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MULTIPLE DRAWING MULTI-COLOUR URNS BY STOCHASTIC APPROXIMATION

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Abstract

A classical Pólya urn scheme is a Markov process where the evolution is encoded by a replacement matrix $(R_{i,j})_{1\le i,j\le d}$. At every discrete time-step, we draw a ball uniformly at random, denote its colour c, and replace it in the urn together with $R_{c,j}$ balls of colour j (for all $1 \le j \le d$). We study multiple drawing Pólya urns, where the replacement rule depends on the random drawing of a set of m balls from the urn (with or without replacement). Many particular examples of this situation have been studied in the literature, but the only general results are due to Kuba and Mahmoud (2017). These authors proved second-order asymptotic results in the two-colour case, under the so-called *balance* and *affinity* assumptions, the latter being somewhat artificial. The main idea of this work is to apply stochastic approximation methods to this problem, which enables us to prove analogous results to Kuba and Mahmoud, but without the artificial *affinity* hypothesis, and, for the first time in the literature, in the d-colour case ($d \ge 3$). We also provide some partial results in the two-colour nonbalanced case, the novelty here being that the only results for this case currently in the literature are for particular examples.

Keywords: Multiple drawing Pólya urn; limit theorem; stochastic approximation; reinforced process; discrete-time martingale

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1. Introduction

1.1. Classical Pólya urns

Pólya urn schemes are the simplest example of a stochastic process with reinforcement. Understanding these objects is thus the first step in understanding many more intricate models such as reinforced random walks, preferential attachment networks, interacting urn models, and so on. Furthermore, reinforced stochastic processes appear in a wide range of applications; in biology (for example, ant movement, reinforced branching processes), finance and clinical trials, and computer science (preferential attachment networks are used to model the complex networks such as the internet, the World Wide Web, or social networks). We refer the reader to Pemantle's survey [19] for an overview of both the mathematics and applications of reinforced processes.

The classical Pólya urn, first introduced by Eggenberger and Pólya [7], is described as follows. An urn contains initially one white ball and one black ball. At each discrete time-step,

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a ball is picked uniformly at random and then returned to the urn together with another ball of the same colour. Many generalisations of this model have been studied in the literature; namely, urns with more than two colours, or with different, possibly random, replacement rules.

The methods used to study Pólya urns are quite varied: following the seminal work of Athreya and Karlin [2], one successful approach is to embed the urn process in continuous time using exponential clocks and then apply martingale arguments. This method can be very effective; see, for example, [9] and [10]. The main difficulty is to make them applicable to the discrete-time framework.

Following the work of Flajolet *et al.* [8], analytic combinatorics have been used to study Pólya's urn schemes (see [17] and [18]). The main advantage of this method is that, when successful, it yields results for fixed finite time and not only asymptotically when time goes to ∞ as in the embedding-in-continuous-time method. The major drawback is that this method is often nontractable as it relies on solving a nonlinear differential system.

Finally, stochastic approximation (or stochastic algorithms) provides a powerful toolbox to prove strong laws of large numbers and central limit theorems for the composition of a Pólya urn. This method relies on discrete-time martingale methods; a good introduction to stochastic approximation can be found in Duflo [6]. The literature on stochastic algorithms is very wide; in the urn context, stochastic approximation has already been used, for example, by Laruelle and Pagès [14] in the study of Pólya urns with random replacement rules. In this paper we apply the stochastic approximation results of Renlund [20] and Zhang [22] to a model of multiple drawing Pólya urns.

1.2. Multiple drawing Pólya urns

In this paper we focus on the multiple drawing generalisation of Pólya urns: instead of choosing the replacement rule according to the random drawing of one ball in the urn, we pick at random a handful of balls (more precisely, a fixed number m of balls), and the replacement rule then depends on the colours of these m balls. This model has numerous applications, such as degrees in increasing trees and preferential attachment networks (see [12]), leaves in random circuits (see [12]), and tournaments *à la rock–paper–scissors* (see [16]).

Many particular examples have been studied in the literature, mostly in the two-colour case; see, for example, [1], [4], [5], and [13].

