BUCKLING OF CYLINDRICAL SHELLS WITH ARBITRARY CIRCUMFERENTIAL THICKNESS VARIATIONS UNDER EXTERNAL PRESSURE

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ABSTRACT

This paper presents an analytical study on the buckling of cylindrical shells with arbitrary circumferential thickness variations under external pressure. Firstly, based on the thin shell theory and separation of variables, corresponding ordinary differential equations of laterally pressured cylinders are obtained. Secondly, the general asymptotic formula of buckling load, which is in terms of thickness variation parameter up to arbitrary order, is derived by combining the perturbation method and Fourier series expansion. Thirdly, the effects of uniform and circumferential modal thickness variations on the buckling of cylindrical shells under external pressure are investigated, respectively, and the results agree well with those available in literature. Particularly, the buckling load reduction of cylinders with circumferential modal thickness variation obtained by the proposed method coincides with the numerical results presented by Gusic *et al*. This analytical method is applicable for evaluating the stability of laterally pressured cylindrical shells with arbitrary circumferential thickness variations, once the thickness variation is known.

Keywords: Buckling, Cylindrical shell, Circumferential thickness variation, External pressure.

1. INTRODUCTION

There has been a vast literature [1-9] devoted to the study of stability of cylindrical thin shells with constant thickness in the past decades. Nevertheless, cylindrical shells in actual engineering may exist a great variety of thickness imperfections such as linear, local and so on, due to the fabrication procedure (composite shells, rolled steel sheets) or the corrosion in aggressive environments (off-shore structures, pressure vessels in chemical industry, submarines). It is rather significant to study the effect of thickness variation on the stability of these thin shells.

Koiter *et al*. [10] pioneered this area on axially compressed shells with axisymmetric thickness variation using a hybrid perturbation-Galerkin method to solve linearized governing partial differential equations. In their work, the thickness variation is assumed to be axisymmetric, periodic, and sufficiently small compared to the shell thickness. The results indicated that the most detrimental thickness variation mode $2p$ could reduce buckling load remarkably, where p is the buckling mode in axial direction. Based on this method, Li *et al*. [11] and Nguyen *et al*. [12] conducted further researches on the axial buckling of cylindrical shells with axisymmetric thickness variations. Yet, the hybrid perturbation-Galerkin method is limited and only applicable for simple thickness variation, because the equation satisfying the boundary conditions of compatibility must be solved exactly. Recently, Chen *et al*. [13] extended previous works [10-12] and presented a general investigation on the buckling of axially compressed cylindrical shells, by combining the separation of variables and perturbation method. In their work, three different axisymmetric thickness variations were considered and the results were compared with previous studies [10, 12]. Yang *et al*. [14] also studied similar buckling problems due to external pressure by the same method. Besides, Aghajari *et al*. [15] undertook numerical and experimental investigations into the buckling and post- buckling behavior of thin-walled cylindrical steel shells with simple stepwise variable thickness subjected to uniform external pressure. Chen *et al*. [16] studied the same topic with both anchored and unanchored base boundaries.

However, the afore-mentioned studies emphasized on the effects of axisymmetric thickness variations only.

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There have been few studies investigating the effect of circumferential thickness variation on the buckling of cylinders. Following a finite element bifurcation analysis, Gusic *et al*. [17] studied the influence of harmonic thickness variation in circumferential direction on the buckling of thin cylindrical shells under external pressure. Combesure and Gusic [18] investigated, later, the associated effects of both circumferential harmonic thickness variation and geometry imperfection by adopting a new quasi-axisymmetric shell element. Up to now, however, no analytical solution is available for elastic buckling of cylindrical shells with arbitrary circumferential thickness variations.

The objective of this study is to present a general analytical method for the buckling analysis of laterally pressured cylinders with arbitrary circumferential thickness variations. This analysis is based on a system of linearized governing partial differential equations of perfect shells with variable thickness. The asymptotic formula of buckling load, which is in terms of thickness variation parameter up to arbitrary order, is derived by the combination of separation of variables, perturbation method and Fourier series expansion. In order to validate the proposed theory, the influences of uniform and circumferential modal thickness variations on the buckling are investigated, respectively. The results are compared with those available in previous studies [12, 14, 17, 18]. This study is an extension of former investigations, and the obtained analytical solution can be used to evaluate the elastic buckling of cylindrical shells with arbitrary circumferential thickness variations in actual engineering, once the thickness variation is known.

2. BASIC EQUATIONS

Consider a cylindrical shell with small thickness variation subjected to uniform external pressure q as shown in Fig. 1. The radius of cylindrical shell is R, and the length is L. The shell thickness t which varies arbitrarily in the y-direction (circumferential direction) is assumed to be

$$t(y)=t_0+\eta t_1(y)+\eta^2 t_2(y)+\cdots=\sum_{n=0}^{\infty}\eta^n t_n(y) \tag{1}$$

where t_0 is the nominal thickness of cylindrical shell, $\eta(0\le\eta<1)$ is the non-dimensional thickness variation parameter, $t_n(y)\,(n\ge 1)$ represent unknown thickness functions.

The linear strain-displacement relations of cylindrical shells can be expressed as

$$\varepsilon_x=\frac{\partial U}{\partial x},\ \varepsilon_y=\frac{\partial V}{\partial y}-\frac{W}{R},\ \gamma_{xy}=\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y} \tag{2}$$

where U, V, W are displacements of the mid-plane along axial, circumferential and radial directions respectively, ε_x, ε_y, γ_{xy} are strain components.

