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Wiener–Wintner dynamical systems

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Abstract. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system on the finite measure space $(X, \mathcal{B}, \mu)$. Let $\mathcal{K}$ denote the Kronecker factor of T, i.e. the closed linear span in L^2 of the eigenfunctions for T. We say that $(X, \mathcal{B}, \mu, T)$ is a Wiener–Wintner (WW) dynamical system of power type α in L^1 if there exists in $\mathcal{K}^{\perp}$ a dense set of functions f for which the following holds: there exists a finite positive constant C_f such that

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_1 \leq \frac{C_f}{N^{\alpha}}$$

for all positive integers N. Examples of ergodic dynamical systems with this WW property include K automorphisms as well as some skew products over irrational rotations. For WW dynamical systems a simpler proof of the almost everywhere double recurrence property, random weights with a break of duality can be obtained. They also provide naturally almost everywhere continuous random Fourier series related to the spectral measure of the transformation.

1. *Introduction*

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system on the finite measure space $(X, \mathcal{B}, \mu)$. We denote by $\mathcal{K}$ the Kronecker factor of T, i.e. the closed linear span in L^2 of the eigenfunctions for T. The Wiener–Wintner (WW) ergodic theorem announced by Wiener and Wintner [**WW**] states that given any function f in $L^1(\mu)$ there exists a set of full measure X_f such that for each $x \in X_f$ the averages $(1/N)\sum_{n=1}^{N} f(T^n x)e^{2\pi i n\varepsilon}$ converge for all ε. The subject of this paper is the study of the speed of convergence in the WW ergodic theorem. More precisely we will be interested in the following functions.

Definition 1. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. A function f is a WW function of power type α in L^p, $1 \leq p \leq \infty$, if there exist finite positive constants C_f and α such that

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_p \leq \frac{C_f}{N^{\alpha}} \quad \text{for all positive integers } N.$$

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Definition 2. An ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ is a WW dynamical system of power type α in L^p if there exists in $\mathcal{K}^\perp$ a dense set (for the L^2 norm) of WW functions of power type α in L^p.

In this paper unless otherwise specified whenever we mention the words WW dynamical systems or WW functions we will refer to WW dynamical systems or functions of power type in L^1.

It appears that the study of the rate of convergence to zero of the sequence

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_1$$

is a central problem behind two difficult theorems of Bourgain: the return times [**BFKO**] and the almost everywhere double recurrence [**B**]. Knowing for instance that

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_1 = \mathcal{O}\left(N^{-\alpha}\right)$$

allows one to considerably simplify the proof of the almost everywhere convergence of the averages

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) f_2(T^{-n} x)$$

to zero for any function $f_2 \in L^2(\mu)$, as we will see in §3.4. Bourgain's proof of the almost everywhere double recurrence theorem answered a question raised by Furstenberg [**Fu**]. A simpler proof of this theorem would make the basis of the arguments in [**A4**] simpler and provide more hope for solving the subsequent open questions on almost everywhere multiple recurrence. This is one reason for the interest in these functions. Another reason arises from the return times property with a break of duality. In fact it is the study of the validity of the return times for functions breaking duality that led us to the existence of WW functions in a natural way (see [**A2**]).

It seems then a natural question to ask if we can find in $\mathcal{K}^\perp$ a dense set of WW functions of some power type in L^1. By using simple inequalities we will be able to answer this question partially. The inequality used in this final version of the paper was suggested by the referee. We used Van der Corput's inequality in the original version.

In §3.1 we will see that for ergodic dynamical systems with positive entropy we can always find WW functions. Actually we will show that if (X_n) is an L^2 sequence of iid random variables with zero mean then we have

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} X_n e^{2\pi i n\varepsilon} \right| \right\|_2^2 \leq \frac{1+\sqrt{2}}{\sqrt{N}} \mathbb{E}(|X_1|^2).$$

This estimate is not sharp as shown by [**SZ**] and pointed out to us by Weber (see Lemma 1), but it establishes easily the existence of WW functions in ergodic dynamical systems with positive entropy. The sharper estimate for independent (X_n) is the following;

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} X_n e^{2\pi i n\varepsilon} \right| \right\|_2^2 \leq C \frac{\sum_{n=1}^{N} \mathbb{E}(|X_n|^2)}{N^2} \log N.$$

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This estimate will allow us to prove in §3.1 a generalization of one of our previous results in [**A2**] by eliminating the symmetric assumption made in that paper.

We will prove in §3.2 that K automorphisms are WW dynamical systems. We will also show that the product of a WW function for this K automorphism with any bounded function in a measure preserving system is a WW function for the product of these two dynamical systems. From the definition we can conclude immediately that ergodic dynamical systems with discrete spectrum are WW dynamical systems. The set $\mathcal{K}^{\perp}$ for such ergodic transformations is trivial. This provides us with trivial examples of WW dynamical systems with zero entropy. We will also show that most classical skew products on the 2-torus, defined by $(x, y) \mapsto (x + \alpha, x + y)$ for α irrational, are WW dynamical systems. These are non-trivial examples of a WW dynamical system with zero entropy. Thus the entropy does not characterize the class of WW dynamical systems. An easy consequence of the definition is the fact that factors of WW dynamical systems are also WW dynamical systems. This WW property is then preserved by isomorphism.

In §3.5 we will show how the concept of WW functions can also be applied to the production of almost everywhere continuous random Fourier series. We give the following definition.

Definition 3. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and f a function in $L^2(\mu)$. The one-sided rotated ergodic Hilbert transform (for f) is defined as the series

$$\sum_{n=1}^{\infty} \frac{f(T^n x)}{n} e^{2\pi i n\varepsilon}.$$

These series are directly related to the spectrum of the transformation T. We will prove the almost everywhere existence and continuity of the one-sided rotated ergodic Hilbert transform for WW functions. This series was studied by Talagrand [**T**] in the case of iid random variables and a necessary and sufficient condition was given for its almost everywhere continuity. Here we will establish the almost everywhere continuity for stationary processes which are not necessarily iid. In the particular case of K automorphisms we will show that there exists a dense set of WW functions f_1 in $\mathbb{C}^{\perp}$ (equal to $\mathcal{K}^{\perp}$ in this case) for which more is true; for all positive integers H, for all bounded functions f_i, $2 \le i \le H$, it is possible to find a set of full measure $\tilde{X} = X_{f_1,\dots,f_H}$ such that for $x \in \tilde{X}$ we have not only convergence of the series

$$\sum_{n=1}^{\infty} \frac{\prod_{k=1}^{H} f_k(T^{kn} x)}{n} e^{2\pi i n\varepsilon} \quad \text{for all } \varepsilon,$$

but also continuity of the map

$$\varepsilon \mapsto \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{H} f_k(T^{kn} x)}{n} e^{2\pi i n\varepsilon}.$$

