $ah' \to h$, we find

$$|S_4^{err}| \le 4\sqrt{2}\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi} \sum_{\substack{h,k \\ c\nmid k}} k^{-\frac{3}{2}} \max_z \left|\overline{M}(h, \frac{lc}{c_1}, c; q_1)\right|.$$

By (3.18) and (3.20), we have

$$\begin{aligned}
&\frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta(2\tau_2)}\overline{C}\left(h, \frac{lc}{c_1}, c; q_1\right) \\
&= \frac{i}{2} q_1^{-\frac{1}{16}} \frac{(q_1^{\frac{1}{2}}; q_1^{\frac{1}{2}})_\infty}{(q_1; q_1)_\infty^2} \left(\sum_{r\ge 0} e^{-\frac{\pi i h}{c} - \frac{2\pi i r h}{c}} q_1^{\frac{l}{2c_1} + \frac{rl}{c_1}} + \sum_{r\ge 0} e^{\frac{\pi i h}{c} + \frac{2\pi i r h}{c}} q_1^{-\frac{l}{2c_1} - \frac{rl}{c_1} + r + 1} \right. \\
&\quad \left. + \sum_{m\ge 1} \frac{(-1)^m e^{-\frac{\pi i h}{c}} q_1^{\frac{m^2+m}{2} + \frac{l}{2c_1}}}{1 - e^{-\frac{2\pi i h}{c}} q_1^{m + \frac{l}{c_1}}} - \sum_{m\ge 2} \frac{(-1)^m e^{\frac{\pi i h}{c}} q_1^{\frac{m^2+m}{2} - \frac{l}{2c_1}}}{1 - e^{\frac{2\pi i h}{c}} q_1^{m - \frac{l}{c_1}}} \right).
\end{aligned}$$

We bound the contributions of $\overline{M}$ term by term beginning with the first one. We find that

$$\left|\frac{(q_1^{1/2}; q_1^{1/2})_\infty}{(q_1; q_1)_\infty^2}\right| = \left|\frac{(q_1^{1/2}; q_1)_\infty}{(q_1; q_1)_\infty}\right| \le \frac{2(-|q_1|; |q_1|)_\infty}{(|q_1|; |q_1|)_\infty} = \sum_{m\ge 0} 2\overline{p}(m)|q_1|^m.$$

Then we gain the following contribution to $\overline{M}$

$$\begin{aligned}
&\left| q_1^{-\frac{1}{16} - \frac{l^2}{2c_1^2} + \frac{l}{2c_1}} \sum_{m=0}^{\infty} \overline{p}(m)|q_1|^m \sum_{r\ge r_0} q_1^{\frac{rl}{c_1}} \right| \\
&\le e^{\frac{\pi}{16} + \frac{\pi l^2}{2c_1^2} - \frac{\pi l}{2c_1}} \left(\sum_{r\ge r_0} e^{-\frac{\pi r l}{c_1}} + \sum_{r\ge 0} e^{-\frac{\pi r l}{c_1}} \sum_{m\ge 1} \overline{p}(m) e^{-\pi m} \right),
\end{aligned}$$

where $r_0 := \lceil \frac{l}{2c_1} + \frac{c_1}{16l} - \frac{1}{2} \rceil$ and $r \ge r_0$ is a solution of $\delta^+_{a,c,k,r} < 0$. Using

$$\sum_{r\ge r_0} e^{-\frac{\pi r l}{c_1}} = \frac{e^{-\frac{\pi r_0 l}{c_1}}}{1 - e^{-\frac{\pi l}{c_1}}},$$

and the usual geometric series we can bound the term further by

$$\frac{e^{\frac{\pi}{16} + \frac{\pi l^2}{2c_1^2} - \frac{\pi l}{2c_1} - \frac{\pi r_0 l}{c_1}}}{\left(1 - e^{-\frac{\pi l}{c_1}}\right)} + \frac{e^{\frac{\pi}{16} + \frac{\pi l^2}{2c_1^2} - \frac{\pi l}{2c_1}} \varepsilon_1}{\left(1 - e^{-\frac{\pi l}{c_1}}\right)} = \frac{e^{\frac{\pi l}{c_1}\left(\frac{c_1}{16l} + \frac{l}{2c_1} - \frac{1}{2} - r_0\right)} \left(1 + \varepsilon_1 e^{\frac{\pi r_0 l}{c_1}}\right)}{\left(1 - e^{-\frac{\pi l}{c_1}}\right)} \le \frac{1 + \varepsilon_1 e^{\pi\delta_0}}{\left(1 - e^{-\frac{\pi}{c}}\right)}.$$

For $0 < l < c_1$, one can easily verify that $\delta^i_{j,c,k,r} < \delta^+_{j,c,k,0} = \frac{1}{2}(\frac{l}{c_1} - \frac{1}{2})^2 - \frac{1}{16}$. Assuming $\frac{l}{c_1} < \frac{1}{2}$, which we may do by the symmetry of the parabola in the argument $\frac{l}{c_1}$, we see that $\delta^+_{j,c,k,0}$ arrives the maximum value $\delta = \frac{1}{2c_1{}^2} - \frac{1}{2c_1} + \frac{1}{16}$ at $l = 1$. Since $\delta \le \delta_0$, we have $\delta^i_{j,c,k,r} \le \delta_0$. Therefore, we can bound the second term by the same way. In the third and fourth summand all the terms will contribute to the error as shown in

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Theorem 1.1. We obtain

$$\begin{aligned}&\left|\frac{i}{2}q_1^{-\frac{1}{16}-\frac{l^2}{2c_1^2}+\frac{l}{2c_1}}\frac{(q_1^{\frac{1}{2}};q_1^{\frac{1}{2}})_\infty}{(q_1;q_1)_\infty^2}\sum_{m=1}^{\infty}\frac{(-1)^m e^{-\frac{\pi i h}{c}}q_1^{\frac{m^2+m}{2}}}{1-e^{-\frac{2\pi i h}{c}}q_1^{m+\frac{l}{c_1}}}\right|\\&\le e^{\frac{\pi}{16}+\frac{\pi l^2}{2c_1^2}-\frac{\pi l}{2c_1}}(1+\varepsilon_1)\sum_{m=1}^{\infty}\frac{e^{-\frac{\pi m(m+1)}{2}}}{1-e^{-\pi m-\frac{\pi l}{c_1}}}\\&\le e^{\pi\delta_0}(1+\varepsilon_1)\varepsilon_2.\end{aligned}$$

Proceeding by repeating this step for the fourth sum in the expansion of $\frac{\eta(\frac{\tau_2}{2})}{\eta(2\tau_2)}\overline{C}\left(h,\frac{lc}{c_1},c;q_1\right)$, we can get a bound for $\overline{M}$

$$e^{\pi\delta_0+\pi}(1+\varepsilon_1)\varepsilon_3.$$

Based on the above estimates, we find that $\overline{M}$ can be bounded by the function $f(c)$ that just depends on c. Therefore the error can be bounded by

$$|S_4^{err}|\le 4\sqrt{2}e^{2\pi}\sum_{\substack{h,k\\ c\nmid k}}k^{-\frac{3}{2}}f(c)\le 8\sqrt{2}e^{2\pi}n^{\frac{1}{4}}f(c). \tag{3.24}$$

Here it used that the sum over h can be bounded trivially by k because the sum runs over the residue class modulo k and we estimated the sum over k by an integral expression.

