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Model sets with precompact Borel windows

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Abstract. Given a cut and project scheme and a precompact Borel window, we show that almost surely all positions of the window give rise to point sets with Besicovitch almost periodic Dirac combs. In particular, all those positions lead to pure point diffractive point sets with autocorrelation given by the covariogram of the window and diffraction given by the inverse Fourier transform of the window squared. Moreover, the Fourier–Bohr coefficients exist and the intensity of the Bragg peaks is given by their absolute value squared. We show the existence of an ergodic measure with pure point dynamical spectrum such that all those window positions give rise to point sets which are generic for this measure.

1 Introduction

In 1984, Schettmann discovered new solids with a clear diffraction pattern and 5-fold symmetry [30]. This discovery led to a paradigm shift in physics, as for more than 100 years, clear diffraction patterns have been thought to be the identifying signature of periodic crystals, and thus incompatible with 5-fold rotational symmetry. The newly discovered materials are now called quasicrystals.

The standard method for producing mathematical models for quasicrystals is via a cut and project scheme (or on short CPS). This method was introduced in the 1970s by Meyer [22] in order to produce point sets with certain harmonic properties and was rediscovered by deBruijn [8] when studying the construction of the Penrose tiling [26] via the Ammann bars. Meyer's pioneering work in the study of CPSs was popularized and expanded in the 1990s by Moody [23, 25] and Lagarias [17, 18]. The basic idea beyond CPSs is the following: to produce a model for a three-dimensional quasicrystal, we start with a lattice $\mathcal{L}$ in a higher dimensional space which sits at an irrational angle with respect to our space, cut a strip of the lattice which is within bounded distance (the "window") of the three-dimensional space, and project the lattice points in this strip on our space. If the shape of the strip is not too bad, the point set produced via the CPS should still show some residue of long-range order from the lattice $\mathcal{L}$ (see, e.g., [10, 11, 29, 31]). In general, if the window

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is "bad," the diffraction can have a continuous component but the pure point and (singular/absolutely) continuous components still show high coherence [33–36].

Recently, results about model sets have been extended to a larger class of such models called weak model sets of maximal density [4, 13–15]. This class contains many models of number theoretic origin, such as square free integers, visible points of a lattice, $\mathcal{B}$-free integers, and coprime families of lattices [3, 6, 9]. A key result in this settings is a theorem by Moody [24] which states that for any CPS and any compact window, almost surely all positions of the window give maximal density weak model sets, and hence point sets with pure point diffraction. Similar results can be shown for open precompact windows [4], but it was not clear so far how to extend these results beyond topological windows. It is our goal to extend these results to precompact Borel windows in this article, by using some new developments into the theory of Besicovitch almost periodic measures [20, 21], and the connection between pure point diffraction and dynamical spectra and mean, Besicovitch and Weyl almost periodicity.

In this article, given a CPS $(G, H, \mathcal{L})$, a precompact Borel window B, and a tempered van Hove sequence $\mathcal{A}$, we prove the existence of a set $\mathcal{MG}(B) \subseteq \mathbb{T}$ of full Haar measure such that, for all $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, the set $-s + \curlywedge(t + W)$ is Besicovitch almost periodic with respect to $\mathcal{A}$ and has autocorrelation γ and diffraction $\widehat{\gamma}$ (with respect to $\mathcal{A}$) given by

$$\gamma = \operatorname{dens}(\mathcal{L})\omega_{c(B)} = \operatorname{dens}(\mathcal{L}) \sum_{(x,x^\star)\in\mathcal{L}} c_B(x^\star)\delta_x, \tag{1.1}$$

$$\widehat{\gamma} = \operatorname{dens}(\mathcal{L})^2 \omega^{\star}_{\widetilde{|1_B|^2}} = \operatorname{dens}(\mathcal{L})^2 \sum_{(\chi,\chi^\star)\in\mathcal{L}^0} \left| \int_B \chi^\star(t)\mathrm{d}t \right|^2 \delta_\chi, \tag{1.2}$$

where $c(B)$ denotes the covariogram of B (see below for definitions and notations). We further show that for all $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, the set $-s + \curlywedge(t + B)$ has well-defined Fourier–Bohr coefficients $a^{\mathcal{A}}_\chi$ with respect to $\mathcal{A}$, and

$$a^{\mathcal{A}}_\chi(-s + \curlywedge(t + B)) = \begin{cases} \operatorname{dens}(\mathcal{L})\chi(s)\int_{t+B}\chi^\star(z)\mathrm{d}z, & \text{if } \chi \in \pi_{\widehat{G}}(\mathcal{L}^0), \\ 0, & \text{otherwise .} \end{cases}$$

In particular, the consistent phase property (CPP) holds.

Furthermore, we show that there exists an ergodic measure $\mathfrak{m}$ on the extended hull $\mathcal{M}^G_B = \overline{\{-s + \curlywedge(t + B) : s \in G, t \in H\}}$ such that, for all $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, the set $-s + \curlywedge(t + B)$ is generic for $\mathfrak{m}$. In particular, $\mathfrak{m}$ has pure point spectrum, autocorrelation γ, and diffraction $\widehat{\gamma}$ given by (1.1) and (1.2), respectively.

We should also note that recently, the same results have been established via dynamical systems in [16]. The approach we use below is via Besicovitch almost periodic measures and can be seen as a complement to [16].

2 Preliminaries

For the entire article, G denotes a second countable locally compact Abelian group (LCAG). We will denote by $C_{\mathrm{u}}(G)$ the space of uniformly continuous bounded functions on G, and by $C_{\mathrm{c}}(G)$ the subspace of compactly supported continuous

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functions. The Haar measure on G is denoted by θ_G. For a Borel set $B \subseteq G$, we will often use the notation $|B| := \theta_G(B)$.

We collect, in this section, some of the necessary background and standard notations needed to read this article. Since this article builds on [4] and uses many results from [20, 21], for brevity, we refer the reader to those papers for most standard definitions and notations.

2.1 Cut and project schemes

We briefly review the definition of cut and project schemes and properties of weak model sets. For more details, we recommend [1, 2, 19, 23, 24, 27, 35, 36], just to name a few.

Definition 2.1 By a *cut and project scheme* (or simply *CPS*), we understand a triple $(G, H, \mathcal{L})$ consisting of two LCAG G, H, and a lattice $\mathcal{L} \subseteq G \times H$ with the following properties:

- The restriction of the first projection $\pi_G|_{\mathcal{L}}$ to $\mathcal{L}$ is one-to-one.
- $\pi_H(\mathcal{L})$ is dense in H.

Given a CPS $(G, H, \mathcal{L})$, we denote by $\mathbb{T} := (G \times H)/\mathcal{L}$ and by $L := \pi_G(\mathcal{L})$. There exists a mapping, call the *star mapping*, $\star : L \to H$ so that $\mathcal{L} = \{(x, x^\star) : x \in L\}$. Next, for each $W \subseteq H$, we can define

$$\curlywedge(W) = \{x \in L : x^\star \in W\}.$$

We are interested in $\curlywedge(W)$, when W is a precompact Borel set. For such set $W \subseteq H$, we define its *covariogram function* $c_W : H \to \mathbb{C}$ via $c_W(t) := \theta_H(W \cap (t + W))$.

More generally, for each $h : H \to \mathbb{C}$, we can define a formal sum

$$\omega_h := \sum_{x \in L} h(x^\star)\delta_x.$$

If h is compactly supported and bounded, then ω_h is a measure. There are many other conditions on h which ensure that ω_h is a measure; see, for example, [19, 27, 36, 37].

Note here that, for all $s \in L$, we have

$$\curlywedge(s^\star + W) = \{x \in L : x^\star \in s^\star + W\} = s + \curlywedge(W) \tag{2.1}$$

$$\tau_s \omega_h = \sum_{x \in L} h(x^\star)\delta_{s+x} = \sum_{y \in L} h(y^\star - s^\star)\delta_y = \omega_{\tau_{s^\star} h}. \tag{2.2}$$

We will often use these two formulas.