In the recent works of Kuba and Mahmoud [11] and Kuba and Sulzbach [12], the authors considered the general case of two-colour Pólya urns with multiple drawing and proved third-order asymptotics of the composition of the urn under two hypothesis. The standard *balance* hypotheses means that the total number in the urn is deterministic. The *affinity* hypothesis is the assumption that, if we denote by W_n the number of white balls in the urn at time n, then there exist two deterministic sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ such that, for all integers n, $\mathbb{E}[W_{n+1} \mid \mathcal{F}_n] = \alpha_n W_n + \beta_n$.

1.3. The main contributions of this paper

The main idea of this paper is to apply the stochastic approximation methods to the multiple drawing problem. Our main motivation for doing this was Laruelle and Pagès' [14] successful application by these techniques to the classical Pólya urn model with random replacement matrices.

Using these methods we prove results in the d-colour ($d \geq 2$) multiple drawing case. In the $d \geq 3$ case, to the best of the authors' knowledge, these are the first general results in the literature. In the $d \geq 2$ case, our results are analogous to [11], but importantly do not require the somewhat artificial affinity hypothesis.

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Furthermore, we prove partial results for the two-colour *nonbalanced* case; these are the first results in the literature for this situation, apart from the study of particular examples in, for example, [1].

1.4. Definition of the model and main assumptions

The model we study is defined as follows. An urn contains balls of $d \geq 2$ different colours. At time 0, the urn contains $U_{0,i} \geq 0$ balls of colour i for all $1 \leq i \leq d$. We assume that the urn is originally nonempty; namely, $\sum_{i=1}^{d} U_{0,i} > 0$. We fix an integer $m \geq 1$ and a replacement rule $R\colon \Sigma_m^{(d)} \to \mathbb{Z}^d$, where

$$\Sigma_m^{(d)} = \left\{(v_1, \ldots, v_d) \in \mathbb{N}^d \colon \sum_{i=1}^{d} v_i = m\right\}.$$

We denote by $R_1, \ldots, R_d$ the coefficient functions of R. At each (discrete) time-step $n \geq 1$, we draw m balls in the urn *with or without replacement*, and denote by $\xi_{n,i}$ the number of balls of colour i among these m balls. Let $\xi_n = (\xi_{n,1}, \ldots, \xi_{n,d})$. Conditionally on $\xi_n = v$, we then add into the urn $R_i(v)$ balls of type i for all $1 \leq i \leq d$. More precisely, we have

$$U_n = U_{n-1} + R(\xi_n) \quad \text{for all } n \geq 1,$$

where $U_{n,i}$ is the number of balls of colour i in the urn at time n, and $U_n = (U_{n,1}, \ldots, U_{n,d})$.

We denote by T_n the total number of balls and $Z_{n,i} = U_{n,i}/T_n$ the proportion of balls of type i in the urn at time n. Note that, with this notation, in the with-replacement case,

$$P_n(v) := \mathbb{P}(\xi_{n+1} = v \mid \mathcal{F}_n) = \binom{m}{v_1, \ldots, v_d} \prod_{i=1}^{d} Z_{n,i}^{v_i},$$

while in the without-replacement case,

$$P_n(v) := \mathbb{P}(\xi_{n+1} = v \mid \mathcal{F}_n) = \binom{T_n}{m}^{-1} \prod_{i=1}^{d} \binom{U_{n,i}}{v_i}.$$

In the literature, it is often assumed that the urn is *balanced*, meaning that, for all $v \in \Sigma_m^{(d)}$, $r(v) := \sum_{i=1}^{d} R_i(v) = S$, where S is a positive integer. Without this assumption, we need to control the speed of convergence of T_n/n to its limit, and this is not yet understood for $m \geq 2$ (i.e. in the multiple drawing case). It is also standard to assume that the urn is *tenable*, meaning that balls not in the urn are never needed to be removed: we give necessary and sufficient conditions for the tenability to be achieved. Since these assumptions will be used throughout this paper, we denote them as follows.

(B) For all $v \in \Sigma_m^{(d)}$, $r(v) := \sum_{i=1}^{d} R_i(v) = S$.

(T) The urn scheme is tenable.

Although the tenability assumption is natural (an alternative would be to work conditionally on the event that no impossible configuration can occur), we believe it is an interesting and challenging open question to remove the balance hypothesis altogether. Note that in the classical $m = 1$ case, strong results can be proved without the balance assumption (see [9]); the reason being that the urn scheme embedded in continuous time is a multi-type Galton–Watson process, which is no longer the case when $m \geq 2$. Although our main results for d-colour urns require the balance assumption, we are able to prove partial results for two-colour nonbalanced urns.