By taking the second derivative of Eq. (2) and combining the resulting expressions, we obtain the compatibility equation

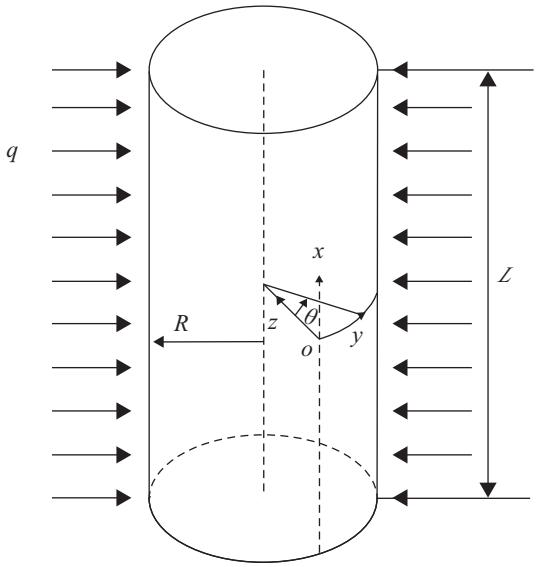


Fig. 1 Geometry of the laterally pressured cylindrical shell.

$$\frac{\partial^2\varepsilon_x}{\partial y^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}+\frac{\partial^2\varepsilon_y}{\partial x^2}=-\frac{1}{R}\frac{\partial^2 W}{\partial x^2} \tag{3}$$

Based on the thin shell theory, the equilibrium equations of a cylindrical shell under uniform external pressure are as follows

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0 \tag{4}$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0 \tag{5}$$

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+\frac{N_y}{R}-qR\frac{\partial^2 W}{\partial y^2}=0 \tag{6}$$

where N_x, N_y, N_{xy} are force components, M_x, M_y, M_{xy} are moment components.

Since the shell thickness varies along the circumferential direction, we express the relations of strain components and membrane forces increments N_x, N_y, N_{xy} by the known formulas representing the Hooke law [12]

$$\varepsilon_x=\frac{N_x-\nu N_y}{Et(y)},\quad \varepsilon_y=\frac{N_y-\nu N_x}{Et(y)},\quad \gamma_{xy}=\frac{2(1+\nu)N_{xy}}{Et(y)} \tag{7}$$

And the components of moment can be expressed by

$$\begin{aligned} M_x&=-D(y)\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)\\ M_y&=-D(y)\left(\frac{\partial^2 W}{\partial y^2}+\frac{W}{R^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)\\ M_{xy}&=-D(y)(1-\nu)\frac{\partial^2 W}{\partial x\partial y}\end{aligned} \tag{8}$$

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where $D(y)=\dfrac{Et^3(y)}{12\left(1-v^2\right)}$, v is the Poisson ratio, and E is the modulus of elasticity.

The solution of equilibrium equations is greatly simplified by introducing the Airy stress function F defined as

$$N_x=\frac{\partial^2 F}{\partial y^2},\quad N_y=\frac{\partial^2 F}{\partial x^2},\quad N_{xy}=-\frac{\partial^2 F}{\partial x\partial y} \tag{9}$$

Substituting Eq. (7) into Eq. (3) and taking into account Eq. (9) yields

$$t^2\nabla^2\nabla^2 F-2t\frac{\partial t}{\partial y}\left(\frac{\partial^3 F}{\partial y^3}+\frac{\partial^3 F}{\partial x^2\partial y}\right)+\left(2\frac{\partial t}{\partial y}\frac{\partial t}{\partial y}-t\frac{\partial^2 t}{\partial y^2}\right)\left(\frac{\partial^2 F}{\partial y^2}-v\frac{\partial^2 F}{\partial x^2}\right)=-\frac{Et^3}{R}\frac{\partial^2 W}{\partial x^2} \tag{10}$$

with $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$, $\nabla^2\nabla^2=\dfrac{\partial^4}{\partial x^4}+2\dfrac{\partial^4}{\partial x^2\partial y^2}+\dfrac{\partial^4}{\partial y^4}$.

Substituting Eqs. (8) and (9) into Eq. (6), we obtain

$$\begin{aligned}&\frac{Et^3}{12\left(1-v^2\right)}\left(\nabla^2\nabla^2 W+\frac{1}{R^2}\frac{\partial^2 W}{\partial y^2}\right)+\frac{6Et^2}{12\left(1-v^2\right)}\frac{\partial t}{\partial y}\times\left(\frac{\partial}{\partial y}\nabla^2 W+\frac{1}{R^2}\frac{\partial W}{\partial y}\right)\\&+\frac{6Et}{12\left(1-v^2\right)}\frac{\partial t}{\partial y}\frac{\partial t}{\partial y}\left(\frac{\partial^2 W}{\partial y^2}+\frac{W}{R^2}+v\frac{\partial^2 W}{\partial x^2}\right)+\frac{3Et^2}{12\left(1-v^2\right)}\frac{\partial^2 t}{\partial y^2}\left(\frac{\partial^2 W}{\partial y^2}+v\frac{\partial^2 W}{\partial x^2}+\frac{W}{R^2}\right)-\frac{1}{R}\frac{\partial^2 F}{\partial x^2}+qR\frac{\partial^2 W}{\partial y^2}=0\end{aligned} \tag{11}$$

Eqs. (10) and (11) are the compatibility equation and equilibrium equation, respectively.

To make the resulting solutions more general, we transform Eqs. (10) and (11) by introducing the following non-dimensional parameters

$$\xi=\frac{x}{L},\ \theta=\frac{y}{R},\ w=\frac{W}{t_0},\ f=\frac{F}{D_0},\ T=\frac{t}{t_0} \tag{12}$$

where θ (θ changes from 0 to 2π) is the angle coordinate shown in Fig. 1, D_0 is the flexural rigidity with nominal shell thickness t_0.

Correspondingly, the governing Eqs. (10) and (11) can be rewritten in the non-dimensional forms

$$\begin{aligned}&T^2\left(\frac{\partial^4 f}{\partial\xi^4}+2\frac{L^2}{R^2}\frac{\partial^4 f}{\partial\xi^2\partial\theta^2}+\frac{L^4}{R^4}\frac{\partial^4 f}{\partial\theta^4}\right)-2\frac{L^2}{R^2}T\frac{\partial T}{\partial\theta}\times\frac{\partial}{\partial\theta}\left(\frac{L^2}{R^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial\xi^2}\right)+\frac{L^2}{R^2}\left(2\frac{\partial T}{\partial\theta}\frac{\partial T}{\partial\theta}-T\frac{\partial^2 T}{\partial\theta^2}\right)\\&\times\left(\frac{L^2}{R^2}\frac{\partial^2 f}{\partial\theta^2}-v\frac{\partial^2 f}{\partial\xi^2}\right)=-\frac{12\left(1-v^2\right)L^2}{Rt_0}T^3\frac{\partial^2 w}{\partial\xi^2}\end{aligned} \tag{13}$$