Given a CPS $(G, H, \mathcal{L})$, we can define

$$\mathcal{L}^0 := \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \chi(x)\psi(x^\star) = 1 \forall x \in L\}.$$

Then, $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$ is a CPS[23], called the *CPS dual to* $(G, H, \mathcal{L})$. This dual CPS plays the central role in the study of diffraction of model sets.

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To avoid confusion, for each $h : \widehat{H} \to \mathbb{C}$, we will denote the comb given by h in the dual CPS $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$ by $\omega_h^\star$, that is,

$$\omega_h^\star := \sum_{(\chi,\chi^\star)\in\mathcal{L}^0} h(\chi^\star)\delta_\chi.$$

When dealing with functions, one can use Følner sequences to calculate averages (means). Below, we will often calculate the average (density) of point sets, and in this situation, Følner sets do not suffice. Indeed, the sequence

$$F_n = \left[-n-\frac{1}{2}, n+\frac{1}{2}\right] \setminus \cup_{k=-n}^{n} \left[k-\frac{1}{2^n}, k+\frac{1}{2^n}\right],$$

which is the interval $\left[-n-\frac{1}{2}, n+\frac{1}{2}\right]$ with small holes around the integers, is a Følner sequence which cannot be used to average the Delone set $\mathbb{Z}$, or weighted Dirac combs on $\mathbb{Z}$. Similarly, for each Delone subset $\Lambda \subseteq \mathbb{R}$, it is easy to construct a Følner sequence which has an empty intersection with Λ.

As it was observed by Schlottmann [29], to avoid this issue when dealing with point sets or measures, one needs to work instead with van Hove sequences (see [20, Definition 1.6] for definition and properties). Any van Hove sequence is a Følner sequence, and any σ-compact LCAG G admits a (tempered) van Hove sequence.

Given a uniformly discrete set $\Lambda \subseteq G$ and a van Hove sequence $\mathcal{A} = \{A_k\}$, we define

$$\underline{\mathrm{dens}}_{\mathcal{A}}(\Lambda) := \liminf_k \frac{\mathrm{card}(\Lambda \cap A_k)}{|A_k|}$$
$$\overline{\mathrm{dens}}_{\mathcal{A}}(\Lambda) := \limsup_k \frac{\mathrm{card}(\Lambda \cap A_k)}{|A_k|}.$$

If the following limit exists, or equivalently if $\underline{\mathrm{dens}}_{\mathcal{A}}(\Lambda) = \overline{\mathrm{dens}}_{\mathcal{A}}(\Lambda)$, we define

$$\mathrm{dens}_{\mathcal{A}}(\Lambda) := \lim_k \frac{\mathrm{card}(\Lambda \cap A_k)}{|A_k|}.$$

Let us next recall the following result of [12].

Theorem 2.1 [12] *Let $(G, H, \mathcal{L})$ be a CPS and $W \subseteq H$ be a precompact. Then, for all van Hove sequences $\mathcal{A}$, we have*

$$\mathrm{dens}(\mathcal{L})\theta_H(W^\circ) \leq \underline{\mathrm{dens}}(\curlywedge(W)) \leq \overline{\mathrm{dens}}(\curlywedge(W)) \leq \mathrm{dens}(\mathcal{L})\theta_H(\overline{W}).$$

We will use the following definitions.

Definition 2.2 Let $(G, H, \mathcal{L})$ be a CPS and let $W \subseteq H$ be a precompact.

(a) If W is compact, we call $\curlywedge(W)$ a *weak model set*.
(b) If W is open, we call $\curlywedge(W)$ a *open model set*.
(c) We say that $\curlywedge(W)$ has *maximal density with respect to* $\mathcal{A}$ if

$$\mathrm{dens}_{\mathcal{A}}(\curlywedge(W)) = \mathrm{dens}(\mathcal{L})\theta_H(\overline{W}).$$

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(d) We say that $\curlywedge(W)$ has *minimal density with respect to* $\mathcal{A}$ if

$$\operatorname{dens}_{\mathcal{A}}(\curlywedge(W)) = \operatorname{dens}(\mathcal{L})\theta_H(W^\circ).$$

Remark 2.2

(a) If $\curlywedge(W)$ has maximal density with respect to $\mathcal{A}$, then $\curlywedge(\overline{W})$ is a maximal density weak model set. Such models have been studied in [4, 13–15].
(b) If $\curlywedge(W)$ has minimal density with respect to $\mathcal{A}$, then $\curlywedge(W^\circ)$ is a minimal density open model set. We derive the theory of such models in the next section, from the theory of maximal density weak model sets.

Let us complete this section by recalling the following result from [25].

Theorem 2.3 [25, Theorem 1] *Let $(G, H, \mathcal{L})$ be a CPS, let $W \subseteq H$ be a measurable precompact set, and let $\mathcal{A}$ be a tempered van Hove sequence in G. Then, for almost all $(s,t) + \mathcal{L} \in (G \times H)/\mathcal{L} =: \mathbb{T}$, we have*[1]

$$\operatorname{dens}_{\mathcal{A}}(-s + \curlywedge(t + W)) = \operatorname{dens}(\mathcal{L})\theta_H(W). \tag{2.3}$$

2.2 Dynamical systems

Next, we review few basic facts about dynamical systems of translation bounded measures. Let us start by recalling the following result of [32] (compare [5]).

Lemma 2.4 [32] *Let $\mathbb{X} \subseteq \mathcal{M}^\infty(G)$ be any G invariant vaguely closed set. Then, $\mathbb{X}$ is vaguely compact if and only if there exists some C and an open precompact set $\varnothing \neq U \subseteq G$ such that*

$$\mathbb{X} \subseteq \{\mu \in \mathcal{M}^\infty(G) : \|\mu\|_U \leq C\} =: \mathcal{M}_{C,U}(G).$$

Moreover, in this case, the vague topology on $\mathbb{X}$ is metrizable.

Next, let us review the following definition, which is inspired by the general Birkhoff ergodic theorem for continuous group actions [7, 38].

Definition 2.3 Let $(X, G, \mathfrak{m})$ be an ergodic dynamical system and let $\mathcal{A}$ be a van Hove sequence. An element $x \in X$ is called *generic* for $\mathfrak{m}$ if

$$\lim_k \frac{1}{|A_k|} \int_{A_k} f(\tau_t x)\mathrm{d}t = \int_X f(y)\mathrm{d}\mathfrak{m}(y)$$

holds for all $f \in C(X)$.

We will need the following simple characterizations of generic elements. This has been used implicitly in [5, 20, 21] just to name a few. Since the result is well known and trivial to prove, we skip the proof.

[1] Note that in [25] the Haar measure on H is renormalized such that $\operatorname{dens}(\mathcal{L}) = 1$.

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Proposition 2.5 *Let $(X, G, \mathfrak{m})$ be an ergodic dynamical system. Then, $x \in X$ is generic for $\mathfrak{m}$ if and only if the set of $f \in C(X)$ for which*

$$\int_X f(y)\mathrm{d}\mathfrak{m}(y) = \lim_k \frac{1}{|A_k|}\int_{A_k} f(\tau_t x)\mathrm{d}t$$

is dense in $C(X)$.

By combining this result with Stone–Weierstrass theorem, we immediately get the following proposition.

