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An example of non-unique g-measures

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Abstract. An example of non-unique g-measures is constructed. The function g satisfies a certain summability condition, thereby providing a counterexample to the claim in an earlier paper that all such g have a unique g-measure.

1. *Introduction*

The concept of a g-measure was introduced by Keane [**9**]; an equivalent notion, chains with complete connections, was studied earlier by Doeblin and Fortet [**4**]. We consider g-measures for the one-sided shift on a sequence space.

Let S be a finite set with the discrete topology, $X = S^{\mathbb{Z}^+}$ have the product topology, $\mathcal{B}$ denote the Borel σ-algebra on X, and T be the shift on X; that is, $(Tx)_i = x_{i+1}$. (Here and later, x_i denotes the ith coordinate of x.) Let $\mathcal{G}_X$ be the set of all non-negative Borel functions g on X such that $\sum_{y\in T^{-1}x} g(y) = 1$ for all $x \in X$, and let $g \in \mathcal{G}_X$. A probability measure μ on X is a *g-measure* if μ is T-invariant and $\mu(\{x \in X : x_0 = s\} \mid T^{-1}\mathcal{B})(x) = g(sx_1x_2\dots)$ for all $s \in S$, or equivalently,

$$\int_X f\,d\mu = \int_X \sum_{y\in T^{-1}Tx} g(y)f(y)\,d\mu$$

for all $f \in L^1(\mu)$.

A fundamental question concerns the existence and uniqueness of g-measures. It is usual to assume that g is continuous and positive, as this not only guarantees the existence of at least one g-measure μ, but at the same time rules out trivial examples of non-uniqueness based on the fact that the conditional probabilities $\mu(\,\cdot \mid T^{-1}\mathcal{B})$ are only unique modulo sets of μ-measure 0: if g is positive, any set of μ-measure 1 is dense, and g is determined by its values on such a set. It is well known that uniqueness holds when g depends on only finitely many coordinates, but various more general conditions have been obtained. Most, such as those in [**1, 11**], involve some restriction on $\mathrm{var}_n(g)$, where

$$\mathrm{var}_n(f) = \sup\{|f(x) - f(y)| : x, y \in X,\ x_i = y_i\ \forall\ 0 \le i \le n-1\} \quad (f \in C(X))$$

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(although see [**5**] for an exception). For example, Johansson and Öberg [**8**] proved uniqueness when $\sum(\mathrm{var}_n(g))^2 < \infty$.

In general, however, g-measures are not unique. Bramson and Kalikow [**3**] gave the first example of a positive, continuous g for which there is more than one g-measure. More recently, Berger, Hoffman and Sidoravicius [**2**] showed that Johansson and Öberg's result is optimal, in that the power of 2 cannot be replaced by a larger power.

Given a function $f \in C(X)$, let

$$v_n(f) = \sup\{|f(x) - f(y)| : x, y \in X,\ x_i = y_i\ \forall\ i \neq n\}.$$

(Note that $v_n(g) \leq \mathrm{var}_n(g)$.) We consider those positive, continuous $g \in \mathcal{G}(X)$ such that $\sum v_n(g) < \infty$. For an example of such a g, consider the following. Let $S = \{-1, 1\}$; given a convergent series $\sum_{n=1}^{\infty} a_n$ of non-negative terms, and $\beta \geq 0$, define $g \in \mathcal{G}_X$ by

$$g(x) = \psi\left(x_0 \sum_{n=1}^{\infty} a_n x_n + x_0\beta\right), \tag{1.1}$$

where

$$\psi(t) = e^t(e^t + e^{-t})^{-1} \quad (t \in \mathbb{R}).$$

Note that $0 \leq \psi' \leq \frac{1}{2}$, and so $v_n(g) \leq a_n$ $(n \geq 1)$. We have the following result (see [**6, 7**]).

THEOREM 1. *Let $g \in \mathcal{G}_X$ be positive and continuous. If $\sum_{n=0}^{\infty} \sum_{i \in S} v_n(g(i\,\cdot)) < 2$, there is a unique g-measure μ. The natural extension of (T, μ) is Bernoulli.*

When $|S| = 2$, the hypothesis is equivalent to $\sum_{n=1}^{\infty} v_n(g) < 1$, and it follows that g of the form (1.1) have a unique g-measure when $\sum_{n=1}^{\infty} a_n < 1$. By way of comparison with previously known results, let $a_n = cn^{-\alpha}$ $(c > 0,\ \alpha > 1)$. In this case, the best such result is Johansson and Öberg's [**8**], which implies uniqueness for all $c > 0$ and $\alpha > \frac{3}{2}$, whereas Theorem 1 implies uniqueness when $\alpha > 1$, provided c is small enough.