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The following function from $\Sigma^{(d)} = \{(x_1, \dots, x_d) \in [0,1]^d : \sum_{i=1}^d x_i = 1\}$ onto $\{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_{i=1}^d x_i = 0\}$ (two $(d-1)$-dimensional spaces) and its zeros will play a crucial role in this paper (recall that $r(v) := \sum_{i=1}^d R_i(v)$):

$$h(x) = \sum_{v \in \Sigma_m^{(d)}} \binom{m}{v_1, \dots, v_d} \left(\prod_{i=1}^d x_i^{v_i} \right) (R(v) - r(v)\, x).$$

We will specifically focus on the stable zeros of h, i.e. the zeros at which all eigenvalues of ∇h (the Jacobian matrix of h) have a negative real part.

In this paper we focus on the renormalised composition vector $Z_n := (Z_{n,1}, \dots, Z_{n,d})$. In view of the definition above, the process $(Z_n)_{n \geq 0}$ is a Markov process that depends on two parameters, the initial composition Z_0 and the replacement function R. Before stating them in full detail, in the remainder of this section we summarise our main results. We prove the following results for balanced urns.

- The *limit set* (see Definition 1) of the renormalised composition vector Z_n is almost surely a compact connected set of $\Sigma^{(d)}$ stable by flow of the differential equation $\dot{x} = h(x)$ (see Theorem 1(i)). Note that, given a function h, it is a nontrivial question to determine the compact connected sets stable by the flow of $\dot{x} = h(x)$ (we provide some examples in Section 4). Also, the limit set of Z_n *a priori* depends on the initial composition vector Z_0. Reducing the number of colours to $d = 2$ makes the matter much simpler and, in this particular case, we state almost sure convergence to a constant vector θ as long as the function h is not constant equal to 0 (see Corollary 1(i)).

- Assuming that Z_n converges almost surely to a stable zero of h, we prove convergence of the fluctuations around this almost sure limit (see Theorem 1(ii)).

- When $h \equiv 0$ (we call this case the *diagonal case*), we prove almost sure convergence of the renormalised composition vector to a random vector Z_∞ (see Theorem 2).

In the two-colour nonbalanced case, we prove almost sure convergence to a zero of h (as long as $h \not\equiv 0$), and some partial result for the fluctuations around this almost sure limit (see Theorem 3).

1.5. Main result for d-colour balanced urns

Definition 1. (*See, for example, [19, Definition 2.11].*) Given a stochastic process $(\Pi_n)_{n \geq 0}$, we define its limit set as

$$L(\Pi) := \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \Pi_m}.$$

Theorem 1. (Balanced and tenable d-colour urns.) *Under assumptions (B) and (T), we have the following.*

(i) *The limit set of the renormalised composition vector $Z_n = (Z_{n,1}, \dots, Z_{n,d})$ is almost surely a compact connected set of $\Sigma^{(d)}$ stable by the flow of the differential equation $\dot{x} = h(x)$. If there exists $\theta \in \Sigma^{(d)}$ such that $h(\theta) = 0$ and, for all $n \geq 0$, $\langle h(Z_n), Z_n - \theta \rangle < 0$, then Z_n converges almost surely to θ.*

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(ii) *Assume that there exists a stable zero θ of h such that Z_n converges almost surely to θ when n goes to ∞. Let Λ be the eigenvalue of $-\nabla h(\theta)$ with the smallest real part, let*

$$\Gamma = \frac{1}{S^2} \sum_{v \in \Sigma_m^{(d)}} \binom{m}{v_1, \ldots, v_d} \left(\prod_{i=1}^{d} \theta_i^{v_i} \right) (R(v) - S\theta)(R(v) - S\theta)^t.$$

Then if $\mathrm{Re}(\Lambda) > S/2$ then $\sqrt{n}(Z_n - \theta) \to \mathcal{N}(0, \Sigma)$ in distribution when $n \to \infty$, where

$$\Sigma = \int_0^{+\infty} \left(\exp\left(\left(\frac{\nabla h(\theta)}{S} + \frac{\mathrm{id}}{2} \right) u \right) \right)^t \Gamma \exp\left(\left(\frac{\nabla h(\theta)}{S} + \frac{\mathrm{id}}{2} \right) u \right) \mathrm{d}u$$