This allows us to establish the existence and continuity of the rotated multiple ergodic Hilbert transform and the one sided rotated multiple ergodic Hilbert transform for this dense set of functions. The rotated multiple ergodic Hilbert transform for the functions f_i,

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$1 \leq i \leq H$, is defined as

$$\sum_{n=-\infty}^{\infty}{}' \frac{(\prod_{i=1}^{H} f_i(T^{in}x))}{n} e^{2\pi in\varepsilon}.$$

As far as the author knows the almost everywhere existence of the multiple ergodic Hilbert transform (corresponding to $\varepsilon = 0$) has not yet been established even for the case $H = 2$. The existence of a dense set for which we could have the pointwise convergence of such series is also unknown to the author. Here we will show that if the system $(X, \mathcal{B}, \mu, T)$ is ergodic then for all WW functions $f \in L^2(\mu)$ and for all $g \in L^2(\mu)$, the series

$$\sum_{n=1}^{\infty} \frac{f(T^n x).g(T^{2n}x)}{n} \quad \text{and} \quad \sum_{n=-1}^{-\infty} \frac{f(T^n x).g(T^{2n}x)}{n}$$

converge almost everywhere.

2. *Preliminary*

We will use the following elementary inequality in this paper.

If $(u_n)_{0\leq n\leq N-1}$ is a family of complex numbers then

$$\sup_{\varepsilon} \left| \sum_{n=0}^{N-1} u_n e^{2\pi in\varepsilon} \right|^2 \leq \sum_{n=0}^{N-1} |u_n|^2 + 2 \sum_{h=1}^{N-1} \left| \sum_{n=0}^{N-h-1} \overline{u_{n+h}} u_n \right|. \tag{1}$$

3. *Results*

3.1. *The case of iid random variables.*

We want to prove the following results.

THEOREM 1. *Let (X_n) be a sequence of iid random variables with mean zero and finite second moment. Then we have*

$$\int \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} X_n e^{2\pi in\varepsilon} \right|^2 dP \leq (1+\sqrt{2}) \frac{\mathbb{E}(|X_1|^2)}{N^{1/2}} \quad \textit{for all } N \textit{ positive integers.}$$

THEOREM 2. *Let (X_n) be a sequence of iid random variables with finite pth moment, $1 < p < \infty$. Then we can find a set of full measure $\widetilde{\Omega}$ such that for (ω) in $\widetilde{\Omega}$ we have*

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} X_n(\omega) g(S^n y) = \mathbb{E}(X_1) \int g \, dv,$$

v almost everywhere, for all ergodic dynamical systems $(Y, \mathcal{Y}, v, S)$ and all functions g in L^r, $1 < r \leq \infty$.

Remark. As indicated in the Introduction, the estimate obtained in Theorem 1 is not sharp. However the proof of Theorem 1 will illustrate the method we will be using throughout this paper. It also establishes in a simple way the existence of WW functions. The interested reader may wish to know that Theorem 2 can be proved with the estimate in Theorem 1 for (X_n) a sequence of iid random variables with finite pth moment, where $\frac{3}{2} < p < \infty$.

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Proof of Theorem 1. We apply the inequality (1) to the sequence $u_n = X_n(\omega)$. We obtain

$$\sup_{\varepsilon}\left|\sum_{n=0}^{N-1} X_n(\omega)e^{2\pi in\varepsilon}\right|^2 \le \sum_{n=0}^{N-1}|X_n(\omega)|^2 + 2\sum_{h=1}^{N-1}\left|\sum_{n=0}^{N-h-1}\bar{X}_{n+h}(\omega)X_n(\omega)\right|.$$

Hence

$$\int \sup_{\varepsilon}\left|\sum_{n=0}^{N-1} X_n e^{2\pi in\varepsilon}\right|^2 dP \le N\mathbb{E}(|X_1|^2) + 2\sum_{h=1}^{N-1}\int\left|\sum_{n=0}^{N-h-1}\bar{X}_{n+h}X_n\right| dP.$$

We focus our attention on the integral in the second term; we have

$$\int\left|\sum_{n=0}^{N-h-1}\bar{X}_{n+h}X_n\right| dP \le \left(\int\left|\sum_{n=0}^{N-h-1}\bar{X}_{n+h}X_n\right|^2 dP\right)^{1/2}.$$

Since

$$\left|\sum_{n=0}^{N-h-1}\bar{X}_{n+h}(\omega)X_n(\omega)\right|^2 = \sum_{n=0}^{N-h-1}|\bar{X}_{n+h}(\omega)|^2|X_n(\omega)|^2 + 2\,\mathrm{Re}\sum_{n<j}\bar{X}_{n+h}(\omega)X_n(\omega)X_j(\omega)\bar{X}_{j+h}(\omega),$$

by using the independence of the variables X_n and the fact that their mean is zero, we just need to estimate the integral $(\int\sum_{n=0}^{N-h-1}|\bar{X}_{n+h}|^2|X_n|^2\,dP)^{1/2}$. We obtain the bound

$$(N-h-1)^{1/2}\mathbb{E}(|X_1|^2).$$

Hence

$$\int \sup_{\varepsilon}\left|\sum_{n=0}^{N-1} X_n e^{2\pi in\varepsilon}\right|^2 dP \le N\mathbb{E}(|X_1|^2) + 2\sum_{h-1}^{N-1}(N-h-1)^{1/2}\mathbb{E}(|X_1|^2).$$

As

$$\sum_{h-1}^{N-1}(N-h-1)^{1/2} \le \sqrt{N-1}\left(\sum_{h=1}^{N-1}h\right)^{1/2} \le \frac{N^{3/2}}{\sqrt{2}},$$

the proof of the theorem is complete. □

For the proof of Theorem 2 we use the estimate given by the following lemma.

LEMMA 1. *Let X_n be a sequence of independent random variables with finite second moment and mean zero. Then there exists a finite constant C such that*

$$\left\|\sup_x\left|\frac{1}{N}\sum_{n=1}^{N}X_n e^{2\pi inx}\right|\right\|_2^2 \le C\sum_{n=1}^{N}\mathbb{E}(|X_n|^2)\frac{\log N}{N^2}.$$

Proof of Lemma 1. First we assume that X_n is a sequence of independent Rademacher random variables and r_n a sequence of real numbers. Let $R_N = \sum_{n=1}^{N} r_n^2$, $M_N(t) = \sup_x|\sum_{n=1}^{N} r_n X_n(t)e^{2\pi inx}|$ and $\lambda = (\beta R_N^{-1}\log N)^{1/2}$, where $\beta > 0$. The proof of Theorem 4.3.1 in [**SZ**] shows that for $0 < \theta < 1$ we have

$$\int_0^1 e^{\lambda\theta M_N(t)}\,dt < \frac{4\pi}{1-\theta}e^{((\frac{1}{2}\beta+1)\log N)}.$$