As a next step we want to bound the second part error that comes from symmetrizing the integral, which is the associated sums of the last two integrals in (3.12), that is

$$\begin{aligned}\tilde{S}_4^{err}:=-\sqrt{2}\sin\left(\frac{\pi a}{c}\right)&\sum_{\substack{c\nmid k,\ 2\nmid k\\ r\ge 0,\ i\in\{-,+\}}}(-1)^{ak+l}\sum_h\omega_{h,k}^2\omega_{2h,k}^{-1}e^{\frac{2\pi i}{k}(-nh+m^i_{a,c,k,r}h')}\\&\cdot\left(\int_{-\frac{1}{kN}}^{-\frac{1}{k(\tilde{k}_1+k)}}+\int_{\frac{1}{k(\tilde{k}_2+k)}}^{\frac{1}{kN}}\right)z^{-\frac{1}{2}}e^{\frac{2\pi nz}{k}+\frac{2\pi}{kz}\delta^i_{a,c,k,r}}d\Phi.\end{aligned}$$

Completely analogous to Σ_1, it is possible to show

$$|\tilde{S}_4^{err}|\le 4\sqrt{2}e^{2\pi}\sum_{\substack{r,k\\ \delta^i_{a,c,k,r}>0\\ i\in\{+,-\}}}k^{-\frac{1}{2}}e^{2\pi\delta^i_{a,c,k,r}}$$

As we can bound the term with $i=-$ and $i=+$ in a same way, we restrict to the case $i=+$. The sum over r gives

$$\begin{aligned}|\tilde{S}_4^{err}|&\le 8\sqrt{2}e^{2\pi}\sum_{\substack{r,k\\ \delta^+_{j,c,k,r}>0}}k^{-\frac{1}{2}}e^{2\pi\delta^+_{j,c,k,r}}\\&=8\sqrt{2}e^{2\pi}\sum_k k^{-\frac{1}{2}}\sum_{r\le r_0-1}e^{\frac{\pi l^2}{c_1^2}+\frac{\pi}{8}-\frac{2\pi rl}{c_1}-\frac{\pi l}{c_1}}\end{aligned}$$

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$$= 8\sqrt{2}e^{2\pi}\sum_{k} k^{-\frac{1}{2}} \frac{e^{\frac{\pi l^2}{c_1^2}+\frac{\pi}{8}-\frac{\pi l}{c_1}}\left(e^{-\frac{2\pi l r_0}{c_1}}-1\right)}{e^{-\frac{2\pi l}{c_1}}-1}$$

$$\leq 8\sqrt{2}e^{2\pi}\sum_{k} k^{-\frac{1}{2}} \frac{e^{2\pi\delta_0}}{1-e^{-\frac{2\pi}{c}}}$$

$$\leq 16\sqrt{2}e^{2\pi}\frac{e^{2\pi\delta_0}}{1-e^{-\frac{2\pi}{c}}} n^{\frac{1}{4}}. \tag{3.25}$$

Here we have used that the summation over r is an error term if it starts with r_0, so here we have to sum over all the r-terms where $r \leq r_0 - 1$.

The last error we have to bound comes from the evaluation of the integral. There it is used that it is possible to change the path of integration if one accounts the integral over the smaller arc. This term has to be made explicit. Applying (3.5), and estimation of the sum over k, we bound the third part error by

$$4\sqrt{2}\left(\frac{4}{3}+2^{\frac{5}{4}}\right)\frac{e^{2\pi\delta_0+2\pi}}{1-e^{-\frac{2\pi}{c}}} n^{\frac{3}{8}}. \tag{3.26}$$

Summing up (3.24), (3.25), and (3.26), we obtain the bound of the error coming from Σ_4. This completes the proof. ∎

In the computation of Σ_2, which also contributes a main term, the calculation process differs slightly from that of Σ_4. Therefore, we will detail the differences and omit the similar parts in the calculations.

Lemma 3.6 *Σ_2 gives rise to the main term, as defined in (3.7), which can be expressed specifically as follows:*

$$\Sigma_2 = in^{-\frac{1}{2}}\sin\left(\frac{\pi a}{c}\right)\sum_{h}\sum_{\substack{c|k\\ 2\nmid k}} \frac{(-1)^{ak+1}}{\sin\left(\frac{\pi a h'}{c}\right)}\omega_{h,k}^2\omega_{2h,k}^{-1} e^{-\frac{2\pi i h n}{k}-\frac{\pi i a^2 k_1 h'}{c}} k^{-\frac{1}{2}}\sinh\left(\frac{\pi\sqrt{n}}{k}\right)$$

$$+ O(n^{\varepsilon}), \tag{3.27}$$

where the error term $O(n^{\varepsilon})$ coming from Σ_2 can be bounded by

$$\frac{4\sqrt{2}\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi+\frac{\pi}{8}}\left(3\varepsilon_1+2\varepsilon_1^2\right)\left(1+\log\left(\frac{c-1}{2}\right)\right) n^{\frac{1}{4}}}{c\pi\left(1-\pi^2/24\right)} + \frac{8\sqrt{2}e^{2\pi+\frac{\pi}{8}}\left(1+\log\left(\frac{c-1}{2}\right)\right) n^{\frac{1}{4}}}{c\pi(1-\pi^2/24)}$$

$$+ \frac{2\sqrt{2}e^{2\pi+\frac{\pi}{8}}\left(1+\log\left(\frac{c-1}{2}\right)\right)\left(\frac{4}{3}+2^{\frac{5}{4}}\right) n^{\frac{3}{8}}}{\pi(1-\pi^2/24)}. \tag{3.28}$$

Proof Using (3.9), we obtain

$$\frac{\eta\left(\frac{\tau_2}{2}\right)}{\eta(2\tau_2)}\overline{C}\left(e^{\frac{2\pi i a h'}{c}};q_1\right) = \frac{q_1^{\frac{1}{48}}\left(q_1^{\frac{1}{2}};q_1^{\frac{1}{2}}\right)_\infty}{q_1^{\frac{1}{12}}\left(q_1^2;q_1^2\right)_\infty}\left(1+\sum_{r=1}^{\infty}\sum_{s \pmod c}\overline{a}(r,s)e^{\frac{2\pi i h'}{c}m_{r,s}}q_1^r\right)$$

$$= q_1^{-\frac{1}{16}}(1+\widehat{T}_1),$$

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where $\tau_2 = \frac{h'+iz^{-1}}{k}$, $q_1 = e^{2\pi i\tau_2}$ and

$$\widehat{T}_1 = \frac{(q_1^{\frac{1}{2}};q_1^{\frac{1}{2}})_\infty}{(q_1^2;q_1^2)_\infty}\left(1+\sum_{r=1}^{\infty}\sum_{s \pmod c}\overline{a}(r,s)e^{\frac{2\pi ih'}{c}m_{r,s}}q_1^r\right)-1. \tag{3.29}$$