*with '*id*' denoting the identity matrix. Assume, additionally, that all Jordan blocks of $\nabla h(\theta)$ associated to Λ are of size* 1. *Then*

- *if $\mathrm{Re}(\Lambda) = S/2$, then $\sqrt{n/\log n}(Z_n - \theta) \to \mathcal{N}(0, \Sigma)$ in distribution when $n \to \infty$, where*

$$\Sigma = \lim_{n \to \infty} \frac{1}{\log n} \int_0^{\log n} \left(\exp\left(\left(\frac{\nabla h(\theta)}{S} + \frac{\mathrm{id}}{2} \right) u \right) \right)^t \times \Gamma \exp\left(\left(\frac{\nabla h(\theta)}{S} + \frac{\mathrm{id}}{2} \right) u \right) \mathrm{d}u;$$

- *if $\mathrm{Re}(\Lambda) < S/2$ then $n^{\mathrm{Re}(\Lambda)/S}(Z_n - \theta)$ converges almost surely to a random variable.*

Remark 1. (i) The fact that both Z_n and θ are in $\Sigma^{(d)}$ implies that the matrix Σ satisfies $\Sigma(1, \ldots, 1)^t = 0$.

(ii) When applying this theorem to particular examples, the difficulty is to understand the flow of the differential equation $\dot{x} = h(x)$. It is, in general, nontrivial to describe the limit set of $(Z_n)_{n \geq 0}$ (see, for example, [16] where this was carried out for one particular example), and this limit set is *a priori* not a point and depends on the initial composition Z_0. The strength of our result is that, if one can prove, for a particular replacement rule and for one particular initial composition vector Z_0, that the renormalised composition vector Z_n converges almost surely to a stable zero of h, then the fluctuations are given relatively easily by Theorem 1(ii). We provide numerous examples in Section 4.

In order to discuss our main result, we first apply it to the well-known particular case of $m = 1$. In this case, the replacement rule is encoded by a replacement matrix $R = (R_{i,j})_{1 \leq i,j \leq d}$. At every time step, we pick a ball uniformly at random from the urn, denote by i its colour, and return it to the urn together with $R_{i,j}$ balls of colour j for all $1 \leq i, j \leq d$. Note that this translates into our framework as $R_j(\boldsymbol{e}_i) = R_{i,j}$ for all $1 \leq i, j \leq d$ (where $\boldsymbol{e}_i$ is the vector with coordinates that are all equal to 0 except the ith, which is equal to 1). We thus have $h(x) = (R - S\mathrm{id}_d)x$. If we assume that R is irreducible and that the urn is tenable, the Perron–Frobenius theorem implies that S is the spectral radius of R, the multiplicity of S as an eigenvalue of R is 1, and there exists an eigenvector ϖ associated to S with coordinates that are all nonnegative and such that $\sum_{i=1}^{d} \varpi_i = 1$.

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In particular, ϖ is the unique zero of h on $\Sigma^{(d)}$, and the eigenvalues of $R - S\mathrm{id}_d$ restricted to $\Sigma^{(d)}$ all have negative real parts. Thus, ϖ is a stable zero of h, and, for all $x \neq \varpi$, $\langle h(x), x - \varpi \rangle = \langle (R - S\mathrm{id}_d)(x - \varpi), x - \varpi \rangle < 0$. Theorem 1(i) applies and we obtain almost sure convergence of Z_n to ϖ and Theorem 1(ii) applies straightforwardly, reproving some standard results from the literature (see [9]).

By analogy to the classical $m = 1$ case, we can say that the cases for which Z_n does not converge to a constant vector or converges to a constant vector that depends on Z_0 are the *nonirreducible* cases. Unfortunately, we are not yet able to give a nice characterisation of these cases in terms of the replacement function R. From Theorem 1(i) we obtain a sufficient condition; namely, if there exists a zero θ of h such that $\langle h(x), x - \theta \rangle < 0$ for all $x \in \Sigma^{(d)}$, then Z_n converges almost surely to θ (for all choices of Z_0 satisfying (T)). Note that, in the classical setting of $m = 1$, the nonirreducible cases are also more intricate; see [3] for the d-colour balanced and triangular case and [10] for two-colour triangular urns.

