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PAPERS

Stabilization of arbitrary structures in a three-dimensional doubly degenerate nutrient taxis system

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Abstract

The doubly degenerate nutrient taxis system

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \chi \nabla \cdot (u^\alpha v \nabla v) + \ell uv, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases} \tag{0.1}$$

is considered under zero-flux boundary conditions in a smoothly bounded domain $\Omega \subset \mathbb{R}^3$ where $\alpha > 0$, $\chi > 0$ and $\ell > 0$. By developing a novel class of functional inequalities to address the challenges posed by the doubly degenerate diffusion mechanism in (0.1), it is shown that for $\alpha \in (\frac{3}{2}, \frac{19}{12})$, the associated initial-boundary value problem admits a global continuous weak solution for sufficiently regular initial data. Furthermore, in an appropriate topological setting, this solution converges to an equilibrium $(u_\infty, 0)$ as $t \to \infty$. Notably, the limiting profile u_∞ is non-homogeneous when the initial signal concentration v_0 is sufficiently small, provided the initial data u_0 is not identically constant.

1. Introduction

Significant research efforts have been devoted to characterising the conditions under which parabolic systems can accurately model pattern formation in chemotaxis processes, ensuring consistency with experimental observations. A cornerstone of this research is the Keller–Segel model, in which the chemotactic-diffusion mechanism enables the description of collective microbial behaviour driven by chemical gradients [1, 8, 20].

Recent studies have demonstrated the crucial influence of chemotactic sensitivity function on the dynamic prospects of chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

posed on a smoothly bounded n-dimensional domain. Specifically, the ratio $\frac{S(u)}{D(u)}$ critically determines whether solutions to the zero-flux initial-boundary value problem of (1.1) are globally bounded or posses finite-time singularities. Indeed, when the diffusivity and sensitivity functions $D, S \in C^2(\mathbb{R}_+)$ satisfy $S(0) = 0$ and the growth condition $\frac{S(u)}{D(u)} \le Cu^\alpha$ for all $u \ge 1$ with $C > 0$ and $\alpha < \frac{2}{n}$, the system admits a global bounded classical solution [6, 21]. In contrast, when $\frac{S(u)}{D(u)} \ge Cu^\alpha$ for some $C > 0$ and $\alpha > \frac{2}{n}$ at large values, the solution exhibits finite-time blow-up through concentration phenomena [2, 3].

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When chemotaxis directs motion of bacteria (with density u) towards higher concentrations of nutrient (of concentration v), the resulting dynamics are mathematically described by chemotaxis-consumption models of the form

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - vu, & x \in \Omega,\ t > 0. \end{cases} \tag{1.2}$$

In sharp contrast to the situation in (1.1), due to the essentially dissipative character of v-equation in (1.2), the latter system should exhibit some considerably stronger relaxation properties. Notably, when considering the specific case with constant diffusivity $D(u) = 1$ and linear sensitivity $S(u) = u$, subject to no-flux boundary conditions in bounded convex domains, the available literature reveals no evidence of non-trivial large-time dynamics for solutions of (1.2). At the level of rigorous mathematical analysis, it is confirmed that for reasonably regular but arbitrarily large initial data, two-dimensional versions of (1.2) are known to possess global bounded classical solutions (see [7, 22] for example); whereas when $n = 3$, the certain global weak solutions emanating from large initial data at least eventually become bounded and smooth and particularly approach the unique relevant constant steady state in the large time limit [22, 24].

Recent experimental studies indicate that motility constraints in *Bacillus subtilis* strains under nutrient-poor environments play a crucial role in the formation of diverse morphological aggregation patterns, including densely accumulating multiply branches structures [4, 5, 19]. To gain a comprehensive understanding of the underlying mechanism, particularly the hypothesis that bacterial motility decreases under nutrient deprivation, a doubly degenerate nutrient taxis system of the form

$$\begin{cases} u_t = \nabla \cdot (u^{m-1}v\nabla u) - \chi\nabla \cdot (S(u)v\nabla v) + f(u,v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

with $m = 2$, $S(u) = u^2$, was introduced in [9, 12, 17]. A remarkable feature of (1.3) is the doubly degenerate structure of the first equation, arising from two distinct mechanisms: (i) porous-medium-type degeneracy as u approaches zero and (ii) cross-degeneracy induced by the factor v, which vanishes when v tends to zero (as governed by the second equation). This peculiarity brings about noticeable mathematical challenges beyond classical porous medium-type diffusion theory. Hence, current understanding of even fundamental existence questions is still largely restricted to the global solvability in low-dimensional settings.

It is shown in [26] that when $\chi = 0, f(u,v) = 0$ and $\Omega \subset \mathbb{R}^n$ is a bounded convex domain, the associated initial-boundary value problem for (1.3) possesses a global weak solution for the reasonably regular initial data. Moreover, within an appropriate topological setting, the solution can approach the non-homogeneous steady-state $(u_\infty, 0)$ in the large time limit. It is noticed that due to the absence of taxis effects in (1.3), the comparison principle allows for deriving local bounds for $\|u(\cdot,t)\|_{L^\infty(\Omega)}$, and the boundedness of $\int_0^\infty \int_\Omega u^{q-1}v|\nabla u|^2$ for $q \in (0,1)$ becomes the basic regularity property of (1.3). These estimates provide the essential foundation for establishing the global existence of weak solution. However, when $\chi > 0$, the chemoattraction mechanisms in (1.3) may exert considerably destabilising effect, as evidenced by blow-up phenomena observed in analogous modelling contexts of (1.1) [1, 10]. To the best of our knowledge, the available analytical results for (1.3) with $\chi > 0, f(u,v) = \ell uv$ so far remain restricted to low-dimensional settings:

i. In the **one-dimensional** case, when $m = 2$ and $S(u) = u^2$, global existence of continuous weak solutions and their asymptotic stabilisation towards non-homogeneous equilibria was established in [14, 29]. These results were later extended to cases where either $2 \le m < 3$ and $S(u) \le Cu^\alpha$ for $\alpha \in [m-1, \frac{m}{2}+1]$ or $3 \le m < 4$ [32].

ii. In **two-dimensional** counterparts, global existence of weak solutions was first established for $m = 2$ and $S(u) = u^\alpha$ with $1 < \alpha < \frac{3}{2}$ [13]. Subsequent work in [28] extended this result to all $\alpha < 2$, and in particular, proved L^∞-bounds for the solutions. Notably, for the critical case $\alpha = 2$, global existence

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of weak solutions was obtained under a smallness condition solely on the initial data v_0 in [27]. This restriction was later relaxed in [34]. Furthermore, the large-time behavior of solutions to (1.3) has been studied in [31, 33] using techniques based on Harnack inequalities.

In striking contrast to lower-dimensional cases where effective embedding theorems facilitate the analysis, the study of the *three-dimensional* version of (1.3) becomes significantly more subtle. To date, even fundamental questions regarding global solvability and boundedness, including the large-time behaviour, remain largely unestablished in the literature. For instance, the global solvability of (1.3) was established in [13] when $\frac{7}{6} < \alpha < \frac{13}{9}$, which may be relaxed to $1 < \alpha < \frac{3}{2}$ [32]. Apart from that, the stabilising effect of logistic-type source terms $f(u, v) = \rho u - \mu u^k$ on (1.3) is also investigated. For example, global existence of weak solutions was established in arbitrary dimensions $n \geq 2$ under the condition $k > \frac{n+2}{2}$ [16], and particularly the continuity of weak solutions was achieved when $n = k = 2$ [15].

The intention of the present work is to develop an analytical approach that not only establishes the global solvability of system (1.3) but also characterise the non-trivial long-time dynamics in three-dimensional domains. Specially, we shall consider the initial-boundary value problem

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \chi \nabla \cdot (u^\alpha v \nabla v) + \ell uv, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ (uv\nabla u - \chi u^\alpha v \nabla v) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

posed in a smoothly bounded domain $\Omega \subset \mathbb{R}^3$. Here, the parameters satisfy $\chi > 0$, $\ell > 0$ and $\alpha \in (\frac{3}{2}, \frac{19}{12})$. The initial data (u_0, v_0) are assumed to satisfy

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is such that } u_0 \geq 0 \text{ and } u_0 \not\equiv 0, & \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is such that } v_0 > 0 \text{ in } \Omega \end{cases} \tag{1.5}$$

as well as

$$\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} + \|\nabla \ln v_0\|_{L^\infty(\Omega)} \leq K \quad \text{and} \quad \int_\Omega \ln u_0 \geq -K \tag{1.6}$$

for some $K > 0$.

In this framework, we will first address the basic issue of the global existence and boundedness of continuous weak solutions to (1.4) when the convexity of Ω is not necessary.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and suppose that $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ as well as $\chi > 0$ and $\ell > 0$. Then, for all $p > 1$, one can find $C = C(p, K) > 0$ with the property that whenever u_0 and v_0 fulfill (1.5) and (1.6), there exist*

$$\begin{cases} u \in C^0_{loc}(\overline{\Omega} \times [0, \infty)) \cap L^\infty_{loc}(\overline{\Omega} \times [0, \infty)) & \text{and} \\ v \in C^0_{loc}(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}_{loc}(\overline{\Omega} \times (0, \infty)) \cap L^\infty_{loc}([0, \infty); W^{1,\infty}(\Omega)) \end{cases} \tag{1.7}$$

such that $u \geq 0$ and $v > 0$ a.e. in $\Omega \times (0, \infty)$, and that (u, v) forms a continuous global weak solution of (1.4) in the sense of Definition 2.1 below. Moreover, we have

$$\|u(\cdot, t)\|_{L^p(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C(p, K) \qquad \text{for a.e. } t > 0. \tag{1.8}$$

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By carefully tracking respective dependence on time, the estimates gained in the proof of Theorem 1.1 already pave the way for our subsequent asymptotic analysis. Indeed, by means of a duality-based argument, we can achieve the following result on stability property enjoyed by function pairs $(u_0, 0)$ for all suitably regular u_0.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and suppose that $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ as well as $\chi > 0$ and $\ell > 0$. Then for each $\eta > 0$, there exists $\delta_1 = \delta_1(\eta, K) > 0$ whenever u_0 and v_0 fulfill (1.5) and (1.6), as well as*

$$\int_\Omega v_0 \le \delta_1,$$

the solution (u, v) of (1.4) obtained in Theorem 1.1 satisfies

$$\|u(\cdot,t) - u_0\|_{(W^{1,\infty}(\Omega))^*} \le \eta \qquad \text{for all } t > 0. \tag{1.9}$$

Beyond that, we undertake a deeper qualitative analysis of the asymptotic properties of solutions. Although Theorem 1.2 guarantees that the solution component u remains close to its initial data u_0 in the weak-$*$ topology of $(W^{1,\infty}(\Omega))^*$ for small v_0, the fundamental question of whether the limiting profile u_∞ must be constant remains open. We resolve this question by proving that when $u_0 \not\equiv const$, sufficiently small initial concentrations of v_0 necessarily yield nonconstant limiting profiles u_∞.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and suppose that $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ as well as $\chi > 0$ and $\ell > 0$. Then, there exists a nonnegative function $u_\infty \in \bigcap_{p\ge 1} L^p(\Omega)$ such that as $t \to \infty$, the solution (u, v) of (1.4) from Theorem 1.1 satisfies*

$$u(\cdot,t) \rightharpoonup u_\infty \quad \text{in } L^p(\Omega), \qquad v(\cdot,t) \to 0 \quad \text{in } W^{1,p}(\Omega) \text{ for all } p \ge 1. \tag{1.10}$$

Moreover, for the given nonnegative function $u_0 \not\equiv$ const., one can find $\delta_2 = \delta_2(K, u_0) > 0$ such that whenever u_0 and v_0 fulfill (1.5) and (1.6), as well as

$$\int_\Omega v_0 \le \delta_2,$$

the limit function satisfies $u_\infty \not\equiv$ const.

Main ideas. The main challenge in developing a mathematical theory for system (1.4) lies in effectively quantifying its dissipative mechanisms, which are substantially weakened by the signal-dependent degeneracy in the u-equation. Accordingly, existing literature has been limited to (1.4) posed on the bounded convex domain $\Omega \subset \mathbb{R}^n, n \le 2$ [13, 14, 26–29]. In non-convex three-dimensional domains, however, the analysis becomes considerably more intricate due to the loss of favourable embedding properties.