Proposition 2.6 *Let $\mathbb{X} \subseteq \mathcal{M}_{C,U}$ be vaguely compact and G-invariant, and let $\mathfrak{m}$ be an ergodic measure on $\mathbb{X}$. If all measures in $\mathbb{X}$ are real valued, then a measure $\mu \in \mathbb{X}$ is generic for $\mathfrak{m}$ if and only if, for all $n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_n \in C_c(G)$, we have*

$$\begin{aligned}\lim_k \frac{1}{|A_k|}\int_{A_k} \mu * \varphi_1(t)\mu * \varphi_2(t) \cdot \cdots \cdot \mu * \varphi_n(t)\mathrm{d}t \\ = \int_{\mathbb{X}(B)} f_{\varphi_1}(\omega) f_{\varphi_2}(\omega) \cdot \cdots \cdot f_{\varphi_n}(\omega)\mathrm{d}\mathfrak{m}(\omega).\end{aligned} \tag{2.4}$$

2.3 The hull and extended hull

Recall that, for a precompact open set $\varnothing \neq U \neq\subseteq G$, we denote by $\mathcal{D}_U$ the set of U-uniformly discrete subsets of G. The mapping $\Lambda \to \delta_\Lambda$ induces an embedding of $\mathcal{D}_U$ into $\mathcal{M}_{1,U}$ with compact range. We can then equip $\mathcal{D}_U$ with the *local rubber topology* τ_{lr}, which is the pullback through this mapping of the vague topology.

Now, let $(G, H, \mathcal{L})$ be a cut and project scheme and $B \subseteq H$ be any precompact set. Then, there exists some precompact open set $\varnothing \neq U \neq\subseteq G$ such that, for all $(s, t) \in G \times H$, we have $s + \curlywedge(t + B) \in \mathcal{D}_U$.

Definition 2.4 The *hull* $\mathbb{X}(\curlywedge(B))$ *of* $\curlywedge(B)$ and *extended hull* $\mathcal{M}_B^G$ *of* B are defined as

$$\mathbb{X}(\curlywedge(B)) := \overline{\{s + \curlywedge(B) : s \in G\}}$$

$$\mathcal{M}_B^G := \overline{\{s + \curlywedge(t + B) : s \in G, t \in H\}},$$

where the closure is taken in the local rubber topology.

Remark 2.7 As observed by Schlottmann [29], we get a natural G-mapping

$$\mathbb{T} := G \times H/\mathcal{L} \ni (s, t) + \mathcal{L} \to -s + \curlywedge(B) \in \mathcal{M}_B^G.$$

Moody's uniform distribution result, Theorem 2.3, implies that almost surely almost all positions $(s, t) + \mathcal{L} \in G \times H/\mathcal{L}$ give us point sets whose distributional average in G matches the corresponding integrals over the window. Since we are interested in those elements, and since the original position may not be one of those generic positions, one needs to work with the extended hull $\mathcal{M}_B^G$ and not just the classical hull $\mathbb{X}(\curlywedge(B))$.

For regular model sets, we have $\mathcal{M}_B^G = \mathbb{X}(\curlywedge(B))$, but this is in general not true. Indeed, consider the CPS $(\mathbb{R}, H, \mathcal{L})$ and the window W of the visible points of the

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lattice [4]. Then, there exists positions $t \in H$ such that $(t + W) \cap L^\star = \varnothing$. Pick one such position t and set $B = t + W$. Then, $\mathbb{X}(\curlywedge(B)) = \{\varnothing\}$ while the extended hull $\mathcal{M}_B^G$ contains the visible points of the lattice.

From a dynamical point of view, the above mapping can be seen as a (generally) discontinuous observation introduced by the window on the compact group $\mathbb{T}$. We will show below that there exists a natural measure (the Mirsky measure) which describes the distribution generated by the observation, and that almost all $(s, t) + \mathcal{L} \in \mathbb{T}$ define observations $-s + \curlywedge(t + B)$ which are generic for this measure.

We complete this section by proving the following simple lemma.

Lemma 2.8 *Let $(G, H, \mathcal{L})$ be a CPS and $h \in C_c(H)$ and let $\mathfrak{m}$ denote the unique ergodic measure on $\mathbb{X}(\omega_h)$. Then,*

(a) *For all $t \in H$, we have $\omega_{\tau_t h} \in \mathbb{X}(\omega_g)$.*
(b) *For all $f \in C(\mathbb{X}(\omega_h))$ and all $(s, t) \in G \times H$, we have*

$$\mathfrak{m}(f) = \lim_k \frac{1}{|A_k|} \int_{A_k} f(\tau_{s+x} \omega_{\tau_t h}) \mathrm{d}x.$$

Proof "**(a)**" Since $L^\star$ is dense in H, we can find a precompact neighborhood $t \in V$ and a net t_α with $t_\alpha \in V$ which converges in H to t. Then, using the uniform continuity of h, we get that

$$\lim_\alpha \|\tau_{t_\alpha} h - \tau_t h\|_\infty = 0. \tag{2.5}$$

Next, all these functions are supported in the compact set $\overline{V} + \mathrm{supp}(h)$. Therefore, all the measures $\omega_{\tau_{t_\alpha} h}, \omega_{\tau_t h}$ are supported inside the uniformly discrete set $\curlywedge(\overline{V} + \mathrm{supp}(h))$. Moreover, (2.5) implies that for all $x \in \curlywedge(\overline{V} + \mathrm{supp}(g))$, we have

$$\omega_{\tau_t h}(\{x\}) = \lim_\alpha \omega_{\tau_{t_\alpha} h}(\{x\}).$$

This combined with the equi uniform discreteness of their support implies that $\omega_{\tau_{t_\alpha} h}$ converges vaguely to $\omega_{\tau_t h}$.

Finally, since $t_\alpha \in L^\star$, there exists some $s_\alpha \in L$ such that $t_\alpha = s_\alpha^\star$. Then, by (2.2), we have

$$\omega_{\tau_{t_\alpha} h} = \tau_{s_\alpha} \omega_h \in \mathbb{X}(\omega_h).$$

The claim follows.

"**(b)**" follows immediately from (a) and the unique ergodic theorem. ∎

2.4 Besicovitch almost periodicity

Consider a Følner sequence $\mathcal{A}$ on G. On the space of a locally bounded measurable function, we can then define a semi-norm, called the *Besicovitch p-seminorm for functions*

$$\|f\|_{\mathrm{B},p,\mathcal{A}} := \left(\limsup_k \frac{1}{|A_k|} \int_{A_k} |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

With this in mind, let us recall the following definition.

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Definition 2.5 A function $f \in L^1_{loc}(G)$ is called *Besicovitch* 1*-almost periodic* if for each $\varepsilon > 0$ there exists a Bohr almost periodic function g such that

$$\|f - g\|_{\mathrm{B},1,\mathcal{A}} < \varepsilon.$$

A translation bounded measure μ is called *Besicovitch almost periodic* if $\mu * \varphi$ is a Besicovitch 1-almost periodic function for all $\varphi \in C_c(G)$.

The space of Besicovitch almost periodic measures is denoted by $\mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$.

In this article, we will be interested in point sets Λ for which $\delta_\Lambda \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$. Because of this, we introduce the following notation:

$$\mathcal{AP}\text{-}Bes_{\mathcal{A},ps}(G) := \{\Lambda \subseteq G : \delta_\Lambda \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)\}.$$

Below, we will make use of the following two results.

Theorem 2.9 [20, Theorem 4.39] *Let $\mu \in \mathcal{M}^\infty(G)$ and let $\mathcal{A}$ be a van Hove sequence. Then, $\mu \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$ if and only if the following three conditions hold:*

- *The autocorrelation γ of μ exists with respect to $\mathcal{A}$ and $\widehat{\gamma}$ is pure point.*
- *For each $\chi \in \widehat{G}$, the Fourier–Bohr coefficient*

$$a_\chi^{\mathcal{A}}(\mu) := \lim_k \frac{1}{|A_k|} \int_{A_k} \overline{\chi(t)} \mathrm{d}\mu(t)$$

  exists.
- *For each $\chi \in \widehat{G}$, we have*

$$\widehat{\gamma}(\{\chi\}) = \left|a_\chi^{\mathcal{A}}(\mu)\right|^2.$$

Theorem 2.10 [20, Theorem 7.28] *Let $\mu \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$ and let $\mathcal{A}$ be a van Hove sequence. Then, there exists an ergodic measure $\mathfrak{m}$ on $\mathbb{X}(\mu)$ with pure point spectrum such that μ is generic for $\mathfrak{m}$ with respect to $\mathcal{A}$.*

For a measure $\mu \in \mathcal{M}^\infty$, we will denote by

$$\|\mu\|_{\mathrm{B},\mathcal{A}} := \limsup_k \frac{|\mu|(A_k)}{|A_k|}.$$

For all $\varphi \in C_c(G)$ and $\mu \in \mathcal{M}^\infty(G)$, we have [20, Lemma 2.13]

$$\|\varphi * \mu\|_{\mathrm{B},1,\mathcal{A}} \leq \|\mu\|_{\mathrm{B},\mathcal{A}} \|\varphi\|_1. \tag{2.6}$$

3 Minimal density open model sets

In this section, we review the theory of minimal density open model sets, deducing all important results from the corresponding theory of maximal density weak model sets from [4]. We also derive new results.