In the case when $h \equiv 0$, which we call the *diagonal* case (although in [11] and [12] it is termed *triangular* – we choose to change the terminology so that it is coherent with the $m = 1$ case), we prove the following theorem.

Theorem 2. (Diagonal balanced case.) *Under assumptions (B) and (T), if $h \equiv 0$ then Z_n converges almost surely to a random vector Z_∞ when n tends to ∞.*

1.6. Main results for the two-colour case

In the two-colour case, the renormalised composition vector Z_n of the urn at time n is characterised by its first coordinate since $Z_{n,1} + Z_{n,2} = 1$. Thus, it is enough to study the first coordinate and the problem becomes unidimensional and easier. We introduce the simplified notation for this special case.

In the two-colour case, we say that the urn contains white balls and black balls. We denote by W_n and $X_n = W_n/T_n$, respectively, the number and the proportion of white balls in the urn at time n. The replacement function can be seen as a replacement matrix

$$R = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix},$$

where the a_i's, and b_i's are integers (such that $a_i + b_i = S$ for all $1 \leq i \leq m$ under assumption (B)). At each (discrete) time-step n, we draw m balls from the urn *with* or *without* replacement, and denote by ζ_n the number of white balls among these m. Conditionally on $\zeta_n = k$, we then add into the urn a_{m-k} white balls and b_{m-k} black balls (we retain the notation of [11] and [12]).

Note that, with this notation, in the with-replacement case,

$$\mathbb{P}(\zeta_{n+1} = k \mid \mathcal{F}_n) = \binom{m}{k} X_n^k (1 - X_n)^{m-k},$$

and in the without-replacement case,

$$\mathbb{P}(\zeta_{n+1} = k \mid \mathcal{F}_n) = \binom{m}{k} \frac{(W_n)_k (T_n - W_n)_{m-k}}{(T_n)_m},$$

where $(n)_q = n(n-1)\cdots(n-q+1)$ for all integers q and n.

The following result is a corollary of Theorem 1.

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Corollary 1. (Two-colour balanced urns.) *Assume that $d = 2$, (B) and (T) hold, and let*

$$g(x) := \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} (a_{m-k} - Sx).$$

(i) *If $g \not\equiv 0$ then there exists a random variable $\theta_\star \in [0,1]$ such that the proportion of white balls in the urn at time n, denoted by X_n, converges almost surely to $\theta_\star$. Furthermore, we have $g(\theta_\star) = 0$ and $g'(\theta_\star) \leq 0$ almost surely.*

(ii) *Furthermore, conditionally on $\theta_\star$, the following hold with $\Lambda = -g'(\theta_\star)/S$ and*

$$\Gamma = \frac{1}{S^2} \sum_{k=0}^{m} \binom{m}{k} \theta_\star^k (1-\theta_\star)^{m-k} (a_{m-k} - S\theta_\star)^2 :$$

- *if $\Lambda > \frac{1}{2}$ then $\sqrt{n}(X_n - \theta_\star) \to \mathcal{N}(0, \Gamma/(2\Lambda - 1))$ in distribution when $n \to \infty$;*
- *if $\Lambda = \frac{1}{2}$ then $\sqrt{n/\log n}(X_n - \theta_\star) \to \mathcal{N}(0, \Gamma)$ in distribution when $n \to \infty$;*
- *if $\Lambda < \frac{1}{2}$ then $n^\Lambda (X_n - \theta_\star)$ converges almost surely to a finite random variable.*

In the two–colour case, we are also able to obtain some partial results about the nonbalanced case. The following result is not a corollary of Theorem 1, which only applies to balanced urn schemes. Note that in the nonbalanced case, we need a stronger assumption than the tenability assumption: we are only able to study urns under the assumption that the total number of balls in the urn grows linearly in n, i.e. $\liminf_{n\to\infty} T_n/n > 0$. A way to ensure that this holds is, for example, to assume that $\min_{0\leq k\leq m}(a_k + b_k) \geq 1$.

