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A NOTE ON ALTERNATING KNOTS IN HANDLEBODIES

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Abstract

We establish a Kauffman–Murasugi–Thistlethwaite-type theorem for alternating knots in a solid torus. Specifically, we show that any *dotted-reduced* alternating diagram of a knot in a handlebody realises the minimal crossing number. The proof relies on a generalisation of the Jones polynomial to the setting of handlebodies. A stronger version of this result was already proved by Boden *et al.* ['Adequate links in thickened surfaces and the generalized Tait conjectures', *Algebr. Geom. Topol.* **23**(5) (2023), 2271–2308] using a different generalised Jones polynomial; this paper largely expands on one of the main proof tools.

2020 *Mathematics subject classification*: primary 57K10; secondary 57K12, 57K14, 57K31.

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1. Introduction

In the late 19th century, Tait formulated several conjectures concerning alternating knot diagrams, motivated by empirical observations and early attempts to classify knots. One of these, the reduced alternating diagram conjecture, asserts that any two reduced alternating diagrams of the same knot have the same number of crossings. Equivalently, the minimal crossing number of an alternating knot is achieved by any of its reduced alternating diagrams [7].

This conjecture remained unresolved for more than a hundred years, until it was finally proved independently by Kauffman [4], Murasugi [6] and Thistlethwaite [8] in 1987. Their proofs made essential use of the Jones polynomial, showing that its span determines the crossing number for alternating knots. The resolution of Tait's conjecture not only confirmed his geometric intuition, but also marked a key moment in the development of knot theory, demonstrating the power of quantum invariants in low-dimensional topology.

Every 3-manifold admits a Heegaard decomposition, dividing it into two handlebodies of genus g. The decomposition represents the manifold as the union of these two handlebodies, glued along the common boundary surface. This decomposition

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provides a framework to project knots in M^3 onto the handlebodies. We have studied reduced alternating knots in genus-1 handlebodies and a notion of *dotted-reduced* diagrams (see Definition 3.3). For such diagrams, we prove the following result.

COROLLARY 3.6. *An alternating dotted-reduced diagram of a link in a solid torus is minimal (with respect to crossing number).*

We prove this by studying a generalisation of the Jones polynomial developed by Bataineh and Hajij in [1] for links in a solid torus. This polynomial is very similar to, but not the same as, the generalised Jones polynomial appearing in [3] or the Jones–Krushkal polynomial of [2, 5].

We note that Corollary 3.6 is already proved in [3] as a more general result, but we specialise to handlebodies, studying the links in this setting so that we can later use these tools and this language to study links in arbitrary 3-manifolds.

We define generalised reduced diagrams (which can be detected from the projection) for knots in handlebodies and use the span of the generalised Jones polynomial for this study. These same ideas extend to studying alternating knots in genus-g handlebodies.

2. Preliminary

In this section, we give a description of a Jones polynomial for links in a solid torus; we refer the reader to [1] for more details.

Let $M = S^1 \times D^2$ be a solid torus, viewed as a handlebody of genus one. In classical knot theory, a link in S^3 can be studied via its projection onto a disk. Turning to the theory of links in solid handlebodies, a link in M can be studied via its projection onto a punctured disk $D^2 \setminus \{0\}$, where the puncture corresponds to the core $S^1 \times \{0\}$ of the solid torus.

To contrast with the classical case, we will often call such a projection a *dotted diagram*. An example of a knot sitting in a solid torus and a corresponding dotted diagram appear in Figure 1. We generally conflate the notion of a 'link in a solid torus' with its 'link diagram in the punctured disk' and simply speak of a *link diagram in the solid torus*.