Our approach is to improve the regularity of (1.4) via the bootstrap argument. The most crucial step is to obtain the global boundedness of $\|u(\cdot,t)\|_{L^{p_0}(\Omega)}$ for some $p_0 > \frac{3}{2}$. To this end, we shall appropriately utilise the diffusion-induced contributions of the form $\int_\Omega u^k v|\nabla u|^2$ (for suitable k) to counteract the unfavourable terms typified by $\int_\Omega u^\beta v$. As the starting point of our analysis, we certify that the integrals $\int_\Omega u^{1-\alpha} v|\nabla u|^2$ and $\int_\Omega u\frac{|\nabla v|^2}{v}$ are dissipated by tracking the time evolution of $\int_\Omega u^{2-\alpha}$ and $F(t) := a\int_\Omega \frac{|\nabla v(\cdot,t)|^2}{v(\cdot,t)} - \int_\Omega \ln u(\cdot,t)$ (with some $a > 0$) along trajectories of (1.4), as detailed in Lemmas 2.3 and 2.5, respectively, which distinguishes our approach from that in [13]. Moreover, differing from [13, 32], our derivation of bounds for quantities such as

$$\int_0^T \int_\Omega u^{\frac{5}{3}} v \tag{1.11}$$

is to make use of the aforementioned dissipation terms and an interpolation inequality (Lemma 2.6). Building upon (1.11), we can estimate the unfavourable terms arising from the time evolution of the

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functional

$$\mathcal{G}(t) := \int_\Omega u(\cdot,t)\ln u(\cdot,t) + b\int_\Omega \frac{|\nabla v(\cdot,t)|^4}{v^3(\cdot,t)}$$

for some $b>0$, and thereby derive the dissipated quantity of the form

$$\int_\Omega u^{p_*} v|\nabla u|^2 \quad \text{and} \quad \int_\Omega u\frac{|\nabla v|^4}{v^3} \tag{1.12}$$

for all $p_* \in [1-\alpha, 0]$ (Lemmas 3.3 and 3.4). A further crucial ingredient in our analysis is the novel observation: Assume that $\Omega \subset \mathbb{R}^3$ is a smoothly bounded domain, $L>0, k\in(-1,-\frac{1}{3})$ and $\beta \in [1, k+\frac{8}{3})$, then there exists $C(L,k,\beta)>0$ such that

$$\int_\Omega \varphi^\beta \psi \le \int_\Omega \varphi^k \psi |\nabla\varphi|^2 + \int_\Omega \varphi\frac{|\nabla\psi|^4}{\psi^3} + C(L,k,\beta)\int_\Omega \varphi\psi \tag{1.13}$$

holds for all sufficiently regular positive functions φ and ψ fulfilling $\int_\Omega \varphi \le L$ (see Lemma 3.5), which is somewhat different from that in [32]. Thanks to the estimates (1.12), we then employ inequality (1.13) to achieve the desired L^{p_0} bounds for u. This improved integrability of u allows us to derive a refined version of (1.13) (see Lemma 3.7) and accordingly certify an energy-like property for the coupled quantity

$$\mathcal{H}(t) := \int_\Omega \frac{|\nabla v(\cdot,t)|^q}{v^{q-1}(\cdot,t)} + \int_\Omega u^p(\cdot,t)$$

for any fixed $p>1$. This property thereby yields the bounds for $\|u(\cdot,t)\|_{L^p(\Omega)}$ as detailed in Lemma 3.8.

By suitably taking into account respective dependence on time, we intend to characterise the large-time behaviour of solution (u,v) to system (1.4). To achieve this, we employ a duality-based estimate for the time derivative u_t, establishing that

$$\int_0^\infty \|u_t(\cdot,t)\|_{(W^{1,\infty}(\Omega))^*}dt \le C(K)\cdot\left\{\int_\Omega v_0\right\}^\sigma, \tag{1.14}$$

for some $C(K)>0$ and $\sigma>0$ (see Lemma 4.1). Thanks to (1.14), we prove that the solution of (1.4) converges to equilibrium state $(u_\infty, 0)$ as $t\to\infty$ (see Lemmas 4.3 and 4.6). Moreover, we establish that for sufficiently small initial data v_0, the long-time limit u_∞ exhibits spatial heterogeneity (Lemma 4.7).

The structure of this paper is as follows: In Section 2, we specify the weak solutions of system (1.4) and establishes fundamental a priori estimates for the regularised systems. In Section 3, crucial L^p-estimates for u_ε are derived through careful analysis of energy functionals and bootstrap arguments, which needs to effectively quantifying the interplay between the weaken smoothing effects of random diffusion and the potentially destabilisng influence of taxis-diffusion. Building upon these estimates and further higher regularity properties of the regularised systems, we then prove Theorem 1.1 via compactness arguments. In Section 4, we investigates the stability properties of (u,v) by means of a duality-based argument.

2. Preliminaries

In view of the considered diffusion degeneracy in (1.4), the notation of weak solutions specified below seems fairly natural [28], but it is somewhat stronger than those in [13, 26], as it requires the integrability of ∇u.

Definition 2.1. *Assume that (1.5) and (1.6) hold, suppose that $\chi>0$ and $\ell>0$, and that $\alpha\ge 1$. Then, a pair of nonnegative functions*

$$\begin{cases} u\in C^0(\overline{\Omega}\times[0,\infty)) \quad \textit{and} \\ v\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \end{cases} \tag{2.1}$$

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such that

$$u^2 \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \quad \text{and} \quad u^\alpha \nabla v \in L^1_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^3) \tag{2.2}$$

will be called a continuous weak solution of (1.4) if

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\frac{1}{2}\int_0^\infty \int_\Omega v\nabla u^2 \cdot \nabla\varphi + \chi \int_0^\infty \int_\Omega u^\alpha v \nabla v \cdot \nabla\varphi + \ell \int_0^\infty \int_\Omega uv\varphi \tag{2.3}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ *fulfilling* $\frac{\partial\varphi}{\partial\nu} = 0$ *on* $\partial\Omega \times (0,\infty)$, *and if*

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega uv\varphi. \tag{2.4}$$

for any $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

To construct the defined weak solutions above through approximation by the classical solutions to the conveniently regularised problems, we consider the the regularised variants of (1.4) given by

$$\begin{cases} u_{\varepsilon t} = \nabla \cdot \left(u_\varepsilon v_\varepsilon \nabla u_\varepsilon\right) - \chi \nabla \cdot (u_\varepsilon^\alpha v_\varepsilon \nabla v_\varepsilon) + \ell u_\varepsilon v_\varepsilon, & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x) + \varepsilon,\ v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.5}$$

for $\varepsilon \in (0,1)$. According to Lemma 2.1 of [27], we have the following statement regarding the existence, extensibility and basic properties of solutions to (2.5).

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary,* $\chi > 0$, $\ell > 0$ *and* $\alpha \geq 1$, *and assume that (1.5) and (1.6) hold. Then, for each* $\varepsilon \in (0,1)$, *there exist* $T_{max,\varepsilon} \in (0,\infty]$ *and functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{max,\varepsilon}) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})) \\ v_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})) \end{cases} \tag{2.6}$$

such that $u_\varepsilon > 0$ *and* $v_\varepsilon > 0$ *in* $\overline{\Omega} \times [0, T_{max,\varepsilon})$, *that* $(u_\varepsilon, v_\varepsilon)$ *solves (2.5) classically in* $\Omega \times (0, T_{max,\varepsilon})$, *and that*

$$\text{if } T_{max,\varepsilon} < \infty, \ \text{ then } \limsup_{t \nearrow T_{max,\varepsilon}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.7}$$

Furthermore, this solution satisfies

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{2.8}$$

and

$$\int_\Omega u_0 \leq \int_\Omega u_\varepsilon(\cdot,t) \leq \int_\Omega u_0 + \ell \int_\Omega v_0 + |\Omega| \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{2.9}$$

as well as

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \leq \int_\Omega v_0. \tag{2.10}$$

Throughout the sequel, unless explicitly stated otherwise, we shall assume that $\Omega \subset \mathbb{R}^3$ is a smoothly bounded domain (the convexity of Ω is not required), $\chi > 0$, $\ell > 0$ and $\alpha \geq 1$. Moreover, whenever functions u_0 and v_0 fulfilling (1.5)–(1.6) have been selected, we shall denote by $(u_\varepsilon, v_\varepsilon)$ the solutions given by Lemma 2.1, with $T_{max,\varepsilon}$ representing their maximal existence time.

Due to the nonnegativity of u_ε and v_ε, the evolution of v_ε yields the following estimate.

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Lemma 2.2. *For any $K > 0$ with the property that (1.6) is valid, one can find $C = C(K) > 0$ such that*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \le C. \tag{2.11}$$

Proof. Multiplying the second equation in (2.5) by v_ε^2, we have

$$\frac{d}{dt}\int_\Omega v_\varepsilon^3 + 6\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 = -3\int_\Omega u_\varepsilon v_\varepsilon^3 \le 0 \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{2.12}$$

Integrating (2.12) over $(0, T_{max,\varepsilon})$ and using (1.5), we then obtain (2.11). □

It is noticed that in [13], the bounds of $\int_0^{T_{\max,\varepsilon}} \int_\Omega u_\varepsilon^{2-2\alpha} v_\varepsilon |\nabla u_\varepsilon|^2$ were utilised as a basic piece of information in deriving the bounds for $\int_\Omega u_\varepsilon^{p_0}(\cdot, t)$ with some $p_0 > 1$. In contrast, our approach in this paper takes $\int_0^{T_{\max,\varepsilon}} \int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2$ as the starting point for the bootstrap argument, which is achieved by analysing the temporal evolution of $-\frac{1}{2-\alpha}\int_\Omega u_\epsilon^{2-\alpha} + \frac{\chi(\alpha-1)}{2}\int_\Omega |\nabla v_\varepsilon|^2$ rather $-\frac{1}{3-2\alpha}\int_\Omega u_\varepsilon^{3-2\alpha}$.

Lemma 2.3. *Let $\alpha \in (1, 2]$. Then, for all $K > 0$ with the property that (1.6) is valid, one can find $C = C(K) > 0$ such that*

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 \le C. \tag{2.13}$$

Proof. Multiplying the equations in (2.5) by $u_\epsilon^{1-\alpha}$ and Δv_ε, respectively, we obtain

$$-\frac{1}{2-\alpha}\frac{d}{dt}\int_\Omega u_\varepsilon^{2-\alpha} + (\alpha-1)\int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 = (\alpha-1)\chi\int_\Omega v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \ell\int_\Omega u_\varepsilon^{2-\alpha} v_\varepsilon \tag{2.14}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\Delta v_\varepsilon|^2 = \int_\Omega u_\varepsilon v_\varepsilon \Delta v_\varepsilon = -\int_\Omega v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \int_\Omega u_\varepsilon |\nabla v_\varepsilon|^2 \tag{2.15}$$

for all $t \in (0, T_{\max,\varepsilon})$. From (2.14) and (2.15), we have

$$\begin{aligned}
&\frac{d}{dt}\left\{-\frac{1}{2-\alpha}\int_\Omega u_\epsilon^{2-\alpha} + \frac{\chi(\alpha-1)}{2}\int_\Omega |\nabla v_\varepsilon|^2\right\} + (\alpha-1)\int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 \\
&\le -\ell\int_\Omega u_\varepsilon^{2-\alpha} v_\varepsilon - \chi(\alpha-1)\int_\Omega u_\varepsilon |\nabla v_\varepsilon|^2
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. Integrating the above differential inequality on $(0, t)$, along with (1.6) and (2.9), we then get

$$\begin{aligned}
&\frac{1}{2-\alpha}\int_\Omega u_{0\epsilon}^{2-\alpha} + \frac{\chi(\alpha-1)}{2}\int_\Omega |\nabla v_\varepsilon|^2 + (\alpha-1)\int_0^t\int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 \\
&+ \ell\int_0^t\int_\Omega u_\varepsilon^{2-\alpha} v_\varepsilon + \chi(\alpha-1)\int_0^t\int_\Omega u_\varepsilon |\nabla v_\varepsilon|^2 \\
&= \frac{1}{2-\alpha}\int_\Omega u_\varepsilon^{2-\alpha} + \frac{\chi(\alpha-1)}{2}\int_\Omega |\nabla v_0|^2 \\
&\le \frac{1}{2-\alpha}\int_\Omega (u_\varepsilon + 1) + \frac{\chi(\alpha-1)}{2}\int_\Omega |\nabla v_0|^2 \\
&\le C(K)
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, due to $2 - \alpha \in [0, 1)$, which implies (2.13) immediately. □

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When the convexity of Ω is not assumed, the following estimate indicates how the related boundary integral over $\partial\Omega$ can be controlled by the corresponding higher-order integrals on Ω involving singular weights.

Lemma 2.4. [25] *Let $q \geq 2$ and $\eta > 0$. Then, there exists $C = C(\eta, q) > 0$ with the property that*

$$\int_{\partial\Omega} \psi^{-q+1} |\nabla\psi|^{q-2} \cdot \frac{\partial|\nabla\psi|}{\partial\nu} \leq \eta \int_\Omega \psi^{-q+1} |\nabla\psi|^{q-2} |D^2\psi|^2 + \eta \int_\Omega \psi^{-q-1} |\nabla\psi|^{q+2} + C\int_\Omega \psi \tag{2.16}$$

for any $\psi \in C^2(\overline{\Omega})$, which is such that $\psi > 0$ in $\overline{\Omega}$ and $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$.

By analysing the temporal evolution of

$$\|v_0\|^2_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} - \int_\Omega \ln u_\varepsilon,$$

we establish spatio-temporal integral bounds for both v_ε and $\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2$, thanks to the estimates (2.13) and (2.16).