It was originally claimed in [**6**] that Theorem 1 holds whenever $\sum v_n(g) < \infty$. However, this is not the case, as the following result shows; indeed the bound on $\sum v_n(g)$ is at least close to being optimal.

THEOREM 2. *There exists g of the form (1.1) with more than one g-measure. Moreover, given $\lambda > 1$, g can be chosen so that $\sum_{n=1}^{\infty} v_n(g) < \lambda$.*

The construction of g is in §3, and is based on Bramson and Kalikow's approach. Section 2 contains various preliminary results and observations used in the construction.

2. *Couplings and other preliminaries*

Given a sequence space X with state space S, it is assumed that S is well ordered, and X is partially ordered in the usual way: $x \leq y$ if $x_i \leq y_i$ for all $i \geq 0$. If W, W' are subsets of an ordered set, let $W \times_{\geq} W'$ denote the set $\{(w_1, w_2) \in W \times W' : w_1 \geq w_2\}$. We identify ${S_1}^{\mathbb{Z}^+} \times {S_2}^{\mathbb{Z}^+}$ with the sequence space $(S_1 \times S_2)^{\mathbb{Z}^+}$ in the obvious way; similarly, $X \times_{\geq} X = (S \times_{\geq} S)^{\mathbb{Z}^+}$. If $s \in S$, then $s^\infty \in X$ is the sequence such that $(s^\infty)_i = s$ for all $i \geq 0$; if $s \in \{-1, 1\}$ then $\bar{s}$ denotes $-s$.

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A useful tool in the study of g-measures is the *Ruelle operator* $\mathcal{L}_g$, defined by

$$\mathcal{L}_g f(x) = \sum_{y \in T^{-1}x} g(y) f(y) \quad (g \in \mathcal{G}_X,\ f \in C(X),\ x \in X).$$

A probability measure μ is a g-measure if and only if μ is $\mathcal{L}_g$-invariant; that is, $\mu(\mathcal{L}_g f) = \mu(f)$ for all $f \in C(X)$. Hence, for continuous g, uniqueness of the g-measure is equivalent to the uniform convergence of $\frac{1}{n}\sum_{i=0}^{n-1} \mathcal{L}_g^i f$ to a constant for all $f \in C(X)$; if there is a unique g-measure μ, it follows from the Dominated Convergence Theorem that the constant is $\mu(f)$.

In general, if $g \in \mathcal{G}_X$ and μ is any g-measure, then

$$\inf \mathcal{L}_g^n f \le \mu(f) \le \sup \mathcal{L}_g^n f$$

for all $f \in C(X)$ and $n \ge 1$. The set of g-measures is convex and closed (in the weak-* topology), and its extreme points are the ergodic g-measures. In particular, a unique g-measure is ergodic.

Now, let X, Y be two sequence spaces with state spaces S_X, S_Y respectively. If $g \in \mathcal{G}_X$ and $h \in \mathcal{G}_Y$, then a *coupling* of g and h is a $p \in \mathcal{G}_{X\times Y}$ such that

$$\sum_{z \in T^{-1}y} p(x,z) = g(x), \qquad \sum_{z \in T^{-1}x} p(z,y) = h(y) \tag{2.1}$$

for all $x \in X$, $y \in Y$. Equivalently, it is a set of probability measures $\{P(\cdot \mid x,y) : x \in X,\ y \in Y\}$ on $S_X \times S_Y$ such that

$$\sum_{u \in S_Y} P((s,u) \mid x,y) = g(sx), \qquad \sum_{u \in S_X} P((u,t) \mid x,y) = h(ty) \quad (s \in S_X,\ t \in S_Y).$$

The correspondence between P and p is given by $P((s,t) \mid x,y) = p(sx,ty)$. The measures $P(\cdot \mid x,y)$ generate probability measures $P^n(\cdot \mid x,y)$ on $S_X^n \times S_Y^n$ $(n \ge 1)$ given by

$$P^n((\xi,\eta) \mid x,y) = \prod_{i=0}^{n-1} P((\xi_i,\eta_i) \mid T^{i+1}\xi x,\ T^{i+1}\eta y),$$

so that if $\phi \in C(X \times Y)$ then

$$\int \phi(\cdot\, x, \cdot\, y)\, dP^n(\cdot \mid x,y) = \mathcal{L}_p^n \phi(x,y).$$