To define a Jones-type polynomial invariant in this setting, we work within the *Kauffman bracket skein module* of a link diagram D in the solid torus developed by Bataineh and Hajij in [1]. The bracket $\langle\cdot\rangle$ in the variables A and t is defined by

$$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle =\!= \rangle$$

$$\langle D \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle$$

$$\langle D \cup \odot \rangle = (-A^2 - A^{-2})\, t\, \langle D \rangle$$

$$\langle \bigcirc \rangle = 1$$

$$\langle \odot \rangle = t.$$

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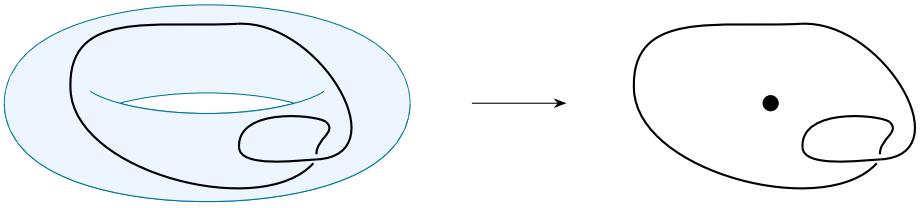


FIGURE 1. A link in a solid torus (left) and its dotted diagram (right).

Let $R = \mathbb{Z}[A^{\pm 1}, t]$. The *Kauffman bracket skein module* $\mathcal{S}(M)$ is the R-module generated by isotopy classes of framed unoriented links in M, modulo the Kauffman bracket relations shown above.

The module $\mathcal{S}(M)$ has a natural basis indexed by the number of parallel copies of the core curve of the torus. That is, any link can be resolved via the skein relations into a linear combination of powers of a generator t, where t denotes a longitude of the torus.

The skein module of the solid torus is the free $\mathbb{Z}[A^{\pm 1}]$-module generated by powers of t and the above sum gives a well-defined skein element. For an oriented diagram D with writhe $w(D)$, we normalise as in the case of S^3 to obtain the Jones polynomial.

The *Jones polynomial* $V_L(q, t)$ for a framed link $L \subset S^1 \times D^2$ is derived from the normalised Kauffman bracket as follows.

- Let $\langle D \rangle \in \mathcal{S}(S^1 \times D^2)$ be the Kauffman bracket of a diagram D of a link L, expressed in the basis $\{1, t, t^2, \ldots\}$.
- Each basis element t^n corresponds to n parallel longitudes (a cable of the core).
- Apply the writhe correction and change variables to obtain a Jones polynomial for links in the thickened surface, invariant among all diagrams of the link L,

$$V_L(q, t) = (-A^3)^{-w(D)} \langle D \rangle|_{A=q^{-1/4}},$$

  where $w(D)$ is the *writhe* of the diagram (the number of positive crossings  minus the number of negative crossings).

When we only consider one link L, we will generally write simply V instead of V_L.

Equivalently, let S be a Kauffman state of D, that is, a choice of A- or B-smoothing at each crossing. Write $a(S)$ and $b(S)$ for the number of A- and B-smoothings, respectively (see Figure 2).

Then, we express $\langle D \rangle$ as a sum over the contributions from each state S_σ as

$$\langle D \rangle \;=\; \sum_{\sigma} A^{\,a(S_\sigma) - b(S_\sigma)} \, (-A^2 - A^{-2})^{|S_\sigma| - 1} \, t^{|T_\sigma|}, \tag{2.1}$$

where $|S_\sigma|$ is the total number of circles in the state, $|T_\sigma|$ is the number of circles that 'go around the dot', $a(S_\sigma)$ is the number of A-smoothings in the state and $b(S_\sigma)$ is the number of B-smoothings in the state.

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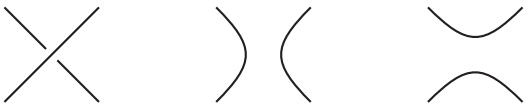


FIGURE 2. A crossing (left), its A-smoothing (middle) and its B-smoothing (right).

The Jones polynomial in a solid torus is sensitive to winding: links that are isotopic in S^3, but differ in their winding around the core of $S^1 \times D^2$, generally have different Jones polynomials. If a link lies in a 3-ball embedded in a solid torus, its Jones polynomial agrees with the classical Jones polynomial in S^3.

