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Horizontal partitions and Kleshchev's algorithm

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1. *A conjecture in modular representation theory*

We shall work with rational $\mathbf{M}_n(K)$-modules, where K is an infinite field of prime characteristic $p > 0$, and $\mathbf{M}_n(K)$ is the full matrix semigroup of $n \times n$ matrices over K. Recall that equivalence classes of simple rational $\mathbf{M}_n(K)$-modules are parametrized by the set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of length $l = l(\lambda) \leqslant n$, and that the socle $L(\lambda)$ of the Schur module (or dual Weyl module) $H^0(\lambda)$ is a simple $\mathbf{M}_n(K)$-module whose highest weight corresponds to the partition λ.

Here we are concerned with simple quotients of Schur modules, rather than their socles. By a theorem of Donkin [**1**] and Kouwenhoven [**6**], if λ is row regular then $H^0(\lambda)$ has a unique simple quotient, which is isomorphic to $L(\mu)$ for $\mu' = \iota(\lambda)$. Here ι is the involution on the set of row regular partitions of degree $d = \sum_i \lambda_i$ which is defined by multiplying the corresponding simple modular character of the symmetric group Σ_d by the alternating character, and μ' is the (column regular) partition conjugate to μ. Here we give a direct construction for μ in the case where λ is horizontal [**9**], so proving [**2**, conjecture 2·12]. For this purpose we use an algorithm discovered by Kleshchev [**5**] which describes ι. (In a more recent preprint [**3**], Ford and Kleshchev have proved that the famous algorithm of Mullineux [**8**] also describes ι.)

In order to explain our result, some definitions are needed. First, the partition λ is (p-)*horizontal* if and only if, for each column of the diagram of λ, either all or none of the hook lengths is divisible by p [**9**, §0]. Equivalently [**9**, remark 3·7], every sequence of removals of (rim) p-hooks from the diagram of λ consists of successive removals of p nodes from the same row. Secondly, with any partition λ we associate the column regular partition λ^C obtained as follows by conjugation from the row regularization construction λ^R of [**4**, p. 282]. Consider the nodes of the diagram of λ as points on a family of straight line segments (or 'ladders') of slope $1/(p-1)$ in the plane. To obtain the diagram of the column regularization λ^C from this, slide the nodes down the ladders as far as possible. For example, when $p = 3$ and $\lambda = (7, 1)$, $\lambda^C = (3, 2^2, 1)$. Similarly, for any partition μ, the row regular partition μ^R is obtained by sliding the nodes of the diagram of μ up ladders of slope $p-1$ as far as possible. For example, when $p = 3$ and $\mu = (2, 1^6)$, $\lambda^R = (4, 3, 1)$ [**4**, example 6·3·49]. Clearly $(\lambda^C)' = (\lambda')^R$ for any λ.

The purpose of this note is to prove the following result [**2**, conjecture 2·12].

PROPOSITION 1. *For any horizontal partition* λ,
(i) $\iota(\lambda)' = \lambda^C$,
(ii) $L(\lambda^C)$ *is the unique simple quotient of* $H^0(\lambda)$.

The equivalence of (i) and (ii) follows immediately from the theorem of Donkin and

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Kouwenhoven. If λ satisfies the Carter condition, (namely, if for each column of the diagram of λ, and for each positive integer c, either all or none of the hook lengths is divisible by p^c), then the result is proved in [**2**, corollary 2·11]. This, of course, includes the case where λ is a core.

2. *Kleshchev's algorithm*

The algorithm of [**5**] requires the concepts of *normal* and *good* nodes in the diagram of a partition.

The following preliminary definitions are needed. A *removable* (or corner) node is a node of the diagram of λ whose removal gives the diagram of a partition of degree $d-1$. An *indent* node is a node not in the diagram of λ whose addition gives the diagram of a partition of degree $d+1$. The (p-)*residue* of the node in row i and column j of a diagram is the residue class of $j-i \bmod p$ [**4**, §2·7]. For (Young) diagrams of partitions, we follow the conventions of [**7**, chapter I], so that the top row of the diagram of λ is row 1 and the bottom row is row $l(\lambda)$.

Given a removable node (i,j) in row i with residue r, form a sequence of 1's and 2's by reading up the diagram from row $i-1$ to row 1, using 1 to denote each removable node with residue r, and 2 to denote each indent node with residue r. Then the node (i,j) is *normal* if and only if this sequence is a lattice permutation [**7**, I·9], *i.e.* each initial segment of the sequence contains at least as many 1's as 2's. For example, the normal nodes of $\lambda = (6,4,4,2,1)$ are $(1,6)$ and $(5,1)$ [**5**, example 1·2].