Lemma 2.5. *Assume $\alpha \in \left(\frac{3}{2}, 2\right]$ and $K > 0$ such that (1.6) holds. Then, there exists $C > 0$ independent of $T_{\max,\varepsilon}$ such that*

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega v_\varepsilon \leq C \tag{2.17}$$

and

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 \leq C \tag{2.18}$$

as well as

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} |\nabla u_\varepsilon|^2 \leq C \qquad \text{and} \qquad \int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq C. \tag{2.19}$$

Proof. According to the second equation in (2.5), several integrations by parts show that for all $t \in \left(0, T_{\max,\varepsilon}\right)$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} &= \int_\Omega \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla \{\Delta v_\varepsilon - u_\varepsilon v_\varepsilon\} - \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon^2} |\nabla v_\varepsilon|^2 \cdot \{\Delta v_\varepsilon - u_\varepsilon v_\varepsilon\} \\
&= \int_\Omega \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon - \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 - \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&\quad - \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon^2} |\nabla v_\varepsilon|^2 \cdot \Delta v_\varepsilon + \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 \\
&= \int_\Omega \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon - \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon^2} |\nabla v_\varepsilon|^2 \cdot \Delta v_\varepsilon - \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 - \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&= \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon} \Delta |\nabla v_\varepsilon|^2 - \int_\Omega \frac{1}{v_\varepsilon} |D^2 v_\varepsilon|^2 + \frac{1}{2}\int_\Omega \nabla\left(\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\right) \cdot \nabla v_\varepsilon \\
&\quad - \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 - \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&= \frac{1}{2}\int_{\partial\Omega} \frac{1}{v_\varepsilon} \frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} + \int_\Omega \frac{1}{v_\varepsilon^2} \nabla v_\varepsilon \cdot \nabla |\nabla v_\varepsilon|^2 - \int_\Omega \frac{1}{v_\varepsilon} |D^2 v_\varepsilon|^2 - \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\quad - \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 - \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon,
\end{aligned}$$

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where $\nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon = \frac{1}{2}\Delta|\nabla v_\varepsilon|^2 - |D^2 v_\varepsilon|^2$ is used. Thanks to the identity [25, Lemma 3.2]

$$|D^2 v_\varepsilon|^2 = v_\varepsilon^2 |D^2 \ln v_\varepsilon|^2 + \frac{1}{v_\varepsilon}\nabla v_\varepsilon \cdot \nabla |\nabla v_\varepsilon|^2 - \frac{1}{v_\varepsilon^2}|\nabla v_\varepsilon|^4, \tag{2.20}$$

we then get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega v_\varepsilon \left|D^2 \ln v_\varepsilon\right|^2 + \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 = \frac{1}{2}\int_{\partial\Omega}\frac{1}{v_\varepsilon}\frac{\partial|\nabla v_\varepsilon|^2}{\partial \nu} - \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \tag{2.21}$$

for all $t \in \left(0, T_{\max,\varepsilon}\right)$. From Lemma 3.4 in [25], it follows that there exists $c_1 > 0$ such that

$$\int_\Omega v_\varepsilon \left|D^2 \ln v_\varepsilon\right|^2 \geq c_1 \int_\Omega \frac{\left|D^2 v_\varepsilon\right|^2}{v_\varepsilon} + c_1 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}. \tag{2.22}$$

As an application of Lemma 2.4 with $\eta := c_1, q := 2$, we arrive at

$$\int_{\partial\Omega}\frac{1}{v_\varepsilon}\cdot\frac{\partial\left|\nabla v_\varepsilon\right|^2}{\partial \nu} \leq c_1 \int_\Omega \frac{\left|D^2 v_\varepsilon\right|^2}{v_\varepsilon} + c_1 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + C(c_1, 2)\int_\Omega v_\varepsilon. \tag{2.23}$$

Apart from that, by Young's inequality, we infer that for all $t \in (0, T_{\max,\varepsilon})$

$$\begin{aligned}
-\int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon &\leq \int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{4}\int_\Omega \frac{u_\varepsilon^{\alpha-1}}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\
&\leq \int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{4}\int_\Omega (u_\varepsilon + 1)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}. \\
&\leq \int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{4}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{c_1}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2c_1}\int_\Omega v_\varepsilon.
\end{aligned} \tag{2.24}$$

Summing up (2.21)–(2.24), we conclude that

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{c_1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq 2\int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 + c_2 \int_\Omega v_\varepsilon \tag{2.25}$$

with $c_2 := C(c_1, 2) + \frac{1}{c_1}$ for all $t \in (0, T_{\max,\varepsilon})$. On the other hand, multiplying the first equation in (2.5) by $-\frac{1}{u_\varepsilon}$ and using (2.8) and Young's inequality, we have

$$\begin{aligned}
&-\frac{d}{dt}\int_\Omega \ln u_\varepsilon + \ell \int_\Omega v_\varepsilon + \int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2 \\
&= \chi \int_\Omega u_\varepsilon^{\alpha-2} v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&\leq \frac{1}{2}\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2 + \frac{\chi^2}{2}\int_\Omega u_\varepsilon^{2\alpha-3} v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\leq \frac{1}{2}\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2 + \frac{\chi^2}{2}\int_{\{u_\varepsilon \geq 1\}} u_\varepsilon^{2\alpha-3} v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{\chi^2}{2}\int_{\{u_\varepsilon < 1\}} v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\leq \frac{1}{2}\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2 + \frac{\chi^2 \eta}{2}\int_\Omega u_\varepsilon v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{\chi^2}{2}(\eta^{-1})^{\frac{2\alpha-3}{4-2\alpha}}\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 + \frac{\chi^2}{2}\int_{\{u_\varepsilon<1\}} v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\leq \frac{1}{2}\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2 + \frac{\chi^2 \eta}{2}\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{\chi^2}{2}(\eta^{\frac{3-2\alpha}{4-2\alpha}} + 1)\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2
\end{aligned} \tag{2.26}$$

for all $t \in (0, T_{\max,\varepsilon})$, due to $\alpha \in \left(\frac{3}{2}, 2\right]$. Letting

$$\eta := \frac{\ell}{4\chi^2 c_2 \|v_0\|^2_{L^\infty(\Omega)}} \quad \text{and} \quad C(\alpha) := \frac{\chi^2}{2}\left(\eta^{\frac{3-2\alpha}{4-2\alpha}} + 1\right).$$

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Then, combining (2.25) with (2.26) yields

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\ell}{2c_2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} - \int_\Omega \ln u_\varepsilon\right\} + \frac{\ell}{4c_2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{\ell c_1}{4c_2}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{\ell}{2}\int_\Omega v_\varepsilon + \frac{1}{2}\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2 \\
&\le \frac{\chi^2\eta}{2}\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{\ell}{c_2}\int_\Omega u_\varepsilon^{1-\alpha}v_\varepsilon|\nabla u_\varepsilon|^2 + C(\alpha)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 \\
&= \frac{\ell}{8c_2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{\ell}{c_2}\int_\Omega u_\varepsilon^{1-\alpha}v_\varepsilon|\nabla u_\varepsilon|^2 + C(\alpha)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. On integration in time, according to $\ln \xi \le \xi$ for all $\xi > 0$ entails that

$$\begin{aligned}
&\frac{\ell}{2c_2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \frac{\ell}{8c_2}\int_0^t\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{\ell c_1}{4c_2}\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{\ell}{2}\int_0^t\int_\Omega v_\varepsilon + \frac{1}{2}\int_0^t\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2 \\
&\le -\int_\Omega \ln u_{0\varepsilon} + \int_\Omega \ln u_\varepsilon + \frac{\ell}{2c_2}\int_\Omega \frac{|\nabla v_{0\varepsilon}|^2}{v_{0\varepsilon}} + \frac{\ell}{c_2}\int_0^t\int_\Omega u_\varepsilon^{1-\alpha}v_\varepsilon|\nabla u_\varepsilon|^2 + C(\alpha)\int_0^t\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 \\
&\le -\int_\Omega \ln u_0 + \int_\Omega u_\varepsilon + \frac{\ell}{2c_2}\int_\Omega \frac{|\nabla v_{0\varepsilon}|^2}{v_{0\varepsilon}} + \frac{\ell}{c_2}\int_0^t\int_\Omega u_\varepsilon^{1-\alpha}v_\varepsilon|\nabla u_\varepsilon|^2 + C(\alpha)\int_0^t\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2 \qquad (2.27)
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. Therefore, (2.27) implies (2.17)–(2.19) in view of (1.6), (2.9), (2.11) and (2.13). □

With the help of the following class of interpolation inequalities, the results derived from (2.13) and (2.18) can be used to estimate $\int_0^{T_{\max}}\int_\Omega u_\varepsilon^{\frac{5}{3}}v_\varepsilon$ in the next section.

Lemma 2.6. [26] *Let $p^* \ge 1$ and $p := \frac{2p^*+3}{3}$. Then, for all $M > 0$, one can find $C(M, p^*) > 0$ such that for any $q \in [0, \frac{2p^*}{3}]$ and functions $\phi, \psi \in C^1(\overline{\Omega})$ satisfying $\phi > 0$, $\psi > 0$ in $\overline{\Omega}$ as well as*

$$\int_\Omega \phi^{p^*} \le M,$$

we have

$$\int_\Omega \phi^p\psi \le C(M, p^*)\left\{\int_\Omega \phi^{q-1}\psi|\nabla\phi|^2 + \int_\Omega \frac{\phi}{\psi}|\nabla\psi|^2 + \int_\Omega \phi\psi\right\}. \qquad (2.28)$$

Since the convexity requirement for the domain Ω is not assumed, we establish a refined results stated in Lemma 2.3 of [28].

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $q \ge 2$. Then, for all $t \in (0, T_{max,\epsilon})$, there exist $\Gamma(q) > 0$ and $\gamma(q) > 0$ such that*

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\epsilon|^q}{v_\epsilon^{q-1}} + \Gamma(q)\int_\Omega \frac{|\nabla v_\epsilon|^{q-2}}{v_\epsilon^{q-3}}|D^2\ln v_\epsilon|^2 + \Gamma(q)\int_\Omega u_\epsilon\frac{|\nabla v_\epsilon|^q}{v_\epsilon^{q-1}} \le \frac{1}{\Gamma(q)}\int_\Omega v_\epsilon\left(u_\epsilon^{\frac{q+2}{2}} + 1\right) \qquad (2.29)$$

and

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} + \gamma(q)\int_\Omega \frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2 + \gamma(q)\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} \le \frac{1}{\gamma(q)}\int_\Omega v_\varepsilon\left(|\nabla u_\varepsilon|^{\frac{q+2}{3}} + 1\right). \qquad (2.30)$$

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Proof. According to the second equation in (2.5) and the identity $\nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon = \frac{1}{2}\Delta|\nabla v_\varepsilon|^2 - |D^2 v_\varepsilon|^2$, we can obtain that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}} &= q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon \cdot \{\nabla \Delta v_\varepsilon - \nabla(u_\varepsilon v_\varepsilon)\} - (q-1)\int_\Omega v_\varepsilon^{-q}|\nabla v_\varepsilon|^q \cdot \{\Delta v_\varepsilon - u_\varepsilon v_\varepsilon\} \\
&= q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon - q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon \cdot \nabla(u_\varepsilon v_\varepsilon) \\
&\quad - (q-1)\int_\Omega v_\varepsilon^{-q}|\nabla v_\varepsilon|^q \cdot \Delta v_\varepsilon + (q-1)\int_\Omega u_\varepsilon v_\epsilon^{-q+1}|\nabla v_\varepsilon|^q \\
&= \frac{q}{2}\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\Delta|\nabla v_\varepsilon|^2 - q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|^2 - (q-1)\int_\Omega v_\varepsilon^{-q}|\nabla v_\varepsilon|^q \Delta v_\varepsilon \\
&\quad - q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon \cdot \nabla(u_\varepsilon v_\varepsilon) + (q-1)\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q \\
&= \frac{q}{2}\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\cdot\frac{\partial|\nabla v_\varepsilon|}{\partial \nu} - \frac{q(q-2)}{4}\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-4}|\nabla|\nabla v_\varepsilon|^2|^2 \\
&\quad - q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|^2 + q(q-1)\int_\Omega v_\varepsilon^{-q}|\nabla v_\varepsilon|^{q-2}\nabla|\nabla v_\varepsilon|^2 \cdot \nabla v_\varepsilon \\
&\quad - q(q-1)\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2} - q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon \cdot \nabla(u_\varepsilon v_\epsilon) \\
&\quad + (q-1)\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q
\end{aligned} \tag{2.31}$$

for all $t \in (0, T_{\max,\varepsilon})$. Thanks to (2.16), one can see that for any $\eta > 0$, there exists $c_1 = C(\eta, q) > 0$ such that

$$\int_{\partial\Omega} v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\cdot\frac{\partial|\nabla v_\varepsilon|}{\partial \nu} \le \eta\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|^2 + \eta\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2} + c_1\int_\Omega v_\varepsilon. \tag{2.32}$$