Theorem 3. (Nonbalanced two-colour case.) *Assume that* $\liminf_{n\to\infty} T_n/n > 0$ *and let*

$$\tilde{g}(x) := \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} ((1-x)a_{m-k} - xb_{m-k}).$$

(i) *If $\tilde{g} \not\equiv 0$ then there exists a random variable $\theta_\star \in [0,1]$ such that the proportion of white balls in the urn, denoted by X_n, converges almost surely to $\theta_\star$. Furthermore, we have $\tilde{g}(\theta_\star) = 0$ and $\tilde{g}'(\theta_\star) \leq 0$ almost surely.*

(ii) *Conditionally on $\theta_\star$, let*

$$\omega := \sum_{k=0}^{m} \binom{m}{k} \theta_\star^k (1-\theta_\star)^{m-k} c_{m-k},$$

where $c_i := a_i + b_i$ for all $0 \leq i \leq m$,

$$H(x) := \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} ((1-x)a_{m-k} - xb_{m-k})^2,$$

and

$$\lambda := \frac{|h'(\theta_\star)|}{\omega}, \qquad \sigma^2 := \frac{H(\theta_\star)}{\omega^2}.$$

Assume that $\sigma^2 > 0$. Then, if $\lambda > \frac{1}{2}$,

$$\sqrt{n}(Z_n - \theta_\star) \to \mathcal{N}\left(0, \frac{\sigma^2}{2\lambda - 1}\right) \quad \text{in distribution when } n \to \infty.$$

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Remark 2. Note that in both Corollary 1 and Theorem 3, part (i) yields almost sure convergence to a random variable $\theta_\star$ which belongs to the set of zeros of the function g (respectively, $\tilde{g}$). The fact that g (respectively, $\tilde{g}$) is a nonzero polynomial function ensures that this set is a set of at most m isolated points of $[0, 1]$. If g admits a unique zero θ_0 in $[0, 1]$ such that $g'(\theta_0) \leq 0$, then the proportion of white balls converges almost surely to this zero independently of the initial composition of the urn; we provide examples in Section 4. This particular case is what we called the *irreducible* case when discussing Theorem 1(i). If g admits at least two such zeros, then the almost sure limit of X_n depends on the initial composition X_0; we refer the reader to Example 4.1.5 for such a case.

1.7. Outline of the paper

In Section 1.8 we state an algebraic necessary and sufficient condition for the urn to be tenable: it is a straightforward generalisation of the two-colour case studied by Kuba and Sulzbach [12, Lemma 1]. Section 2 contains the proofs of Theorems 1 and 2. In Section 3 we treat the two-colour balanced case and state the proof of Corollary 1. In Section 4 we provide numerous two-colour and three-colour examples and show how to apply our main results to them. Finally, in Section 5 we treat the two-colour nonbalanced case and provide the proof of Theorem 3 as well as some examples.

1.8. Tenability

Here we state a lemma in order to provide necessary and sufficient conditions for tenability. This lemma and its proof are very similar to those stated in [12] in the case of two-colour urns. The proof in the d-colour case is very similar to the $d = 2$ case and does not give additional insight. Therefore, we do not detail it.

Lemma 1. *Consider the urn process with initial composition U_0 and replacement function R. For all $1 \leq i \leq d$, we denote by ν_i the greatest common divisor of $\{R_i(v) \colon v \in \Sigma_m^{(d)} \setminus \{m\boldsymbol{e}_i\}\}$. For all integers ℓ, we denote by $[\ell]_i$ the remainder of division of ℓ by ν_i.*

- *In the with-replacement case, the urn scheme is tenable if and only if, for all $1 \leq i \leq d$, $R_i(v) \geq 0$ for all $v \neq m\boldsymbol{e}_i$, and $R_i(m\boldsymbol{e}_i) \geq 0$ or $-R_i(m\boldsymbol{e}_i)$ is a divisor of $U_{0,i}$ and ν_i.*
- *In the without-replacement case, the urn scheme is tenable if and only if, for all $1 \leq i \leq d$, $R_i(v) \geq -\nu_i$ for all $v \neq m\boldsymbol{e}_i$ and*

$$\begin{aligned} R_i(m\boldsymbol{e}_i) \in\ & [-m, \infty] \\ & \cup \big([-m - \nu_i + 1, -m) \\ & \quad \cap \big\{\ell \in -\mathbb{N} \colon [U_{0,i}]_i \in \{[-\ell]_i, [-\ell + 1]_i, \dots, [m + \nu_i - 1]_i\}\big\}\big). \end{aligned}$$