A normal node is *good* if and only if it is the lowest normal node with a given residue. Thus a diagram can have at most one good node with a given residue. Note that if λ is row regular and $\lambda \neq (0)$, then the highest removable node of λ is normal [**5**, remark 4·9], and so λ has at least one good node. For such a λ, we shall call the residue $r(\lambda)$ of the highest removable node the *highest residue* of λ.

We now describe Kleshchev's algorithm. The first step is to construct a sequence $r_1, r_2, \ldots, r_d$ of residues, where $d = \sum_i \lambda_i$, by the following rule. Starting from the diagram of λ, delete any one of its good nodes. Let this node have residue r_1. Now delete any good node from the diagram that remains, and let r_2 be its residue. Continue in this way until the diagram is empty, and a sequence $r_1, r_2, \ldots, r_d$ of residues is generated. Such a sequence is called a λ-sequence in [**5**]; we shall call it a *Kleshchev sequence* for λ. In general, the sequence is not unique.

By reversing a Kleshchev sequence and replacing each residue r by the complementary residue $s = -r$, we obtain a new sequence of residues $s_d, s_{d-1}, \ldots, s_1$, which we call the *complementary sequence*. The second step of the algorithm uses this sequence to build up the diagram of a partition, by the following rules: (1) at each stage, the new node added to the diagram must become a good node in the resulting diagram, and (2) this node must have the residue given by the corresponding term of the complementary sequence.

Kleshchev proves that there is a unique way to carry out this construction, that the resulting diagram is independent of the choice of the sequence $r_1, r_2, \ldots, r_d$, and that it is the diagram of the partition $\iota(\lambda)$. In other words, the complementary sequence is a Kleshchev sequence for $\iota(\lambda)$.

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3. *The case of a horizontal partition*

The following fact explains why Kleshchev's algorithm is easy to apply in the case where λ is a horizontal partition.

LEMMA 2. *Every removable node in a horizontal partition λ is normal. Thus a node of λ is good if and only if it is the lowest among all removable nodes with the same residue.*

Proof. Suppose that λ is horizontal, and let its diagram have a removable node (i,j). Since a removable node has hook length 1, none of the jth column hook lengths is divisible by p. Hence no indent node in a row $i' < i$ can have the same residue as (i,j), and the result follows. ∎

We next describe a specific way to select a Kleshchev sequence for a horizontal partition λ. We shall call this the *standard* Kleshchev sequence of λ.

Note that the removal of a node with a given residue r does not affect the normality of any other removable nodes with residue r. It follows that we can successively strip off all removable nodes with the same residue, from the lowest to the highest, and this will give the initial segment of a Kleshchev sequence. (More generally, for an arbitrary partition, one can remove in this way all the normal nodes with the same residue [**5**, remark 4·9].) We shall call this procedure 'stripping the residue r from λ'.

In terms of the abacus representation of λ [**4**, §2·7], removable nodes correspond to beads which have a gap immediately below them, and nodes with the same residue r correspond to beads on the same runner. If we choose the number of beads used on the abacus to be divisible by p, then this runner is the rth runner. Thus, in terms of the abacus, stripping the residue r from λ corresponds to transferring, whenever possible, a bead on the rth runner to a gap immediately below on the $(r-1)$th runner. (As usual, the runners are numbered cyclically to correspond to residue classes mod p, so that if $r = 0$ then $r-1 = p-1$).

If λ is horizontal and is not a core, then by [**9**, lemma 3·6] its quotient has a unique non-zero component, which is a partition α. The highest $l(\alpha)$ beads in the abacus representation of λ correspond to removable nodes with the same residue, and this is the highest residue $r(\lambda)$ of λ. When we strip the highest residue from λ, all these $l(\alpha)$ beads, and possibly others on the $r(\lambda)$th runner, will be moved. However, if a bead is moved, then any higher bead on the $r(\lambda)$th runner is also moved, because the component of the quotient of λ on the $(r(\lambda)-1)$th runner is (0). It follows from [**9**, remark 3·7] that the partition μ remaining after the stripping operation is also a horizontal partition. If μ is not a core, then it is clear that the highest residue of μ is $r(\lambda)-1$. If λ is a non-zero core, the same argument shows that μ is a core, but, in this case, the highest residue of μ may have any value other than $r(\lambda)$.