Furthermore, by two well-known inequalities (Lemma 3.4 of [25])

$$\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2} \le (q+\sqrt{3})^2\int_\Omega v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}|D^2 \ln v_\varepsilon|^2 \tag{2.33}$$

and

$$\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|^2 \le (q+\sqrt{3}+1)^2\int_\Omega v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}|D^2 \ln v_\varepsilon|^2, \tag{2.34}$$

we can conclude that

$$\begin{aligned}
&-\frac{q(q-2)}{4}\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-4}|\nabla|\nabla v_\varepsilon|^2|^2 - q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|^2 \\
&+ q(q-1)\int_\Omega v_\varepsilon^{-q}|\nabla v_\varepsilon|^{q-2}\nabla|\nabla v_\varepsilon|^2 \cdot \nabla v_\varepsilon - q(q-1)\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2} \\
&= -\frac{q(q-2)}{4}\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-4}|\nabla|\nabla v_\varepsilon|^2|^2 - q\int_\Omega v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}|D^2 \ln v_\varepsilon|^2 \\
&+ q(q-2)\int_\Omega v_\varepsilon^{-q}|\nabla v_\varepsilon|^{q-2}\nabla|\nabla v_\varepsilon|^2 \cdot \nabla v_\varepsilon - q(q-2)\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2} \\
&\le -q\int_\Omega v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}|D^2 \ln v_\varepsilon|^2 \\
&- \frac{q(q-2)}{4}\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-4}|\nabla|\nabla v_\varepsilon|^2 - \frac{2}{v_\varepsilon}|\nabla v_\varepsilon|^2\nabla v_\varepsilon|^2 \\
&\le -q\int_\Omega v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}|D^2 \ln v_\varepsilon|^2.
\end{aligned} \tag{2.35}$$

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By the Young inequality, there exist $c_2 > 0$ and $c_3 > 0$ such that

$$\begin{aligned}
&-q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon\cdot\nabla(u_\varepsilon v_\varepsilon)+(q-1)\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q\\
&=-q\int_\Omega v_\varepsilon^{-q+2}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon\cdot\nabla u_\varepsilon-\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q\\
&\le q\int_\Omega v_\varepsilon^{-q+2}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon\cdot\nabla u_\varepsilon-\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q\\
&\le c_3\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2}+c_2\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^{\frac{q+2}{3}}-\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q. \qquad (2.36)
\end{aligned}$$

On the other hand, since the identities $\nabla|\nabla v_\varepsilon|^2 = 2D^2 v_\varepsilon\cdot\nabla v_\varepsilon$ and $|\Delta v_\varepsilon| = \sqrt{3}|D^2 v_\varepsilon|$, it follows that for all $t\in(0,T_{\max,\varepsilon})$

$$\begin{aligned}
&-q\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon\cdot\nabla(u_\varepsilon v_\varepsilon)+(q-1)\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q\\
&=q(q-2)\int_\Omega u_\varepsilon v_\varepsilon^{-q+2}|\nabla v_\varepsilon|^{q-4}\nabla v_\varepsilon\cdot\left(D^2 v_\varepsilon\cdot\nabla v_\varepsilon\right)\\
&+q\int_\Omega u_\varepsilon v_\varepsilon^{-q+2}|\nabla v_\varepsilon|^{q-2}\Delta v_\varepsilon-(q-1)^2\int_\Omega u_\varepsilon v_\epsilon^{-q+1}|\nabla v_\varepsilon|^q\\
&\le q(q-2+\sqrt{n})\int_\Omega u_\varepsilon v_\varepsilon^{-q+2}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|-(q-1)^2\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q\\
&\le c_4\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|^2+c_5\int_\Omega u_\varepsilon^2 v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}-(q-1)^2\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q\\
&\le c_4\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|^2+c_6\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2}\\
&+c_7\int_\Omega u_\varepsilon^{\frac{q+2}{2}}v_\varepsilon-(q-1)^2\int_\Omega u_\varepsilon v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q, \qquad (2.37)
\end{aligned}$$

where $c_i > 0, i = 4,\cdots,7$, are constants. Therefore, (2.29) follows from (2.31)–(2.36) and (2.30) results from (2.31)–(2.35) and (2.37). □

By means of the Gagliardo–Nirenberg inequality and the Ehrling lemma, we deduce the following functional inequality, which plays an important role in estimating an integral of the type $\int_\Omega \phi^\beta\psi$ below.

Lemma 2.8. *Let $\Omega\subset\mathbb{R}^3$, and supposed that $p > 0$ and $1 < r < 6$. Then, for any $\eta > 0$, one can find $C(\eta,p) > 0$ such that*

$$\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^r(\Omega)}\le\eta\int_\Omega\phi^{p-1}\psi|\nabla\phi|^2+\eta\int_\Omega\phi^{p+1}\psi^{-1}|\nabla\psi|^2+C(\eta,p)\cdot\left\{\int_\Omega\phi\right\}^p\left\{\int_\Omega\phi\psi\right\} \qquad (2.38)$$

for all $\phi\in C^1(\bar\Omega)$ and $\psi\in C^1(\overline{\Omega})$ with $\phi > 0$ and $\psi > 0$ in $\overline{\Omega}$.

Proof. From the Gagliardo–Nirenberg inequality, it follows that for $1 < r < 6$, there exist constants $a\in(0,1)$ and $c_1 > 0$ such that

$$\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^r(\Omega)}\le c_1\|\nabla(\phi^{\frac{p+1}{2}}\sqrt{\psi})\|^{2a}_{L^2(\Omega)}\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^{2(1-a)}_{L^1(\Omega)}+c_1\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^1(\Omega)}.$$

Furthermore, by Young's inequality, one can see that for all $\eta > 0$ there is $c_2(\eta) > 0$ such that

$$c_1\|\nabla(\phi^{\frac{p+1}{2}}\sqrt{\psi})\|^{2a}_{L^2(\Omega)}\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^{2(1-a)}_{L^1(\Omega)}\le\eta\|\nabla(\phi^{\frac{p+1}{2}}\sqrt{\psi})\|^2_{L^2(\Omega)}+c_2(\eta)\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^1(\Omega)}. \qquad (2.39)$$

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For $p \geq 1$, the compact embedding $L^r \hookrightarrow L^1$ and continuous embedding $L^1 \hookrightarrow L^{\frac{2}{p+1}}$ allow us to apply Ehrling lemma, which yields

$$c_2(\eta)\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^1(\Omega)} \leq \frac{1}{2}\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^r(\Omega)} + c_3(\eta)\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^{\frac{2}{p+1}}(\Omega)}$$

with some $c_3(\eta) > 0$. Therefore, for all $p \geq 1$, we have

$$\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^r(\Omega)} \leq \eta\|\nabla\big(\phi^{\frac{p+1}{2}}\sqrt{\psi}\big)\|^2_{L^2(\Omega)} + c_3(\eta)\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^{\frac{2}{p+1}}(\Omega)}.$$

On the other hand, for $p \in (0, 1)$, one can see that

$$\begin{aligned}\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^r(\Omega)} &\leq \eta\|\nabla(\phi^{\frac{p+1}{2}}\sqrt{\psi})\|^2_{L^2(\Omega)} + c_2(\eta)\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^1(\Omega)}\\ &\leq \eta\|\nabla(\phi^{\frac{p+1}{2}}\sqrt{\psi})\|^2_{L^2(\Omega)} + c_4(\eta)\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^{\frac{2}{p+1}}(\Omega)}.\end{aligned}$$

Hence, from the Hölder inequality, it follows that

$$\begin{aligned}&\eta\|\nabla(\phi^{\frac{p+1}{2}}\sqrt{\psi})\|^2_{L^2(\Omega)} + (c_3(\eta)+c_4(\eta))\|\phi^{\frac{p+1}{2}}\sqrt{\psi}\|^2_{L^{\frac{2}{p+1}}(\Omega)}\\ &\leq \eta\int_\Omega \phi^{p-1}\psi|\nabla\phi|^2 + \eta\int_\Omega \phi^{p+1}\psi^{-1}|\nabla\psi|^2 + C(\eta,p)\cdot\Big\{\int_\Omega\phi\Big\}^p\cdot\Big\{\int_\Omega\phi\psi\Big\}. \qquad (2.40)\end{aligned}$$

□

3. Estimates of $\|u_\epsilon(\cdot,t)\|_{L^p(\Omega)}$

3.1. Space-time estimate of $u_\varepsilon^{p_*-1}v_\varepsilon|\nabla u_\varepsilon|^2$

The purpose of this subsection is to derive the space-time estimate of $u_\varepsilon^{p_*}v_\varepsilon|\nabla u_\varepsilon|^2$ with $p^* \in [1-\alpha, 0]$. This refines the space-time estimate of $u_\varepsilon^{1-\alpha}v_\varepsilon|\nabla u_\varepsilon|^2$ provided in Lemma 2.3 and allows us to obtain the bounds for $\|u_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)}$ with $p_0 > \frac{3}{2}$ in the subsequent subsection. To this end, as outlined in the introduction, we shall make sure that $\mathcal{G}_\varepsilon(t)$ enjoys the feature of the energy-like quantity.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^3$. Then, one can find constants $a > 0$ and $b > 0$ such that for all $t \in (0, T_{max,\varepsilon})$ the functional*

$$\mathcal{G}_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) + a\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^4}{v_\varepsilon^3(\cdot,t)}$$

satisfies

$$\begin{aligned}&\mathcal{G}_\varepsilon'(t) + b\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2\ln v_\varepsilon|^2 + b\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\\ &\leq \chi^2\int_\Omega u_\varepsilon^{2\alpha-2}v_\varepsilon|\nabla v_\varepsilon|^2 + \ell\int_\Omega u_\varepsilon v_\varepsilon\ln u_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon + \frac{1}{4}\int_\Omega v_\varepsilon. \qquad (3.1)\end{aligned}$$

Proof. Multiplying the first equation in (2.5) by $1+\ln u_\varepsilon$, we use the Young inequality and $\ln\xi \leqslant \xi$ for all $\xi > 0$ to infer that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_\varepsilon\ln u_\varepsilon + \int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2\\ &= \chi\int_\Omega u_\varepsilon^{\alpha-1}v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon\ln u_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon\\ &\leq \frac{1}{4}\int_\Omega v_\varepsilon|\nabla u_\varepsilon|^2 + \chi^2\int_\Omega u_\varepsilon^{2\alpha-2}v_\varepsilon|\nabla v_\varepsilon|^2 + \ell\int_\Omega u_\varepsilon v_\varepsilon\ln u_\varepsilon + \ell\int_\Omega u_\varepsilon v_\varepsilon \qquad (3.2)\end{aligned}$$

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for all $t \in (0, T_{max,\varepsilon})$. On the other hand, applying (2.30) to $q := 4$, we have

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \gamma(4)\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \gamma(4)\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le \frac{1}{\gamma(4)}\int_\Omega v_\varepsilon(|\nabla u_\varepsilon|^2 + 1). \tag{3.3}$$

Combining (3.2) with (3.3), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\gamma(4)}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}\right\} \\
&+ \frac{\gamma^2(4)}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \frac{\gamma^2(4)}{4}\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\
&\le \chi^2 \int_\Omega u_\varepsilon^{2\alpha-2} v_\varepsilon |\nabla v_\varepsilon|^2 + \ell \int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon + \ell \int_\Omega u_\varepsilon v_\varepsilon + \frac{1}{4}\int_\Omega v_\varepsilon
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$. This readily leads to (3.1) with $a = \frac{\gamma(4)}{4}$ and $b = \frac{\gamma^2(4)}{4}$. □

By Lemmas 2.3, 2.5 and 2.6, we can obtain the estimate of $\int_0^{T_{\max}} \int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon$, which allows us to address the unfavourable contributions on the right-hand side of (3.1) whenever the exponent $\alpha \in \left(\frac{3}{2}, 2\right]$.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^3$, suppose that $\alpha \in \left(\frac{3}{2}, 2\right]$ and $K > 0$ with the property (1.6) be valid. Then, there exists $C = C(K) > 0$ such that*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon \le C. \tag{3.4}$$

Proof. Since (2.9) and (1.6), there exists $c_1 = c_1(K) > 0$ such that

$$\int_\Omega u_\varepsilon(\cdot, t) \le c_1 \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Due to $\alpha \in (\frac{3}{2}, 2]$, then $q := 2 - \alpha \in [0, \frac{2}{3}]$. As an application of Lemma 2.6 to $p^* := 1$, there exists $c_2 > 0$ such that

$$\int_0^t \int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon \le c_2 \left\{\int_0^t \int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2 + \int_0^t \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \int_0^t \int_\Omega u_\varepsilon v_\varepsilon\right\}$$

for all $t \in (0, T_{max,\varepsilon})$. Hence from (2.10), (2.13) and (2.18), there exists $C(K) > 0$ such that (3.4) is valid. □