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Lemma 3.1 *Let $(G, H, \mathcal{L})$ be a CPS and let $U \subseteq H$ be a precompact open set and K a regular window such that $U \subseteq K$. Then,*

(a) $\curlywedge(K) = \curlywedge(K\backslash U) \dot{\bigcup} \curlywedge(U)$.
(b) *$\curlywedge(K\backslash U)$ has maximal density with respect to $\mathcal{A}$ if and only if $\curlywedge(U)$ has minimal density with respect to $\mathcal{A}$.*

Proof "**(a)**" is obvious.

"**(b)**" First note that since $U \subseteq K$, we have

$$\theta_H(K\backslash U) = \theta_H(K) - \theta_H(U).$$

Now, by (a), we have

$$\frac{1}{|A_k|}\mathrm{card}(\curlywedge(K\backslash U) \cap A_k) = \frac{1}{|A_k|}\mathrm{card}(\curlywedge(K) \cap A_k) - \frac{1}{|A_k|}\mathrm{card}(\curlywedge(U) \cap A_k).$$

Therefore,

$$\begin{aligned}
&\frac{1}{|A_k|}\mathrm{card}(\curlywedge(K\backslash U) \cap A_k) - \mathrm{dens}(\mathcal{L})\theta_H(K\backslash U) \\
&\quad = \frac{1}{|A_k|}\mathrm{card}(\curlywedge(K) \cap A_k) - \frac{1}{|A_k|}\mathrm{card}(\curlywedge(U) \cap A_k) \\
&\qquad - \mathrm{dens}(\mathcal{L})\theta_H(K) + \mathrm{dens}(\mathcal{L})\theta_H(U) \\
&\quad = \left(\frac{1}{|A_k|}\mathrm{card}(\curlywedge(K) \cap A_k) - \mathrm{dens}(\mathcal{L})\theta_H(K)\right) \\
&\qquad + \left(\mathrm{dens}(\mathcal{L})\theta_H(U) - \frac{1}{|A_k|}\mathrm{card}(\curlywedge(U) \cap A_k)\right).
\end{aligned}$$

Next, since K is regular model set, we have

$$\lim_k \frac{1}{|A_k|}\mathrm{card}(\curlywedge(K) \cap A_k) - \mathrm{dens}(\mathcal{L})\theta_H(K) = 0.$$

The claim follows. ∎

As consequence, we get (compare [4]).

Theorem 3.2 *Let $\mathcal{A}$ be a van Hove sequence, $(G, H, \mathcal{L})$ be a CPS, and let $\curlywedge(U)$ be a minimal density open model set with respect to $\mathcal{A}$. Then,*

(a) $\curlywedge(U) \in \mathcal{AP}\text{-}Bes_{\mathcal{A},ps}(G)$.
(b) *The autocorrelation γ of $\curlywedge(U)$ exists and satisfies*

$$\gamma = \mathrm{dens}(\mathcal{L})\omega_{c_U} = \mathrm{dens}(\mathcal{L}) \sum_{(x,x^\star)\in\mathcal{L}} \theta_H(U \cap (x^\star + U))\delta_x.$$

(c) *The diffraction $\widehat{\gamma}$ is given by*

$$\widehat{\gamma} = \mathrm{dens}(\mathcal{L})^2 \sum_{(\chi,\chi^\star)\in\mathcal{L}^0} \left|\int_U \chi^\star(t)\mathrm{d}t\right|^2 \delta_\chi = (\mathrm{dens}(\mathcal{L}))^2 \omega^\star_{\widetilde{|1_U|^2}}.$$

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(d) *The Fourier–Bohr coefficients*

$$a_\chi^{\mathcal{A}}(\delta_{\curlywedge(U)}) = \lim_k \frac{1}{|A_k|} \sum_{x \in \curlywedge(U) \cap A_k} \overline{\chi(x)}$$

exist and satisfy

$$a_\chi^{\mathcal{A}}(\delta_{\curlywedge(U)}) = \begin{cases} \mathrm{dens}(\mathcal{L}) \widetilde{1_U}(\chi^\star) = \mathrm{dens}(\mathcal{L}) \int_U \chi^\star(t) \mathrm{d}t & \text{if } \chi \in \pi_{\widehat{G}}(\mathcal{L}^0) \\ 0 & \text{otherwise.} \end{cases}$$

(e) $\delta_{\curlywedge(U)}$ *satisfies the CPP*

$$\widehat{\gamma}(\{\chi\}) = \left| a_\chi^{\mathcal{A}}(\delta_{\curlywedge(U)}) \right|^2 \qquad \forall \chi \in \widehat{G}.$$

(f) *There exists a G-invariant ergodic measure* $\mathfrak{m}$ *on* $\mathbb{X} = \mathbb{X}(\curlywedge(U))$ *such that* $\curlywedge(U)$ *is generic for* $\mathfrak{m}$.

(g) $(\mathbb{X}, G, \mathfrak{m})$ *has a pure point dynamical spectrum.*

(h) γ *is the autocorrelation of* $(\mathbb{X}, G, \mathfrak{m})$.

Proof "**(a)**" Pick some regular set K such that $U \subseteq K$. Then,

$$\delta_{\curlywedge(U)} = \delta_{\curlywedge(K)} - \delta_{\curlywedge(K \setminus U)}.$$

By [20, Corollary 5.35], $\delta_{\curlywedge(K)}$ is Weyl almost periodic and hence Besicovitch almost periodic with respect to $\mathcal{A}$. Moreover, $\curlywedge(K \setminus U)$ has maximal density and hence $\delta_{\curlywedge(K\setminus U)} \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$ [20, Proposition 4.45]. Thus, $\delta_{\curlywedge(U)} = \delta_{\curlywedge(K)} - \delta_{\curlywedge(K\setminus U)} \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$.

"**(b),(c), (d)**" follow from [4, Theorem 9].

"**(e)**" follows from [4, Theorem 9] or [20, Theorem 4.39].

"**(f),(g)**" follow from Theorem 2.10.

"**(h)**" follows from [20, Theorem 7.20]. ∎

4 Model sets with Borel windows

For the entire section, $(G, H, \mathcal{L})$ is a fixed CPS and $B \subseteq H$ is a fixed Borel precompact set. Since B is precompact, $\theta_H(B) < \infty$. We introduce now some important players for the construction of the ergodic measure and of its generic elements.

4.1 Construction of U_n, K_n, and g_n

For each n, by [28, Theorem 2.14(c) and (d)], there exists some compact set K_n and open precompact set $U_n \subseteq U$ such that

$$K_n \subseteq B \subseteq U_n \text{ and } \theta_H(U_n \setminus K_n) < \frac{1}{n}.$$

We fix one such choice of K_n and U_n for each n and, by eventually replacing K_n by $K_1 \cup K_2 \cup \cdots \cup K_n$ and U_n by $U_1 \cap U_2 \cap \cdots \cap U_n$, we can assume without loss of generality that

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots \subseteq B \subseteq \cdots \subseteq U_n \subseteq \cdots \subseteq U_2 \subseteq U_1.$$

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Next, for each n, fix some $g_n \in C_c(G)$ such that $1_{K_n} \le g_n \le 1_{U_n}$. The sets K_n, U_n and the functions g_n are fixed for the remaining of the section.