$$\begin{aligned}&T^3\left(\frac{\partial^4 w}{\partial\xi^4}+2\frac{L^2}{R^2}\frac{\partial^4 w}{\partial\xi^2\partial\theta^2}+\frac{L^4}{R^4}\frac{\partial^4 w}{\partial\theta^4}+\frac{L^4}{R^4}\frac{\partial^2 w}{\partial\theta^2}\right)+\frac{qRL^2}{D_0}\frac{L^2}{R^2}\frac{\partial^2 w}{\partial\theta^2}-\frac{L^2}{Rt_0}\frac{\partial^2 f}{\partial\xi^2}\\&+6\frac{L^2}{R^2}T^2\frac{\partial T}{\partial\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{L^2}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right)+\frac{L^2}{R^2}\frac{\partial w}{\partial\theta}\right]+6\frac{L^2}{R^2}T\frac{\partial T}{\partial\theta}\frac{\partial T}{\partial\theta}\times\left(\frac{L^2}{R^2}\frac{\partial^2 w}{\partial\theta^2}+v\frac{\partial^2 w}{\partial\xi^2}+\frac{L^2}{R^2}w\right)\\&+3\frac{L^2}{R^2}T^2\frac{\partial^2 T}{\partial\theta^2}\times\left(\frac{L^2}{R^2}\frac{\partial^2 w}{\partial\theta^2}+v\frac{\partial^2 w}{\partial\xi^2}+\frac{L^2}{R^2}w\right)=0\end{aligned} \tag{14}$$

The thickness function $t(y)$ can also be transformed into the non-dimensional form

$$T(\theta)=1+\eta T_1(\theta)+\eta^2 T_2(\theta)+\ldots\ldots=1+\sum_{n=1}^{\infty}\eta^n T_n(\theta) \tag{15}$$

If the cylindrical shell is simply-supported at its both ends, the boundary conditions will be

$$\begin{aligned}&f\left(\pm\frac{1}{2},\theta\right)=0,\quad f''\left(\pm\frac{1}{2},\theta\right)=0\\&w\left(\pm\frac{1}{2},\theta\right)=0,\quad w''\left(\pm\frac{1}{2},\theta\right)=0\end{aligned} \tag{16}$$

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Then, in the manner of separation of variables, we seek solutions of Eqs. (13) and (14) in the following forms which satisfy Eq. (16) automatically

$$\begin{aligned} f(\xi,\theta) &= \bar{f}(\theta)\cos(2p+1)\pi\xi = \bar{f}(\theta)\cos\alpha_p\xi \\ w(\xi,\theta) &= \bar{w}(\theta)\cos(2p+1)\pi\xi = \bar{w}(\theta)\cos\alpha_p\xi \end{aligned} \tag{17}$$

where $\alpha_p = (2p+1)\pi$ and p denotes the number of axial buckling waves.

Substituting Eq. (17) into Eqs. (13) and (14), the corresponding ordinary differential equations are given as follows

$$\begin{aligned} &\frac{L^4}{R^4}T^2\bar{f}^{(4)} - 2\frac{L^4}{R^4}T\frac{\partial T}{\partial\theta}\bar{f}''' + \left[-2\alpha_p^2\frac{L^2}{R^2}T^2 + \frac{L^4}{R^4}\times\left(2\frac{\partial T}{\partial\theta}\frac{\partial T}{\partial\theta} - T\frac{\partial^2 T}{\partial\theta^2}\right)\right]\bar{f}'' + 2\alpha_p^2\frac{L^2}{R^2}T\frac{\partial T}{\partial\theta}\bar{f}' \\ &+\left[\alpha_p^4T^2 + \nu\alpha_p^2\frac{L^2}{R^2}\left(2\frac{\partial T}{\partial\theta}\frac{\partial T}{\partial\theta} - T\frac{\partial^2 T}{\partial\theta^2}\right)\right]\bar{f} = \frac{12\left(1-\nu^2\right)L^2}{Rt_0}\alpha_p^2T^3\bar{w} \end{aligned} \tag{18}$$

$$\begin{aligned} &\frac{L^4}{R^4}T^3\bar{w}^{(4)} + 6\frac{L^4}{R^4}T^2\frac{\partial T}{\partial\theta}\bar{w}''' + 6\frac{L^2}{R^2}\left(\frac{L^2}{R^2} - \alpha_p^2\right)T^2\frac{\partial T}{\partial\theta}\bar{w}' + \left[\left(\frac{L^4}{R^4} - 2\alpha_p^2\frac{L^2}{R^2}\right)T^3 + 6\frac{L^4}{R^4}T\frac{\partial T}{\partial\theta}\frac{\partial T}{\partial\theta} + 3\frac{L^4}{R^4}T^2\frac{\partial^2 T}{\partial\theta^2}\right. \\ &\left.+\frac{qRL^2}{D_0}\frac{L^2}{R^2}\right]\bar{w}'' + \left[\alpha_p^4T^3 + 3\frac{L^2}{R^2}\left(\frac{L^2}{R^2} - \nu\alpha_p^2\right)T\times\left(2\frac{\partial T}{\partial\theta}\frac{\partial T}{\partial\theta} + T\frac{\partial^2 T}{\partial\theta^2}\right)\right]\bar{w} + \frac{L^2}{Rt_0}\alpha_p^2\bar{f} = 0 \end{aligned} \tag{19}$$

3. ASYMPTOTIC SOLUTIONS

In this part, the solutions of Eqs. (18) and (19) will be presented, and then the generalized expressions of buckling load q, dimensionless radial displacement $\bar{w}(\theta)$ and stress function $\bar{f}(\theta)$ will be derived.