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As by Jensen's inequality we have

$$\exp\left(\lambda\int_0^1 \theta M_N(t)\,dt\right) \le \int_0^1 \exp(\lambda\theta M_N(t))\,dt,$$

we also have

$$\theta\lambda\int_0^1 M_N(t)\,dt \le \log\frac{4\pi}{1-\theta} + \left(\frac{1}{2}\beta+1\right)\log N.$$

Thus

$$\begin{aligned}\int_0^1 M_N(t)\,dt &\le \frac{1}{\theta}\left[\left(\frac{R_N}{\beta\log N}\right)^{1/2}\log\frac{4\pi}{1-\theta} + \left(\frac{1}{2}\beta+1\right)\frac{1}{\sqrt{\beta}}\sqrt{R_N\log N}\right]\\ &\le C(\beta,\theta)\sqrt{R_N\log N}.\end{aligned}$$

So we have shown that

$$\left\|\sup_x\left|\frac{1}{N}\sum_{n=1}^N r_n X_n e^{2\pi inx}\right|\right\|_1 \le \frac{C(\beta,\theta)}{N}\sqrt{\left(\sum_{n=1}^N r_n^2\right)\sqrt{\log N}}.$$

To move from the L^1 norm to the L^2 norm we can use the Khintchine–Kahane inequalities. These inequalities tell us that

$$\left\|\sup_x\left|\frac{1}{N}\sum_{n=1}^N r_n X_n e^{2\pi inx}\right|\right\|_2 \le C_2\left\|\sup_x\left|\frac{1}{N}\sum_{n=1}^N r_n X_n e^{2\pi inx}\right|\right\|_1.$$

This allows us to obtain for X_n symmetric iid random variables the estimate

$$\left\|\sup_x\left|\frac{1}{N}\sum_{n=1}^N X_n e^{2\pi inx}\right|\right\|_2 \le \frac{C(\beta,\theta)}{N}\mathbb{E}\left(\sqrt{\left(\sum_{n=1}^N r_n^2\right)\log N}\right).$$

The right-hand side is bounded by

$$C(\beta,\theta)\sqrt{\sum_{n=1}^N \mathbb{E}(|X_n|^2)}\frac{\sqrt{\log N}}{N}.$$

From this we can deduce the same estimate for mean zero iid random variables with finite second moment. □

Proof of Theorem 2. We use the estimate given in the previous lemma for independent mean zero random variables. As the arguments are similar to the ones used in [**A2**] we only sketch them. It is enough to consider the cases where $1 < r < 2$. We can assume that the variables X_n are real and non-negative on L^p. Using the classical method of truncation we introduce the variables $Z_n = X_n \wedge n - \mathbb{E}(X_n \wedge n)$. Clearly we have $Z_n \in L^2$, $\mathbb{E}(Z_n) = 0$ and $\lim_{n\to\infty}\mathbb{E}(X_n \wedge n) = \mathbb{E}(X_1)$. For any ergodic dynamical system $(Y, \mathcal{Y}, \nu, S)$ and any $g \in L^1(\nu)$ we have

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^N X_n(\omega)g(S^n y) &= \frac{1}{N}\sum_{n=1}^N Z_n(\omega)g(S^n y) + \frac{1}{N}\sum_{n=1}^N \mathbb{E}(X_n\wedge n)g(S^n y)\\ &\quad + \frac{1}{N}\sum_{n=1}^N [X_n(\omega) - (X_n\wedge n)(\omega)]g(S^n y).\end{aligned}$$

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The third term on the right-hand side converges easily as $X_n(\omega) - \mathbb{E}(X_n \wedge n)(\omega)$ is non-zero for only finitely many terms. The second term on the right-hand side of this equality converges almost everywhere (ν) to $\mathbb{E}(X_1) \int g \, d\nu$ by Birkhoff's pointwise ergodic theorem. So we focus our attention on the averages $(1/N) \sum_{n=1}^{N} Z_n(\omega) g(S^n y)$. We apply the estimate obtained in Lemma 1 to the sequence Z_n of independent random variables. We have

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} Z_n e^{2\pi i n\varepsilon} \right| \right\|_2^2 \leq C \frac{\sum_{n=1}^{N} \mathbb{E}(|Z_n|^2)}{N^2} \log N.$$

We obtain from the previous inequality the following estimate:

$$\int \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} Z_n e^{2\pi i n\varepsilon} \right|^2 dP \leq \frac{C}{N^{p-1}} [\mathbb{E}(|X_1|^p)] \log N.$$

Now we can conclude by applying the method used in [**A2**] for symmetric iid random variables. □

As indicated in the Introduction for WW functions we have the analog of Theorem 2 with a break of duality.

THEOREM 3. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and f a WW function. Then there exists a set of full measure X_f such that for each x in this set, for all ergodic dynamical systems $(Y, \mathcal{G}, \nu, S)$ the averages $(1/N) \sum_{n=1}^{N} f(T^n x) g(S^n y)$ converge almost everywhere (ν) for all functions $g \in L^r(\nu)$ for $\frac{3}{2} < r \leq \infty$.*

Proof. The arguments are mainly the same as in [**A2**] hence we omit them. □

Remarks. (1) In the first version of this paper we had in Theorem 1 the constant 4 instead of $(1 + \sqrt{2})$. Pisier pointed out to us then that Theorem 1 was obtained by him and Marcus [**PM**] with an arbitrary constant C. He also indicated that he and Marcus did not compute this constant C. Theorem 1 can be derived from Corollary 1.1.2 of Theorem 1.1 in [**PM**]; take the compact subgroup $K = G$ the unit circle and A the set of natural numbers. The corollary is applied to the sequence $a_k = 1$ for $1 \leq k \leq n$ and $a_k = 0$ for $k > n$. Identifying the circle with $[0, 1)$, the translation invariant pseudo-metric $d_n(t_1, t_2) = \sigma_n(t_1 - t_2)$, on p. 8, is defined as

$$\sigma_n(t) = \left(\sum_{k=1}^{n} |1 - e^{2\pi i k t}|^2 \right)^{1/2} \leq 2 \left(\sum_{k=1}^{n} |\pi k t|^{\alpha} \right)^{1/2}$$

for any $0 < \alpha < 1$. For the last bound we can use the inequality $|\sin t|^2 \leq |t|^{\alpha}$.

(2) Theorem 1 shows that there exist WW functions in any ergodic dynamical system with positive entropy. In the next section we want to show that K automorphisms are WW dynamical systems.

3.2. *K automorphisms are WW dynamical systems.* In this section we will prove the following result, which as an immediate corollary shows that K automorphisms are WW dynamical systems.