It follows from (2.1) that if $f_1 \in C(X)$, $f_2 \in C(Y)$ and $\phi(x,y) = f_1(x) + f_2(y)$, then

$$\mathcal{L}_p^n \phi(x,y) = \mathcal{L}_g^n f_1(x) + \mathcal{L}_h^n f_2(y) \quad (n \ge 1). \tag{2.2}$$

From here on it is assumed that $S_X = S_Y = S$, and the coupling is defined by

$$P(\{(\xi,\eta) : \xi \ge s,\ \eta \ge t\} \mid x,y) = \min\left\{\sum_{\xi \ge s} g(\xi x),\ \sum_{\eta \ge t} h(\eta y)\right\} \quad (s,\ t \in S). \tag{2.3}$$

Since the right-hand side decreases as either s or t increases, the coupling is well defined. Note that, given x and y,

$$\sum_{\xi \ge s} g(\xi x) \ge \sum_{\xi \ge s} h(\xi y) \quad \text{for all } s \in S \quad \Longrightarrow \quad P(S \times_{\ge} S \mid x,y) = 1. \tag{2.4}$$

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In particular, if the left-hand side of (2.4) holds for all $x \geq y$, we say that g *stochastically dominates* h. In such cases, we assume that the corresponding $p \in \mathcal{G}_{X\times X}$ is restricted to $X \times_{\geq} X$, in which case $p \in \mathcal{G}_{X\times_{\geq}X}$. Moreover, if $f \in C(X)$ is increasing, then $f(x) - f(y) \geq 0$ for all $(x,y) \in X \times_{\geq} X$. Therefore, it follows from (2.2) that if g stochastically dominates h and $f \in C(X)$ is increasing, then

$$\mathcal{L}_g^n f(x) \geq \mathcal{L}_h^n f(y) \quad \text{for all } x \geq y \quad (n \geq 1). \tag{2.5}$$

More generally, suppose $x, y \in X$, $j \in \mathbb{N}$ and $x \geq y$. Then

$$\sum_{\xi \geq s} g(\xi z) \geq \sum_{\xi \geq s} h(\xi w) \quad \text{for all } (z,w) \in T^{-i}x \times_{\geq} T^{-i}y,\ 0 \leq i \leq j-1,\ s \in S$$
$$\Longrightarrow \quad \mathcal{L}_g^j f(x) \geq \mathcal{L}_h^j f(y) \quad \text{for all increasing } f \in C(X). \tag{2.6}$$

This is because (2.4) implies $P(S \times_{\geq} S \mid z,w) = 1$ for all such (z,w), and hence $P^j(S^j \times_{\geq} S^j \mid x,y) = 1$.

A $g \in \mathcal{G}_X$ is said to be *attractive* if $\sum_{\xi \geq s} g(\xi x)$ is an increasing function of x for all $s \in S$; let $\mathcal{F}_X$ denote the set of all such g. (Note that g of the form (1.1) are attractive.) If $g \in \mathcal{F}_X$ and p is the coupling of g with itself, it follows from (2.4) that $p \in \mathcal{G}_{X\times_{\geq}X}$. The properties of attractive g are discussed in more detail in [**5**], but in particular we note the following, which can be deduced from (2.5).

LEMMA 1. *Let $g \in \mathcal{F}_X$ be continuous.*

(i) *If $f \in C(X)$ is increasing, then $\mathcal{L}_g^n f$ is increasing for all $n \geq 1$.*

(ii) *If there is a unique g-measure μ, then $\mathcal{L}_g^n f \to \mu(f)$ uniformly as $n \to \infty$, for all $f \in C(X)$.*

We can also use (2.5) to prove the following continuity result due to Lacroix [**10**], who proved it without the restriction that $g \in \mathcal{F}_X$. We include the proof for completeness.

LEMMA 2. *Let $g \in \mathcal{F}_X$ be continuous and positive, with a unique g-measure μ. Then, given $f \in C(X)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$h \in \mathcal{G}_X,\ \|g - h\| < \delta \quad \Longrightarrow \quad |\mu_h(f) - \mu(f)| < \varepsilon$$

for all h-measures μ_h.