Assume q, $\bar{f}(\theta)$ and $\bar{w}(\theta)$ take the following forms

$$q = q_0 + \eta q_1 + \eta^2 q_2 + \cdots = q_0 + \sum_{n=1}^{\infty}\eta^n q_n \tag{20}$$

$$\begin{aligned} \bar{f}(\theta) &= \bar{f}_0(\theta) + \eta\bar{f}_1(\theta) + \eta^2\bar{f}_2(\theta) + \cdots \\ &= \bar{f}_0(\theta) + \sum_{n=1}^{\infty}\eta^n\bar{f}_n(\theta) \end{aligned} \tag{21}$$

$$\begin{aligned} \bar{w}(\theta) &= \bar{w}_0(\theta) + \eta\bar{w}_1(\theta) + \eta^2\bar{w}_2(\theta) + \cdots \\ &= \bar{w}_0(\theta) + \sum_{n=1}^{\infty}\eta^n\bar{w}_n(\theta) \end{aligned} \tag{22}$$

where q_0, $q_n\ (n\ge 1)$ are constants; q_0, $\bar{f}_0(\theta)$ and $\bar{w}_0(\theta)$ represents the solutions of the cylindrical shell with constant thickness t_0; q_n, $\bar{f}_n(\theta)$, $\bar{w}_n(\theta)$ denotes the solutions which consider the thickness variations.

Substituting Eqs. (15), (20), (21) and (22) into Eqs. (18) and (19), and collecting the like terms in η, we obtain

$$\Phi_1(\bar{f}_0) = 4c^2r^2\alpha_p^2\bar{w}_0 \tag{23}$$

$$\Phi_2(\bar{w}_0) = -r^2\alpha_p^2\bar{f}_0 \tag{24}$$

$$\Phi_1(\bar{f}_n) = 4c^2r^2\alpha_p^2\bar{w}_n + H_n \tag{25}$$

$$\Phi_2(\bar{w}_n) = -r^2\alpha_p^2\bar{f}_n - \frac{q_nRL^2}{D_0}N^2\bar{w}_0'' + Q_n \tag{26}$$

Where the operator $\Phi_1(.)$ and $\Phi_2(.)$ are defined as

$$\Phi_1(f) = N^4f^{(4)} - 2\alpha_p^2N^2f'' + \alpha_p^4f \tag{27}$$

$$\Phi_2(w) = N^4w^{(4)} - \left(2\alpha_p^2N^2 - N^4 - \frac{q_0RL^2}{D_0}N^2\right)w'' + \alpha_p^4w \tag{28}$$

where $c = \sqrt{3\left(1-\nu^2\right)}$, $r = L/\sqrt{Rt_0}$, $N = L/R$. $H_n\ (n\ge 1)$ is an expression containing $\bar{w}_{g1}$, $\bar{f}_{g1}^{(4)}$, $\bar{f}_{g1}'''$, $\bar{f}_{g1}''$, $\bar{f}_{g1}'$, $\bar{f}_{g1}\ (g1 = 0,1,2,\ldots,n-1)$, T_{l1}, T_{l1}', T_{l1}'' $(l1 = 1,2,\ldots n)$; $Q_n\ (n\ge 1)$ is an expression containing $\bar{w}_{g2}^{(4)}$, $\bar{w}_{g2}'''$, $\bar{w}_{g2}''$, $\bar{w}_{g2}'$, $\bar{w}_{g2}\ (g2 = 0,1,2,\ldots,n-1)$, T_{l2}, T_{l2}', $T_{l2}''\ (g2 = 0,1,2,\ldots,n-1)$ and q_{g3} $(g3 = 1,2,\ldots,n-1)$.

Particularly, the terms H_1, Q_1 are given as follows

$$\begin{aligned} H_1 = {} & 12c^2r^2\alpha_p^2T_1\bar{w}_0 - 2N^4T_1\bar{f}_0^{(4)} + 2N^4T_1'\bar{f}_0''' + \left(4\alpha_p^2N^2T_1\right. \\ & \left.+N^4T_1''\right)\bar{f}_0'' - 2\alpha_p^2N^2T_1'\bar{f}_0' - \left(2\alpha_p^4T_1 - \nu\alpha_p^2N^2T_1''\right)\bar{f}_0 \end{aligned} \tag{29}$$

$$\begin{aligned} Q_1 = {} & -3N^4T_1\bar{w}_0^{(4)} - 6N^4T_1'\bar{w}_0''' + \left[\left(6\alpha_p^2N^2 - 3N^4\right)T_1 - 3N^4T_1''\right]\bar{w}_0'' \\ & -6N^2\left(N^2 - \alpha_p^2\right)T_1'\bar{w}_0' - \left[3\alpha_p^4T_1 + 3N^2\left(N^2 - \nu\alpha_p^2\right)T_1''\right]\bar{w}_0 \end{aligned} \tag{30}$$

Eqs. (23) and (24) are the basic equations for cylin-

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drical shells with constant thickness under uniform external pressure, which have been investigated in detail. Here, we assume $\bar{w}_0(\theta)$ and $\bar{f}_0(\theta)$ take the form of

$$\begin{aligned}\bar{w}_0(\theta) &= A_{0m}\cos m\theta \\ \bar{f}_0(\theta) &= B_{0m}\cos m\theta\end{aligned} \tag{31}$$

where $A_{0m} \neq 0$ and $B_{0m} \neq 0$ are undetermined constants, m denotes the circumferential buckling mode.

Substituting Eq. (31) into Eqs. (23) and (24) yields

$$B_{0m} = \frac{4c^2r^2\alpha_p^2}{\left(N^2m^2+\alpha_p^2\right)^2}A_{0m} \tag{32}$$

$$\frac{q_0RL^2}{D_0} = \frac{\left(N^2m^2+\alpha_p^2\right)^2}{N^2m^2} + \frac{4c^2r^4\alpha_p^4}{\left(N^2m^2+\alpha_p^2\right)^2N^2m^2} - N^2 \tag{33}$$

It should be pointed out that, the shell buckles when $p = 0$ and this fact is confirmed by experiments. Substituting it into Eq. (33) yields

$$q_0 = D_0R\left(\frac{m}{R^2}+\frac{\pi^2}{mL^2}\right)^2 - \frac{D_0}{R^3} + \frac{\pi^4Et_0R^3}{L^4m^6\left(1+\frac{\pi^2R^2}{m^2L^2}\right)^2} \tag{34}$$

Particularly, for a cylindrical shell with moderate length, whose Poisson ratio v is 0.3, Eq. (34) reaches minimum value with

$$m = m_{cr} = \sqrt[4]{6\pi^2\sqrt{1-v^2}}\sqrt{\frac{R}{L}}\sqrt[4]{\frac{R}{t_0}} \approx 2.7415\sqrt{\frac{R}{L}}\sqrt[4]{\frac{R}{t_0}} \tag{35}$$

Eq. (35) is called the Windenburg equation [19].