By iterating the procedure of stripping the highest residues, we obtain a Kleshchev sequence for λ. In this sequence, residues are stripped in descending cyclic order, until a core is reached. We shall refer to the sequence obtained by this rule as the *standard* Kleshchev sequence of the horizontal partition λ.

We break this standard sequence into segments $K_1(\lambda)K_2(\lambda)\dots K_k(\lambda)$, where each segment represents a *stripping cycle* by $p-1$ successive residues $r, r-1, \dots, r-p+2$. Note that one residue is omitted from each cycle, so that the following cycle will begin by stripping $r+1$. Each segment $K_j(\lambda)$ has the form $(r^{a_1}, (r-1)^{a_2}, \dots, (r-p+2)^{a_{p-1}})$,

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where $a_i \geqslant 0$, and where r^a means that the residue r is repeated a times. If the partition remaining at the end of the jth stripping cycle is not a core, then, by the discussion above, all the a_i are non-zero.

We shall prove that the lengths of the segments $K_j(\lambda)$, $1 \leqslant j \leqslant k$, give a partition of d, and that this partition is $\iota(\lambda)'$.

LEMMA 3. *Let λ be a horizontal partition, and let $c_j(\lambda)$ be the length of the segment $K_j(\lambda)$ of the standard Kleshchev sequence of λ. Then*

(i) *the nodes removed from the diagram of λ during the first stripping cycle $K_1(\lambda)$ are the nodes which lie at the bottom of a column and which have hook length $< p$,*

(ii) *the number of such nodes is $c_1(\lambda) = \nu_{l(\alpha)+1} + (p-1)\,l(\alpha)$, and*

(iii) *the sequence $(c_j(\lambda))$ is monotone decreasing*, i.e. *$c(\lambda)$ is a partition.*

Proof. (i) Since residues are in ascending cyclic order as we read up a column, at most one node can be removed from a column during a stripping cycle.

Let the first row indent node of λ have residue $r+1$. (If λ is not a core, then $r = r(\lambda)$, the highest residue of λ.) The node (i,j) at the bottom of the jth column has residue $r+1$ if and only if the hook length of the $(1,j)$ node is divisible by p. Since λ is horizontal, this is true if and only if the hook length of the node (i,j) is itself divisible by p. Thus successive stripping of all the removable nodes of residues $r, r-1, \ldots, r-p+2$ will remove precisely those bottom nodes of λ with hook length $< p$.

(ii) In a horizontal partition with quotient α and core ν, each node lies either in one of the first $l(\alpha)$ rows or in one of the first $\nu_{l(\alpha)+1}$ columns [**9**, lemma 3·6]. Thus it follows from (i) that $p-1$ nodes are removed from each of the first $l(\alpha)$ rows, and one node from each column of length $> l(\alpha)$.

(iii) Let μ be the partition remaining after the first stripping cycle. Then μ is horizontal, with core ρ and quotient β, say. The partition β is either the same as α or is obtained from α by removal of the first column, so $l(\beta) \leqslant l(\alpha)$. Also $\rho_{l(\alpha)+1} \leqslant \nu_{l(\alpha)+1}$, their difference being the number of nodes removed from row $l(\alpha)+1$ during the first stripping cycle. Since ρ is a core, it is column regular, and hence $\rho_{l(\beta)+1} - \rho_{l(\alpha)+1} \leqslant (p-1)(l(\alpha)-l(\beta))$. From (ii) we have

$$c_1(\lambda) - c_2(\lambda) = (p-1)(l(\alpha)-l(\beta)) - (\rho_{l(\beta)+1} - \nu_{l(\alpha)+1}),$$

and it follows that $c_1(\lambda) \geqslant c_2(\lambda)$. By repeating this argument, we obtain (iii). ∎

We next consider the second stage of Kleshchev's algorithm, in which the diagram for $\iota(\lambda)$ is built up in stages corresponding to the stripping cycles. For this, we have first to reverse the sequence and replace each residue r by the complementary residue $s = -r$. We shall refer to the segment of this sequence which corresponds to the jth stripping cycle $K_j(\lambda)$ as the jth *complementary cycle*.

LEMMA 4. *During each complementary cycle, exactly one node is added to each existing row, and up to $p-1$ new rows are started by placing a single node in each.*

Proof. We show that (1) two nodes cannot be added to the same row, (2) not more than $p-1$ new rows can be started, and (3) one node is added to each existing row.

For (1), observe that nodes are added in descending cyclic order of residues, while the residues along the row are in ascending cyclic order. Similarly, (2) follows by considering the residues in the first column.