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THEOREM 4. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and let $\mathcal{P}$ be the Pinsker σ algebra of T. There exists in $(L^2(X, \mathcal{P}, \mu))^{\perp}$ a dense set of WW functions.*

Proof. We recall a property of the Pinsker σ algebra:

(a) there exists a σ algebra $\mathcal{A}$ such that

$$T^{-1}\mathcal{A} \subset \mathcal{A}; \quad \bigcap_n T^{-n}\mathcal{A} = \mathcal{P}; \quad \bigcup_n T^n\mathcal{A} \text{ is dense in } \mathcal{B}.$$

The next property is valid for all invertible ergodic dynamical systems:

(b) for all L^1 integrable functions g,

$$\mathbb{E}(g \circ T^m | T^{-n}\mathcal{A}) = \mathbb{E}(g | T^{-(n-m)}\mathcal{A}) \circ T^m.$$

It is enough to show that any function of the form $\mathbf{1}_A - \mathbb{E}(\mathbf{1}_A|\mathcal{P})$, for $A \in T^l\mathcal{A}$, can be approximated in L^2 norm by WW functions. Our goal is to show that functions of the form $f_k^A = \mathbf{1}_A - \mathbb{E}(\mathbf{1}_A|T^{-k}\mathcal{A})$ are WW functions. The martingale convergence theorem will guarantee the approximation. For simplicity we will only treat the case $l = 0$. So we fix A and k a positive integer and apply the inequality (1), for $N > k$. We get

$$\sup_{\varepsilon} \left| \sum_{n=0}^{N-1} f_k^A(T^n x) e^{2\pi i n\varepsilon} \right|^2 \leq \sum_{n=0}^{N-1} |f_k^A(T^n x)|^2 + 2\sum_{h=1}^{N-1} \left| \sum_{n=0}^{N-h-1} f_k^A(T^{n+h}x) f_k^A(T^n x) \right|.$$

The function f_k^A being uniformly bounded by 2, the first term in this last inequality is bounded after integration by $4N$.

After integration we dominate the second term in the following way:

$$2\sum_{h=1}^{k-1} \int \left| \sum_{n=0}^{N-h-1} f_k^A \circ T^{n+h} f_k^A \circ T^n \right| d\mu + 2\sum_{h=k}^{N-1} \left[\int \left| \sum_{n=0}^{N-h-1} f_k^A \circ T^{n+h} f_k^A \circ T^n \right|^2 d\mu \right]^{1/2}.$$

The first sum is bounded by $8kN$. The second sum is equal to

$$2\sum_{h=k}^{N-1} \left[\int \sum_{n=0}^{N-h-1} |f_k^A \circ T^{n+h}|^2 |f_k^A \circ T^n|^2 \, d\mu + 2\sum_{n<j} \int f_k^A \circ T^{n+h} f_k^A \circ T^n f_k^A \circ T^{h+j} f_k^A \circ T^j \, d\mu \right]^{1/2}.$$

We focus our attention on the averages

$$2\sum_{n<j} \int f_k^A \circ T^{n+h} f_k^A \circ T^n f_k^A \circ T^{h+j} f_k^A \circ T^j \, d\mu. \tag{2}$$

We want to show that if $h \geq k$ then for $j > n + k$ we have

$$\int f_k^A \circ T^{n+h} f_k^A \circ T^n f_k^A \circ T^{h+j} f_k^A \circ T^j \, d\mu = 0.$$

We distinguish two cases.

(i) If $j > n + h$ then the function

$$f_k^A \circ T^{n+h} f_k^A \circ T^{h+j} f_k^A \circ T^j \quad \text{is } T^{-(n+h)}\mathcal{A} \text{ measurable.}$$

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Hence we have

$$\mathbb{E}[f_k^A \circ T^{n+h} f_k^A \circ T^n f_k^A \circ T^{h+j} f_k^A \circ T^j \mid T^{-(n+h)}\mathcal{A}] \\ = \mathbb{E}[f_k^A \circ T^n | T^{-(n+h)}\mathcal{A}] f_k^A \circ T^{n+h} f_k^A \circ T^{h+j} f_k^A \circ T^j.$$

Let us estimate $\mathbb{E}[f_k^A \circ T^n \mid T^{-(n+h)}\mathcal{A}]$. It is equal to

$$\mathbb{E}[\mathbf{1}_A \circ T^n | T^{-(n+h)}\mathcal{A}] - \mathbb{E}[\mathbb{E}(\mathbf{1}_A | T^{-k}\mathcal{A}) \circ T^n \mid T^{-(n+h)}\mathcal{A}].$$

Using the property (b) mentioned at the beginning of this §2, we obtain

$$\mathbb{E}[\mathbf{1}_A \circ T^n | T^{-(n+h)}\mathcal{A}] = \mathbb{E}[\mathbf{1}_A | T^{-h}\mathcal{A}] \circ T^n$$

and

$$\mathbb{E}[\mathbb{E}(\mathbf{1}_A | T^{-k}\mathcal{A}) \circ T^n | T^{-(n+h)}\mathcal{A}] = \mathbb{E}[\mathbb{E}(\mathbf{1}_A | T^{-k}\mathcal{A}) \mid T^{-h}\mathcal{A}] \circ T^n \\ = \mathbb{E}[\mathbf{1}_A | T^{-h}\mathcal{A}] \circ T^n \quad \text{as } h \geq k.$$

This proves the case $j > n + h$.

(ii) If $n + k < j \leq n + h$ then the function

$$f_k^A \circ T^{n+h} f_k^A \circ T^{h+j} f_k^A \circ T^j \quad \text{is } T^{-j}\mathcal{A} \text{ measurable.}$$

By taking the conditional expectation with respect to the sigma algebra $T^{-j}(\mathcal{A})$ we are left with estimating the function

$$\mathbb{E}[f_k^A \circ T^n | T^{-j}\mathcal{A}].$$

The same computations as above give us the difference

$$[\mathbb{E}[\mathbf{1}_A | T^{-(j-n)}\mathcal{A}] - \mathbb{E}[\mathbb{E}(\mathbf{1}_A | T^{-k}\mathcal{A}) \mid T^{-(j-n)}\mathcal{A}]\,] \circ T^n,$$

which is equal to zero because $j - n > k$. This proves the case $n + k < j \leq n + h$.

We can now return to (2). Using (i) and (ii) we can conclude that for $h \geq k$

$$\left| 2 \sum_{n<j} \int f_k^A \circ T^{n+h} f_k^A \circ T^n f_k^A \circ T^{h+j} f_k^A \circ T^j \, d\mu \right| \leq 2 \cdot 16 \cdot kN = 32kN,$$

as the sum $\sum_{n<j}$ contains at most kN non-zero elements, each of them being less than 16 in absolute value.