Then we seek solutions of Eqs. (25) and (26). Similarly, $\bar{w}_n(\theta)$ and $\bar{f}_n(\theta)(n \geq 1)$ are assumed to be

$$\begin{aligned}\bar{w}_n(\theta) &= \sum_{k=0}^{\infty}A_{nk}\cos k\theta \\ \bar{f}_n(\theta) &= \sum_{k=0}^{\infty}B_{nk}\cos k\theta\end{aligned} \tag{36}$$

where A_{nk} and B_{nk} are undetermined constants.

Substitution of Eq. (36) into Eq. (25) yields

$$\sum_{k=0}^{\infty}B_{nk}\left(N^2k^2+\alpha_p^2\right)^2\cos k\theta = 4c^2r^2\alpha_p^2\sum_{k=0}^{\infty}A_{nk}\cos k\theta + H_n \tag{37}$$

The coefficients of $\cos k\theta$ on the both sides of Eq. (37) should be equal. Multiplying the both sides by $\cos k\theta$ and integrating from 0 to 2π, we acquire

$$B_{nk} = \frac{4c^2r^2\alpha_p^2}{\left(N^2k^2+\alpha_p^2\right)^2}A_{nk} + \frac{\int_0^{2\pi}H_n\cos k\theta d\theta}{\pi\left(N^2k^2+\alpha_p^2\right)^2} \tag{38}$$

Substituting Eqs. (36) and (38) into Eq. (26) yields

$$\begin{aligned}&\sum_{k=0}^{\infty}A_{nk}\left[\left(N^2k^2+\alpha_p^2\right)^2 + \frac{4c^2r^4\alpha_p^4}{\left(N^2k^2+\alpha_p^2\right)^2} - \frac{q_0RL^2}{D_0}N^2k^2 - N^4k^2\right]\times\cos k\theta = -\sum_{k=0}^{\infty}\frac{r^2\alpha_p^2\int_0^{2\pi}H_n\cos k\theta d\theta}{\pi\left(N^2k^2+\alpha_p^2\right)^2}\cos k\theta \\ &+\frac{q_nRL^2}{D_0}N^2A_{0m}m^2\times\cos m\theta + Q_n\end{aligned} \tag{39}$$

Multiplying the both sides of Eq. (39) by $\cos k\theta$ and integrating from 0 to 2π, we obtain

$$\begin{aligned}&A_{nk}\left[\left(N^2k^2+\alpha_p^2\right)^2 + \frac{4c^2r^4\alpha_p^4}{\left(N^2k^2+\alpha_p^2\right)^2} - \frac{q_0RL^2}{D_0}N^2k^2 - N^4k^2\right] = -\frac{r^2\alpha_p^2\int_0^{2\pi}H_n\cos k\theta d\theta}{\pi\left(N^2k^2+\alpha_p^2\right)^2} \\ &+\frac{q_nRL^2}{D_0}N^2A_{0m}m^2\delta_{mk} + \frac{1}{\pi}\int_0^{2\pi}Q_n\cos k\theta d\theta\end{aligned} \tag{40}$$

where $\delta_{mk} = \begin{cases}1 & m=k \\ 0 & m\neq k\end{cases}$ is Kronecker Delta.

Two conditions of Eq. (40) will be discussed in the following.

When $m = k$, the left-hand side of Eq. (40) becomes zero according to Eq. (33), then we obtain

$$\frac{q_nRL^2}{D_0} = \frac{r^2\alpha_p^2\int_0^{2\pi}H_n\cos k\theta d\theta}{\pi N^2k^2A_{0m}\left(N^2k^2+\alpha_p^2\right)^2} - \frac{\int_0^{2\pi}Q_n\cos k\theta d\theta}{\pi N^2k^2A_{om}} \tag{41}$$

Simply, Eq. (40) may be satisfied by choosing $A_{nm} = 0\ (n \geq 1)$.

When $m \neq k$, it can be derived from Eq. (40) that

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$$A_{nk}=\frac{-r^2\alpha_p^2\int_0^{2\pi}H_n\cos k\theta d\theta+\left(N^2k^2+\alpha_p^2\right)^2\int_0^{2\pi}Q_n\cos k\theta d\theta}{\pi\left[\left(N^2k^2+\alpha_p^2\right)^4+4c^2r^4\alpha_p^4-N^2k^2\left(\frac{q_0RL^2}{D_0}+N^2\right)\left(N^2k^2+\alpha_p^2\right)^2\right]} \tag{42}$$

Therefore, A_{nk} can be expressed as 0 ($m = k$) or Eq. (42) ($m \neq k$), and B_{nk} can also be determined.

In conclusion, we acquire the following asymptotic solutions in terms of the thickness variation parameter η up to arbitrary order

$$\begin{aligned} q&=q_0+\sum_{n=1}^{\infty}\eta^n q_n\\ \bar{w}(\theta)&=A_{om}\cos m\theta+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}A_{nk}\eta^n\cos k\theta\\ \bar{f}(\theta)&=B_{0m}\cos m\theta+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}B_{nk}\eta^n\cos k\theta \end{aligned} \tag{43}$$

All the constants in Eq. (43) can be determined by Eqs. (32), (33), (38), (41) and (42).

4. EXAMPLE, COMPARISON AND DISCUSSION

In this section, we will verify the presented method by comparing the results of two examples with previous studies. The non-dimensional thickness variation parameter η is assumed to be relatively small so that the terms of η^2 can be ignored, and the buckling modes of cylindrical shells with constant thickness remain good approximation for those with thickness variations. The buckling waves of shells with moderate length can be determined by Eq. (35). Here, we introduce the buckling load reduction factor λ as Eq. (44) to evaluate the buckling effect

$$\lambda=\frac{q}{q_0}=1+\sum_{n=1}^{\infty}\eta^n\frac{q_n}{q_0} \tag{44}$$

where q_0 represents the buckling load with t_0, q is the buckling load with t.