We can also look at this as $\langle D \rangle = \sum_\sigma \langle D|S_\sigma\rangle$, where the contribution of a single state S_σ to $\langle D\rangle$ is

$$\textit{contribution of state } S_\sigma = \langle D|S_\sigma\rangle = A^{a(S_\sigma)-b(S_\sigma)}(-A^2 - A^{-2})^{|S_\sigma|-1} t^{|T_\sigma|}. \tag{2.2}$$

DEFINITION 2.1. We say that two states S and S' are *adjacent* if S' is obtained from S by changing the smoothing at exactly one crossing.

As with the classical Kauffman bracket, we have the following result for the bracket polynomial for links in a solid torus.

PROPOSITION 2.2. *For any link diagram D in a solid torus,*

$$\max \deg_A \langle D\rangle \leq \max \deg_A(\textit{contribution of the all-}A\textit{ state}) = \max \deg_A \langle D|S_A\rangle$$

and

$$\min \deg_A \langle D\rangle \geq \min \deg_A(\textit{contribution of the all-}B\textit{ state}) = \min \deg_A \langle D|S_B\rangle.$$

PROOF. The proof is the same as that of the classical case and hence we omit it. □

PROPOSITION 2.3. *For any link diagram D in a solid torus with n crossings,*

$$\operatorname{span}\langle D\rangle \leq 4n.$$

PROOF. From Proposition 2.2,

$$\begin{aligned}
\operatorname{span} V_D &= \max \deg_A \langle D\rangle - \min \deg_A \langle D\rangle \\
&\leq \max \deg_A \langle D|S_A\rangle - \min \deg_A \langle D|S_B\rangle \\
&= a(S_A) - b(S_A) + 2(|S_A| - 1) - (a(S_B) - b(S_B) - 2(|S_B| - 1)) \\
&= n + 2(|S_A| - 1) - (-n - 2(|S_B| - 1)) \\
&= 2n + 2(|S_A| + |S_B| - 2) \\
&\leq 2n + 2n = 4n.
\end{aligned}$$

□

REMARK 2.4. This is exactly the same proof as in the classical case. It is included because we will look more closely at some of the inequalities later.

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3. Dotted-reduced diagrams

We now introduce the notions of reduced and dotted-reduced diagrams for knots in a solid torus.

3.1. Reduced and dotted-reduced diagrams. In classical knot theory, a crossing c in a knot diagram is *nugatory* if we can draw a simple closed curve in the projection disk such that the curve intersects the diagram only at the double point of the crossing c. We analogously define nugatory crossings appearing in diagrams of links in a solid torus, described at the start of Section 2.

DEFINITION 3.1. In a diagram of a link in a solid torus, a crossing c is called *nugatory* if we can draw a simple closed curve in the projection punctured disk such that the curve intersects the diagram only at the double point of the crossing c.

In classical knot theory, a diagram is called *reduced* if it does not have any nugatory crossings. The idea is that any nugatory crossing can be 'removed' by twisting one half of the diagram. In the setting of a solid torus, not all nugatory crossings can actually be undone that way, as the structure of the solid torus may prevent such twists.

DEFINITION 3.2. Let D be a link diagram in a solid torus, represented as a dotted diagram, with a crossing c that is nugatory as in Definition 3.1. If we can draw a simple closed curve in the projection punctured disk that intersects D only in the double point of the crossing c such that the curve is contractible in the projection punctured disk, then we say the crossing is *dotted-reducible* (see Figure 3). If instead all such curves are not contractible, then we say the crossing is *dotted-irreducible*.

What we call *dotted-irreducible* crossings for diagrams in a solid torus are comparable to Boden, Karimi and Sikora's *essential* crossings for diagrams in a thickened surface in [3].

DEFINITION 3.3. A link diagram in a solid torus is *dotted-reduced* if it has no nugatory crossings or if its only nugatory crossings are dotted-irreducible crossings.

We can now state our main result, which is proved in the next section.

THEOREM 3.4. *Let D be an alternating dotted-reduced link diagram in a solid torus with n crossings. Then,* $\operatorname{span}\langle D\rangle = 4n$.

We then relate this Kauffman bracket polynomial for links in a solid torus to the corresponding Jones polynomial for links in a solid torus, giving two corollaries.