2. Law of large numbers and central limit theorem for balanced urn schemes

2.1. The balanced urn scheme as a stochastic algorithm

Lemma 2. *Under assumptions (B) and (T), the renormalised composition vector of the urn at time n, denoted by Z_n, satisfies the following recursion:*

$$Z_{n+1} = Z_n + \frac{1}{T_{n+1}}(h(Z_n) + \Delta M_{n+1} + \varepsilon_{n+1}),$$

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where

$$h(x) = \sum_{v \in \Sigma_m^{(d)}} \binom{m}{v_1, \ldots, v_d} \left(\prod_{i=1}^{d} x_i^{v_i} \right) (R(v) - r(v)x) \quad \textit{and} \quad \Delta M_{n+1} = Y_{n+1} - \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n]$$

with

$$Y_{n+1} = R(\xi_{n+1}) - r(\xi_{n+1}) Z_n,$$

where ξ_{n+1} denotes the (random) vector of balls drawn at time $n+1$, and, finally, $\varepsilon_{n+1} = 0$ in the with-replacement case and ε_{n+1} an $\mathcal{F}_{n+1}$-adapted term satisfying $\varepsilon_n \to 0$ almost surely when n tends to ∞ in the without-replacement case.

Proof. Recall that $U_{n+1} = U_n + R(\xi_{n+1})$ and $T_{n+1} = T_n + r(\xi_{n+1})$, implying that

$$\begin{aligned} Z_{n+1} - Z_n &= \frac{U_n + R(\xi_{n+1})}{T_{n+1}} - \frac{U_n}{T_n} \\ &= \frac{1}{T_{n+1}} \left(U_n + R(\xi_{n+1}) - \frac{T_n + r(\xi_{n+1})}{T_n} U_n \right) \\ &= \frac{1}{T_{n+1}} (R(\xi_{n+1}) - r(\xi_{n+1}) Z_n) \\ &= \frac{1}{T_{n+1}} (\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] + \Delta M_{n+1}). \end{aligned}$$

Note that, in the with-replacement case,

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \sum_{v \in \Sigma_m^{(d)}} P_n(v)(R(v) - r(v) Z_n) = h(Z_n),$$

since $P_n(v) = \binom{m}{v_1, \ldots, v_d} \prod_{i=1}^{d} Z_{n,i}^{v_i}$, which concludes the proof of the with-replacement case.

In the without-replacement case,

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \sum_{v \in \Sigma_m^{(d)}} \frac{\prod_{i=1}^{d} \binom{U_{n,i}}{v_i}}{\binom{T_n}{m}} (R(v) - r(v) Z_n) = h(Z_n) + \varepsilon_{n+1},$$

with

$$\begin{aligned} \varepsilon_{n+1} &:= \sum_{v \in \Sigma_m^{(d)}} \left(\frac{\prod_{i=1}^{d} \binom{U_{n,i}}{v_i}}{\binom{T_n}{m}} - \binom{m}{v_1, \ldots, v_d} \prod_{i=1}^{d} Z_{n,i}^{v_i} \right) (R(v) - r(v) Z_n) \\ &= \sum_{v \in \Sigma_m^{(d)}} \binom{m}{v_1, \ldots, v_d} (R(v) - r(v) Z_n) \left(\frac{\prod_{i=1}^{d} \prod_{j=0}^{v_i - 1} (Z_{n,i} - j/T_n)}{\prod_{j=0}^{m-1} (1 - j/T_n)} - \prod_{i=1}^{d} Z_{n,i}^{v_i} \right). \end{aligned}$$

For any two sequences of real-valued random variables $(A_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$, we denote $A_n = \mathcal{O}(B_n)$ if and only if there exists a deterministic constant M such that, for all $n \geq 1$, $|A_n| \leq M B_n$ almost surely. Using the fact that $Z_{n,i} \leq 1$ almost surely for all $1 \leq i \leq d$ and $n \geq 0$, and that $T_n = T_0 + nS \to \infty$ when n goes to ∞, we have, for all $v \in \Sigma_m^{(d)}$,

$$\frac{\prod_{i=1}^{d} \prod_{j=0}^{v_i - 1} (Z_{n,i} - j/T_n)}{\prod_{j=0}^{m-1} (1 - j/T_n)} = \prod_{i=1}^{d} Z_{n,i}^{v_i} + \mathcal{O}\left(\frac{1}{T_n} \right),$$