Setting $\frac{(q_1^{1/2};q_1^{1/2})_\infty}{(q_1^2;q_1^2)_\infty} := 1+\sum_{n=1}^{\infty} b(n)q_1^{n/2}$ in (3.29), we obtain that

$$\begin{aligned}\widehat{T}_1 &= \left(1+\sum_{n=1}^{\infty}b(n)q_1^{n/2}\right)\left(1+\sum_{r=1}^{\infty}\sum_{s \pmod c}\overline{a}(r,s)e^{\frac{2\pi ih'}{c}m(r,s)}q_1^r\right)-1\\ &= \sum_{n=1}^{\infty}b(n)q_1^{n/2}+\sum_{r=1}^{\infty}\sum_{s \pmod c}\overline{a}(r,s)e^{\frac{2\pi ih'}{c}m(r,s)}q_1^r\\ &\quad +\left(\sum_{n=1}^{\infty}b(n)q_1^{n/2}\right)\cdot\left(\sum_{r=1}^{\infty}\sum_{s(\mathrm{mod}\ c)}\overline{a}(r,s)e^{\frac{2\pi ih'}{c}m(r,s)}q_1^r\right).\end{aligned} \tag{3.30}$$

Consequently, Σ_2 can be expressed as follows:

$$\Sigma_2 = \widetilde{S}_1+\widetilde{S}_2,$$

where

$$\begin{aligned}\widetilde{S}_1 &= \frac{\sqrt{2}}{2}i\sin\left(\frac{\pi a}{c}\right)\sum_h\sum_{\substack{c|k\\2\nmid k}}\frac{(-1)^{ak+1}}{\sin\left(\frac{\pi ah'}{c}\right)}\omega_{h,k}^2\omega_{2h,k}^{-1}e^{-\frac{2\pi ihn}{k}+\frac{\pi ih'}{8k}-\frac{\pi ia^2k_1h'}{c}}\\ &\quad\cdot\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}z^{-\frac{1}{2}}e^{\frac{2\pi nz}{k}}q_1^{-\frac{1}{16}}\mathrm{d}\Phi,\\ \widetilde{S}_2 &= \frac{\sqrt{2}}{2}i\sin\left(\frac{\pi a}{c}\right)\sum_h\sum_{\substack{c|k\\2\nmid k}}\frac{(-1)^{ak+1}}{\sin\left(\frac{\pi ah'}{c}\right)}\omega_{h,k}^2\omega_{2h,k}^{-1}e^{-\frac{2\pi ihn}{k}+\frac{\pi ih'}{8k}-\frac{\pi ia^2k_1h'}{c}}\\ &\quad\cdot\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}z^{-\frac{1}{2}}e^{\frac{2\pi nz}{k}}q_1^{-\frac{1}{16}}\widehat{T}_1\mathrm{d}\Phi.\end{aligned} \tag{3.31}$$

Using (3.2), it is possible to demonstrate that $\widetilde{S}_1$ contributes to the main term and $\widetilde{S}_2 = O(n^\varepsilon)$, that is

$$\begin{aligned}\Sigma_2 &= \frac{\sqrt{2}}{2}i\sin\left(\frac{\pi a}{c}\right)\sum_h\sum_{\substack{c|k\\2\nmid k}}\frac{(-1)^{ak+1}}{\sin\left(\frac{\pi ah'}{c}\right)}\omega_{h,k}^2\omega_{2h,k}^{-1}e^{-\frac{2\pi ihn}{k}-\frac{\pi ia^2k_1h'}{c}}\\ &\quad\cdot\int_{-\frac{1}{kN}}^{\frac{1}{kN}}z^{-\frac{1}{2}}e^{\frac{2\pi nz}{k}+\frac{\pi}{8kz}}\mathrm{d}\Phi+O(n^\varepsilon).\end{aligned}$$

Then by (3.4), we obtain (3.27).

We next estimate the error coming from $\widetilde{S}_1$, which consists of two parts: the error arises from integrating over the remaining parts of the interval and the error coming

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from integrating along a smaller arc. Using (3.5), (3.16) and $k \le \sqrt{n}$, we can bound the error coming from integrating along a smaller arc:

$$\widetilde{S}_{11}^{err} \le \frac{2\sqrt{2}e^{2\pi+\frac{\pi}{8}}\left(1+\log\left(\frac{c-1}{2}\right)\right)\left(\frac{4}{3}+2^{\frac{5}{4}}\right)n^{\frac{3}{8}}}{\pi(1-\pi^2/24)}. \tag{3.32}$$

Utilizing (3.16) again, the error arises from integrating over the remaining parts of the interval can be bounded by

$$\widetilde{S}_{1err} \le \frac{8\sqrt{2}e^{2\pi+\frac{\pi}{8}}\left(1+\log\left(\frac{c-1}{2}\right)\right)n^{\frac{1}{4}}}{c\pi(1-\pi^2/24)}. \tag{3.33}$$

We now give a bound for $\widetilde{S}_2$. Applying (3.31), we deduce that

$$\left|\widetilde{S}_2\right| \le \sqrt{2}\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi} \sum_{\substack{1\le k\le\sqrt{n}\\ c|k}} k^{-\frac{3}{2}} \sum_{\substack{h=1\\(h,k)=1}}^{k} \frac{1}{\left|\sin\left(\frac{\pi h}{c}\right)\right|} \max_{z}\left|e^{\frac{\pi}{8kz}}\widehat{T}_1\right|.$$

Here we also use the change of variables $ah' \to h$. Utilizing the triangle inequality and $|q_1| < 1$, we get that

$$\left|\frac{(q_1^{1/2};q_1^{1/2})_\infty}{(q_1^2;q_1^2)_\infty}\right| = \left|\frac{(q_1^{1/2};q_1)_\infty}{(-q_1;q_1)_\infty}\right| \le \frac{2(-|q_1|;|q_1|)_\infty}{(|q_1|;|q_1|)_\infty}.$$

Combining this with the identity

$$\frac{(-q_1;q_1)_\infty}{(q_1;q_1)_\infty} = \sum_{r=0}^{\infty}\overline{p}(r)q_1^r$$

and (3.30), we can infer that

$$\left|\widehat{T}_1\right| \le 3\sum_{r=1}^{\infty}\overline{p}(r)e^{-\pi r} + 2\left(\sum_{r=1}^{\infty}\overline{p}(r)e^{-\pi r}\right)^2 = 3\varepsilon_1 + 2\varepsilon_1^2.$$

Applying (3.16), we have

$$\left|\widetilde{S}_2\right| \le \frac{4\sqrt{2}\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi+\frac{\pi}{8}}(3\varepsilon_1+2\varepsilon_1^2)\left(1+\log\left(\frac{c-1}{2}\right)\right)n^{\frac{1}{4}}}{c\pi\,(1-\pi^2/24)}. \tag{3.34}$$

Summing up (3.32), (3.33), and (3.34) gives (3.28). This completes the proof. ∎

The proof for Σ_3 is very similar to that for Σ_1, so we have omitted the detailed calculations.