Proof. We can assume f is increasing, as linear combinations of such functions are dense in $C(X)$. For $\delta \in \mathbb{R}$, define $g_\delta \in \mathcal{F}_X$ by

$$g_\delta(sx) = g(sx) + \delta \quad (s \in S,\ s \neq \inf S)$$

(g_δ is well defined provided $|\delta|$ is small enough). Let

$$L(n,\delta) = \sup \mathcal{L}_{g_\delta}^n f, \quad l(n,\delta) = \inf \mathcal{L}_{g_\delta}^n f.$$

Then it follows from (2.5) that $L(n,\delta)$ and $l(n,\delta)$ are increasing functions of δ. Moreover, if $\delta > 0$ and $h \in \mathcal{G}_X$ is such that $\|g - h\| < \delta$, then

$$\sum_{\xi \geq s} g_{-\delta}(\xi z) \leq \sum_{\xi \geq s} h(\xi y) \leq \sum_{\xi \geq s} g_\delta(\xi x) \quad \text{for all } x \geq y \geq z \in X.$$

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Thus, by (2.5) again,

$$\mathcal{L}^n_{g_{-\delta}} f \le \mathcal{L}^n_h f \le \mathcal{L}^n_{g_\delta} f$$

for all $n \ge 1$, and so

$$\lim_{n\to\infty} l(n,-\delta) \le \mu_h(f) \le \lim_{n\to\infty} L(n,\delta)$$

for all h-measures μ_h. (Note that the limits exist.) It is enough to show that the left- and right-hand sides converge to $\mu(f)$ as $\delta \to 0^+$. As n increases, $L(n,\delta)$ decreases; therefore,

$$\begin{aligned}\lim_{\delta\to 0^+}\lim_{n\to\infty} L(n,\delta) &= \lim_{n\to\infty}\lim_{\delta\to 0^+} L(n,\delta)\\ &= \lim_{n\to\infty} L(n,0)\\ &= \mu(f)\end{aligned}$$

by Lemma 1(ii). Similarly, $l(n,-\delta)$ increases as n increases, and so

$$\lim_{\delta\to 0^+}\lim_{n\to\infty} l(n,-\delta) = \lim_{n\to\infty} l(n,0) = \mu(f). \qquad \square$$

3. *Proof of Theorem 2*

In this section, $S = \{-1, 1\}$. Let $\chi \in C(X)$ be the function $\chi(x) = x_0$. Recall that

$$\psi(t) = e^t(e^t + e^{-t})^{-1} \quad (t \in \mathbb{R}),$$

and note that ψ is (strictly) increasing, $\psi(t) + \psi(-t) = 1$ and $\psi(t) - \psi(-t)$ takes all values in $(-1, 1)$. Fix $0 < \delta < \frac{1}{4}$ and an integer $k \ge 4$. Given sequences

$$\{\alpha_m\}_{m=0}^\infty, \quad \{\beta_m\}_{m=0}^\infty, \quad \{b_m\}_{m=1}^\infty \subseteq \mathbb{R}^+, \quad \{n_m\}_{m=0}^\infty \subseteq \mathbb{Z}^+, \tag{3.1}$$

define $g_m,\ h_m \in \mathcal{F}_X$ $(m \ge 0)$ by

$$g_m(1x) = \psi\left(\sum_{i=1}^m k b_i \bar\sigma_{n_i}(x) + \beta_m\right),$$

$$h_m(1x) = \psi\left(\sum_{i=1}^m k b_i \bar\sigma_{n_i}(x) + \alpha_m\right),$$

where $\bar\sigma_n(x) = n^{-1}\sum_{i=0}^{n-1} x_i$. (For $m = 0$, the empty sum is taken to be 0.) Let μ_m, ν_m be the (unique) g_m-, h_m-measures respectively. We construct the sequences in (3.1) inductively so that $\{\beta_m\}$ is decreasing, $\alpha_m < \beta_m$ and

$$\mu_m(\chi) > k^{-1} + 2\delta, \quad \nu_m(\chi) = k^{-1} + \delta \quad (m \ge 0) \tag{3.2}$$

as follows.