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For (3), we use the condition that nodes are always to be added so that they become good nodes in the resulting diagram. This implies that nodes of a given residue are added by filling indent nodes downwards from the top of the diagram, without any omissions. Moreover, at any stage an indent node cannot have the residue s that is missing from the complementary cycle. This follows by induction on the cycles, since the previous node in that row would then have the residue $s-1$, so that it could not have been added during the preceding complementary cycle. Finally, by Lemma 3(iii), the number of nodes to be added increases monotonically from one complementary cycle to the next, so that there are always enough new nodes to supply one to each existing row.

PROPOSITION 5. *For any horizontal partition* λ, $\iota(\lambda)' = c(\lambda)$.

Proof. By Lemma 4, the partition $c(\lambda)$ gives the column lengths of the partition which is built up by applying the second stage of Kleshchev's algorithm to the complementary sequence. The result follows from [**5**, theorem 4·7].

Remark 6. The above proof gives a simple algorithm for constructing the partition $\iota(\lambda)$ from the diagram of λ in the special case of a horizontal partition. Note that $c(\lambda)$ may be regarded as the content of the tableau formed by numbering with j each node of λ which is removed from the diagram during the jth stripping cycle. By Lemma 3(ii), this tableau can be obtained as follows. Number with 1 all nodes at the foot of a column which have hook length $< p$. Remove these nodes, and number with 2 all nodes satisfying the same condition in the diagram which remains, and so on. For example, the tableau below shows that for $p = 3$ we have $\iota(11, 6, 1) = (6, 5, 3^2, 1)$.

6	5	5	4	4	3	3	2	2	1	1
3	3	2	2	1	1					
1										

Proof of Proposition 1. Since $(\lambda^C)' = (\lambda')^R$, statement (i) is equivalent to $\iota(\lambda) = (\lambda')^R$. By comparing Lemma 3(ii) with formula (7) of [**2**], we have $c_1(\lambda) = \lambda_1^C$. Hence the lengths of the two partitions agree, i.e. $l(\iota(\lambda)) = l((\lambda')^R)$.

Let λ be a horizontal partition, with standard Kleshchev sequence constructed as above. We shall argue by induction on the number of stripping cycles. Thus if we write μ for the partition which remains after the first stripping cycle on λ, we may assume by the induction hypothesis that $\iota(\mu) = (\mu')^R$, this partition having length $c_1(\mu) = c_2(\lambda)$. By Lemma 4, $\iota(\lambda)$ is obtained by adding a new first column of length $c_1(\lambda)$ to $\iota(\mu)$.

Thus it remains to prove a parallel result to Lemma 4 for the row regularization construction, namely, that the diagram of $(\lambda')^R$ is obtained from that of $(\mu')^R$ by adding exactly one node to each existing row, and starting $c_1(\lambda) - c_2(\lambda)$ new rows by placing a single node in each.

To prove this, we first show that the residues of the nodes to be added correspond to the complementary cycle of the first stripping cycle $K_1(\lambda)$. The residue content of the difference $\lambda - \mu$ of the diagrams is that of $K_1(\lambda)$. Since the nodes (i, j) and (j, i) have complementary residues, the residue content of the conjugate $\lambda' - \mu'$ is that of the complementary cycle. Now row regularization does not change the residue content of the diagram, since all the nodes on a ladder have the same residue. Hence,

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this is also the residue content of $(\lambda')^R - (\mu')^R$. (Note that since the diagram of μ' is contained in the diagram of λ', the diagram of $(\mu')^R$ is certainly contained in that of $(\lambda')^R$.)

Secondly, we recall that we may strip residues in the cycle one at a time, obtaining a horizontal partition after each residue is stripped. Thus we may assume that the residues in the complementary cycle are added to $(\mu')^R$ in descending cyclic order. As in the proof of Lemma 4, this ensures that two nodes cannot be added to the same row of $(\mu')^R$ in constructing $(\lambda')^R$. Since the total number of nodes to be added is the length $c_1(\lambda)$ of $(\lambda')^R$, no row can be omitted, and so the new nodes must be placed as claimed. Thus $\iota(\lambda) = (\lambda')^R$, completing the induction and the proof of Proposition 1.

We conclude by recording the following incidental consequence of our work. It would be interesting to have a direct combinatorial proof of this, but the author has not succeeded in finding one.

COROLLARY 7. *For any horizontal partition* λ, $c(\lambda) = \lambda^C$.

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