Lemma 3.7 *Σ_3 leads to an error term, as defined in (3.7). Additionally, the upper bound for Σ_3 is as follows:*

$$|\Sigma_3| \le 8e^{\frac{5}{2}\pi}(1+\varepsilon_1)(\varepsilon_2+e^{\frac{\pi}{2}}\varepsilon_3)n^{\frac{1}{4}}.$$

Proof By (3.18), it is easy to see that

$$e^{\frac{2\pi i a l h'}{cc_1}-\frac{\pi i a^2 k_1 h'}{cc_1}} q_1^{-\frac{l^2}{2c_1^2}}\overline{C}\left(ah', \frac{lc}{c_1}, c; q_1\right) = \sum_{r=r_0}^{\infty}\sum_{s(\mathrm{mod}\ c)}\overline{b}(r,s)e^{\frac{2\pi i m_{r,s}h'}{k}}q_1^r, \tag{3.35}$$

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where $m_{r,s} \in \mathbb{Z}$. By the same argument as for Σ_4, we can find that the there is no part contributing to the main term, resulting in

$$\Sigma_3 = O(n^{\varepsilon}).$$

Now we consider the bound of Σ_3. Similar to the proof for Σ_1, we have

$$|\Sigma_3| \le 8\left|\sin\left(\frac{\pi a}{c}\right)\right| e^{2\pi} \sum_{\substack{h,k\\ c\nmid k}} k^{-\frac{3}{2}} \max_{z} \left| q_1^{-\frac{l^2}{2c_1^2}} \overline{C}\left(h, \frac{lc}{c_1}, c; q_1\right)\right|.$$

Combining (3.18), we have

$$\left|\overline{C}\left(h, \frac{lc}{c_1}, c; q_1\right)\right| \le \frac{1}{2}\sum_{n=0}^{\infty} \overline{p}(n) e^{-\pi n}\left(\sum_{m=0}^{\infty} e^{-\pi m(m+1)/2 - \frac{\pi l}{2c_1}} \left|\frac{e^{-\frac{\pi i h}{c}}}{1 - e^{-\frac{2\pi i h}{c}} q_1^{m+\frac{l}{c_1}}}\right| + \sum_{m=1}^{\infty} e^{-\pi m(m+1)/2 + \frac{\pi l}{2c_1}} \left|\frac{e^{\frac{\pi i h}{c}}}{1 - e^{\frac{2\pi i h}{c}} q_1^{m-\frac{l}{c_1}}}\right|\right).$$

Since

$$\left|\frac{e^{-\frac{\pi i h}{c}}}{1 - e^{-\frac{2\pi i h}{c}} q_1^{m+\frac{l}{c_1}}}\right| \le \frac{1}{1 - e^{-\pi m - \frac{\pi}{c}}}, \qquad \left|\frac{e^{\frac{\pi i h}{c}}}{1 - e^{\frac{2\pi i h}{c}} q_1^{m-\frac{l}{c_1}}}\right| \le \frac{1}{1 - e^{-\pi m + \pi\left(1 - \frac{1}{c}\right)}},$$

we have

$$|\Sigma_3| \le 4e^{\frac{5}{2}\pi}(1+\varepsilon_1)(\varepsilon_2 + e^{\frac{\pi}{2}}\varepsilon_3) \sum_{\substack{1\le k\le\sqrt{n}\\ c\nmid k}} k^{-\frac{1}{2}} \le 8e^{\frac{5}{2}\pi}(1+\varepsilon_1)(\varepsilon_2 + e^{\frac{\pi}{2}}\varepsilon_3) n^{\frac{1}{4}}.$$

This completes the proof. ∎

Given the established estimates for Σ_1, Σ_2, Σ_3, and Σ_4, the proof of Theorem 1.1 follows readily.

Proof of Theorem 1.1 In light of Lemmas 3.3–3.7, we derive the asymptotic formula for $\overline{A}\left(\frac{a}{c}; n\right)$, including an initial estimation of the error term. Given that this initial error bound is overly broad and lacks precision, we proceed to provide a more detailed and specific error bound.

We first give the estimates of constants ε_1, ε_2 and ε_3 in (3.8). Using the fact that for $n \ge 1$,

$$\overline{p}(n) < e^{\pi\sqrt{n}},$$

we have

$$\varepsilon_1 \le \int_1^{\infty} e^{\pi\sqrt{r}} \cdot e^{-\pi r} dr < 0.5488.$$

It can be easily verify by derivation that,

$$\frac{1}{1 - e^{-\frac{k\pi}{c}}} \le k\pi c, \quad k = 1, 2. \tag{3.36}$$

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Hence, we also can compute that

$$\varepsilon_2 < \frac{1}{1-e^{-\frac{\pi}{c}}} + \int_1^{\infty} \frac{e^{-\pi r(r+1)/2}}{1-e^{-\pi r}} \mathrm{d}r < \pi c + 0.0085.$$

Similarly, $\varepsilon_3 < \pi c + 0.0001$. Moveover, we have the following inequality [20, Eq. (3.13)]

$$\frac{1+\log\left(\frac{c-1}{2}\right)}{\pi\left(1-\frac{\pi^2}{24}\right)} < 0.2704c. \tag{3.37}$$

In view of Lemma 3.4 and (3.37), we have

$$|\Sigma_1| \leq 8e^{2\pi}(1+\varepsilon_1)\cdot 0.2704 n^{\frac{1}{4}} < 1794.0921 n^{\frac{1}{4}}. \tag{3.38}$$

Similarly, by Lemma 3.7, we have

$$|\Sigma_3| \leq (582623.3677c + 286.6509) n^{\frac{1}{4}}. \tag{3.39}$$

By Lemma 3.6 and (3.37), we obtain the bound of the error arising from Σ_2:

$$5153.9942 n^{\frac{1}{4}} + 2251.2819 c n^{\frac{3}{8}}. \tag{3.40}$$

Recall that

$$\delta_0 = \frac{1}{2c^2} - \frac{1}{2c} + \frac{1}{16} < \frac{1}{16}.$$

Hence, in view of Lemma 3.5 and (3.36), we can bound the error arising from Σ_4 by

$$(123.4854 + 1042254.902c) n^{\frac{1}{4}} + 104624.4148 c n^{\frac{3}{8}}. \tag{3.41}$$

Summing up (3.38), (3.39), (3.40), and (3.41) yields

$$\begin{aligned} \overline{R}_c(n) &\leq (1624878.2697c + 7358.2224) n^{\frac{1}{4}} + 106875.6967 c n^{\frac{3}{8}} \\ &\leq (1700009.4994c + 7358.2224) n^{\frac{3}{8}}. \end{aligned}$$

This completes the proof. ∎

4 Asymptotic formula for $\overline{M}(a,c,n)$

In this section, we will employ Theorem 1.1 to derive the asymptotic formula for $\overline{M}(a,c,n)$, as indicated by Corollary 1.2. Following that, we will proceed to estimate the error term of $\overline{M}(a,c,n)$, enabling us to establish a more precise asymptotic formula, as presented in Theorem 1.3.