With the help of two lemmas above, the quasi-energy feature of $G_\varepsilon(t)$ can be verified and thereby space-time estimates of $u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}$ and $v_\varepsilon |\nabla u_\varepsilon|^2$ are established.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^3$ and $\alpha \in \left(\frac{3}{2}, \frac{5}{3}\right]$, suppose that $K > 0$ with the property (1.6) be valid. Then, there exists $C = C(K) > 0$ such that*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le C \tag{3.5}$$

and

$$\int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \le C. \tag{3.6}$$

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Proof. In light of condition $\alpha \in \left(\frac{3}{2}, \frac{5}{3}\right]$, $4\alpha - 5 \in (0, \frac{5}{3}]$. Therefore, from Lemma 3.1, it follows that for all $t \in (0, T_{max,\varepsilon})$,

$$\begin{aligned}
&\mathcal{G}_\varepsilon'(t) + b\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + b\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\
&\le \chi^2 \int_\Omega u_\varepsilon^{2\alpha-2} v_\varepsilon |\nabla v_\varepsilon|^2 + \ell \int_\Omega u_\varepsilon v_\varepsilon \ln u_\varepsilon + \ell \int_\Omega u_\varepsilon v_\varepsilon + \frac{1}{4}\int_\Omega v_\varepsilon \\
&\le c_1 \int_\Omega u_\varepsilon^{4\alpha-5} v_\varepsilon + \frac{b}{2}\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \ell \int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon + \ell \int_\Omega u_\varepsilon v_\varepsilon + \frac{1}{4}\int_\Omega v_\varepsilon \\
&\le (c_1 + \ell)\int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon + \frac{b}{2}\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \ell \int_\Omega u_\varepsilon v_\varepsilon + (c_1+1)\int_\Omega v_\varepsilon
\end{aligned} \tag{3.7}$$

with $c_1 := \frac{\chi^2 \|v_0\|_{L^\infty}^4}{b}$ and hence

$$\begin{aligned}
&\mathcal{G}_\varepsilon'(t) + b\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 + \frac{b}{2}\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 \\
&\le c_2 \int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon + \ell \int_\Omega u_\varepsilon v_\varepsilon + c_2 \int_\Omega v_\varepsilon
\end{aligned} \tag{3.8}$$

with $c_2 := c_1 + 1 + \ell$. After an integration over time, this together with (3.4), (2.10), (2.17) and (1.6) entails that

$$\begin{aligned}
&\frac{b}{2}\int_0^t\int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_0^t\int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 + b\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}|D^2 \ln v_\varepsilon|^2 \\
&\le c_2 \int_0^t\int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon + \ell \int_0^t\int_\Omega u_\varepsilon v_\varepsilon + c_2 \int_0^t\int_\Omega v_\varepsilon \\
&\quad + \int_\Omega u_{0\varepsilon} \ln u_{0\varepsilon} + a\int_\Omega \frac{|\nabla v_{0\varepsilon}|^4}{v_{0\varepsilon}^3} - \int_\Omega u_\varepsilon \ln u_\varepsilon \\
&\le c_2 \int_0^t\int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon + \ell \int_0^t\int_\Omega u_\varepsilon v_\varepsilon + c_2 \int_0^t\int_\Omega v_\varepsilon \\
&\quad + \int_\Omega (u_0+1)^2 + a\int_\Omega \frac{|\nabla v_0|^4}{v_0^3} + \frac{|\Omega|}{e} \\
&\le C(K)
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$, and thereby completes the proof. □

As a direct consequence of (3.6) and (2.13), we have

Lemma 3.4. *Let $\alpha \in \left(\frac{3}{2}, \frac{5}{3}\right]$ and $K > 0$ with the property (1.6) be valid. Then for any $p_* \in [1-\alpha, 0]$ and $C = C(K) > 0$ such that*

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^{p_*} v_\varepsilon |\nabla u_\varepsilon|^2 \le C. \tag{3.9}$$

Proof. Since $u_\varepsilon^{p_*} \le 1 + u_\varepsilon^{1-\alpha}$ for any $p_* \in [1-\alpha, 0]$, we have

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^{p_*} v_\varepsilon |\nabla u_\varepsilon|^2 \le \int_0^{T_{max,\varepsilon}} \int_\Omega v_\varepsilon |\nabla u_\varepsilon|^2 + \int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^{1-\alpha} v_\varepsilon |\nabla u_\varepsilon|^2,$$

which along with (3.6) and (2.13) entails (3.9) readily. □

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3.2. L^{p_0}-estimates of u_ε with some $p_0 > \frac{3}{2}$

This subsection establishes the L^{p_0}-bounds for u_ε with some $p_0 > \frac{3}{2}$, which serves as the foundation for a bootstrap argument to achieve the L^p bounds of u_ε for arbitrary $p > 1$. As a prerequisite for the derivation of estimate on L^{p_0}-bounds for u_ε, we apply Lemma 2.8 to show that the integral of the form $\int_\Omega \varphi^\beta \psi$ can be dominated by $\int_\Omega \varphi^k \psi |\nabla \varphi|^2$ when added $\int_\Omega \varphi \frac{|\nabla \psi|^4}{\psi^3}$ and term $\int_\Omega \varphi \psi$.

Lemma 3.5. *Let $L > 0$, $k \in (-1, -\frac{1}{3})$ and $\beta \in [1, k + \frac{8}{3})$. There exists $C = C(L, k, \beta) > 0$ such that if $\int_\Omega \varphi \le L$, then*

$$\int_\Omega \varphi^\beta \psi \le \int_\Omega \varphi^k \psi |\nabla \varphi|^2 + \int_\Omega \varphi \frac{|\nabla \psi|^4}{\psi^3} + C \int_\Omega \varphi \psi \tag{3.10}$$

is valid for all $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$ fulfilling $\varphi > 0$ and $\psi > 0$ in $\overline{\Omega}$.

Proof. Let $\theta := \frac{1}{k+3-\beta}$ and $\theta_* := \frac{1}{\beta-k-2}$. Then,

$$1 \le \theta < 3 \tag{3.11}$$

can be warranted by $\beta \in [k+2, k+\frac{8}{3})$. Consequently, by the Hölder inequality and applying Lemma 2.8 to $\eta = \frac{1}{2}$, we conclude that there exists $c_1 = c_1(\beta, k, L)$ such that for all $t \in (0, T_{max,\varepsilon})$,

$$\begin{aligned}
\int_\Omega \varphi^\beta \psi &= \int_\Omega \left(\varphi^{\frac{k+2}{2}} \psi^{\frac{1}{2}}\right)^2 \varphi^{\beta-k-2} \\
&\le \|\varphi^{\frac{k+2}{2}} \psi^{\frac{1}{2}}\|^2_{L^{2\theta}(\Omega)} \cdot \|\varphi^{\beta-k-2}\|_{L^{\theta_*}(\Omega)} \\
&\le L^{\frac{1}{\theta_*}} \|\varphi^{\frac{k+2}{2}} \psi^{\frac{1}{2}}\|^2_{L^{2\theta}(\Omega)} \\
&\le \frac{1}{2} \int_\Omega \varphi^k \psi |\nabla \varphi|^2 + \frac{1}{2} \int_\Omega \varphi^{k+2} \frac{|\nabla \psi|^2}{\psi} + c_1 \int_\Omega \varphi \psi.
\end{aligned} \tag{3.12}$$

Furthermore, let $\lambda := -\frac{1}{k}$ and $\lambda_* := \frac{1}{k+1}$. Then, the restriction $k \in (-1, -\frac{1}{3})$ warrants that $1 < \lambda < 3$. Applying Lemma 2.8 to $p := k+1$, $r := 2\lambda$, $\eta := \min\{L^{-\frac{1}{\lambda_*}}, 1\}$ and Young's inequality once more, we obtain that there exists positive constant $c_2 = c_2(k, \beta, L)$ such that for all $t \in (0, T_{max,\varepsilon})$,

$$\begin{aligned}
\int_\Omega \varphi^{k+2} \frac{|\nabla \psi|^2}{\psi} &\le \frac{1}{2} \int_\Omega \varphi \frac{|\nabla \psi|^4}{\psi^3} + \frac{1}{2} \int_\Omega \varphi^{2k+3} \psi \\
&= \frac{1}{2} \int_\Omega \varphi \frac{|\nabla \psi|^4}{\psi^3} + \frac{1}{2} \int_\Omega \left(\varphi^{\frac{k+2}{2}} \psi^{\frac{1}{2}}\right)^2 \varphi^{k+1} \\
&\le \frac{1}{2} \int_\Omega \varphi \frac{|\nabla \psi|^4}{\psi^3} + \frac{1}{2} \|\varphi^{\frac{k+2}{2}} \psi^{\frac{1}{2}}\|^2_{L^{2\lambda}(\Omega)} \|\varphi^{k+1}\|_{L^{\lambda_*}(\Omega)} \\
&\le \frac{1}{2} \int_\Omega \varphi \frac{|\nabla \psi|^4}{\psi^3} + \frac{L^{\frac{1}{\lambda_*}}}{2} \|\varphi^{\frac{k+2}{2}} \psi^{\frac{1}{2}}\|^2_{L^{2\lambda}(\Omega)} \\
&\le \frac{1}{2} \int_\Omega \varphi \frac{|\nabla \psi|^4}{\psi^3} + \frac{1}{2} \int_\Omega \varphi^{k+2} \frac{|\nabla \psi|^2}{\psi} + \frac{1}{2} \int_\Omega \varphi^k \psi |\nabla \varphi|^2 + c_2 \int_\Omega \varphi \psi.
\end{aligned} \tag{3.13}$$

Therefore, if $\beta \in [k+2, k+\frac{8}{3})$, it follows from (3.12) and (3.13) that

$$\int_\Omega \varphi^\beta \psi \le \eta \int_\Omega \varphi^k \psi |\nabla \varphi|^2 + \int_\Omega \varphi \frac{|\nabla \psi|^4}{\psi^3} + C(\beta, k, L) \int_\Omega \varphi \psi \tag{3.14}$$

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for all $t \in (0, T_{max,\varepsilon})$. Apart from that, it is observed that for $\beta \in [1, k+2)$,

$$\int_\Omega \varphi^\beta \psi \le \int_\Omega \varphi^{k+2}\psi + \int_\Omega \varphi\psi \quad \text{for all } t \in (0, T_{max,\varepsilon}).$$

This together with (3.14) implies (3.10) readily. □

The combination of Lemmas 3.3, 3.4 and 3.5 yields the result of follows.

Lemma 3.6. *Assume $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ and $K > 0$ be parameter such that property (1.6) is valid. Then there exist $p_0 > \frac{3}{2}$ and $C = C(K) > 0$ such that*

$$\int_\Omega u_\varepsilon^{p_0}(\cdot, t) \le C \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.15}$$

Proof. According to (2.9), there exists $c_1 = c_1(K) > 0$ such that

$$\int_\Omega u_\varepsilon(\cdot, t) \le c_1 \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Let $\delta := \min\{\frac{1}{2}(\alpha - \frac{4}{3}), \frac{19}{12} - \alpha, \frac{1}{2}\} > 0$ and $p_0 := \frac{3}{2} + \frac{\delta}{2}$. Then, multiplying the first equation in (2.5) by $u_\varepsilon^{p_0-1}$ and by the Young inequality, we infer that

$$\begin{aligned}
&\frac{1}{p_0}\frac{d}{dt}\int_\Omega u_\varepsilon^{p_0} + \frac{p_0-1}{2}\int_\Omega u_\varepsilon^{p_0-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\
&\le \frac{(1-p_0)\chi^2}{2}\int_\Omega u_\varepsilon^{p_0+2\alpha-3} v_\varepsilon |\nabla v_\varepsilon|^2 + \ell \int_\Omega u_\varepsilon^{p_0} v_\varepsilon \\
&\le \frac{(1-p_0)\chi^2 \|v_0\|_{L^\infty}^4}{2}\int_\Omega u_\varepsilon^{2p_0+4\alpha-7} v_\varepsilon + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \ell \int_\Omega u_\varepsilon^{\frac{5}{3}} v_\varepsilon + \ell \int_\Omega u_\varepsilon v_\varepsilon.
\end{aligned} \tag{3.16}$$

Let $k := -\frac{1}{3} - \delta$. Then, $k \in (-1, -\frac{1}{3})$ due to the fact $\delta < \frac{2}{3}$. In addition, by the definition of δ, we can see that $k \in (1-\alpha, 0)$. Therefore, $k \in (1-\alpha, 0) \bigcap (-1, -\frac{1}{3})$, and thus

$$\int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^k v_\varepsilon |\nabla u_\varepsilon|^2 \le c_1(K, T_{max,\varepsilon}) \tag{3.17}$$

for some $c_1(K, T_{max,\varepsilon}) > 0$ by Lemma 3.4.

On other hand, according to the definitions of δ and p_0, we have $\delta < \frac{19}{12} - \alpha$ and thereby $2p_0 + 4\alpha - 7 < \frac{7}{3} - \delta = k + \frac{8}{3}$. Apart from that, it is claimed that

$$2p_0 + 4\alpha - 7 > 1. \tag{3.18}$$

Indeed, according to the definition of p_0, it is easy to see that $2p_0 + 4\alpha - 8 = \delta + 4\alpha - 5 > 0$ due to $\alpha > \frac{3}{2}$.