We can now introduce the following definition.

Definition 4.1 We denote by

$$\mathcal{MG}(B) := \{\{(s,t)+\mathcal{L} \in \mathbb{T} : \lim_n \|\tau_{-s}\omega_{\tau_t g_n} - \delta_{-s+\curlywedge(t+B)}\|_{B,\mathcal{A}} = 0\},$$

and we will refer to this as the set of *fb-generic* points.

Remark 4.1 While the definition of $\mathcal{MG}(B)$ seems to depend on the choice of g_n, Lemma 4.8 shows that it does not. Lemma 4.8 will also explain the choice of name.

Lemma 4.2 *Let $(s,t)+\mathcal{L} \in \mathbb{T}$. Then, $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$ if and only if, for all $\varphi \in C_c(G)$, we have*

$$\lim_n \|\tau_{-s}\omega_{\tau_t g_n} * \varphi - \delta_{-s+\curlywedge(t+B)} * \varphi\|_{B,1,\mathcal{A}} = 0.$$

Proof $\Longrightarrow$ follows from (2.6).

$\Longleftarrow$ We know that $\operatorname{supp}(\tau_{-s}\omega_{\tau_t g_n} - \delta_{-s+\curlywedge(t+B)}) \subseteq -s+\curlywedge(t+U_1)$ which is uniformly discrete. Let $V \subseteq G$ be any open set such that $\operatorname{supp}(\tau_{-s}\omega_{\tau_t g_n} - \delta_{-s+\curlywedge(t+B)})$ is V-uniformly discrete. Then, $\operatorname{supp}(\tau_{-s}\omega_{\tau_t g_n} - \delta_{-s+\curlywedge(t+B)})$ is $v+V$-uniformly discrete for all $v \in G$.

Let $C := \{\varphi \in C_c(G) : \exists v \in G \text{ s.t. } \operatorname{supp}(\varphi) \subseteq v+V\}$. Strungaru [35, Lemma 5.8.3] implies that for all $\varphi \in C$, we have

$$\left|\tau_{-s}\omega_{\tau_t g_n} * \varphi - \delta_{-s+\curlywedge(t+B)} * \varphi\right| = \left|\tau_{-s}\omega_{\tau_t g_n} - \delta_{-s+\curlywedge(t+B)}\right| * |\varphi|.$$

This immediately implies that

$$\lim_n \|\tau_{-s}\omega_{\tau_t g_n} * \varphi - \delta_{-s+\curlywedge(t+B)} * \varphi\|_{B,\mathcal{A}} = 0$$

for all $\varphi \in C$. As $\operatorname{Span}(C) = C_c(G)$, the claim follows. ∎

4.2 The main result

For each n, define

$$X_n := \{(s,t)+\mathcal{L} \in \mathbb{T} : \lim_k \frac{1}{|A_k|}\operatorname{card}((-s+\curlywedge(t+K_n)) \cap A_k) = \operatorname{dens}(\mathcal{L})\theta_H(K_n)\}$$

$$Y_n := \{(s,t)+\mathcal{L} \in \mathbb{T} : \lim_k \frac{1}{|A_k|}\operatorname{card}((-s+\curlywedge(t+U_n)) \cap A_k) = \operatorname{dens}(\mathcal{L})\theta_H(U_n)\},$$

and let

$$\mathcal{T} := \bigcap_n (X_n \cap Y_n). \tag{4.1}$$

By Theorem 2.3, $X_n, Y_n \subseteq \mathbb{T}$ contain subsets of full measure in $\mathbb{T}$. In particular, since we assume the Haar measure to be complete, X_n and Y_n have full measure in $\mathbb{T}$.

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It follows that $\theta_{\mathbb{T}}(\mathcal{T}) = 1$. Note here that $(s,t) + \mathcal{L} \in \mathcal{T}$ if and only if for all n, we have

$$\lim_k \frac{1}{|A_k|}\mathrm{card}((-s + \curlywedge(t + K_n)) \cap A_k) = \mathrm{dens}(\mathcal{L})\theta_H(K_n)\,, \tag{4.2}$$

$$\lim_k \frac{1}{|A_k|}\mathrm{card}((-s + \curlywedge(t + U_n)) \cap A_k) = \mathrm{dens}(\mathcal{L})\theta_H(U_n). \tag{4.3}$$

We can now prove the following result (compare [16, Theorem 4.6]).

Theorem 4.3 *Let $(G, H, \mathcal{L})$ be a CPS, let $B \subseteq H$ be a Borel precompact set, and let $\mathcal{T} \subseteq \mathbb{T}$ be as in (4.1). Then,*

(a) *We have $\mathcal{T} \subseteq \mathcal{MG}(B)$. In particular, $\mathcal{MG}(B)$ has full measure in $\mathbb{T}$.*
(b) *For all $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, we have $-s + \curlywedge(t + B) \in \mathcal{AP}\text{-}Bes_{\mathcal{A},ps}(G)$.*
(c) *For each $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, we have*

$$\lim_k \frac{1}{|A_k|}\mathrm{card}((-s + \curlywedge(t + B)) \cap A_k) = \mathrm{dens}(\mathcal{L})\theta_H(B).$$

(d) *For each $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, the autocorrelation γ of $-s + \curlywedge(t + B)$ exists, is independent of the choice of $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, and satisfies*

$$\gamma = \mathrm{dens}(\mathcal{L})\omega_{c(B)} = \mathrm{dens}(\mathcal{L}) \sum_{(x,x^\star)\in\mathcal{L}} \theta_H(B \cap (x^\star + B))\delta_x.$$

(e) *The diffraction $\widehat{\gamma}$ is given by*

$$\widehat{\gamma} = \mathrm{dens}(\mathcal{L})^2 \sum_{(\chi,\chi^\star)\in\mathcal{L}^0} \left|\int_B \chi^\star(t)\mathrm{d}t\right|^2 \delta_\chi = (\mathrm{dens}(\mathcal{L}))^2 \omega^\star_{\widetilde{|1_B|^2}}.$$

(f) *For each $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, the Fourier–Bohr coefficients*

$$a^{\mathcal{A}}_\chi(-s + \curlywedge(t + B)) = \chi(s)\lim_k \frac{1}{|A_k|} \sum_{x\in\curlywedge(t+B)\cap A_k} \overline{\chi(x)}$$

exist and satisfy

$$a^{\mathcal{A}}_\chi(-s + \curlywedge(t + B)) = \begin{cases} \mathrm{dens}(\mathcal{L})(\chi(s)\chi^\star(t)) \int_B \chi^\star(r)\mathrm{d}r & \text{if } \chi \in \pi_{\widehat{G}}(\mathcal{L}^0) \\ 0 & \text{otherwise.} \end{cases}$$

(g) *For all $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, the set $-s + \curlywedge(t + B)$ satisfies the CPP*

$$\widehat{\gamma}(\{\chi\}) = \left|a^{\mathcal{A}}_\chi(-s + \curlywedge(t + B))\right|^2 \qquad \forall \chi \in \widehat{G}.$$

Proof "**(a)**" Let $(s,t) + \mathcal{L} \in \mathcal{T}$. We have

$$\delta_{-s+\curlywedge(t+K_n)} \le \delta_{-s+\curlywedge(t+B)} \le \delta_{-s+\curlywedge(t+U_n)}$$

$$\delta_{-s+\curlywedge(t+K_n)} \le \tau_{-s}\omega_{\tau_t g_n} \le \delta_{-s+\curlywedge(t+U_n)}\,,$$

which together with (4.2) and (4.3) gives

$$\left|\tau_{-s}\omega_{\tau_t g_n} - \delta_{-s+\curlywedge(t+B)}\right| \le \delta_{-s+\curlywedge(t+U_n)} - \delta_{-s+\curlywedge(t+K_n)}.$$

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Thus,

$$\|\tau_{-s}\omega_{\tau_t g_n} - \delta_{-s+\curlywedge(t+B)}\|_{\mathrm{B},\mathcal{A}} \le \mathrm{dens}(\mathcal{L})\,(\theta_H(U_n) - \theta_H(K_n)) < \frac{\mathrm{dens}(\mathcal{L})}{n},$$

which proves (a).