Proof of Corollary 1.2 By the orthogonality of roots of unity, we obtain

$$\sum_{n=0}^{\infty} \overline{M}(a,c,n) q^n = \frac{1}{c} \sum_{n=0}^{\infty} \overline{p}(n) q^n + \frac{1}{c} \sum_{j=1}^{c-1} \zeta_c^{-aj} \overline{C}\left(\zeta_c^j; q\right). \tag{4.1}$$

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Recall the following asymptotic formula for $\overline{p}(n)$ in [21]:

$$\overline{p}(n) = \frac{1}{2\pi}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\sqrt{k}\sum_{\substack{h=0\\(h,k)=1}}^{k}\frac{\omega_{h,k}^2}{\omega_{2h,k}}e^{-\frac{2\pi inh}{k}}\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{\sinh\frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right). \tag{4.2}$$

By substituting Theorem 1.1 and (4.2) into (4.1), the proof is concluded. ∎

Our next objective is to give an asymptotic formula of $\overline{M}(a,c,n)$ with an effective bound. By Corollary 1.2 and (4.2), we obtain

$$\begin{aligned}\overline{M}(a,c,n) = {}& \frac{\overline{p}(n)}{c} + \frac{1}{c}\sum_{j=1}^{c-1}\zeta_c^{-aj}in^{-\frac{1}{2}}\sum_{\substack{1\le k\le\sqrt{n}\\c'\mid k,\ 2\nmid k}}\frac{\overline{B}_{j',c',k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{n}}{k}\right)\\ &+\frac{1}{c}\sum_{j=1}^{c-1}\zeta_c^{-aj}2n^{-\frac{1}{2}}\sin\left(\frac{\pi j}{c}\right)\\ &\cdot\sum_{\substack{1\le k\le\sqrt{n}\\c'\nmid k,\ 2\nmid k\\ \delta^i_{j',c',k,r}>0\\ r\ge0,\ i\in\{+,-\}}}\frac{\overline{D}_{j',c',k}(-n,m^i_{j',c',k,r})}{\sqrt{k}}\sinh\left(\frac{4\pi}{k}\sqrt{n\delta^i_{j',c',k,r}}\right)+R_c(n).\end{aligned} \tag{4.3}$$

We denote the second and third summand as $\overline{G}_1$ and $\overline{G}_2$, respectively. To complete the proof of Theorem 1.3, we require the bounds for $\overline{G}_1$ and $\overline{G}_2$.

Lemma 4.1 *Let c be an odd integer and n be a positive integer. We have*

$$|\overline{G}_1| \le 0.1803n^{\frac{1}{4}}\cdot e^{\frac{\pi\sqrt{n}}{3}},$$

and

$$|\overline{G}_2| \le \frac{c}{12}n^{\frac{1}{4}}e^{4\pi\sqrt{n\delta_0}}. \tag{4.4}$$

Proof With the fact that $|\exp(\pi ix)| = 1$ for $x\in\mathbb{R}$, letting $jh'\to h$ in the summation of sine, we have

$$\left|\sum_{\substack{1\le k\le\sqrt{n}\\c'\mid k,\ 2\nmid k}}\frac{\overline{B}_{j',c',k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{n}}{k}\right)\right| \le \sinh\left(\frac{\pi\sqrt{n}}{3}\right)\sum_{\substack{1\le k\le\sqrt{n}\\c\mid k,\ 2\nmid k}}\frac{1}{\sqrt{k}}\sum_{\substack{h=1\\(h,k)=1}}^{k}\frac{1}{\left|\sin\left(\frac{\pi h}{c}\right)\right|}.$$

Using the estimate (3.16), we obtain that

$$\begin{aligned}\left|\sum_{\substack{1\le k\le\sqrt{n}\\c'\mid k,\ 2\nmid k}}\frac{\overline{B}_{j,c,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{n}}{k}\right)\right| &\le \sinh\left(\frac{\pi\sqrt{n}}{3}\right)\frac{2\left(1+\log\left(\frac{c-1}{2}\right)\right)}{\pi(1-\pi^2/24)}\sum_{\substack{1\le k\le\sqrt{n}\\c\mid k,\ 2\nmid k}}k^{\frac{1}{2}}\\ &\le \frac{4n^{3/4}\left(1+\log\left(\frac{c-1}{2}\right)\right)}{3c\pi(1-\pi^2/24)}\sinh\left(\frac{\pi\sqrt{n}}{3}\right).\end{aligned} \tag{4.5}$$

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Here it is used that following estimation of the sum:

$$\sum_{\substack{1\le k\le\sqrt{n}\\ c|k,\ 2\nmid k}} k^{\frac{1}{2}} \le c^{\frac{1}{2}} \sum_{1\le j\le\lfloor\frac{N}{c}\rfloor} j^{\frac{1}{2}} \le c^{\frac{1}{2}} \int_1^{\lfloor\frac{N}{c}\rfloor} x^{\frac{1}{2}}dx \le \frac{2}{3c}n^{\frac{3}{4}}.$$

By (4.5) and (3.37), we have

$$|\overline{G}_1| \le \frac{2n^{\frac{1}{4}}\left(1+\log(\frac{c-1}{2})\right)}{3\pi c\left(1-\frac{\pi^2}{24}\right)}\cdot e^{\frac{\pi\sqrt{n}}{c}} \le 0.1803n^{\frac{1}{4}}\cdot e^{\frac{\pi\sqrt{n}}{3}}.$$

We next turn to $\overline{G}_2$. Recall that $\delta^i_{j,c,k,r} \le \delta_0$. Combining $|\overline{D}_{j,c,k}(-n, m^i_{j,c,k,r})| \le k$, we have

$$|\overline{G}_2| \le n^{-\frac{1}{2}} e^{4\pi\sqrt{n\delta_0}} \sum_{\substack{1\le k\le\sqrt{n}\\ c'\nmid k,\ 2\nmid k\\ \delta^i_{j',c',k,r}>0\\ r\ge 0,\ i\in\{+,-\}}} \sqrt{k}. \tag{4.6}$$

Noting that the number of r satisfying $\delta^i_{j,c,k,r} > 0$ is less than $\frac{c}{16}$, we have

$$\sum_{\substack{r\ge 0,\ i\in\{+,-\}\\ \delta^i_{j',c',k,r}>0}} 1 \le \frac{c}{8}. \tag{4.7}$$

Moreover, one can see that

$$\sum_{\substack{1\le k\le\sqrt{n}\\ c'\nmid k,\ 2\nmid k}} \sqrt{k} \le \int_1^{\sqrt{n}} x^{\frac{1}{2}}dx \le \frac{2}{3}n^{\frac{3}{4}}. \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.6), we can obtain (4.4). This completes the proof. ∎

Now we are in a position of proving Theorem 1.3.