Therefore, applying Lemma 3.5 to $\beta := 2p_0 + 4\alpha - 7$, we can conclude that there exists $c_2 > 0$ such that

$$\frac{(1-p_0)\chi^2 \|v_0\|_{L^\infty}^4}{2}\int_\Omega u_\varepsilon^{2p_0+4\alpha-7} v_\varepsilon \le \int_\Omega u_\varepsilon^k v_\varepsilon |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_2 \int_\Omega u_\varepsilon v_\varepsilon \tag{3.19}$$

for $t \in (0, T_{max,\varepsilon})$. Combining (3.19), (3.16) with (3.17) and integrating the results over time, together with (3.5), (1.6), (2.17) and (2.10), we have

$$\begin{aligned}
&\frac{1}{p_0}\int_\Omega u_\varepsilon^{p_0} + \frac{p_0-1}{2}\int_0^t \int_\Omega u_\varepsilon^{p_0-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\
&\le \int_0^t \int_\Omega u_\varepsilon^k v_\varepsilon |\nabla u_\varepsilon|^2 + 2\int_0^t \int_\Omega u_\varepsilon \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_2 \int_0^t \int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega (u_0+1)^{p_0} + \int_0^t \int_\Omega v_\varepsilon \\
&\le C(K, T_{max,\varepsilon})
\end{aligned} \tag{3.20}$$

for all $t \in (0, T_{max,\varepsilon})$, and thus complete the proof. □

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3.3. L^p-estimates of u_ε for all $p > 1$

The aim of this subsection is to achieve L^p-bounds of u_ε for all $p > 1$. Of crucial importance to our approach in this direction is to refine (3.10) in Lemma 3.5, which involves the term $\int_\Omega \varphi \frac{|\nabla\psi|^q}{\psi^{q-1}}$ instead of $\int_\Omega \varphi \frac{|\nabla\psi|^4}{\psi^3}$, when the enhanced integrability properties of φ are taken into account.

Lemma 3.7. *For $L > 0, p_0 > \frac{3}{2}$, $p > 1$, $q > 2 + \frac{3p}{p_0}$, $\beta \in [1, \frac{2p_0}{3} + p + 1)$ and any $\eta \in (0,1)$, there exists $C = C(\eta, L, p, \beta) > 0$ such that whenever $\int_\Omega \varphi^{p_0} \le L$,*

$$\int_\Omega \varphi^\beta \psi \le \eta \int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \eta \int_\Omega \varphi \frac{|\nabla\psi|^q}{\psi^{q-1}} + C\int_\Omega \varphi\psi \tag{3.21}$$

holds for all $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$ fulfilling $\varphi > 0$ and $\psi > 0$ in $\overline{\Omega}$.

Proof. It is observed that the restriction $q > 2 + \frac{3p}{p_0}$ ensures that $\frac{qp+q-2}{q-2} < \frac{2p_0}{3} + p + 1$. Now we first verify (3.21) in the case

$$\beta \in \left[\frac{qp+q-2}{q-2}, \frac{2p_0}{3} + p + 1\right). \tag{3.22}$$

To this end, let

$$\rho := \frac{p_0}{p_0 - \beta + p + 1} \quad \text{and} \quad \rho_* := \frac{p_0}{\beta - p - 1}.$$

Due to (3.22), it follows that

$$\rho \ge 1 \quad \text{and} \quad 1 < 2\rho < 6. \tag{3.23}$$

Hence, thanks to $\int_\Omega \varphi^{p_0} \le L$ for $p_0 > \frac{3}{2}$, the Hölder inequality, the Young inequality and Lemma 2.8, we deduce that for any $\eta > 0$, there exists positive constant $c_1 = c_1(\eta, p, \beta, L)$ such that

$$\begin{aligned}
\int_\Omega \varphi^\beta\psi &= \int_\Omega (\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}})^2 \varphi^{\beta-p-1} \\
&\le \|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^2_{L^{2\rho}(\Omega)} \cdot \|\varphi^{\beta-p-1}\|_{L^{\rho_*}(\Omega)} \\
&= \|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^2_{L^{2\rho}(\Omega)} \cdot \|\varphi\|^{\beta-p-1}_{L^{p_0}(\Omega)} \\
&\le L^{\frac{\beta-p-1}{p_0}} \|\varphi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\|^2_{L^{2\rho}(\Omega)} \\
&\le \frac{\eta}{2}\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \frac{\eta}{2}\int_\Omega \varphi^{p+1}\frac{|\nabla\psi|^2}{\psi} + c_1\int_\Omega \varphi\psi \\
&\le \frac{\eta}{2}\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \frac{\eta}{2}\int_\Omega \varphi\frac{|\nabla\psi|^q}{\psi^{q-1}} + \frac{1}{2}\int_\Omega \varphi^{\frac{qp+q-2}{q-2}}\psi + c_1\int_\Omega \varphi\psi \\
&\le \frac{\eta}{2}\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \frac{\eta}{2}\int_\Omega \varphi\frac{|\nabla\psi|^q}{\psi^{q-1}} + \frac{1}{2}\int_\Omega \varphi^\beta\psi + (c_1+1)\int_\Omega \varphi\psi.
\end{aligned}$$

and thus

$$\int_\Omega \varphi^\beta\psi \le \eta\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + \eta\int_\Omega \varphi\frac{|\nabla\psi|^q}{\psi^{q-1}} + 2(c_1+1)\int_\Omega \varphi\psi. \tag{3.24}$$

Furthermore, it is observed that for $\beta \in [1, \frac{qp+q-2}{q-2})$, we have

$$\int_\Omega \varphi^\beta\psi \le \int_\Omega \varphi^{\frac{qp+q-2}{q-2}}\psi + \int_\Omega \varphi\psi,$$

which along with (3.24) complete the proof readily. □

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We are now in the position to achieve L^p-estimates for u_ε for any $p>1$ through tracing the evolution of $\mathcal{H}(t):=\int_\Omega u_\varepsilon^p+\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}$, thanks to the L^{p_0}-estimates of u_ε with some $p_0>\frac{3}{2}$ established in Lemma 3.6.

Lemma 3.8. *Let $\Omega\subset\mathbb{R}^3$ and $K>0$ with the property that (1.6) holds. Supposed that $\alpha\in\left(\frac{3}{2},\frac{19}{12}\right)$ and $p>1$, there exists $C=C(K,p)>0$ such that for all $t\in(0,T_{max,\varepsilon})$,*

$$\int_\Omega u_\varepsilon^p\le C \quad \text{and} \quad \int_0^{T_{max,\varepsilon}}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2\le C \tag{3.25}$$

as well as

$$\int_0^{T_{max,\varepsilon}}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon\le C. \tag{3.26}$$

Proof. From (2.29) in Lemma 2.7, it follows that

$$\frac{d}{dt}\int_\Omega\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}+\Gamma(q)\int_\Omega\frac{|\nabla v_\varepsilon|^{q-2}}{v_\varepsilon^{q-3}}|D^2\ln v_\varepsilon|^2+\Gamma(q)\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}\le\frac{1}{\Gamma(q)}\int_\Omega v_\varepsilon\left(u_\varepsilon^{\frac{q+2}{2}}+1\right) \tag{3.27}$$

for all $t\in(0,T_{max,\varepsilon})$.

Multiplying the first equation in (2.5) by u_ε^{p-1}, applying Young's inequality and (1.6), there exists $c_1>0$ such that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p+\frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \\
&\le\frac{p-1}{2}\int_\Omega u_\varepsilon^{p+2\alpha-3}v_\varepsilon|\nabla v_\epsilon|^2+\ell\int_\Omega u_\varepsilon^p v_\varepsilon \\
&\le(p-1)\int_\Omega u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2+(p-1)\int_\Omega v_\varepsilon|\nabla v_\epsilon|^2+\ell\int_\Omega u_\varepsilon^{p+1}v_\varepsilon+\ell\int_\Omega u_\varepsilon v_\varepsilon \\
&\le\frac{\Gamma(q)}{2}\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}+(p-1)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2+\ell\int_\Omega u_\varepsilon^{p+1}v_\varepsilon+c_1\int_\Omega u_\varepsilon^{\frac{qp+q-2}{q-2}}v_\varepsilon^{\frac{3q-2}{q-2}}+\ell\int_\Omega u_\varepsilon v_\varepsilon \\
&\le\frac{\Gamma(q)}{2}\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}+(p-1)\int_\Omega v_\varepsilon|\nabla v_\varepsilon|^2+\ell\int_\Omega u_\varepsilon^{p+1}v_\varepsilon+c_1K^{\frac{2q}{q-2}}\int_\Omega u_\varepsilon^{\frac{qp+q-2}{q-2}}v_\varepsilon+\ell\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned} \tag{3.28}$$

for all $t\in(0,T_{max,\varepsilon})$.

Noting that if $p_0>\frac{3}{2}$, then for any fixed $p>1$, one can find $q>2$ satisfying

$$q\in\left(2+\frac{3p}{p_0},\frac{4p_0}{3}+2p\right) \tag{3.29}$$

which ensures that

$$\frac{qp+q-2}{q-2}<\frac{2p_0}{3}+p+1 \tag{3.30}$$

and

$$\frac{q+2}{2}<\frac{2p_0}{3}+p+1. \tag{3.31}$$

Therefore, thanks to (3.15), (3.30) and (3.31), the application of Lemma 3.7 to $\eta:=\min\{\frac{\Gamma(q)}{4},\frac{p-1}{4}\}$ yields

$$\begin{aligned}
&\ell\int_\Omega u_\varepsilon^{p+1}v_\varepsilon+c_1K^{\frac{2q}{q-2}}\int_\Omega u_\varepsilon^{\frac{qp+q-2}{q-2}}v_\varepsilon+\frac{1}{\Gamma(q)}\int_\Omega u_\varepsilon^{\frac{q+2}{2}}v_\varepsilon \\
&\le\eta\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2+\eta\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}+C(\eta,K,p)\int_\Omega u_\varepsilon v_\varepsilon
\end{aligned} \tag{3.32}$$

for all $t\in(0,T_{max,\varepsilon})$.

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From (3.27), (3.28) and (3.32), there exists $C(p,K)>0$ such that

$$\frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u_\varepsilon^p+\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}\right\}+\frac{\Gamma(q)}{4}\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}+\frac{p-1}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2$$
$$\le C(p,K)\int_\Omega\left(v_\varepsilon|\nabla v_\varepsilon|^2+u_\varepsilon v_\varepsilon+v_\varepsilon\right). \tag{3.33}$$

Integrating (3.33) with respect to time, we have

$$\int_\Omega u_\varepsilon^p+\int_\Omega \frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}+\frac{\Gamma(q)}{4}\int_0^t\int_\Omega u_\varepsilon\frac{|\nabla v_\varepsilon|^q}{v_\varepsilon^{q-1}}+\frac{p-1}{4}\int_0^t\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2$$
$$\le C(p,K)\int_0^t\int_\Omega\left(v_\varepsilon|\nabla v_\varepsilon|^2+u_\varepsilon v_\varepsilon+v_\varepsilon\right)+\int_\Omega(u_0+1)^p+\int_\Omega\frac{|\nabla v_0|^q}{v_0^{q-1}}$$

for all $t\in(0,T_{max,\varepsilon})$. Hence, from (2.11), (2.17), (1.6) and (2.10), it follows that

$$\int_\Omega u_\varepsilon^p+\frac{p-1}{2}\int_0^t\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2\le C(p,K). \tag{3.34}$$

Furthermore, (3.26) results from (3.32), (3.34) and (2.10), and hence completes the proof. □

As a natural consequence of (3.25), the application of standard smoothing estimates for the Neumann heat semigroup directly yields lower bounds for v_ε.

Lemma 3.9. *Let $\Omega\subset\mathbb{R}^3$, $\alpha\in(\frac{3}{2},\frac{19}{12})$ and $K>0$ with the property that (1.6) holds. Assuming that $T_{max,\varepsilon}<\infty$, then there exists $C=C(K,T_{max,\varepsilon})>0$ such that*

$$v_\varepsilon(x,t)\ge C\qquad \textit{for all } (x,t)\in\Omega\times(0,T_{max,\varepsilon}). \tag{3.35}$$

Proof. First, it is observed from Lemma 2.1 that $v_\varepsilon\in C^{2,1}(\overline{\Omega}\times(0,T_{\max,\varepsilon}))$ with $v_\varepsilon>0$ in $\overline{\Omega}\times(0,T_{\max,\varepsilon})$. Defining the transformed variable $w_\varepsilon:=\ln\frac{\|v_0\|_{L^\infty(\Omega)}}{v_\varepsilon}\in C^{2,1}(\overline{\Omega}\times[0,T_{\max,\varepsilon}))$ and invoking (2.5), we derive

$$w_{\varepsilon t}=\Delta w_\varepsilon-|\nabla w_\varepsilon|^2+u_\varepsilon\le\Delta w_\varepsilon+u_\varepsilon\quad\text{in }\Omega\times(0,T_{\max,\varepsilon}).$$

Using the comparison principle along with known regularisation features of the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ on Ω [30], one can see that there is constant $c_1>0$ such that for all $t\in(0,T_{\max,\varepsilon})$,

$$w_\varepsilon(\cdot,t)\le e^{t\Delta}w_{0\varepsilon}+\int_0^t e^{(t-s)\Delta}u_\varepsilon(\cdot,s)\,ds$$
$$\le\sup_{x\in\Omega}w_{0\varepsilon}(x)+c_1\int_0^t(1+(t-s)^{-\frac{3}{4}})\|u_\varepsilon(\cdot,s)\|_{L^2(\Omega)}\,ds$$
$$\le\sup_{x\in\Omega}w_{0\varepsilon}(x)+c_1(T_{\max,\varepsilon}+4T_{\max,\varepsilon}^{\frac{1}{4}})\cdot\sup_{s\in(0,T_{\max,\varepsilon})}\|u_\varepsilon(\cdot,s)\|_{L^2(\Omega)}.$$

This along with (3.25) implies that $w_\varepsilon(\cdot,t)\le c_2(K,T_{\max,\varepsilon})$ for some $c_2(K,T_{\max,\varepsilon})$ and thus $v_\varepsilon(x,t)\ge\|v_0\|_{L^\infty(\Omega)}e^{-c_2(K,T_{\max,\varepsilon})}$. □

3.4. Proof of Theorem 1.1

On the basis of the result stated by Lemma 3.8, we further derive the higher regularity properties of the solution components to (2.5) and then obtain a global continuous weak solution of (1.4).