"**(b)**" Let $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$. Then, (2.6) gives that for all $\varphi \in C_{\mathrm{c}}(G)$, we have

$$\lim_n \|\tau_{-s}\omega_{\tau_t g_n} * \varphi - \delta_{-s+\curlywedge(t+B)} * \varphi\|_{\mathrm{B},1,\mathcal{A}} = 0. \tag{4.4}$$

Since $\tau_{-s}\omega_{\tau_t g_n} \in \mathcal{AP}\text{-}Strong(G)$, (b) follows.

"**(c,d,e,f,g)**" Let $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$. Since $\delta_{-s+\curlywedge(t+B)} \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$, for each character χ, the Fourier–Bohr coefficient $a_\chi^{\mathcal{A}}(\delta_{-s+\curlywedge(t+B)})$ exists by [20, Proposition 4.32].

Moreover, for all $\varphi \in C_{\mathrm{c}}(G)$, we have

$$\begin{aligned}(\widehat{\varphi}(\chi))\, a_\chi^{\mathcal{A}}(\delta_{-s+\curlywedge(t+B)}) &= a_\chi^{\mathcal{A}}(\varphi * \delta_{-s+\curlywedge(t+B)}) \\ &= \lim_n a_\chi^{\mathcal{A}}(\varphi * \tau_{-s}\omega_{\tau_t g_n}) = (\widehat{\varphi}(\chi)) \lim_n a_\chi^{\mathcal{A}}(\tau_{-s}\omega_{\tau_t g_n})\end{aligned}$$

with the second inequality following from (4.4) the inequality $\left|a_\chi^{\mathcal{A}}(f)\right| \le \|f\|_{\mathrm{B},1,\mathcal{A}}$. This shows that

$$a_\chi^{\mathcal{A}}(\delta_{-s+\curlywedge(t+B)}) = \lim_n a_\chi^{\mathcal{A}}(\omega_{g_n}). \tag{4.5}$$

Now, when $\chi = 0$ is the trivial character, (4.5) yields (c).

Next, since $\delta_{-s+\curlywedge(t+B)} \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$, the autocorrelation γ of $\delta_{-s+\curlywedge(t+B)}$ exists with respect to $\mathcal{A}$, is strongly almost periodic, and satisfies [20, Theorem 4.39].

$$\widehat{\gamma}(\{\chi\}) = \left|a_\chi^{\mathcal{A}}(-s+\curlywedge(t+B))\right|^2 \qquad \forall \chi \in \widehat{G},$$

which is (g).

Moreover, since

$$a_\chi^{\mathcal{A}}(\tau_{-s}\omega_{\tau_t g_n}) = \begin{cases} \mathrm{dens}(\mathcal{L})(\chi(s)\chi^\star(t)) \int_H g_n(r)\chi^\star(r)\mathrm{d}r & \text{if } \chi \in \pi_{\widehat{G}}(\mathcal{L}^0) \\ 0 & \text{otherwise} \end{cases},$$

(f) follows. Moreover, combining (f) with the fact that γ is strongly almost periodic and satisfies the CPP, we get (e).

Finally, the measure $\eta := \mathrm{dens}(\mathcal{L})\omega_{c(B)}$ is positive definite and hence Fourier transformable. Moreover,

$$\widehat{\eta} = (\mathrm{dens}(\mathcal{L}))^2 \omega^{\star}_{\underset{\smile}{|1_B|^2}} = \widehat{\gamma}$$

with the second equality following from (e). This yields (d). ∎

By combining Theorem 4.3 with Theorem 2.10, we get the following corollary.

Corollary 4.4 *Let $(G,H,\mathcal{L})$ be a CPS, let $B \subseteq H$ be a Borel precompact set, and let $\mathcal{T} \subseteq \mathbb{T}$ be as in (4.1). Then, for each $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$, there exists an ergodic measure $\mathfrak{m}_{(s,t)+\mathcal{L}}$ on $\mathbb{X}(\curlywedge(B))$ such that $\delta_{\curlywedge(B)}$ is generic for $\mathfrak{m}_{(x,t)+\mathcal{L}}$.*

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In the next section, we are going to prove that $\mathfrak{m}_{(s,t)+\mathcal{L}}$ does not depend on the choice of $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$.

4.3 The Mirsky measure

We can now introduce the Mirsky measure associated with the window B. Since ω_{g_n} are strongly almost periodic, the hull $\mathbb{X}(\omega_{g_n})$ is uniquely ergodic. Let $\mathfrak{m}_n$ be the unique ergodic measure on $\mathbb{X}(\omega_{g_n})$. We will show that for all $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$, the sequence $\mathfrak{m}_n$ converges vaguely to $\mathfrak{m}_{(s,t)+\mathcal{L}}$.

By construction, all measures ω_n are supported inside the model set $\curlywedge(U_1)$. It follows immediately that there exists some open precompact set U and some $C > 0$ such that all the hulls $\mathbb{X}(\omega_n)$ and $\mathcal{M}_B^G$ are subsets of $\mathcal{M}_{C,U}$. This set is G-invariant and vaguely compact [5]. Let $\mathbb{Y}$ denote the set of real valued measures in $\mathcal{M}_{C,K}$. Then, $\mathbb{Y}$ is G invariant, vaguely compact, and contains $\mathbb{X}(g_n)$ and $\mathcal{M}_B^G$. Also, we can consider the measures $\mathfrak{m}_n$ and $\mathfrak{m}_{(s,t)+\mathcal{L}}$ as measures on $\mathbb{Y}$.

As usual, for $\phi \in C_c(G)$, we define $F_\phi : \mathbb{Y} \to \mathbb{C}$ via

$$F_\phi(\omega) = \omega * \phi(0).$$

Since $\mathbb{Y}$ only consists of real valued measures, we have $\overline{F_\phi} = F_{\overline{\phi}}$.

Let us first prove the following preliminary lemma.

Lemma 4.5 *Let $\phi_1, \dots, \phi_j \in C_c(G)$. Then, for all $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$, we have*

$$\lim_k \frac{1}{|A_k|} \int_{A_k} \prod_{i=1}^{j} \phi_i * \delta_{-s+\curlywedge(t+B)}(t)\mathrm{d}t = \lim_n \int_{\mathcal{M}_{C,K}} \prod_{i=1}^{j} F_{\phi_i}(\omega)\mathrm{d}\mathfrak{m}_n(\omega).$$

Proof The limit

$$\lim_k \frac{1}{|A_k|} \int_{A_k} \prod_{i=1}^{j} \phi_i * \delta_{-s+\curlywedge(t+B)}(t)\mathrm{d}t$$

exists by the Besicovitch almost periodicity of $\phi_i * \delta_{-s+\curlywedge(t+B)}$. A short computation shows

$$\begin{aligned}
&\left\| \left(\prod_{i=1}^{j} \phi_i * \delta_{-s+\curlywedge(t+B)} \right) - \prod_{i=1}^{j} \left(\phi_i * (\tau_{-s}\omega_{\tau_t g_n}) \right) \right\|_{\mathrm{B},1,\mathcal{A}} \\
&\quad \le \sum_{l=1}^{j} \left\| \left(\prod_{i=1}^{l-1} \phi_i * \delta_{-s+\curlywedge(t+B)} \right) \cdot \left(\phi_l * \delta_{-s+\curlywedge(t+B)} - \phi_l * (\tau_{-s}\omega_{\tau_t g_n}) \right) \right. \\
&\quad \left. \cdot \left(\prod_{i=l+1}^{j} \phi_i * (\tau_{-s}\omega_{\tau_t g_n}) \right) \right\|_{\mathrm{B},1,\mathcal{A}} \\
&\quad \le jC^{j-1} \| \phi_l * \delta_{-s+\curlywedge(t+B)} - \phi_l * (\tau_{-s}\omega_{\tau_t g_n}) \|_{\mathrm{B},1,\mathcal{A}},
\end{aligned}$$

where

$$C = \max\{ \| \phi_i * \delta_{-s+\curlywedge(t+B)} \|_\infty, \| \phi_i * (\tau_{-s}\omega_{\tau_t g_n}) \|_\infty : 1 \le i \le j \}.$$