Proof of Theorem 1.3 From [20, Theorem 6], we get the following lower bound for $\overline{p}(n)$, that is, for $n \ge 1$,

$$\overline{p}(n) > \frac{1}{8n}\left(1-\frac{1}{\sqrt{n}}\right)e^{\pi\sqrt{n}}. \tag{4.9}$$

Combining with (1.2), (4.3), (4.9), and Lemma 4.1, for odd c and positive integer n, we have

$$\left|\overline{R}(a,c,n)\right| \le \left(0.1803n^{\frac{1}{4}}e^{\frac{\pi\sqrt{n}}{3}} + \frac{c}{12}n^{\frac{1}{4}}e^{4\pi\sqrt{n\delta_0}} + (1700009.4994c + 7358.2224)n^{\frac{3}{8}}\right) \cdot \frac{8ne^{-\pi\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}.$$

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Since for $n \geq 2$,

$$\frac{1}{1-\frac{1}{\sqrt{n}}} \leq \frac{1}{1-\frac{1}{\sqrt{2}}} \leq 3.415,$$

which yields that

$$\begin{aligned}\left|\overline{R}(a,c,n)\right| \leq\ & 4.93n^{\frac{5}{4}}e^{-\frac{2}{3}\pi\sqrt{n}} + 2.28cn^{\frac{5}{4}}e^{(4\sqrt{\delta_0}-1)\pi\sqrt{n}} \\ & + 10^5(464.443c + 2.011)n^{\frac{11}{8}}e^{-\pi\sqrt{n}}.\end{aligned}$$

We take the sum of all three coefficients, the highest order exponential, and the highest power of n from the three terms and put them together in one term. Note that $4\sqrt{\delta_0} > \frac{1}{3}$ for $c \geq 9$. Therefore, we get the bounds of $\overline{R}(a,c,n)$, which completes the proof. ∎

5 Inequalities of $\overline{M}(a,c,n)$

In this section, utilizing the asymptotic formulas obtained earlier, we establish several inequality relationships concerning $\overline{M}(a,c,n)$, including crank differences, positivity, and strict log-subadditivity. First, we will utilize Theorem 1.1 to prove Theorem 1.4.

Proof of Theorem 1.4 By (4.1) we have

$$\sum_{n=0}^{\infty}\left(\overline{M}(a,c,n) - \overline{M}(b,c,n)\right)q^n = \frac{2}{c}\sum_{j=1}^{\frac{c-1}{2}}\rho_j(a,b,c)\overline{C}(\zeta_c^j;q), \tag{5.1}$$

where

$$\rho_j(a,b,c) = \cos\left(\frac{2\pi aj}{c}\right) - \cos\left(\frac{2\pi bj}{c}\right).$$

We use Theorem 1.1, with a change of variables that does not effect the theorem. If $c \nmid k$, we redefine $0 < l < c$ to be the unique solution to the congruence $l \equiv jk \pmod{c}$. Inserting Theorem 1.1 into (5.1), we obtain

$$\overline{M}(a,c,n) - \overline{M}(b,c,n) = \sum_{j=1}^{\frac{c-1}{2}}\left(\overline{S}_j(a,b,c;n) + \sum_{i\in\{+,-\}}\overline{T}_j^i(a,b,c;n) + O(n^{\varepsilon})\right),$$

where

$$\overline{S}_j(a,b,c;n) = \rho_j(a,b,c)\frac{2i}{c\sqrt{n}}\sum_{\substack{1\leq k\leq\sqrt{n}\\ c\mid k,\ 2\nmid k}}\frac{\overline{B}_{j,c,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{n}}{k}\right),$$

$$\begin{aligned}\overline{T}_j^i(a,b,c;n) = \rho_j(a,b,c)\frac{4}{c\sqrt{n}}\sin\left(\frac{\pi j}{c}\right)&\sum_{\substack{1\leq k\leq\sqrt{n}\\ c\nmid k,\ 2\nmid k\\ \delta^i_{j,c,k,r}>0\\ r\geq 0}}\frac{\overline{D}_{j,c,k}(-n,m^i_{j,c,k,r})}{\sqrt{k}}\\ &\times\sinh\left(\frac{4\pi}{k}\sqrt{n\delta^i_{j,c,k,r}}\right).\end{aligned}$$

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Then, we need to find the main term in $\overline{M}(a,c,n) - \overline{M}(b,c,n)$.

The main term:

Firstly, we detect the main contribution coming from sinh. In $\overline{S}_j$, the main term occurs for $k = c$, giving $\sinh\left(\frac{\pi\sqrt{n}}{c}\right)$ as the main hyperbolic sine argument, whereas in $\overline{T}_j^i$ it is given by $k = 1$. Recall that $\delta_{j,c,k,r}^i < \delta_{j,c,k,0}^+$ and $\delta_{j,c,k,0}^+$ arrives the maximum value $\delta_0 = \frac{1}{2c^2} - \frac{1}{2c} + \frac{1}{16}$ at $l = 1$. Moreover, $l \equiv jk \pmod{c}$ together with $k = l = 1$ gives that $j = l = 1$. These imply that the largest argument occurs for $k = 1, r = 0$ and $j = 1$. For $k = 1, r = 0$, and any j, we easily see that $\overline{D}_{j,c,1}(-n, m_{j,c,1,0}^+) = \overline{D}_{j,c,1}(-n, 0) = 1$. Now, since

$$4\sqrt{\delta_0} > \frac{1}{c}$$

for $c \geq 9$, the main term is

$$\overline{T}_1^+ = \rho_1(a,b,c)\frac{4}{c\sqrt{n}}\sinh\left(4\pi\sqrt{n\delta_0}\right),$$

and this is positive for all $0 \leq a < b \leq \frac{c-1}{2}$. Therefore, for sufficiently large n, we obtain $\overline{M}(a,c,n) > \overline{M}(b,c,n)$. The following step entails providing a clear definition of "sufficiently large" by establishing bounds for the remaining terms.

Bounding contribution of $\overline{S}_j$:

Since $|\rho_j(a,b,c)| < 2$, by (4.5), we have

$$|\overline{S}_j(a,b,c;n)| \leq \frac{16n^{1/4}\left(1+\log\left(\frac{c-1}{2}\right)\right)}{3c^2\pi(1-\pi^2/24)}\sinh\left(\frac{\pi\sqrt{n}}{c}\right).$$

Next we want to bound the $\overline{T}_j^i$ for $j \geq 2$ and $\overline{T}_1^-$.