Finally going back to

$$\int \sup_{\varepsilon} \left| \sum_{n=0}^{N-1} f_k^A \circ T^n e^{2\pi i n\varepsilon} \right|^2 d\mu$$

we have

$$\int \sup_{\varepsilon} \left| \sum_{n=0}^{N-1} f_k^A \circ T^n e^{2\pi i n\varepsilon} \right|^2 d\mu \leq 4N + 8kN + \sum_{h=k}^{N-1} [16N + 32kN]^{1/2}.$$

This shows that

$$\int \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=0}^{N-1} f_k^A \circ T^n e^{2\pi i n\varepsilon} \right|^2 d\mu \leq \frac{C_{f_k^A}}{N^{1/2}}$$

for some finite constant $C_{f_k^A}$ depending on k and A. This proves Theorem 3. □

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An immediate corollary of the previous theorem is the following result announced in the Introduction.

THEOREM 5. *Let $(X, \mathcal{B}, \mu, T)$ be a K automorphism. There exists in $\mathbb{C}^{\perp}$ a dense set of WW functions.*

We now give a multidimensional version of Theorem 4 for a K automorphism. This result can also be reformulated in the setting of a general ergodic dynamical system by using the Pinsker σ algebra.

THEOREM 6. *Let $(X, \mathcal{B}, \mu, T)$ be a K automorphism and let H be a positive integer $H \geq 2$. We denote by $\mathcal{A}$ the σ algebra such that $T^{-1}\mathcal{A} \subset \mathcal{A}$, $\bigcap_n T^{-n}\mathcal{A} = \{\phi, X\}$ and $\bigcup_n T^n\mathcal{A}$ is dense in $\mathcal{B}$. Consider $(H-1)$ bounded functions f_ℓ, $T^j\mathcal{A}$ measurable for some integer j. Then there exists a dense set of WW functions f in $\mathbb{C}^{\perp}$ such that*

$$\int \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^n x) \left(\prod_{i=2}^{H} f_i(T^{in}x) \right) e^{2\pi i n\varepsilon} \right|^2 d\mu(x) \leq \frac{C_f}{N^{1/2}}. \tag{3}$$

Proof. We can assume without loss of generality that the functions f_i for $2 \leq i \leq H$ are $\mathcal{A}$ measurable and bounded by one. We use the same idea as in the proof of Theorem 3 for the method and the choice of the functions f. We apply the inequality (1) pointwise to the sequence

$$u_n = f_k^A(T^n x) \left(\prod_{i=2}^{H} f_i(T^{in}x) \right) e^{2\pi i n\varepsilon}, \quad \text{where } f_k^A = \mathbf{1}_A - \mathbb{E}(\mathbf{1}_A | T^{-k}\mathcal{A}).$$

We obtain

$$\sup_{\varepsilon} \left| \sum_{n=0}^{N-1} f_k^A(T^n x) \left(\prod_{i=2}^{H} f_i(T^{in}x) \right) e^{2\pi i n\varepsilon} \right|^2 \leq \sum_{n=0}^{N-1} 2^2 \cdot 1 + 2 \sum_{h=1}^{N-1} \left| \sum_{n=0}^{N-h-1} u_{n+h} u_n \right|.$$

We denote by $g_n(x)$ the function $\prod_{i=2}^{H} f_i(T^{in}x)$. Once again the bound is mainly given by evaluating the integrals

$$\sum_{n<j} \int f_k^A \circ T^{n+h} f_k^A \circ T^n f_k^A \circ T^{h+j} f_k^A \circ T^j \cdot g_{n+h} g_n g_j g_{j+h} \, d\mu. \tag{4}$$

As before, many of these integrals have zero value. Indeed:

(iii) if $j \geq n+h$ and $n+h \leq 2n$ then the function

$$R_{n,j,h} = f_k^A \circ T^{n+h} f_k^A \circ T^{h+j} f_k^A \circ T^j g_{n+h} \cdot g_n \cdot g_j \cdot g_{j+h} \quad \text{is } T^{-(n+h)}(\mathcal{A}) \text{ measurable.}$$

Hence the conditional expectation of the function

$$R_{n,j,h} \cdot f_k^A \circ T^n$$

with respect to the σ algebra $T^{-(n+h)}\mathcal{A}$ is equal to

$$R_{n,j,h} \cdot \mathbb{E}[f_k^A \circ T^n | T^{-(n+h)}\mathcal{A}] = R_{n,j,h} \cdot [\mathbb{E}[f_k^A | T^{-h}\mathcal{A}] \circ T^n] = 0 \quad \text{for } h \geq k.$$

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If $j \geq n+h$ and $n+h > 2n$ then the function $R_{n,j,h}$ is this time $T^{-2n}\mathcal{A}$ measurable and

$$\begin{aligned}\mathbb{E}[R_{n,j,h} \cdot f_k^A \circ T^n | T^{-2n}\mathcal{A}] &= R_{n,j,h}\mathbb{E}[f_k^A \circ T^n | T^{-2n}\mathcal{A}] \\ &= 0 \quad \text{for } n \geq k.\end{aligned}$$

(iv) If $n+k < j < n+h$ and $n+k < j \leq 2n < n+h$ then the function $R_{n,j,h}$ is $T^{-j}(\mathcal{A})$ measurable. By taking the conditional expectation of $R_{n,j,h} \cdot f_k^A \circ T^n$ with respect to $T^{-j}(\mathcal{A})$ we obtain 0 as $j - n > k$.

If $n+k < j < n+h < 2n$ then $R_{n,j,h}$ is $T^{-j}(\mathcal{A})$ measurable and the same conditional expectation gives us 0.

Finally if $n+k \leq 2n < j < n+h$ then the function $R_{n,j,h}$ is $T^{-2n}(\mathcal{A})$ measurable. We have

$$\begin{aligned}\mathbb{E}[R_{n,j,h} \cdot f_k^A \circ T^n | T^{-2n}\mathcal{A}] &= R_{n,j,h} \cdot \mathbb{E}[f_k^A \circ T^n | T^{-2n}\mathcal{A}] \\ &= R_{n,j,h} \cdot \mathbb{E}[f_k^A | T^{-n}\mathcal{A}] \circ T^n = 0 \quad \text{as } n \geq k.\end{aligned}$$

In conclusion if $h \geq k$, $n \geq k$ and $n+k \leq j$ then the corresponding terms in (4) are null. Hence (4) is bounded by

$$\begin{aligned}&2 \cdot 16kN \quad \text{(for the terms corresponding to } n < j < n+k\text{)}, \\ &+2 \cdot 16kN \quad \text{(for the terms corresponding to } n < k\text{)}.\end{aligned}$$

Thus dividing by N^2 we obtain the bound

$$2\frac{32kN}{N^2} = \frac{64k}{N}.$$

We can now finish the proof of Theorem 6 with the estimate $C_f/N^{1/2}$. □

Remark. Analogous computations show that we also have

$$\int \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^{-n}x) \left(\prod_{i=2}^{H} f_i(T^{-in}x) \right) e^{2\pi i n\varepsilon} \right|^2 d\mu(x) \leq \frac{C_f}{N^{1/2}}$$

for the same set of WW functions f.

WW functions in the product of a K automorphism and a measure preserving system. We have the following result.