Example 1: Uniform thickness variation

Firstly, we investigate the buckling of a cylindrical shell with uniform thickness variation under external pressure. The shell thickness is assumed to be

$$t=t_0(1-\eta) \tag{45}$$

Noting $T_1=-1$, $T_n=0\left(n\geq 2\right)$ and combining Eqs. (33) and (41), the following expressions can be acquired after related calculation

$$\frac{q_0RL^2}{D_0}=\frac{\left(N^2m^2+\alpha_p^2\right)^2}{N^2m^2}+\frac{4c^2r^4\alpha_p^4}{\left(N^2m^2+\alpha_p^2\right)^2N^2m^2}-N^2 \tag{46}$$

$$\frac{q_1RL^2}{D_0}=-\frac{3\left(N^2m^2+\alpha_p^2\right)^2}{N^2m^2}-\frac{4c^2r^4\alpha_p^4}{\left(N^2m^2+\alpha_p^2\right)^2N^2m^2}+3N^2 \tag{47}$$

Substituting corresponding parameters into Eqs. (46) and (47) and combining Eq. (44), we obtain

$$\lambda=\frac{q}{q_0}=1+\frac{q_1}{q_0}\eta+O\left(\eta^2\right)=1-\frac{3R\left(\frac{m}{R^2}+\frac{\pi^2}{mL^2}\right)^2-3\frac{1}{R^3}+\frac{12\left(1-v^2\right)\pi^4R^3}{L^4m^6t_0^2\left(1+\frac{\pi^2R^2}{m^2L^2}\right)^2}}{R\left(\frac{m}{R^2}+\frac{\pi^2}{mL^2}\right)^2-\frac{1}{R^3}+\frac{12\left(1-v^2\right)\pi^4R^3}{L^4m^6t_0^2\left(1+\frac{\pi^2R^2}{m^2L^2}\right)^2}}\eta+O\left(\eta^2\right) \tag{48}$$

Eq. (48) is a generalized expression of buckling load reduction for elastic cylinders with uniform thickness variation, and the coefficient of η is not equal to -3 due to the terms with π^4. It is totally consistent with the recent result presented by Yang *et al*. [14]. In their work, the computed results of Eq. (48) are confirmed by previous study [19] with same geometry properties of cylinders.

When L approaches infinity, Eq. (48) becomes

$$\lambda=1-\frac{\frac{3\left(m^2-1\right)}{R^3}}{\frac{\left(m^2-1\right)}{R^3}}\eta+O\left(\eta^2\right)=1-3\eta+O\left(\eta^2\right) \tag{49}$$

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Eq. (49) has also been confirmed by [12, 14].

Example 2: Circumferential modal thickness variation

The buckling of a laterally pressured cylindrical shell with circumferential modal thickness variation in the form of Eq. (50) is discussed in this section

$$t = t_0(1-\eta\cos 2m\theta) \tag{50}$$

where θ is the circumferential angle that changes from *0* to 2π, m is the number of circumferential buckling waves determined by Eq. (35).

We transform Eq. (50) by introducing non-dimensional parameters like Eq. (12) and obtain

$$T(\theta)=1-\eta\cos 2m\theta \tag{51}$$

Noting $T_1 = -\cos 2m\theta$, $T_n = 0\ (n \geq 2)$ and p=0 when shell buckles, the following expression can be derived from Eq. (41)

$$q_1 = -\frac{2c^2r^4\pi^4 D_0}{N^2m^2\left(N^2m^2+\pi^2\right)^2 RL^2} + \frac{8c^2r^4\pi^6(1+\nu)D_0}{\left(N^2m^2+\pi^2\right)^4 RL^2} - \frac{3}{2}\frac{\left(N^2m^2+\pi^2\right)^2 D_0}{N^2m^2RL^2} + \frac{3}{2}\frac{N^2D_0}{RL^2} - 6\pi^2(\nu-1)\frac{D_0}{RL^2} \tag{52}$$

According to Eq. (33), the buckling pressure of a cylindrical shell with constant thickness t_0 is determined as

$$q_0 = \frac{4c^2r^4\pi^4 D_0}{\left(N^2m^2+\pi^2\right)^2 N^2m^2RL^2} + \frac{\left(N^2m^2+\pi^2\right)^2 D_0}{N^2m^2RL^2} - N^2\frac{D_0}{RL^2} \tag{53}$$

The combination of Eqs. (52), (53) and (44) yields

$$\lambda = \frac{q}{q_0} = 1+\frac{q_1}{q_0}\eta + O\left(\eta^2\right) = 1 - \frac{\dfrac{2c^2r^4\pi^4}{N^2m^2\left(N^2m^2+\pi^2\right)^2} - \dfrac{8c^2r^4\pi^6(1+\nu)}{\left(N^2m^2+\pi^2\right)^4} + \dfrac{3}{2}\dfrac{\left(N^2m^2+\pi^2\right)^2}{N^2m^2} - \dfrac{3}{2}N^2 + 6\pi^2(\nu-1)}{\dfrac{4c^2r^4\pi^4}{N^2m^2\left(N^2m^2+\pi^2\right)^2} + \dfrac{\left(N^2m^2+\pi^2\right)^2}{N^2m^2} - N^2}\eta + O\left(\eta^2\right) \tag{54}$$

Substituting corresponding parameters into Eq. (54) and neglecting the terms of η^2, we obtain

$$\lambda = 1 - \frac{\dfrac{6\left(1-\nu^2\right)R^4\pi^4}{L^2t_0^2m^6\left(1+\dfrac{\pi^2R^2}{m^2L^2}\right)^2} - \dfrac{24\left(1-\nu^2\right)R^6\pi^6(1+\nu)}{L^4t_0^2m^8\left(1+\dfrac{\pi^2R^2}{m^2L^2}\right)^4} + \dfrac{3}{2}R^2L^2\left(\dfrac{m}{R^2}+\dfrac{\pi^2}{mL^2}\right)^2 - \dfrac{3}{2}\dfrac{L^2}{R^2} + 6\pi^2(\nu-1)}{\dfrac{12\left(1-\nu^2\right)R^4\pi^4}{L^2t_0^2m^6\left(1+\dfrac{\pi^2R^2}{m^2L^2}\right)^2} + R^2L^2\left(\dfrac{m}{R^2}+\dfrac{\pi^2}{mL^2}\right)^2 - \dfrac{L^2}{R^2}}\eta \tag{55}$$

Eq. (55) is a generalized expression of buckling load reduction for cylindrical shells with circumferential modal thickness variation of Eq. (50). It can be seen that the terms containing π^4 or π^6 in the numerator and denominator are not negligible, when cylinders are finite in length.