Bounding the contribution of $\overline{T}_j^i$ for $j \geq 2$:

For $k \geq 2$, we trivially bound

$$|\overline{D}_{j,c,k}(-n, m_{j,c,k,r}^+)| \leq k.$$

Since $\sinh(x) = (e^x - e^{-x})/2$, we have

$$\sinh\left(\frac{4\pi}{k}\sqrt{n\delta_{j,c,k,r}^i}\right) \leq \frac{e^{2\pi\sqrt{n\delta_0}}}{2}.$$

Next, recall that the number of r satisfying $\delta_{j,c,k,r}^i > 0$ is less than $\frac{c}{16}$. Thus we can bound $\overline{T}_j^i$ for $k \geq 2$ by the following expression:

$$\frac{1}{6}n^{1/4}e^{2\pi\sqrt{n\delta_0}}.$$

Since $\delta_{j,c,1,0}^i < \delta_0$ for $j \geq 2$, we bound the $k = 1$ contribution by the argument

$$\frac{1}{4\sqrt{n}}e^{4\pi\sqrt{n\delta_{2,c,1,0}^i}}.$$

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Before coming to the error terms of the circle method we have to bound the contribution of $\overline{T}_1^-$.

Bounding the contribution of $\overline{T}_1^-$:

By the same analysis, it is possible to bound this term by a similar expression as the ones before. By bounding the sinh by the exponential function, using $|\rho_j(a,b,c)| < 2$ and showing $\delta^-_{1,c,k,r}$ is smaller than $\delta^+_{2,c,1,0}$ and so choosing the right argument in the exponential function, $\overline{T}_1^-$ can be bounded by

$$\frac{1}{4\sqrt{n}} e^{4\pi\sqrt{n\delta^+_{2,c,1,0}}}.$$

The explicit constant:

Denoting the four different error terms in Lemmas 3.4–3.7, $\overline{S}_j$, $\overline{T}_j^i$ for $j \geq 2$ and $\overline{T}_1^-$ by $\Sigma_{err,m}$ we can conclude that

$$\overline{N}_{a,b,c} = \min\left\{ n \in \mathbb{N} \,\middle|\, \overline{T}_1^+ - \sum_m \Sigma_{err,m}(c,n) > 0 \right\}.$$

This finishes the proof. ∎

For $c < 9$, the S_j will give the main contributions to the circle method as noticed in the last theorem. As the sign of S_j depend on the sign of $\tilde{B}_{j,c,k}$ and the $\tilde{B}_{j,c,k}$ oscillate we will have the following.

Corollary 5.1 *For sufficiently large n, we obtain the following inequalities.*

(1) *The crank differences* $\overline{M}(a,3,3n+d) - \overline{M}(b,3,3n+d)$ *are*

$$\begin{cases} < 0, & \text{if } (a,b,d) \in \{(0,1,1),(0,1,2)\}, \\ > 0, & \text{if } (a,b,d) \in \{(0,1,0)\}. \end{cases}$$

(2) *The crank differences* $\overline{M}(a,5,5n+d) - \overline{M}(b,5,5n+d)$ *are*

$$\begin{cases} < 0, & \text{if } (a,b,d) \in \{(0,2,4),(0,1,2),(0,1,4),(0,b,b),(1,2,0),(1,2,3)\}, \\ > 0, & \text{if } (a,b,d) \in \{(0,b,0),(1,2,1),(0,1,3)\}. \end{cases}$$

(3) *The crank differences* $\overline{M}(a,7,7n+d) - \overline{M}(b,7,7n+d)$ *are*

$$\begin{cases} < 0, & \text{if } (a,b,d) \in \{(0,b,4),(0,1,1),(0,1,2),(0,2,2),(1,2,0), \\ & \quad (1,2,5),(2,3,6)\}, \\ > 0, & \text{if } (a,b,d) \in \{(0,b,0),(0,b,3),(0,1,5),(0,2,6),(0,3,5),(1,2,1), \\ & \quad (1,2,4),(1,2,6),(1,3,1),(1,3,2),(1,3,4),(2,3,0),(2,3,2),(2,3,5)\}. \end{cases}$$

Proof For c being an odd integer less than 9, these inequalities can be readily verified using mathematical software. Specifically, for cases where $c < 7$ and $k = c$, this setting results in the largest argument for the hyperbolic sine function in $\overline{S}j$. Consequently, it is sufficient to determine the sign of $\sum_j \rho_j(a,b,c)\overline{B}j,c,c(-n,0)$ to identify which inequality the crank differences satisfy. For the case where $c = 7$, the arguments of the hyperbolic sines in $\overline{S}_j$ and $\overline{T}_1^+$ may align and potentially cancel

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each other out. Therefore, we compute the sum of $\sum_j i\rho_j(a,b,c)\overline{B}_{j,c,c}(-n,0)/\sqrt{7}$ and $2\rho_1(a,b,c)\sin\frac{\pi}{7}$ to determine which inequality is met by the crank differences. ∎

Therefore, using Theorem 1.3, we can establish Theorem 1.5.

Proof of Theorem 1.5 In light of (1.3), to prove $\overline{M}(a,c,n)>0$, it is sufficient to prove

$$\left|\overline{R}(a,c,n)\right|<\frac{1}{c}$$

if $c<9$ and $n\geq 196$, and

$$\left|\overline{R}(a,c,n)\right|<\frac{1}{2c} \tag{5.2}$$

if $c\geq 9$ and $n>\overline{Q}_c$, where $\overline{Q}_c$ is defined in (1.4). According to the bound of $\overline{R}(a,c,n)$ in Theorem 1.3, we just need to show for $c<9$

$$10^5(464.45c+2.02)n^{\frac{11}{8}}e^{-\frac{2}{3}\pi\sqrt{n}}<\frac{1}{c} \tag{5.3}$$

and for $c\geq 9$

$$10^5(464.45c+2.02)n^{\frac{11}{8}}e^{-\left(1-\sqrt{1-\frac{8}{c}+\frac{8}{c^2}}\right)\pi\sqrt{n}}<\frac{1}{2c}. \tag{5.4}$$

In order to verify (5.3), it suffices to prove that

$$10^5(464.45\times 9+2.02)n^{\frac{11}{8}}e^{-\frac{2}{3}\pi\sqrt{n}}<\frac{1}{9},$$

which holds for $n\geq 196$. We next deal with the case of $c\geq 9$. To prove (5.4), it is equivalent to prove that

$$\left(1-\sqrt{1-\frac{8}{c}+\frac{8}{c^2}}\right)\pi\sqrt{n}-\frac{11}{8}\log n>\log\left(2\times 10^5\times\left(464.45c^2+2.02c\right)\right). \tag{5.5}$$

Since for $x\geq 104$,

$$\log x<\frac{16}{11}x^{\frac{1}{4}},$$

and for $c\geq 9$,

$$\left(1-\sqrt{1-\frac{8}{c}+\frac{8}{c^2}}\right)>\frac{4}{c},$$

we have

$$\left(1-\sqrt{1-\frac{8}{c}+\frac{8}{c^2}}\right)\pi\sqrt{n}-\frac{11}{8}\log n>\frac{4\pi\sqrt{n}}{c}-2n^{\frac{1}{4}}. \tag{5.6}$$

In view of (5.5) and (5.6), it is sufficient to verify

$$\frac{4\pi\sqrt{n}}{c}-2n^{\frac{1}{4}}>\log\left(2\times 10^5\times\left(464.45c^2+2.02c\right)\right),$$

which holds for $n\geq\overline{Q}_c$. This completes the proof. ∎

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We use the bound of $\overline{R}(a,c,n)$ to prove Theorem 1.6 now.