Let $n_0 = 0$ and choose α_0, β_0 so that $\psi(\beta_0) - \psi(-\beta_0) = k^{-1} + 3\delta$ and $\psi(\alpha_0) - \psi(-\alpha_0) = k^{-1} + \delta$. Then $\mu_0(\chi) > k^{-1} + 2\delta$ and $\nu_0(\chi) = k^{-1} + \delta$. Now suppose the terms in (3.1) up to and including the mth have been defined, for some $m \ge 0$. Define

$$b_{m+1} = (\beta_m - \alpha_m)/(k+1) \quad \text{and} \quad \beta_{m+1} = \beta_m - b_{m+1}. \tag{3.3}$$

Let $\varepsilon = \mu_m(\chi) - (k^{-1} + 2\delta) > 0$. Using (3.2) and Lemma 1, we can choose j so that

$$\mathcal{L}^j_{g_m}\chi(\bar 1^\infty) > k^{-1} + 2\delta + \varepsilon/2.$$

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Then by (3.2) and the ergodic theorem, we can choose n_{m+1} so that if

$$E = \{x : \bar{\sigma}_{n_{m+1}}(x) > k^{-1} + 2n_{m+1}^{-1} j\}$$

then $\nu_m(E) > 1 - \varepsilon/4$. It follows from (3.3) that g_{m+1} stochastically dominates h_m, and so, since $\mathbf{1}(E)$ is an increasing function, $\mu_{m+1}(E) > 1 - \varepsilon/4$. Note also that if $x \in E$ and $y \in T^{-i}x$ for some $0 \le i \le j-1$, then $\bar{\sigma}_{n_{m+1}}(y) > k^{-1}$, and so $g_{m+1}(1y) \ge g_m(1y)$. Since $g_m,\ g_{m+1} \in \mathcal{F}_X$, it follows from (2.6) that

$$\mathcal{L}^j_{g_{m+1}}\chi(x) \ge \mathcal{L}^j_{g_m}\chi(\bar{1}^\infty) \quad \text{for all } x \in E.$$

Therefore,

$$\begin{aligned}\mu_{m+1}(\chi) &\ge \mathcal{L}^j_{g_m}\chi(\bar{1}^\infty) - 2\mu_{m+1}(E^c)\\ &> k^{-1} + 2\delta.\end{aligned}$$

Finally, it follows from Lemma 2 that $\nu_{m+1}(\chi)$ is continuous as a function of α_{m+1}, and by symmetry, $\nu_{m+1}(\chi) = 0$ if $\alpha_{m+1} = 0$; thus, we can choose α_{m+1} $(0 < \alpha_{m+1} < \beta_{m+1})$ so that $\nu_{m+1}(\chi) = k^{-1} + \delta$. This completes the induction.

Let $\beta = \lim_{m\to\infty}\beta_m$. Then $\lim_{m\to\infty}\alpha_m = \beta$ also, since (3.3) implies

$$\beta_m - \beta_{m+1} = (\beta_m - \alpha_m)/(k+1),$$

and so $\lim_{m\to\infty}(\beta_m - \alpha_m) = 0$. Define $g \in \mathcal{F}_X$ by

$$g(1x) = \psi\left(\sum_{i=1}^{\infty} k b_i \bar{\sigma}_{n_i}(x) + \beta\right).$$

Note that $\sum_{m=1}^{\infty} b_m = \beta_0 - \beta$, so g is well defined; also, g is of the form (1.1) with

$$a_n = \sum_{i\ge 1, n_i \ge n} k b_i n_i^{-1}.$$

Clearly, $\|g_m - g\| \to 0$ and $\|h_m - g\| \to 0$ as $m \to \infty$, whereas $\mu_m(\chi) - \nu_m(\chi) > \delta$ for all m. Therefore, by Lemma 2, g does not have a unique g-measure.

Note that

$$\sum_{n=1}^{\infty} v_n(g) \le \sum_{n=1}^{\infty} a_n = k\sum_{n=1}^{\infty} b_n \le k\beta_0,$$

and $\lim_{t\to\infty} t^{-1}(\psi(t) - \psi(-t)) = 1$. Therefore, given $\lambda > 1$, we can let $\delta = k^{-2}$, and choose k large enough so that $1 + 3k^{-1} < \lambda^{1/2}$ and

$$\beta_0 < \lambda^{1/2}(\psi(\beta_0) - \psi(-\beta_0)) = \lambda^{1/2}(k^{-1} + 3k^{-2}).$$

Then $\sum_{n=1}^{\infty} v_n(g) < \lambda$.

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