COROLLARY 3.5. *Let D be an alternating dotted-reduced link diagram in a solid torus with n crossings. Then,* $\operatorname{span} V = n(D)$.

PROOF. This follows from the construction of V from $\langle D\rangle$ and Theorem 3.4. □

COROLLARY 3.6. *An alternating dotted-reduced diagram of a link in a solid torus is minimal with respect to crossing number.*

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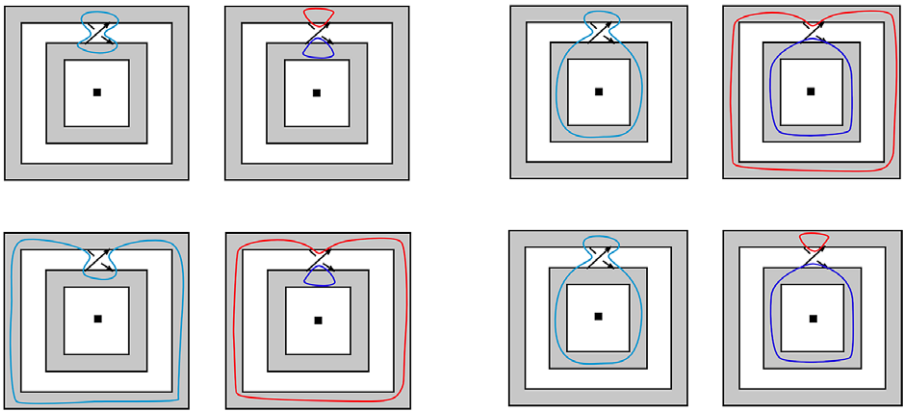


FIGURE 3. Removing dotted-reducible crossings.

PROOF. The proof is analogous to the proof for the classical case. For an alternating dotted-reduced diagram D, by Corollary 3.5, $\operatorname{span} V \leq n(L) \leq n(D) = \operatorname{span} V$, so D must be a minimal diagram. □

4. Proof of the main theorem

Now, we begin to prove Theorem 3.4. In the proof of Proposition 2.3, we obtain

$$\operatorname{span}\langle D\rangle \leq \max \deg_A \langle D|S_A\rangle - \min \deg_A \langle D|S_B\rangle = 2n + 2(|S_A| + |S_B| - 2) \leq 4n.$$

From classical knot theory, if D is an alternating diagram, then $|S_A| + |S_B| = n + 2$ and so the last inequality is an equality [4]. This holds regardless of whether or not the diagram is reduced (in a classical sense), so it still holds in the dotted-reduced setting.

To show that in an alternating dotted-reduced diagram we have $\operatorname{span}\langle D\rangle = 4n$, it remains to show that the first inequality must also be an equality.

THEOREM 4.1. *Let D be an alternating dotted-reduced link diagram in a solid torus. Then,*

$$\max \deg_A \langle D\rangle = \max \deg_A(\textit{contribution of the all-A state}) = \max \deg_A \langle D|S_A\rangle$$

and

$$\min \deg_A \langle D\rangle = \min \deg_A(\textit{contribution of the all-B state}) = \min \deg_A \langle D|S_B\rangle.$$

PROOF. By symmetry, it suffices to prove the first statement. We claim that the contribution of the all-A state to the bracket polynomial is not cancelled out by the contribution of any other state.

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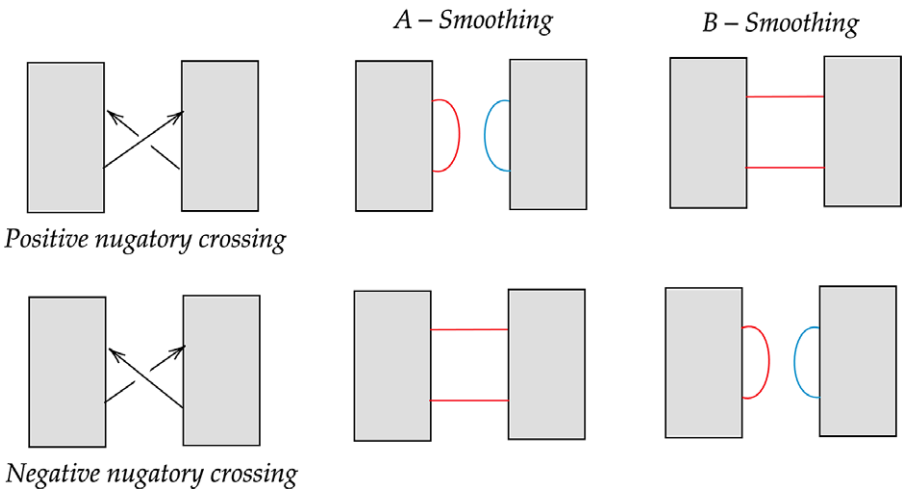