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As a consequence of (3.25), by leveraging standard heat semigroup estimates, one can verify that $\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ is locally bounded for $t \in (0, T_{max,\varepsilon})$.

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^3$ and $K > 0$ with the property that (1.6) holds. Supposed that $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$, there exists $C = C(K,p) > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.36}$$

Proof. According to Lemma 3.8, we have the uniform boundedness of u_ε in $L^\infty((0,T_{max,\varepsilon}); L^p(\Omega))$ for any $p > 1$. Therefore, by invoking the well-established smoothing properties of the Neumann heat semigroup documented in [30], there exist $c_1 > 0$ and $c_2 > 0$ such that for any $p > 3$

$$\begin{aligned}
\|\nabla v_\varepsilon(t)\|_{L^\infty(\Omega)} &= \left\| \nabla e^{t(\Delta-1)} v_0 - \int_0^t \nabla e^{(t-s)(\Delta-1)} \left\{ u_\varepsilon(s) v_\varepsilon(s) - v_\varepsilon(s) \right\} ds \right\|_{L^\infty(\Omega)} \\
&\le c_1 \|v_0\|_{W^{1,\infty}(\Omega)} + c_1 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\right) e^{-(t-s)} \|u_\varepsilon(\cdot,s) v_\varepsilon(\cdot,s) - v_\varepsilon(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le c_1 \|v_0\|_{W^{1,\infty}(\Omega)} + + c_1 |\Omega|^{\frac{1}{p}} \|v_0\|_{L^\infty(\Omega)} \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\right) e^{-(t-s)} ds \\
&\quad + c_1 \|v_0\|_{L^\infty(\Omega)} \sup_{t \in (0, T_{max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\right) e^{-(t-s)} ds \\
&\le c_2
\end{aligned} \tag{3.37}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Then, we finish the proof. □

At this position, we can proceed to prove the L^∞-bounds for u_ε.

Lemma 3.11. *Let $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$. Suppose $T_{max,\varepsilon} < \infty$, then there exists $C(T_{max,\varepsilon}) > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C(T_{max,\varepsilon}) \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.38}$$

Proof. We begin by reformulating the governing equation for u_ε as

$$u_{\varepsilon t} = \nabla \cdot (A_\varepsilon(x,t,u_\varepsilon) \nabla u_\varepsilon) + \nabla \cdot B_\varepsilon(x,t) + D_\varepsilon(x,t) \qquad (x,t) \in \Omega \times (0, T_{max,\varepsilon}),$$

where the nonlinear operators are defined by

$$A_\varepsilon(x,t,\xi) := v_\varepsilon(x,t)\xi \qquad (x,t,\xi) \in \Omega \times (0, T_{max,\varepsilon}) \times [0,\infty),$$

and

$$B_\varepsilon(x,t) := -\chi u_\varepsilon^\alpha(x,t) v_\varepsilon(x,t) \nabla v_\varepsilon(x,t) \qquad (x,t) \in \Omega \times (0, T_{max,\varepsilon}),$$

as well as

$$D_\varepsilon(x,t) := \ell u_\varepsilon(x,t) v_\varepsilon(x,t) \qquad (x,t) \in \Omega \times (0, T_{max,\varepsilon}).$$

Through applications of Lemmas 3.8, 3.9 and 3.10, we derive the following estimates for any $p > 1$

$$A_\varepsilon(x,t) \ge c_1(T_{max,\varepsilon})\xi \qquad \text{for all } (x,t,\xi) \in \Omega \times (0, T_{max,\varepsilon}) \times [0,\infty),$$

and

$$\|B_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le c_2 \quad \text{and} \quad \|D_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le c_2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

with some $c_1 > 0, c_2 > 0$. Thereafter (3.38) can be derived from a Moser-type iteration, as recorded in ([21], Lemma A.1). □

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As an application of (2.7) and Lemma 3.11, one can show that the solutions to (2.5) are in fact global in time.

Lemma 3.12. *Under the assumptions of Theorem 1.1, we have* $T_{max,\varepsilon} = \infty$ *for all* $\varepsilon \in (0,1)$.

Drawing on the a priori estimates from the previous lemmas, we now deduce the Hölder regularity for the global solution to (2.5) by applying standard parabolic regularity theory.

Lemma 3.13. *For all* $T > 0$ *and* $\varepsilon \in (0,1)$, *there exist* $\theta_1 = \theta_1(T) \in (0,1)$ *and* $C(T) > 0$ *such that*

$$\|u_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T])} \le C(T) \tag{3.39}$$

and

$$\|v_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T])} \le C(T). \tag{3.40}$$

Moreover, for all $\tau > 0$ *and* $T > \tau$, *there exist* $\theta_2 = \theta_2(\tau, T) \in (0,1)$ *and* $C(\tau, T) > 0$ *such that*

$$\|v_\varepsilon\|_{C^{2+\theta_2,1+\frac{\theta_2}{2}}(\overline{\Omega}\times[\tau,T])} \le C(\tau,T) \qquad \text{for all } \varepsilon \in (0,1). \tag{3.41}$$

Proof. The Hölder estimates (3.39) and (3.40) are achieved through the combination of Lemmas 3.9–3.11, (2.8) and an application of the parabolic Hölder regularity framework established in [18, Theorem 1.3; Remark 1.4]. The subsequent estimate (3.41) then follows as a direct consequence of classical Schauder theory for scalar parabolic equations [11, Chapter IV], when coupled with the established Hölder continuity from (3.39). □

Upon the above estimates, we can construct a continuous weak solutions to (1.4) through a standard extraction procedure.

Lemma 3.14. *Let* $\alpha \in (\frac{3}{2}, \frac{19}{12})$ *and suppose that* (u_0, v_0) *satisfies (1.5) and (1.6). Then there exist* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ *and functions* u *and* v *fulfilling (1.7), as well as* $u \ge 0$ *and* $v > 0$ *in* $\overline{\Omega} \times (0,\infty)$, *such that*

$$u_\varepsilon \to u \qquad in \quad C^0_{loc}(\overline{\Omega} \times [0,\infty)) \tag{3.42}$$

$$v_\varepsilon \to v \qquad in \quad C^0_{loc}(\overline{\Omega} \times [0,\infty)) \quad and \quad in \quad C^{2,1}_{loc}(\overline{\Omega} \times (0,\infty)) \qquad and \tag{3.43}$$

$$\nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla v \qquad in \quad L^\infty(\Omega \times (0,\infty)) \tag{3.44}$$

as $\varepsilon = \varepsilon_j \searrow 0$ *and that* (u,v) *forms a continuous global weak solutions of (1.4) in the sense of Definition 2.1.*

Proof. By virtue of Lemmas 3.8, and 3.9 and Young's inequality, there exists $C = C(K,T) > 0$ such that for all $T > 0$,

$$\int_0^T \int_\Omega |\nabla u_\varepsilon^2| \le \int_0^T \int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon v_\varepsilon^{-1} \le C,$$

thereby establishing the boundedness of the gradient sequence $(\nabla u_\varepsilon^2)_{\varepsilon\in(0,1)}$ in $L^1((0,T); W^{1,1}(\Omega))$. Subsequently, employing Lemmas 3.10, and 3.13 and the Arzelá-Ascoli compactness theorem, a straightforward extraction procedure allows us to construct a vanishing subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$ with $\varepsilon_j \searrow 0$. This yields nonnegative functions (u,v) satisfying (1.7), (2.2) and (3.42)–(3.44). Finally, (2.3) and (2.4) can be accomplished in the respective weak formulation associated with (2.5) on the basis of these convergence properties. □

Proof of Theorem 1.1. The statements has fully been covered by Lemma 3.14. □

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4. Large time behaviour

This section is devoted to establishing the stability properties and nontrivial stabilisation results stated in Theorems 1.2 and 1.3, respectively. The proofs rely crucially on the temporal decay of $u_{\varepsilon t}$ in suitable dual spaces, which is established via the integral estimates (2.10), (2.11) and (2.19).

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ and $K > 0$ with the property that (1.6) holds. Supposed that $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$, there exist $\sigma > 0$ and $C = C(K) > 0$ such that for all $\varepsilon \in (0,1)$, we have*

$$\int_0^\infty \|u_{\varepsilon t}(\cdot,t)\|_{(W^{1,\infty}(\Omega))^*} dt \le C \cdot \left\{\int_\Omega v_0\right\}^\sigma . \tag{4.1}$$

Proof. Using the first equation in (2.5), one can see that for all $t > 0$ and any $\psi \in W^{1,\infty}(\Omega)$ such that $\|\psi\|_{W^{1,\infty}(\Omega)} \equiv \max\{\|\psi\|_{L^\infty(\Omega)}, \|\nabla\psi\|_{L^\infty(\Omega)}\} \le 1$,

$$\begin{aligned}\left|\int_\Omega u_{\varepsilon t}\psi\right| &= \left|-\int_\Omega u_\varepsilon v_\varepsilon \nabla u_\varepsilon \cdot \nabla\psi + \chi\int_\Omega u_\varepsilon^\alpha v_\varepsilon \nabla v_\varepsilon \cdot \nabla\psi + \ell\int_\Omega u_\varepsilon v_\varepsilon\right| \\ &\le \int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon| + \chi\int_\Omega u_\varepsilon^\alpha v_\varepsilon |\nabla v_\varepsilon| + \ell\int_\Omega u_\varepsilon v_\varepsilon,\end{aligned}$$

so that

$$\|u_{\varepsilon t}(\cdot,t)\|_{(W^{1,\infty}(\Omega))^*} \le \int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon| + \chi\int_\Omega u_\varepsilon^\alpha v_\varepsilon |\nabla v_\varepsilon| + \ell\int_\Omega u_\varepsilon v_\varepsilon. \tag{4.2}$$

Thanks to the Höder inequality and (2.10), for all $T > 0$, we infer that

$$\begin{aligned}\int_0^T\int_\Omega u_\varepsilon v_\varepsilon |\nabla u_\varepsilon| &\le \left\{\int_0^T\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2\right\}^{\frac12} \cdot \left\{\int_0^T\int_\Omega u_\varepsilon^3 v_\varepsilon\right\}^{\frac12} \\ &\le \left\{\int_0^T\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2\right\}^{\frac12} \cdot \left\{\int_0^T\int_\Omega u_\varepsilon^{\frac{3-r}{1-r}} v_\varepsilon\right\}^{\frac{1-r}{2}} \cdot \left\{\int_0^T\int_\Omega u_\varepsilon v_\varepsilon\right\}^{\frac{r}{2}} \\ &\le \left\{\int_0^T\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}|\nabla u_\varepsilon|^2\right\}^{\frac12} \cdot \left\{\int_0^T\int_\Omega u_\varepsilon^{\frac{3-r}{1-r}} v_\varepsilon\right\}^{\frac{1-r}{2}} \cdot \left\{\int_\Omega v_0\right\}^{\frac{r}{2}}\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\int_0^T\int_\Omega u_\varepsilon^\alpha v_\varepsilon |\nabla v_\varepsilon| &\le \left\{\int_0^T\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2\right\}^{\frac12} \cdot \left\{\int_0^T\int_\Omega u_\varepsilon^{2\alpha} v_\varepsilon\right\}^{\frac12} \\ &\le \left\{\int_0^T\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2\right\}^{\frac12} \cdot \left\{\int_0^T\int_\Omega u_\varepsilon^{\frac{2\alpha-r}{1-r}} v_\varepsilon\right\}^{\frac{1-r}{2}} \cdot \left\{\int_0^T\int_\Omega u_\varepsilon v_\varepsilon\right\}^{\frac{r}{2}} \\ &\le \left\{\int_0^T\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2\right\}^{\frac12} \cdot \left\{\int_0^T\int_\Omega u_\varepsilon^{\frac{2\alpha-r}{1-r}} v_\varepsilon\right\}^{\frac{1-r}{2}} \cdot \left\{\int_\Omega v_0\right\}^{\frac{r}{2}}\end{aligned} \tag{4.4}$$

as well as

$$\int_0^T\int_\Omega u_\varepsilon v_\varepsilon \le \int_\Omega v_0 \le \left\{\int_\Omega v_0\right\}^{\frac{r}{2}} \cdot \left(\|v_0\|_{L^\infty(\Omega)} \cdot |\Omega|\right)^{\frac{1-r}{2}}, \tag{4.5}$$

where $r \in (0,1)$ and $\min\{\frac{2\alpha-r}{1-r}, \frac{3-r}{1-r}\} > 2$. It follows from (4.2)–(4.5) that there exists $C(K) > 0$ such that

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|_{(W^{1,\infty}(\Omega))^*} dt \le C(K) \cdot \left\{\int_\Omega v_0\right\}^{\frac{r}{2}}$$

due to (1.6), (2.11), (2.19) and (3.26), and thereby we arrive at (4.1) with $\sigma := \frac{r}{2}$. □

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Making use of Lemma 4.1, we can derive the following consequence that exclusively involves the zero-order expression of u.