THEOREM 7. *Let $(X, \mathcal{B}, \mu, T)$ be a K automorphism and $(Y, \mathcal{G}, \nu, S)$ a measure preserving dynamical system. Let f_k^A be the WW function obtained in the proof of Theorem 5 and let g be any $L^\infty(\nu)$ function. Then the function $f_k^A \otimes g$ is a WW function in the product space.*

Proof. It is enough to show that for any bounded sequence of real numbers (a_n) we have

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=0}^{N-1} f_k^A \circ T^n a_n e^{2\pi i n\varepsilon} \right| \right\|_1 \leq \frac{C_f \|a\|_\infty}{N^\alpha}$$

for all positive integers N. The constant C_f does not depend on the sequence (a_n). Once this is established, it will only remain to set $a_n = g(S^n y)$ for any $L^\infty(\nu)$ function

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g and then integrate the previous inequality with respect to ν. The function f_k^A is the one used in the proof of Theorem 3, $f_k^A = \mathbf{1}_A - \mathbb{E}(\mathbf{1}_A|T^{-k}\mathcal{A})$. Applying the inequality (1), for $N-1 > k$, we get

$$\sup_\varepsilon \left| \sum_{n=0}^{N-1} f_k^A(T^n x) a_n e^{2\pi i n\varepsilon} \right|^2 \le \sum_{n=0}^{N-1} |f_k^A(T^n x)|^2 |a_n|^2 + 2\sum_{h=1}^{N-1} \left| \sum_{n=0}^{N-h-1} f_k^A(T^{n+h}x) a_{n+h} f_k^A(T^n x) a_n \right|.$$

The function f_k^A being uniformly bounded by 2, the first term in this last inequality is bounded after integration with respect to μ by $4\|a\|_\infty$. Following the path of Theorem 3 we can estimate the second term in the same way. We have

$$2\sum_{h=1}^{k-1} \int \left| \sum_{n=0}^{N-h-1} f_k^A \circ T^{n+h} a_{n+h} f_k^A \circ T^n a_n \right| d\mu + 2\sum_{h=k}^{N-1} \left[\int \left| \sum_{n=0}^{N-h-1} f_k^A \circ T^{n+h} a_{n+h} f_k^A \circ T^n a_n \right|^2 d\mu \right]^{1/2}.$$

The first sum is bounded by $8kN\|a\|_\infty$. The second sum is equal to

$$2\sum_{h=k}^{N-1} \left[\int \sum_{n=0}^{N-h-1} |f_k^A \circ T^{n+h}|^2 |a_{n+h}^2| f_k^A \circ T^n|^2 |a_n|^2 \, d\mu + 2\sum_{n<j} \int f_k^A \circ T^{n+h} a_{n+h} f_k^A \circ T^n a_n f_k^A \circ T^{h+j} a_{h+j} f_k^A \circ T^j a_j \, d\mu \right]^{1/2}.$$

The computations made previously on the averages

$$\sum_{n<j} \int f_k^A \circ T^{n+h} f_k^A \circ T^n f_k^A \circ T^{h+j} f_k^A \circ T^j \, d\mu$$

show that the $\sum_{n<j}$ contains at most kN non-zero terms. From this we can conclude that

$$\left\| \sup_\varepsilon \left| \frac{1}{N} \sum_{n=0}^{N} f_k^A \circ T^n a_n e^{2\pi i n\varepsilon} \right| \right\|_1 \le \frac{C_{f_k^A}\|a\|_\infty}{N^\alpha} \quad \text{for } \alpha = \frac{1}{4}.$$

This ends the proof of the theorem. □

3.3. *Example of WW dynamical systems with zero entropy.* The example is given by the classical skew product on $\mathbb{T}^2$, the 2-torus defined by $S_\alpha : (x, y) \to (x+\alpha, x+y)$ where α is irrational. These dynamical systems are known to be ergodic and to have zero entropy. It is also known that the orthocomplement of the Kronecker factor for S_α is spanned by the functions $f_{p,q}(x, y) = e^{2\pi i q x} \cdot e^{2\pi i p y}$ with $p \neq 0$. The powers of S_α can be easily computed. We have $S_\alpha^n(x, y) = (x + n\alpha, nx + \frac{1}{2}\alpha n(n-1) + y)$. We want to prove the following proposition.

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PROPOSITION 8. *There exists a set $\mathcal{I}_\alpha$ of irrational numbers α of Lebesgue measure 1 such that for each α in this set the system $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), S_\alpha, m)$ is a WW dynamical system of power type in L^∞.*

Proof. Evaluating $\sup_\varepsilon |\sum_{n=0}^{N-1} f_{p,q}(S^n(x,y))e^{2\pi i n\varepsilon}|^2$ using the inequality (1), we get

$$\begin{aligned}
\sup_\varepsilon \left| \sum_{n=0}^{N-1} f_{p,q}(S^n(x,y))e^{2\pi i n\varepsilon} \right|^2 &= \sup_\varepsilon \left| \sum_{n=0}^{N-1} e^{2\pi i n[q\alpha+px+\varepsilon]} \cdot e^{2\pi i p\alpha(n(n-1)/2)} \right|^2 \\
&\le N + 2\sum_{h=1}^{N-1} \left| \sum_{n=1}^{N-h-1} e^{-2\pi i p\alpha((n+h)(n+h-1)/2)} \cdot e^{2\pi i p\alpha(n(n-1)/2)} \right| \\
&= N + 2\sum_{h=1}^{N-1} \left| \sum_{n=1}^{N-h-1} e^{2\pi i p\alpha([n^2-n-n^2-h^2-2hn+n+h]/2)} \right| \\
&= N + 2\sum_{h=1}^{N-1} \left| \sum_{n=1}^{N-h-1} e^{-2\pi i phn\alpha} \right|.
\end{aligned}$$

This estimate is independent of x and y. It is valid for all $1 < N$. In particular for $N = M^4$ we obtain the inequality

$$\begin{aligned}
&\sup_{x,y} \sup_\varepsilon \left| \frac{1}{M^4} \sum_{n=0}^{M^4-1} f_{p,q}(S^n(x,y))e^{2\pi i n\varepsilon} \right|^2 \\
&\qquad \le \frac{1}{M^4} + 2\left(\frac{1}{M^4} \sum_{h=1}^{M^4-1} \left| \frac{1}{M^4} \sum_{n=1}^{M^4-h-1} e^{-2\pi i nhp\alpha} \right|^2 \right)^{1/2}.
\end{aligned}$$