Proof of Theorem 1.6 We first consider the case $c < 9$. By Theorem 1.3, we can give a lower and an upper bound for $\overline{M}(a,c,n)$, that is,

$$\overline{L}(c,n) < \overline{M}(a,c,n) < \overline{U}(c,n),$$

where

$$\overline{L}(c,n) := \overline{p}(n)\left(\frac{1}{c} - 10^5(464.45c + 2.02)n^{\frac{11}{8}}e^{-\frac{2\pi}{3}\sqrt{n}}\right),$$
$$\overline{U}(c,n) := \overline{p}(n)\left(\frac{1}{c} + 10^5(464.45c + 2.02)n^{\frac{11}{8}}e^{-\frac{2\pi}{3}\sqrt{n}}\right).$$

For $3 \le c < 9$, we have

$$\overline{L}(c,n) > \overline{p}(n)\left(\frac{1}{9} - 10^5(464.45 \times 9 + 2.02)n^{\frac{11}{8}}e^{-\frac{2\pi}{3}\sqrt{n}}\right),$$

Noting that $n^{\frac{11}{8}}e^{-\frac{2\pi}{3}\sqrt{n}}$ is decreasing for $n \ge 196$, it follows that

$$\overline{L}(c,n) > 0.0017\overline{p}(n).$$

Similarly, we get

$$\overline{U}(c,n) < \overline{p}(n)\left(\frac{1}{3} + 10^5(464.45 \times 9 + 2.02)n^{\frac{11}{8}}e^{-\frac{2\pi}{3}\sqrt{n}}\right) < 0.4428\overline{p}(n).$$

Hence, when $3 \le c < 9$ and $n \ge 196$, we have

$$0.0017\,\overline{p}(n) < \overline{M}(a,c,n) < 0.4428\,\overline{p}(n). \tag{5.7}$$

From [20, Theorem 6], we know that for $n \ge 1$,

$$\frac{1}{8n}\left(1 - \frac{1}{\sqrt{n}}\right)e^{\pi\sqrt{n}} < \overline{p}(n) < \frac{1}{8n}\left(1 + \frac{1}{\sqrt{n}}\right)e^{\pi\sqrt{n}}. \tag{5.8}$$

For $c < 9$, let $n_1 = \lambda n_2$ with $\lambda \ge 1$. Then by (5.7) and (5.8), we have

$$\overline{M}(a,c,n_1)\overline{M}(a,c,n_2) > \frac{0.0017^2}{64\lambda n_2^2} \cdot \left(1 - \frac{1}{\sqrt{\lambda n_2}}\right)\left(1 - \frac{1}{\sqrt{n_2}}\right) \cdot e^{\pi\sqrt{\lambda n_2} + \pi\sqrt{n_2}}$$

and

$$\overline{M}(a,c,n_1 + n_2) < \frac{0.4428}{8(\lambda n_2 + n_2)}\left(1 + \frac{1}{\sqrt{\lambda n_2 + n_2}}\right)e^{\pi\sqrt{\lambda n_2 + n_2}}$$

for $n \ge 196$. Hence it suffices to show that

$$T_{n_2}(\lambda) > \log(V_{n_2}(\lambda)) + \log(S_{n_2}(\lambda)),$$

where

$$T_{n_2}(\lambda) := \pi\sqrt{\lambda n_2} + \pi\sqrt{n_2} - \pi\sqrt{\lambda n_2 + n_2},$$
$$V_{n_2}(\lambda) := \frac{0.4428}{0.0017^2} \cdot \frac{8\lambda n_2}{\lambda + 1},$$

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$$S_{n_2}(\lambda) := \frac{1+\frac{1}{\sqrt{\lambda n_2+n_2}}}{\left(1-\frac{1}{\sqrt{\lambda n_2}}\right)\left(1-\frac{1}{\sqrt{n_2}}\right)}.$$

On the one hand, it can be shown that $T_{n_2}(\lambda)$ is increasing and $S_{n_2}(\lambda)$ is decreasing for $\lambda \geq 1$. On the other hand,

$$V_{n_2}(\lambda) < \frac{0.4428 \cdot 8n_2}{0.0017^2}.$$

So it is sufficient to verify that

$$T_{n_2}(1) = 2\pi\sqrt{n_2} - \pi\sqrt{2n_2} > \log\left(\frac{0.4428 \cdot 8n_2}{0.0017^2}\right) + \log\left(\frac{1+\frac{1}{\sqrt{2n_2}}}{\left(1-\frac{1}{\sqrt{n_2}}\right)^2}\right),$$

which holds for $n_2 \geq 107$.

Finally, we deal with the case $c \geq 9$. In this case, we bound $\overline{M}(a, c, n)$ by a different way. Applying (1.3) and (5.2), we obtain that

$$\frac{1}{2c}\overline{p}(n) < \overline{M}(a, c, n) < \frac{3}{2c}\overline{p}(n)$$

for $n \geq \overline{Q}_c$. By the same argument of the case $c < 9$, it suffices to show that

$$T_{n_2}(\lambda) > \log(\overline{V}_{n_2}(\lambda)) + \log(S_{n_2}(\lambda)),$$

where

$$\overline{V}_{n_2}(\lambda) := 6c \cdot \frac{8\lambda n_2}{\lambda+1}.$$

It is obvious that

$$\overline{V}_{n_2}(\lambda) < 48cn_2.$$

So it is equivalent to verify that

$$T_{n_2}(1) = 2\pi\sqrt{n_2} - \pi\sqrt{2n_2} > \log(48cn_2) + \log\left(\frac{1+\frac{1}{\sqrt{2n_2}}}{\left(1-\frac{1}{\sqrt{n_2}}\right)^2}\right). \tag{5.9}$$

On the one hand, when $n_2 \geq 2$, we get

$$\log(48cn_2) + \log\left(\frac{1+\frac{1}{\sqrt{2n_2}}}{\left(1-\frac{1}{\sqrt{n_2}}\right)^2}\right) < \log(48cn_2) + \log\left(\frac{1+\frac{1}{\sqrt{4}}}{\left(1-\frac{1}{\sqrt{2}}\right)^2}\right) < \log(840cn_2). \tag{5.10}$$

On the other hand, when $n_2 > 840c$, we have

$$T_{n_2}(1) = 2\pi\sqrt{n_2} - \pi\sqrt{2n_2} > 2\log(n_2) > \log(n_2) + \log(840c) = \log(840cn_2). \tag{5.11}$$

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Therefore, in light of (5.9), (5.10), and (5.11), we know (1.5) holds for $c \geq 9$ and $n_1, n_2 \geq \overline{Q}_c$. ■

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