FIGURE 4. Smoothing crossings.

Suppose for a contradiction that it does cancel out. Then, there is some other state S_σ with $\max \deg_A\langle D|S_\sigma\rangle = \max \deg_A\langle D|S_A\rangle$, and so

$$\underbrace{a(S_\sigma) - b(S_\sigma) + 2(|S_\sigma| - 1)}_{=\max \deg_A\langle D|S_\sigma\rangle} = \underbrace{n + 2(|S_A| - 1)}_{=\max \deg_A\langle D|S_A\rangle}.$$

Since $a(S_\sigma) - b(S_\sigma) = n - 2b(S_\sigma)$, this is equivalent to $|S_\sigma| = |S_A| + b(S_\sigma)$.

Consider $\widetilde{D}$, the image of D under the obvious lift from $A \times I$ to $\mathbb{R}^2 \times I$. Since we want to find the total number of A-circles, and the number of A-circles in D is the same as the number of A-circles in $\widetilde{D}$, we can consider $\widetilde{D}$ and leverage what we know about classical diagrams. In a reduced alternating classical diagram $\widetilde{D}$, any state adjacent to S_A will have one fewer circle than S_A. However, if the alternating diagram has nugatory crossings, there may be states adjacent to S_A that have more circles. □

CLAIM 1. *Let $\widetilde{D}$ be a classical (not necessarily reduced) alternating link diagram. State S_σ satisfies $|S_\sigma| = |S_A| + b(S_\sigma)$ if and only if S_σ is obtained from S_A by performing B-smoothings at negative nugatory crossings.*

PROOF. Let S_σ be a state satisfying $|S_\sigma| = |S_A| + b(S_\sigma)$. Then, S_σ is obtained from S_A by changing the smoothings on a sequence of $b(S_\sigma)$ crossings such that at every step in the sequence, the total number of circles in the resulting state increases by one.

Starting with S_A and changing the smoothing of a nonnugatory crossing will decrease the number of circles. So, S_σ must be obtained by changing the smoothing at a nugatory crossing.

In Figure 4, we see that a state S_σ adjacent to S_A will have $|S_\sigma| = |S_A| + 1$ if and only if S_σ is obtained from S_A by performing a B-smoothing on a negative

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nugatory crossing. Similarly, any state S_σ will have $|S_\sigma| = |S_A| + b(S_\sigma)$ if and only if S_σ is obtained from S_A by performing B-smoothings on $b(S_\sigma)$ negative nugatory crossings. □

CLAIM 2. *Let D be a dotted-reduced alternating link diagram in a solid torus. A state S_σ satisfies $|S_\sigma| = |S_A| + b(S_\sigma)$ if and only if S_σ is obtained from S_A by performing B-smoothings on negative dotted-irreducible crossings.*

PROOF. Any nugatory crossing in $\widetilde{D}$ corresponds to a dotted-reducible or dotted-irreducible crossing in D. Since our D is dotted-reduced, any nugatory crossing in $\widetilde{D}$ corresponds to a dotted-irreducible crossing in D. So, if S_σ has the same max A-degree as S_A, then S_σ has B-smoothings at some number of negative dotted-irreducible crossings and has A-smoothings at all other crossings. □

CLAIM 3. *If S_σ is obtained from S_A by performing B-smoothings at negative dotted-irreducible crossings, then $|T_\sigma| > |T_A|$. (As in (2.1) and (2.2), $|T_\sigma|$ is the number of circles that 'go around the dot' in state S_σ.)*

PROOF. Consider two states S'_σ and S_σ that differ only at some negative dotted-irreducible crossing x. If S'_σ is the state with x A-smoothed and S_σ is the state with x B-smoothed, then either $|T_\sigma| = |T'_\sigma|$ or $|T_\sigma| = |T'_\sigma| + 2$. This is shown in Figure 2. So, if S_σ is obtained from S_A by successively changing the smoothing at negative dotted-irreducible crossings (changing from A-smoothings to B-smoothings), then $|T_\sigma| \geq |T_A|$.