By taking the integral of this last term with respect to α we see that

$$\begin{aligned}
&\int \left[\frac{1}{M^4} + 2\left(\frac{1}{M^4} \sum_{h=1}^{M^4-1} \left| \frac{1}{M^4} \sum_{n=1}^{M^4-h-1} e^{-2\pi i nhp\alpha} \right|^2 \right)^{1/2} \right] d\alpha = \int g_M(\alpha)\, d\alpha \\
&\quad \le \frac{1}{M^4} + 2\left(\int \frac{1}{M^4} \sum_{h=1}^{M^4-1} \left| \frac{1}{M^4} \sum_{n=1}^{M^4-h-1} e^{-2\pi i pnh\alpha} \right|^2 d\alpha \right)^{1/2} \\
&\quad \le \frac{1}{M^4} + 2\left[\frac{1}{M^4} M^4 \left[\frac{1}{M^4} \right] \right]^{1/2} \\
&\quad \le \frac{C}{M^2}.
\end{aligned}$$

Hence we have $\sum_{M=1}^{\infty} \int M^{1/2} g_M(\alpha)\, d\alpha < \infty$. Thus we have for almost every α, $g_M(\alpha) \le C(\alpha)/M^{1/2}$. So we have obtained the estimate

$$\sup_{x,y} \sup_\varepsilon \left| \frac{1}{M^4} \sum_{n=0}^{M^4-1} f_{p,q}(S^n(x,y))e^{2\pi i n\varepsilon} \right|^2 \le \frac{C(\alpha)}{M^{1/2}}. \tag{5}$$

To derive an analogous estimate for

$$\sup_{x,y} \sup_\varepsilon \left| \frac{1}{N} \sum_{n=0}^{N-1} f_{p,q}(S^n(x,y))e^{2\pi i n\varepsilon} \right|^2,$$

we can proceed in the following manner.

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For $M^4 < N \le (M+1)^4$, we have

$$\sup_{x,y}\sup_{\varepsilon}\left|\frac{1}{N}\sum_{n=0}^{N-1} f_{p,q}(S^n(x,y))e^{2\pi in\varepsilon}\right|^2$$
$$\le C\left[\left(\frac{M^4}{N}\right)^2\left(\sup_{x,y}\sup_{\varepsilon}\left|\frac{1}{M^4}\sum_{n=0}^{M^4-1} f_{p,q}(S^n(x,y))e^{2\pi in\varepsilon}\right|^2\right)+\frac{1}{M^8}\left|\sum_{n=M^4}^{(M+1)^4} 1\right|^2\right]$$
$$\le C\left[\sup_{x,y}\sup_{\varepsilon}\left|\frac{1}{M^4}\sum_{n=0}^{M^4-1} f_{p,q}(S^n(x,y))e^{2\pi in\varepsilon}\right|^2+\frac{1}{M^2}\right].$$

Using the estimate in (5) we get

$$\sup_{x,y}\sup_{\varepsilon}\left|\frac{1}{N}\sum_{n=1}^{N} f_{p,q}(S^n(x,y))e^{2\pi in\varepsilon}\right|^2 \le \frac{C(\alpha)}{N^{1/8}},$$

$C(\alpha)$ being an absolute constant. □

Remarks. The system $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), S_\alpha, m)$, being a WW dynamical system of power type in L^∞, is then of power type in all L^p spaces for $1 \le p \le \infty$. The set of full measure can be made precise. It is shown in [**AN**] that for α with finite type the system $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), S_\alpha, m)$ is a WW dynamical system of power type in L^1.

3.4. *WW Dynamical systems and almost everywhere double recurrence.* In this section we would like to show that the arguments given in [**B**] can be simplified for WW dynamical systems. Actually, as we will show, for such systems the first two pages in [**B**] are all that is needed.

THEOREM 9. *Let $(X, \mathcal{B}, \mu, T)$ be an (invertible) WW dynamical system of power type α in L^1. Then there exists a dense set of functions $f_1 \in L^2$ such that for all functions $f_2 \in L^\infty(\mu)$ we have*

$$\left\|\frac{1}{N}\sum_{n=1}^{N} f_1\circ T^n f_2\circ T^{2n}\right\|_1 \le \frac{C_f}{N^{2\alpha/3}}.$$

As a consequence for all $f, g \in L^2$ the averages $(1/N)\sum_{n=1}^{N} f(T^n x)g(T^{2n}x)$ converge almost everywhere.

Proof. The difficult part comes when one of the functions, say f_1, is in $\mathcal{K}^\perp$. The first two pages of [**B**] show that we have the following estimates:

$$\|A_N(f_1,f_2)\|_1 \le \frac{2N_1}{N}\|f_1\|_1\|f_2\|_\infty + 2\left(\frac{N}{N_1}\right)^{1/2}\|f_2\|_\infty\left\|\sup_{\varepsilon}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi in\varepsilon} f_1\circ T^n\right|\right\|_1$$

and

$$\|A_N(f_1,f_2)\|_1 \le C\left[\left(\frac{N_1}{N}\right)+\left(\frac{N}{N_1}\right)^{1/2}\left\|\sup_{\varepsilon}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi in\varepsilon} f_1\circ T^n\right|\right\|_1\right], \qquad (6)$$

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where C is a finite positive constant. Assuming $\|f_1\|_1 \leq 1$ we put

$$N_1 = \left[N \left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n\varepsilon} f_1 \circ T^n \right| \right\|_1^{2/3} \right].$$

We get

$$\|A_N(f_1, f_2)\|_1 \leq C \left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n\varepsilon} f_1 \circ T^n \right| \right\|_1^{2/3}. \tag{7}$$

As we assume that f_1 is a WW function we get the following estimate:

$$\|A_N(f_1, f_2)\|_1 \leq \frac{C}{N^{2\alpha/3}}. \tag{8}$$

By choosing subsequences of the form $N = [M^\gamma]$ we can now obtain almost everywhere convergence to zero of the sequence $A_N(f_1, f_2)$. By approximation one can then obtain the last statement of Theorem 8. □

3.5. *WW dynamical systems and almost everywhere continuous random Fourier series.* In this section we want to show that WW functions produce almost everywhere continuous random Fourier series. We recall that a random Fourier series is defined as $F((\omega), \varepsilon) = \sum_{n=1}^{\infty} X_n(\omega) e^{2\pi i n\varepsilon}$ where X_n are random variables on the probability measure space $(\Omega, \mathcal{A}, P)$. We refer the reader to [**Ka, PM**] and [**LT**] for background on random Fourier series where the variables X_n are given by iid random variables. We are interested not only in the convergence of the series $F((\omega), \varepsilon) = \sum_{n=1}^{\infty} X_n(\omega) e^{2\pi i n\varepsilon}$, but also in the existence of a set of full measure $\widetilde{\Omega}$ with the following property: if $(\omega) \in \widetilde{\Omega}$ then the map $\varepsilon \to F((\omega), \varepsilon)$ is continuous. We will also deal with variables X_n which are not iid. More precisely we will look at the series $\sum_{n=1}^{\infty} (f(T^n x)/n) e^{2\pi i n\varepsilon}$ where $(X, \mathcal{B}, \mu, T)$ is an ergodic dynamical system and f a WW function. In the case of WW functions for the examples given in §3.2 such as K automorphisms, we will study also the almost everywhere continuity of the series $\sum_{n=1}^{\infty} ((\prod_{i=1}^{H} f_i(T^{in} x))/n) e^{2\pi i n\varepsilon}$ that we define as the rotated one-sided multiple ergodic Hilbert transform. We will show the almost everywhere continuity of these random Fourier series for a dense set of functions f_1. Let us remark that one can show easily that for almost everywhere x the series $\sum_{n=1}^{\infty} ((\prod_{i=1}^{H} f_i(T^{in} x))/n) e^{2\pi i n\varepsilon}$ represents an L^2 function of ε. The difficulty is to show that for almost everywhere x the series actually represents a continuous function. We have the following results.