By means of FE bifurcation analysis, Gusic *et al*. [17] studied the influence of Eq. (50) on the buckling of cylindrical shells under external pressure. In their work, cylinders are investigated by FE linear perturbation analysis, assuming the material linearly elastic and pre-buckling deformations sufficiently small. The resulting load reductions are then least square fitted for all the analyzed shells. Within parameter Z domain ($50 \leq Z \leq 1000$), an approximation of the buckling pressure reduction as a linear function of η is introduced such that

$$\lambda = \frac{q}{q_0} = 1 - b\eta \tag{56}$$

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The thickness imperfection sensitivity factor b is given by

$$b_1 = -0.086(\log Z)^2 + 0.716(\log Z) - 0.181 \quad (57)$$

$$b_2 = 21.37\left(Z^{-1/2}\right)^2 - 7.20\left(Z^{-1/2}\right) + 1.39 \quad (58)$$

Parameter b is evaluated as a function of $\log(Z)$ and of $Z^{1/2}$ in order to cover two limiting values of Z ($Z \to 0$ and $Z \to \infty$). In [17], the computed results of Eqs. (57) and (58) are within 2% of the FE mean values.

Here, we make a comparison between the analytical results of Eq. (55) and the FE analysis results obtained by Gusic *et al*. [17]. Table 1 lists the geometry properties of different cylindrical shells which have been studied in [17].

Table 1 Geometry properties of the cylinders with finite length.

R/mm	L/mm	t_0/mm	Z
100	35.9	0.247	50
100	51.0	0.247	100
100	71.6	0.247	200
100	113.8	0.247	500
100	160.0	0.247	1000

After substituting the geometry properties from Table 1 into Eqs. (55), (57) and (58) respectively, we list the computed results of coefficient of η and corresponding comparisons in Table 2.

Table 2 Comparisons of coefficient of η with different geometry properties.

Z	Coefficient of η			Error (%)	
	Gusic *et al*. [17]		This paper	With b_1	With b_2
	b_1	b_2			
50	0.7872	0.7992	0.83678	6.3	4.7
100	0.907	0.8837	0.93303	2.9	5.6
200	1.0112	0.9877	1.00981	0.1	2.2
500	1.125	1.1107	1.0891	3.2	1.9
1000	1.193	1.1837	1.13113	5.2	4.4

As shown in Table 2, the coefficients of η obtained by the method presented in this paper are very close to those obtained by FE analysis. The reason why the error between them is so small is that both of the two methods assume the shells are simply-supported at both ends and the material is linearly elastic. Moreover, the results of Eqs. (55), (57) and (58) are plotted in Fig. 2 (a)~(e) for different cylindrical shells.

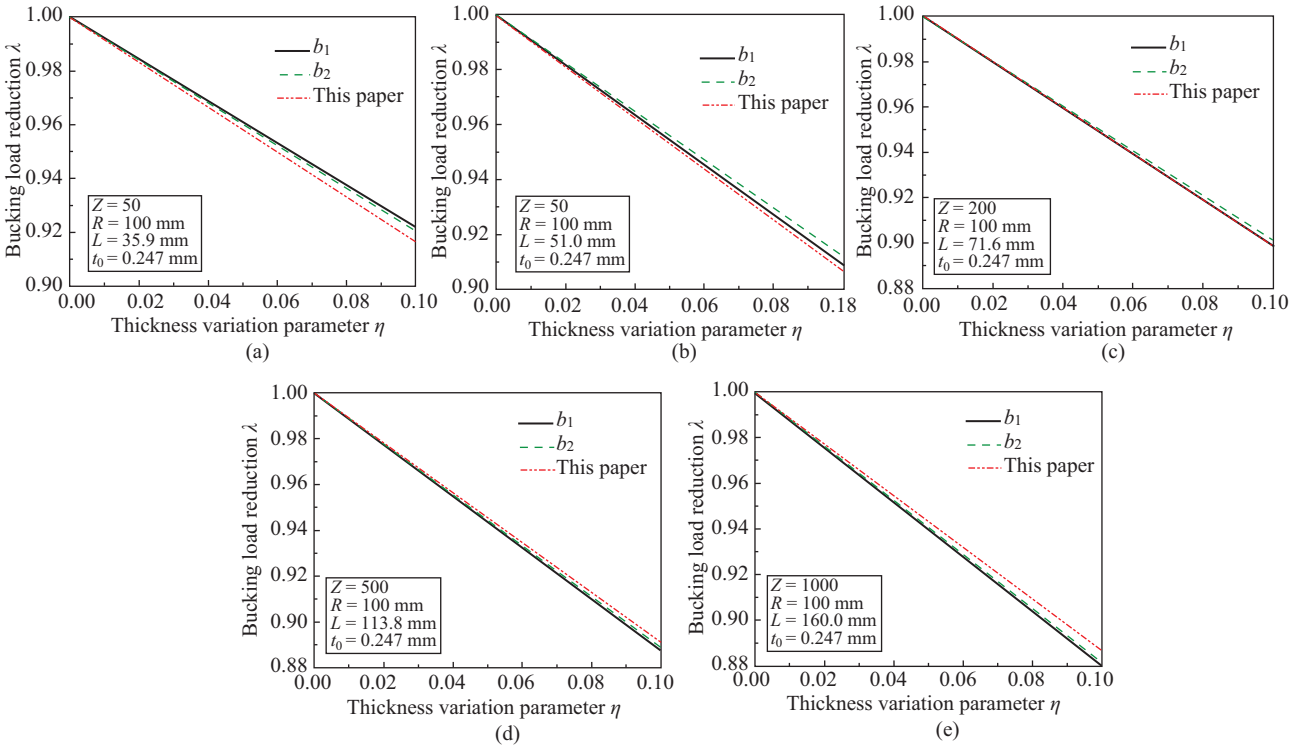


Fig. 2 Buckling load reduction factor vs thickness variation parameter.