Now, consider what happens when we start with the all-A state and change the smoothing at just one negative dotted-irreducible crossing. Call this state S'_σ. Since the diagram is dotted-reduced and alternating, the A-state must look like the example shown in Figure 5: the circle in the A-state that meets itself at this negative dotted-irreducible crossing does not go around the dot. When we change the smoothing at this crossing to be a B-smoothing, we obtain two circles that go around the dot, as in the last row of Figure 5. Then, the number of circles that go around the dot in S'_σ is $|T'_\sigma| = |T_A| + 2$.

So, if S_σ is obtained from S_A by changing some number of negative dotted-irreducible crossings, then $|T_\sigma| \geq |T'_\sigma| = |T_A| + 2 > |T_A|$. □

Thus, any state S_σ which has the same max $\deg_A$ as S_A will not have the same value of $|T|$, so the contributions of S_σ and S_A cannot cancel out in the Kauffman bracket for links in the solid torus. Therefore, the contribution of the all-A state to the Jones polynomial does not cancel out and we have $\max \deg_A \langle D \rangle = \max \deg_A \langle D|S_A \rangle$.

This completes the proof of Theorem 4.1.

REMARK 4.2 (Alternating knots in genus-g handlebodies). The Jones polynomial for alternating knots in genus-g handlebodies can be obtained analogously to the computation in [1]. One will get multiply-dotted diagrams, one dot for each genus. In particular, the same polynomial expression as derived therein applies in our setting and the proof carries over verbatim, but with more variables introduced to record the

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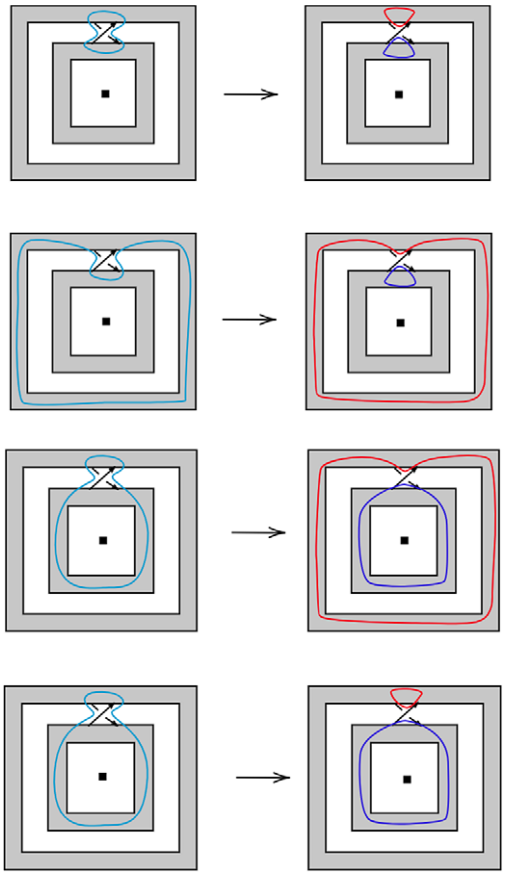


FIGURE 5. In each of the four rows, we have on the left a state S in which our negative dotted-irreducible crossing is given an A-smoothing, and on the right the adjacent state S' which is identical to S except at that one crossing. The four rows show each of the four possibilities for how a state circle involving this crossing could interact with the rest of the diagram.

occurrence of circles going around each or many of these additional dots. Since no new ideas are required, we omit the details. The details of the polynomial can be checked in [1].

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