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Therefore, the definition of $\mathcal{MG}(B)$ and (2.6) give

$$\lim_k \frac{1}{|A_k|}\int_{A_k}\prod_{i=1}^{j}\phi_i * \delta_{-s+\curlywedge(t+B)}(t)\mathrm{d}t = \lim_n \lim_k \frac{1}{|A_k|}\int_{A_k}\prod_{i=1}^{j}\phi_i * (\tau_{-s}\omega_{\tau_t g_n})\mathrm{d}t.$$

Finally, by Lemma 2.8 and the unique ergodicity of m_n, we have

$$\lim_k \frac{1}{|A_k|}\int_{A_k}\prod_{i=1}^{j}\phi_i * (\tau_{-s}\omega_{\tau_t g_n})\mathrm{d}t = \int_{\mathcal{M}_{C,K}}\prod_{i=1}^{j}F_{\phi_i}(\omega)\mathrm{d}\mathfrak{m}_n(\omega)$$

which proves the claim. ∎

We can now prove that all $m_{(s,t)+\mathcal{L}}$ are the same.

Proposition 4.6 *For each $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$, the measures $\mathfrak{m}_n$ converge vaguely on $\mathbb{Y}$ to $\mathfrak{m}_{(s,t)+\mathcal{L}}$. In particular, $\mathfrak{m}_{(s,t)+\mathcal{L}}$ is the same for all $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$.*

Proof Fix an arbitrary $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$

Consider the algebra $\mathbb{A}$ generated by $\{F_\phi : \phi \in C_c(G)\} \cup \{1_{\mathbb{Y}}\}$, which is dense in $C(\mathbb{Y})$.

Since $(s,t)+\mathcal{L} \in \mathcal{MG}(B)$, the measure $\delta_{-s+\curlywedge(t+B)}$ is generic for $\mathfrak{m}_{(s,t)+\mathcal{L}}$ and hence, for all $\varphi_1,\ldots,\varphi_j \in C_c(G)$, we have

$$\lim_k \frac{1}{|A_k|}\int_{A_k}\prod_{i=1}^{j}\delta_{-s+\curlywedge(t+B)} * \varphi_i(t)\mathrm{d}t = \mathfrak{m}_{(s,t)+\mathcal{L}}\Big(\prod_{i=1}^{j}F_{\phi_i}\Big). \tag{4.6}$$

This together with Lemma 4.5 gives that for all $\varphi_1,\ldots,\varphi_j \in C_c(G)$, we have

$$\mathfrak{m}_{(s,t)+\mathcal{L}}\Big(\prod_{i=1}^{j}F_{\phi_i}\Big) = \lim_n \mathfrak{m}_n\Big(\prod_{i=1}^{j}F_{\phi_i}\Big).$$

Now, since $\mathfrak{m}_{(s,t)+\mathcal{L}}(1_{\mathbb{Y}}) = 1 = \mathfrak{m}_n(1_{\mathbb{Y}})$, it follows immediately that

$$\mathfrak{m}_{(s,t)+\mathcal{L}}(F) = \lim_n \mathfrak{m}_n(F) \qquad \forall F \in \mathbb{A}. \tag{4.7}$$

Since G is second countable and $\mathbb{Y}$ is metrizable and compact, the set $\mathrm{Prob}(\mathbb{Y})$ of probability measures on $\mathbb{Y}$ is vaguely compact and metrizable. Therefore, to show that m_n converges vaguely to $\mathfrak{m}_{(s,t)+\mathcal{L}}$, it suffices to show that $\mathfrak{m}_{(s,t)+\mathcal{L}}$ is the unique vague cluster point of $\mathfrak{m}_n$. Let $\mathfrak{m}$ be any vague cluster point of $\mathfrak{m}_n$. Then, (4.7) yields

$$\mathfrak{m}_{(s,t)+\mathcal{L}}(F) = \lim_n \mathfrak{m}_n(F) \qquad \forall F \in \mathbb{A}.$$

Therefore, the denseness of $\mathbb{A}$ gives $\mathfrak{m}_{(s,t)+\mathcal{L}} = \mathfrak{m}$, completing the proof. ∎

This leads to the next definition, compare [16].

Definition 4.2 We define the *Mirsky measure* Q_B^G of B as

$$Q_B^G = \mathfrak{m}_{(s,t)+\mathcal{L}} \qquad \forall (s,t)+\mathcal{L} \in \mathcal{MG}(B).$$

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By combining Theorem 4.3, Proposition 4.6, and [20, 21], we get the following proposition.

Proposition 4.7 *Let $(G, H, \mathcal{L})$ be a CPS and $B \subseteq H$ be a Borel precompact set. Then,*

(a) *For all $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$, the set $-s + \curlywedge(t + B)$ is generic for Q_B^G.*
(b) *$(\mathcal{M}_B^G, G, Q_B^G)$ has a pure point dynamical spectrum. Moreover, the dynamical spectrum is generated by the set $\{\chi : \widehat{\gamma}(\{\chi\}) \neq 0\}$ of Bragg peak positions.*
(c) *γ is the autocorrelation of $(\mathcal{M}_B^G, G, Q_B^G)$.*
(d) *$Q_B^G(\mathcal{M}_B^G \cap \mathcal{AP}\text{-}Bes_{\mathcal{A},ps}(G)) = 1$.*
(e) *For each χ with $\widehat{\gamma}(\{\chi\}) \neq 0$, the function*

$$\mathcal{M}_B^G \cap \mathcal{AP}\text{-}Bes_{\mathcal{A},ps}(G) \ni \omega \to a_\chi^{\mathcal{A}}(\omega)$$

is a measurable eigenfunction for $\mathcal{M}_B^G$.

We complete the section by characterizing $\mathcal{MG}(B)$ in terms of genericity with respect to the Mirsky measure and Fourier–Bohr coefficients.

Lemma 4.8 *Let $(s,t) + \mathcal{L} \in \mathbb{T}$. Then, $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$ if and only if $-s + \curlywedge(t + B) \in \mathcal{AP}\text{-}Bes_{\mathcal{A}}(G)$ and*

$$a_\chi^{\mathcal{A}}(-s + \curlywedge(t + B)) = \begin{cases} \mathrm{dens}(\mathcal{L})(\chi(s)\chi^\star(t)) \int_B \chi^\star(r)\mathrm{d}r & \text{if } \chi \in \pi_{\widehat{G}}(\mathcal{L}^0) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathcal{MG}(B)$ does not depend on the choice of U_n, K_n, and g_n.

Proof $\Longrightarrow$ follows from Theorem 4.3.

$\Longleftarrow$ Let $0 \in V \subseteq G$ be open such that $-s + \curlywedge(t + U_1)$ is V-uniformly discrete. Pick some $\varphi \in C_c(G)$ such that $\varphi \geq 0$, $\int \varphi = 1$, and $\mathrm{supp}(\varphi) \subseteq V$.