THEOREM 10. *Let $(X, \mathcal{B}, \mu, T)$ be an (invertible) ergodic dynamical system and let f be a WW function. Then there exists a set of full measure X_f such that for x in X_f the functions*

$$\varepsilon \mapsto \sum_{n=1}^{\infty} \frac{f(T^n x)}{n} e^{2\pi i n\varepsilon}$$

and

$$\varepsilon \mapsto \sum_{n=1}^{\infty} \frac{f(T^{-n} x)}{n} e^{2\pi i n\varepsilon}$$

are continuous.

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Proof. As the proof and the method in the first version of this paper have been used in a subsequent paper that has already appeared [**AN**], we skip the proof of this theorem. Part of the argument uses Theorem 6 in [**A1**]. □

An immediate corollary is the following.

COROLLARY 11. *Let $(X, \mathcal{B}, \mu, T)$ be an invertible dynamical system. If $(X, \mathcal{B}, \mu, T)$ is a WW dynamical system then there exists in $\mathcal{K}^{\perp}$ a dense set of functions f for which we can find a set of full measure X_f such that the following holds: for $x \in X_f$ the sequence*

$$\sum_{n=-N}^{N}{}' \frac{f(T^n x)}{n} e^{2\pi i n\varepsilon}$$

converges for each $\varepsilon \in \mathbb{R}$. Furthermore the map

$$\varepsilon \to \sum_{n=-\infty}^{\infty}{}' \frac{f(T^n x)}{n} e^{2\pi i n\varepsilon}$$

is continuous.

This corollary is used in [**A3**]. In the multidimensional situation expressed in Theorem 6 we can also obtain the almost everywhere uniform convergence of the rotated multiple ergodic Hilbert transform defined as

$$\sum_{n=-\infty}^{\infty}{}' \frac{(\prod_{i=2}^{H} f_i(T^{in}x)) f(T^n x)}{n} e^{2\pi i n\varepsilon}$$

for the functions f_i. We leave the proof to the interested reader.

THEOREM 12. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and f a WW function of power type α in L^1. Then for all $g \in L^2(\mu)$ the series*

$$\sum_{n=1}^{\infty} \frac{f(T^n x).g(T^{2n}x)}{n^{\gamma}}$$

converges almost everywhere (μ) for each γ such that $2\alpha/3 + \gamma > 1$.

Proof. Theorem 9 shows that the averages

$$A_N(f, g)(x) = \frac{1}{N} \sum_{n=1}^{N} f(T^n x).g(T^{2n}x)$$

satisfy the inequality

$$\|A_N(f_1, f_2)\|_1 \leq \frac{C}{N^{2\alpha/3}}$$

for each N positive integer. Using the fact that the averages $A_N(f, g)(x)$ converge almost everywhere to zero, we just rewrite the series

$$\sum_{n=1}^{\infty} \frac{f(T^n x).g(T^{2n}x)}{n^{\gamma}}$$

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as

$$\lim_K \sum_{n=1}^{K} \frac{(n+1)A_{n+1} - nA_n}{n^\gamma} = \lim_K \frac{A_{K+1}}{K^\gamma} - A_1 + \sum_{n=2}^{K} A_n \left(\frac{1}{(n-1)^\gamma} - \frac{1}{n^\gamma} \right) + \sum_{n=1}^{K} \frac{A_{n+1}}{n^\gamma}.$$

The last sum converges by the previous inequality, the second because $1/(n-1)^\gamma - 1/n^\gamma$ is equivalent to $1/n^{1+\gamma}$. This ends the proof of the theorem. □

4. *Some questions and remarks*

The first version of this paper was written and circulated in 1998. In [**A3**] we used the definition of WW dynamical systems of logarithmic type.

Definition 4. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. A function f is a WW function of logarithmic type α in L^p if there exist finite positive constants C_f and α such that

$$\left\| \sup_\varepsilon \left| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_p \leq \frac{C_f}{(\ln(N+1))^{\alpha+1}} \quad \text{for all positive integers.}$$

Definition 5. An ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ is a WW dynamical system of logarithmic type α in L^p if there exists in $\mathcal{K}^\perp$ a dense set (for the L^2 norm) of WW functions of logarithmic type α in L^p.

But this does not affect the conclusions reached in [**A3**]. In fact all the theorems obtained in this paper could also be proved using this rate of convergence. It is clear that a WW dynamical system of power type is also a WW dynamical system of logarithmic type. For Theorem 9 the estimate for the L^1 norm of $A_N(f_1, f_2)$ is replaced with $C_f/(\ln(N+1))^\gamma$ for some $\gamma > 1$ when $\alpha > 1$. Picking $N = [t^M]$ and taking limits when t goes to one allows us to get the pointwise convergence to zero of $A_N(f_1, f_2)$.

Since the original version of this paper, several results have been obtained on the concept introduced here of WW functions. In [**A5**] and in [**L**] it was shown that the class of WW dynamical systems of power type in L^2 is a strict subset of the set of all ergodic dynamical systems. (All examples here are WW dynamical systems of power type in L^2.) It was also shown in [**AN**] and later in [**L**] that there are WW dynamical systems with singular spectrum. Thus the spectrum does not characterize the WW property studied here. Furthermore in [**A5**] it is shown that WW functions are directly linked to spectral properties of the transformation. They can also be characterized by the almost everywhere uniform convergence of series of the form $\sum_{n=1}^{\infty} (f(T^n x)/n^\gamma) e^{2\pi i n\varepsilon}$, for some $0 < \gamma < 1$.

Several questions were raised in the first version of this paper. Some answers have already been provided.

(1) We gave a spectral characterization in L^2 of the spectral measure associated with WW functions in [**A5**].

(2) We still do not know if all ergodic dynamical systems are WW dynamical systems in L^1. We do not know if there is a WW dynamical system of logarithmic type in L^1

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which is not of power type in L^1 (or in L^2). We produced such an example for the weak WW property in [**A5**]. Such an example is not of power type in L^2.

(3) Is the product of a K automorphism and an ergodic dynamical system a WW dynamical system? As pointed out to us by Thouvenot, such a result would imply among other results another proof of the return times theorem.

(4) It would be interesting to have more examples of WW dynamical systems.

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