Lemma 4.2. *Let $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ and $K > 0$ with the property that (1.6) holds. Given $\sigma > 0$ and $C = C(K) > 0$ as in Lemma 4.1, then for any nondecreasing $(t_k)_{k\in\mathbb{N}} \subset [0,\infty)$, we have*

$$\sum_{k=1}^{\infty} \|u(\cdot,t_{k+1}) - u(\cdot,t_k)\|_{(W^{1,\infty}(\Omega))^*} dt \le C \cdot \left\{\int_\Omega v_0\right\}^\sigma, \tag{4.6}$$

where we set $u(\cdot,0) := u_0$.

Proof. Fixing any such nondecreasing sequence $(t_k)_{k\in\mathbb{N}}$, we infer from Lemma 4.1 that

$$\begin{aligned}\sum_{k\in\mathbb{N}} \|u_\varepsilon(\cdot,t_{k+1}) - u_\varepsilon(\cdot,t_k)\|_{(W^{1,\infty}(\Omega))^*} &= \sum_{k\in\mathbb{N}} \left\| \int_{t_k}^{t_{k+1}} u_{\varepsilon t}(\cdot,t)\,dt \right\|_{(W^{1,\infty}(\Omega))^*} \\ &\le \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} \|u_\varepsilon(\cdot,t)\|_{(W^{1,\infty}(\Omega))^*}\,dt \\ &\le C(K) \cdot \left\{\int_\Omega v_0\right\}^\sigma,\end{aligned}$$

because $(t_k, t_{k+1}) \cap (t_l, t_{l+1}) = \emptyset$ for $k \in \mathbb{N}$ and $l \in \mathbb{N}$ with $k \neq l$. This along with (3.42) implies (4.6) immediately. □

The quantitative dependence on v_0 not only establishes large-time stabilisation of individual trajectories in their first component but also quantifies the proximity between the limiting profile and initial data.

Lemma 4.3. *Let $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ and $K > 0$ with the property that (1.6) holds. Then, the function u obtained in Lemma 4.2 exhibits the convergence*

$$u(\cdot,t) \to u_\infty \qquad \text{in } (W^{1,\infty}(\Omega))^* \qquad \text{as } t \to \infty \tag{4.7}$$

with some $u_\infty \in (W^{1,\infty}(\Omega))^*$ that satisfies

$$\|u_\infty - u_0\|_{(W^{1,\infty}(\Omega))^*} \le C(K) \cdot \left\{\int_\Omega v_0\right\}^\sigma \tag{4.8}$$

with $C(K) > 0$ as in Lemma 4.1.

Proof. Lemma 4.2 implies that $\{u(\cdot,t_k)\}_{k\in\mathbb{N}}$ forms a Cauchy sequence in $(W^{1,\infty}(\Omega))^*$, which establishes (4.7) with some $u_\infty \in (W^{1,\infty}(\Omega))^*$. Now select the sequence $(t_k)_{k\in\mathbb{N}}$ with $t_1 := 0$ and $t_k := t$ for $k \ge 2$ in (4.6), yielding

$$\|u(\cdot,t) - u_0\|_{(W^{1,\infty}(\Omega))^*} \le C(K) \cdot \left\{\int_\Omega v_0\right\}^\sigma \qquad \text{for all } t > 0, \tag{4.9}$$

and thereby derives (4.8) due to (4.7). □

The quantitative forms of the right-hand side in (4.8) and (4.9) allow us to derive the following stability property of function pairs $(u_0, 0)$.

Lemma 4.4. *Let $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ and $K > 0$ with the property that (1.6) holds. Then for each $\eta > 0$, there exists $\delta_1 = \delta_1(K,\eta) > 0$ whenever u_0 and v_0 fulfill (1.5), as well as*

$$\int_\Omega v_0 \le \delta_1, \tag{4.10}$$

the solution (u,v) of (1.4) obtained in Theorem 1.1 satisfies

$$\|u(\cdot,t) - u_0\|_{(W^{1,\infty}(\Omega))^*} \le \eta \qquad \text{for all } t > 0. \tag{4.11}$$

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Moreover, the corresponding limit function from Lemma 4.3 admits

$$\|u_\infty - u_0\|_{(W^{1,\infty}(\Omega))^*} \le \eta. \tag{4.12}$$

Proof. From (4.8) and (4.9), it follows that for any $\eta > 0$, there exists $\delta_1 = \delta_1(K, \eta) > 0$ such that (4.10) warrants (4.11) and (4.12). □

Proof of Theorem 1.2. The claimed result has precisely been asserted by Lemma 4.4. □

The following lemma is used to obtain the decay estimate for v.

Lemma 4.5. [23] *Let $y \in C^1([0,\infty))$ and $h \in L^1_{loc}([0,\infty))$ both be nonnegative, and suppose that*

$$\int_t^{t-1} h(s)ds \to 0 \qquad as\ t \to \infty,$$

and that there exists $\lambda > 0$ such that

$$y'(t) + \lambda y(t) \le h(t) \qquad for\ all\ t > 0.$$

Then,

$$y(t) \to 0 \qquad as\ t \to \infty.$$

The estimates established in (2.17) and (2.19) imply the following asymptotic decay behaviour for the second solution component.

Lemma 4.6. *Let $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ and $K > 0$ with the property that (1.6) holds. Then, we have*

$$v(\cdot,t) \to 0 \qquad in\ W^{1,p}(\Omega)\ for\ all\ p \ge 1 \qquad as\ t \to \infty. \tag{4.13}$$

Proof. Thanks to (2.10), (2.17), (3.42) and (3.43), the application of Fatou's lemma then yields

$$\int_0^\infty \int_\Omega v < \infty \quad \text{and} \quad \int_0^\infty \int_\Omega uv < \infty.$$

Consequently,

$$\int_{t-1}^t \int_\Omega v \to 0 \quad \text{and} \quad \int_{t-1}^t \int_\Omega uv \to 0 \qquad as\ t \to \infty. \tag{4.14}$$

Testing the second equation in (1.4) by $-\Delta v$ and applying Young's inequality together with Hölder's inequality, one can find $c_1 > 0$ such that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^2 &= -2\int_\Omega |\Delta v|^2 + 2\int_\Omega uv\Delta v \\
&\le -\int_\Omega |\Delta v|^2 + \int_\Omega u^2 v^2 \\
&\le -\int_\Omega |\Delta v|^2 + \|v_0\|_{L^\infty(\Omega)} \left(\int_\Omega u^3\right)^{\frac{1}{2}} \left(\int_\Omega uv\right)^{\frac{1}{2}} \\
&\le -\int_\Omega |\Delta v|^2 + c_1 \left(\int_\Omega uv\right)^{\frac{1}{2}}
\end{aligned} \tag{4.15}$$

for all $t > 0$, due to (2.8) and (3.25). Apart from that, integration by parts followed by Young's inequality yields

$$\int_\Omega |\nabla v|^2 = \int_\Omega v\Delta v \le \int_\Omega |\Delta v|^2 + \frac{1}{4}\int_\Omega v^2. \tag{4.16}$$

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Combining (2.8), (4.15) with (4.16), we get

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla v|^2 \le c_1\left(\int_\Omega uv\right)^{\frac{1}{2}} + \frac{\|v_0\|_{L^\infty(\Omega)}}{4}\int_\Omega v. \tag{4.17}$$

Define

$$y(t) := \int_\Omega |\nabla v|^2 \qquad \text{and} \qquad g(t) := c_1\left(\int_\Omega uv\right)^{\frac{1}{2}} + \frac{\|v_0\|_{L^\infty(\Omega)}}{4}\int_\Omega v.$$

Then it follows from (4.17) that

$$y'(t) + y(t) \le g(t).$$

Furthermore, by Hölder's inequality and (4.14), we arrive at

$$\int_{t-1}^t g(t) \le c_1\left(\int_{t-1}^t\int_\Omega uv\right)^{\frac{1}{2}} + \frac{\|v_0\|_{L^\infty(\Omega)}}{4}\int_{t-1}^t\int_\Omega v \to 0 \qquad as \quad t\to\infty. \tag{4.18}$$

As an application of Lemma 4.5, we infer that

$$\int_\Omega |\nabla v|^2 \to 0 \qquad as\, t\to\infty. \tag{4.19}$$

According to the interpolation inequality, (4.19) and (3.36), we obtain for any $p > 2$

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}^{\frac{p-2}{p}}\|\nabla v(\cdot,t)\|_{L^2(\Omega)}^{\frac{2}{p}} \le C\|\nabla v(\cdot,t)\|_{L^2(\Omega)}^{\frac{2}{p}} \to 0 \quad \text{as} \quad t\to\infty. \tag{4.20}$$

Therefore (4.13) follows from (4.20). □

Beyond the stabilisation result established in Theorem 1.2 for solutions to (1.4), we further demonstrate that the limiting profile u_∞ of the solution component u obtained in (4.7) becomes non-homogeneous when the initial signal concentration v_0 is sufficiently small, provided that u_0 is not identically constant.

Lemma 4.7. *Let $\alpha \in \left(\frac{3}{2}, \frac{19}{12}\right)$ and $K > 0$ be such that (1.6) holds and suppose $u_0 \not\equiv$ const. Then there exists $\delta_2 = \delta_2(K, u_0) > 0$ such that if u_0 and v_0 satisfy (1.5) and (1.6), as well as*

$$\int_\Omega v_0 < \delta_2, \tag{4.21}$$

the corresponding limit function $u_\infty \in (W^{1,\infty}(\Omega))^*$ from Lemma 4.3 satisfies $u_\infty \not\equiv$ const.

Proof. Since u_0 is continuous and not constant, we can fix numbers $c_1 > 0$, $c_2 > c_1$, as well as open sets $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$, such that $u_0 \le c_1$ in Ω_1 and $u_0 \ge c_2$ in Ω_2. Furthermore, let functions $0 \le \psi_i \in C_0^\infty(\Omega)$, $i \in \{1,2\}$, such that supp $\psi_i \subset \Omega_i$ and $\|\psi_i\|_{W^{1,\infty}(\Omega)} \le 1$. Choose $\delta_2 = \delta_2(K, u_0) > 0$ sufficiently small so that

$$C(K)\cdot\delta_2^\sigma < \kappa\cdot\min\left\{\int_{\Omega_1}\psi_1, \int_{\Omega_2}\psi_2\right\} \tag{4.22}$$

with $\kappa := \frac{c_2-c_1}{4}$, $C(K) > 0$ and $\sigma > 0$ given as in Lemma 4.1.

Now if $u_\infty \notin L^\infty(\Omega)$, then u_∞ is non-constant because any constant function is bounded, and the proof is thus complete. Therefore, we may assume $u_\infty \in L^\infty(\Omega)$. At this position, we claim that

$$\operatorname{ess\,sup}_\Omega u_\infty \ge c_2 - \kappa \quad \text{and} \quad \operatorname{ess\,inf}_\Omega u_\infty \le c_1 + \kappa. \tag{4.23}$$

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To verify this, suppose that ess $\sup_\Omega u_\infty < c_2 - \kappa$, the properties of ψ_2 imply

$$\begin{aligned} \|u_\infty - u_0\|_{(W^{1,\infty}(\Omega))^*} &\geq \left| \int_\Omega (u_\infty - u_0) \cdot \psi_2 \right| \\ &= \int_{\Omega_2} (u_0 - u_\infty) \cdot \psi_2 \\ &\geq \int_{\Omega_2} \{c_2 + (\kappa - c_2)\} \cdot \psi_2 \\ &= \kappa \int_{\Omega_2} \psi_2, \end{aligned} \tag{4.24}$$

On the other hand, Lemma 4.3 and (4.22) yield

$$\|u_\infty - u_0\|_{(W^{1,\infty}(\Omega))^*} \leq C(K) \cdot \left\{ \int_\Omega v_0 \right\}^\sigma < \kappa \cdot \min \left\{ \int_{\Omega_1} \psi_1, \int_{\Omega_2} \psi_2 \right\}, \tag{4.25}$$

which contradicts (4.24). The second inequality of (4.23) follows analogously, and thereby u_∞ cannot coincide with any constant due to our choice of κ. □

Proof of Theorem 1.3. The non-triviality of the limiting profile u_∞ in Theorem 1.3, along with the properties stated in (1.10), follows directly from Lemmas 4.3–4.7, which completes the proof of the theorem. □

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