Since $\delta_{-s+\curlywedge(t+B)} - \tau_{-s}\omega_{\tau_t g_n}$ is Besicovitch almost periodic, we have by the Parseval identity

$$\begin{aligned}
&\|\varphi * \delta_{-s+\curlywedge(t+B)}(x) - \varphi * (\tau_{-s}\omega_{\tau_t g_n})(x)\|_{\mathrm{B},2,\mathcal{A}}^2 \\
&= \sum_{\chi \in \widehat{G}} \left| a_\chi^{\mathcal{A}}(\varphi * \delta_{-s+\curlywedge(t+B)}) - \varphi * (\tau_{-s}\omega_{\tau_t g_n}) \right|^2 \\
&= \sum_{\chi \in \widehat{G}} |\widehat{\varphi}(\chi)|^2 \left| a_\chi^{\mathcal{A}}(\delta_{-s+\curlywedge(t+B)}) - \tau_{-s}\omega_{\tau_t g_n}) \right|^2 \\
&= \mathrm{dens}(\mathcal{L})^2 \sum_{(\chi,\chi^\star) \in \mathcal{L}^0} \left|\widehat{\phi}(\chi)\right|^2 \left|\widetilde{1_B}(\chi^\star) - \widetilde{g_m}(\chi^\star)\right|^2 \\
&= \mathrm{dens}(\mathcal{L})^2 \delta_{\mathcal{L}^0} \left(\left|\widehat{\phi}\right|^2 \otimes \left|\widetilde{1_B} - \widetilde{g_m}\right|^2 \right) \\
&= \mathrm{dens}(\mathcal{L}) \delta_{\mathcal{L}} \left((\varphi^\dagger) * \widetilde{(\varphi^\dagger)} \otimes (1_B - g_m) * \widetilde{(1_B - g_m)} \right),
\end{aligned}$$

with the last equality following from the Poisson summation formula [27].

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Note here that $\Big((\varphi^\dagger) * \widetilde{(\varphi^\dagger)} \otimes (1_B - g_m) * \widetilde{(1_B - g_m)}\Big)$ are supported inside a fixed compact set, and hence $\delta_{\mathcal{L}}\Big((\varphi^\dagger) * \widetilde{(\varphi^\dagger)} \otimes (1_B - g_m) * \widetilde{(1_B - g_m)}\Big)$ is a finite sum. Since $\Big((\varphi^\dagger) * \widetilde{(\varphi^\dagger)} \otimes (1_B - g_m) * \widetilde{(1_B - g_m)}\Big)$ converge point wisely to 0, we get

$$\lim_n \|\varphi * \delta_{-s+\curlywedge(t+B)}(x) - \varphi * (\tau_{-s}\omega_{\tau_t g_n})(x)\|^2_{\mathrm{B},2,\mathcal{A}} = 0\,,$$

and hence by [20, Lemma 2.7], we have

$$\lim_n \|\varphi * \delta_{-s+\curlywedge(t+B)}(x) - \varphi * (\tau_{-s}\omega_{\tau_t g_n})(x)\|_{\mathrm{B},1,\mathcal{A}} = 0.$$

Finally, the choice of φ and [35, Lemma 5.8.3] give the claim. ∎

Remark 4.9 (Independence of the Mirsky measure of various choices) Here, we explain why the Mirsky measure does not depend on the various choices we made.

(a) The Mirsky measure Q_B^G does not depend on the choice of U_n, K_n, and g_n. Indeed, for a fixed tempered van Hove sequence $\mathcal{A}$, by Lemma 4.8, the set $\mathcal{MG}(B)$ does not depend on the choice of U_n, K_n, and g_n. Since for each $(s,t) + \mathcal{L} \in \mathcal{MG}(B)$ the measure $-s + \curlywedge(t+B)$ is generic for the Mirsky measure, and since any element in the dynamical system can be generic for at most one ergodic measure with respect to $\mathcal{A}$, the Mirsky measure is the same for all choices of U_n, K_n, and g_n.

(b) The Mirsky measure Q_B^G does not depend on the choice of the van Hove sequence $\mathcal{A}$ but the set $\mathcal{MG}(B)$ usually does depend on this choice. In particular, the elements $-s + \curlywedge(t+B)$ which are generic with respect to Q_B^G depend on the choice of $\mathcal{A}$.

The independence of Q_B^G of $\mathcal{A}$ is immediate. Indeed, given some precompact Borel set $B \subseteq H$, we can construct the sets U_n, K_n, and the function g_n, and these constructions are independent of the choice of $\mathcal{A}$. Now, the dynamical system $\mathbb{X}(\omega_{g_n})$ is uniquely ergodic, and hence the choice of the unique ergodic measure $\mathfrak{m}_n$ on $\mathbb{X}(\omega_{g_n})$ does not depend on the choice of $\mathcal{A}$. As Q_B^G is the vague limit of $\mathfrak{m}_n$, it is also independent on the choice of $\mathcal{A}$.

On the other hand, since $\mathcal{MG}(B)$ uniquely determine the generic Besicovitch almost periodic elements in $(\mathcal{M}_B^G, G, Q_B^G)$, and since $\mathcal{M}_B^G$ is usually not uniquely ergodic, it is easy to construct examples where different choices of $\mathcal{A}$ lead to different generic Besicovitch almost periodic elements.

(c) In [16], the Mirsky measure Q_B^G is introduced as the push forward of the Haar measure $\theta_{\mathbb{T}}$ through the measurable function

$$G \times H/\mathcal{L}(s,t) + \mathcal{L} \to -s + \curlywedge(t+B) \in \mathcal{M}_B^G.$$

Lemma 4.8 and [16, Theorem 4.6] imply that the two constructions produce the same measure.

Points (a) and (b) above then follow alternately from the observation that the construction in [16] does not depend on these choices.

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5 Some examples with Borel windows

Consider a CPS $(G, H, \mathcal{L})$ and a precompact Borel window B. Let $\mathbb{Y} := \{\delta_{-s+\curlywedge(t+W)} : (s,t) + \mathcal{L} \in \mathcal{MG}(B)\}$. Theorem 4.3 gives the inclusion

$$\mathbb{Y} \subseteq \{\Gamma \in \mathcal{M}_B^G : \mathrm{dens}(\Gamma) = \mathrm{dens}(\mathcal{L})\theta_H(B)\}.$$

Moreover, when B is topological (meaning open or closed), the above inclusion is equality [4, 14].

We give below a class of examples with Borel window for which this inclusion is strict. First, let us prove the following simple lemma.

Lemma 5.1 *Let $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ be any fully Euclidean CPS with one to one star mapping. Let $W \subset \mathbb{R}^m$ be any regular model set and $\Gamma \subset \curlywedge(W)$ be any subset. Then, for each $0 < a < \infty$, there exists some precompact Borel set B_a such that*

$$\Gamma = \curlywedge(B_a) \text{ and } \lambda(B_a) = a.$$

Here, λ denotes the Lebesgue measure on $\mathbb{R}^m$.

Proof Define

$$W_0 := \Gamma^\star = \{x^\star : x \in \Gamma\} \subseteq W.$$

Then, $\pi_{\mathbb{R}^m}(\mathcal{L})$ is countable. In particular, W_0 is at most countable and hence it is Borel and satisfies $\lambda(W_0) = 0$. Moreover, since $\star$ is one to one we have $\Gamma = \curlywedge(W_0)$.

Next, pick some t such that $t + [0, \sqrt[m]{a}]^m \cap W = \varnothing$. Define

$$W_1 := \left(t + [0, \sqrt[m]{a}]^m\right) \setminus (\pi_{\mathbb{R}^m}(\mathcal{L})).$$

Then, W_1 is Borel, $\lambda(W_1) = a$ and $\curlywedge(W_1) = \varnothing$. The set $B_a = W_0 \cup W_1$ satisfies the desired conditions. ∎

Now, we can easily construct counterexamples. Pick, for example, the Fibonacci CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ and window, and let $\curlywedge(W)$ be the Fibonacci model set. Pick a subset $\Gamma \subset \curlywedge(W)$ which is not pure point diffractive with respect to $A_n = [-n, n]$, for example, a generic point set for the Bernoullisation of Fibonacci. Set $a = \mathrm{dens}_{\mathcal{A}}(\Gamma)$ and pick the window B_a from Lemma 5.1. Then, by the construction of B_a, for this Borel window, we have

$$\Gamma \in \{\Lambda \in \mathcal{M}_B^G : \mathrm{dens}(\Lambda) = \mathrm{dens}(\mathcal{L})\theta_H(B)\}.$$

On the other hand, since every element in $\mathbb{Y}$ has pure point diffraction, we have $\Gamma \notin \mathbb{Y}$.

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