From Fig. 2 (a)~(e), it can be observed that

(1) For the effect of circumferential modal thickness variation on the buckling load reduction, the results of analytical method presented in this paper are in good agreement with those obtained by FE bifurcation analysis;

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(2) The buckling loads of cylindrical shells decrease with the increase of thickness variation parameter η and Batdorf Z parameter;

(3) The influence of circumferential modal thickness variation should be taken into account in engineering applications of cylindrical shells. For example, when Z is 1000 and η reaches 0.10, the buckling load is reduced by nearly 12%.

Besides, if cylinders are infinite in length, Eq. (55) can be simplified to be

$$\lambda = 1 - \frac{\frac{3}{2}\frac{L^2}{R^2}\left(m^2-1\right)}{\frac{L^2}{R^2}\left(m^2-1\right)}\eta + O\left(\eta^2\right) = 1 - \frac{3}{2}\eta + O\left(\eta^2\right) \quad (59)$$

Eq. (59) coincides with the result presented by Combesure and Gusic [18].

In the discussion of section 4, the influences of two circumferential thickness variations on the buckling of cylindrical shells are analyzed, and the results are confirmed by previous studies. However, the proposed method is applicable for arbitary circunferential thickness variations, once the thickness variations can be described by mathematical expressions. And compared with the FE analysis, it is more convenient to evaluate the elastic buckling of cylindrical shells by this analytical method.

5. CONCLUSIONS

1. A unified analytical method for buckling analysis of laterally pressured cylindrical shells with arbitrary circumferential thickness variations is developed in this paper. And the asymptotic formula of buckling load is derived.
2. Based on the proposed method, the effects of uniform and circumferential modal thickness variations on the buckling of cylinders are discussed, respectively, and the results of both cases are confirmed by previous studies. Particularly, the buckling load reduction of cylinders with circumferential modal thickness variation coincides with the numerical results presented by Gusic *et al*.
3. The effect of circumferential thickness variation on the buckling of cylindrical shells in engineering applications should be taken into account, and the analytical method proposed in this paper is applicable for evaluating the elastic buckling problem, once the thickness variation forms are known.

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REFERENCES

1. Yamaki, N., Otorno, K. and Matsuda, K., "Experiments on the post-buckling behavior of circular cylindrical shells under compression," *Experimental Mechanics*, **15**, pp. 23-28 (1975).
2. Rotter, J. M. and Teng, J. G., "Elastic stability of cylindrical shells with weld depressions," *Journal of Structural Engineering*, **115**, pp. 1244-1263 (1989).
3. Shen, H. and Chen, T., "Buckling and postbuckling behaviour of cylindrical shells under combined external pressure and axial compression," *Thin-walled Structures*, **12**, pp. 321-334 (1991).
4. Greiner, R. and Guggenberger, W., "Buckling behavior of axially loaded steel cylinders on local supports—with and without internal pressure," *Thin-Walled Structures*, **31**, pp. 159-167 (1998).
5. Khamlichi, A., Bezzazi, M. and Limam, A., "Buckling of elastic cylindrical shells considering the effect of localized axisymmetric imperfections," *Thin-walled structures*, **42**, pp. 1035-1047 (2004).
6. Hübner, A., Teng, J. G. and Saal, H., "Buckling behaviour of large steel cylinders with patterned welds," *International Journal of Pressure Vessels and Piping*, **83**, pp. 13-26 (2006).
7. Ali, L. and Abdellatif, K., "Effect of multiple localized geometric imperfections on stability of thin axisymmetric cylindrical shells under axial compression," *International Journal of Solids and Structures*, **48**, pp. 1034-1043 (2011).
8. Yu, C. L., Chen, Z. P., Wang, J., Yan, S. J. and Yang L. C., "Effect of weld reinforcement on axial plastic buckling of welded steel cylindrical shells," *Journal of Zhejiang University SCIENCE A*, **13**, pp. 79-90 (2012).
9. Chen, L., "Buckling analysis of short cylinders of uniform thickness under uniform external pressure," *Applied Mechanics and Materials*, **351**, pp. 501-504 (2013).
10. Koiter, W. T., Elishakoff, I., Li, Y. W. and Starnes J. H., "Buckling of an axially compressed cylindrical shell of variable thickness," *International Journal of Solids and Structures*, **31**, pp. 797-805 (1994).
11. Li, Y. W., Elishakoff, I. and Starnes, J. H. "Axial buckling of composite cylindrical shells with periodic thickness variation," *Computers & Structures*, **56**, pp.65-74 (1995).
12. Nguyen, H. L. T., Elishakoff, I. and Nguyen, V. T., "Buckling under the external pressure of cylindrical shells with variable thickness," *International Journal of Solids and Structures*, **46**, pp. 4163-4168 (2009).
13. Chen, Z., Yang, L., Cao, G. and Guo W., "Buckling of the axially compressed cylindrical shells with arbitrary axisymmetric thickness variation," *Thin-Walled Structures*, **60**, pp. 38-45 (2012).
14. Yang, L., Chen, Z., Chen, F., Guo W. and Cao G., "Buckling of cylindrical shells with general axisym-

https://doi.org/10.1017/jmech.2016.59 Published online by Cambridge University Press

metric thickness imperfections under external pressure," *European Journal of Mechanics-A/Solids,* **38**, pp. 90-99 (2013).

15. Aghajari, S., Abedi, K. and Showkati, H., "Buckling and post-buckling behavior of thin-walled cylindrical steel shells with varying thickness subjected to uniform external pressure," *Thin-Walled Structures*, **44**, pp. 904-909 (2006).
16. Chen, L., Rotter, J. M. and Doerich C., "Buckling of cylindrical shells with stepwise variable wall thickness under uniform external pressure," *Engineering Structure*s, **33**, pp. 3570-3578 (2011).
17. Gusic, G., Combescure, A. and Jullien, J. F., "The influence of circumferential thickness variations on the buckling of cylindrical shells under external pressure," *Computers & Structures*, **74**, pp. 461-477 (2000).
18. Combescure, A. and Gusic, G., "Nonlinear buckling of cylinders under external pressure with non-axisymmetric thickness imperfections using the COMI axisymmetric shell element," *International Journal of Solids and Structures*, **38**, pp. 6207-6226 (2001).
19. Yamaki, N., *Elastic Stability of Circular Cylindrical Shells*, Elsevier Science Publishers B. V., Amsterdam (1984).

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