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where the deterministic constant in the $\mathcal{O}$ term depends on v; since there is a finite number of elements in $\Sigma_m^{(d)}$, we can take the maximum of these v-dependent constants so that the constant in the $\mathcal{O}$ term above does not depend on v. We thus have, using, in particular, the fact that $\|Z_n\|$ is bounded by 1 since $Z_n \in \Sigma^{(d)}$ (we use the L^2-norm, i.e. for all $u \in \mathbb{R}^d$, $\|u\|^2 := \sum_{i=1}^d u_i^2$),

$$\begin{aligned}\|\varepsilon_{n+1}\| &\le \sum_{v\in\Sigma_m^{(d)}} \binom{m}{v_1,\dots,v_d}(\|R(v)\| + |r(v)|\|Z_n\|)\left|\prod_{i=1}^d Z_{n,i}^{v_i} + \mathcal{O}\left(\frac{1}{T_n}\right) - \prod_{i=1}^d Z_{n,i}^{v_i}\right| \\ &\le d^m \max_{v\in\Sigma_m^{(d)}}(\|R(v)\| + |r(v)|)\mathcal{O}\left(\frac{1}{T_n}\right) \\ &= \mathcal{O}\left(\frac{1}{T_n}\right) \qquad (1)\end{aligned}$$

almost surely when n goes to ∞. We find, in particular, that $\varepsilon_{n+1} \to 0$ almost surely when n goes to ∞, which concludes the proof of the without-replacement case. □

2.2. Strong law of large numbers (proof of Theorem 1(i))

As mentioned in [19], there are few results for the almost sure convergence of d-dimensional ($d \ge 2$) stochastic algorithms; ; [19, Section 2.5] is dedicated to such higher-dimensional stochastic algorithms. Different versions of the following theorem (originally due to Robbins and Monro [21]) exist in the literature; the version we state here is a combination of [6, Theorem 1.4.26] and [14, Theorem A.1], which are both stated in dimension $d \ge 2$. We also refer the reader to [19, Corollary 2.15], also stated in the multi-dimensional case.

Theorem 4. *Consider the sequence of random vectors $(\theta_n)_{n\ge0}$ defined by the following recursion:*

$$\theta_{n+1} = \theta_n + \gamma_{n+1} f(\theta_n) + \gamma_{n+1}(\Delta\hat{M}_{n+1} + \hat{\varepsilon}_{n+1}) \quad \text{for all } n \ge n_0, \qquad (2)$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a differentiable nonnull function, θ_0 is a deterministic vector, and, for all $n \ge n_0$, $\Delta\hat{M}_n$ is an $\mathcal{F}_n$-increment martingale and $\hat{\varepsilon}_n$ is an $\mathcal{F}_n$-adapted remainder term.

Assume, in addition, that

$$\hat{\varepsilon}_n \xrightarrow{\text{A.S.}} 0, \qquad \sup_{n\ge0}\mathbb{E}[\|\Delta\hat{M}_{n+1}\|^2|\mathcal{F}_n] < +\infty,$$

where '$\xrightarrow{\text{A.S.}}$' denotes almost sure convergence, and that $(\gamma_n)_{n\ge1}$ is a sequence of positive real numbers satisfying

$$\sum_{n\ge0}\gamma_n = \infty \quad \text{and} \quad \sum_{n\ge0}\gamma_n^2 < +\infty.$$

Then the limit set of $(\theta_n)_{n\ge0}$ (see Definition 1) is almost surely a compact connected set stable by the flow of the differential equation $\dot{x} = f(x)$.

If, in addition, f is uniformly bounded, and admits a zero θ such that, for all n, $\langle f(\theta_n), \theta_n - \theta\rangle < 0$, θ_n converges almost surely to θ.

This theorem then applies to our particular case, in the balanced case only (since otherwise, the sequence $(\gamma_n)_n$ is random and not deterministic), and leads us to the proof of Theorem 1(i).

Proof of Theorem 1(i). In view of Lemma 2, and since we assume that the urn is balanced, the vector Z_n satisfies recursion (2) with $\gamma_n = 1/T_n = 1/(T_0 + nS)$, and $f = h$ a differentiable function, $\Delta\hat{M}_{n+1} = \Delta M_{n+1}$ an $\mathcal{F}_n$-martingale difference, and $\hat{\varepsilon}_{n+1} = \varepsilon_{n+1}$ a remainder term tending to 0 when n tends to ∞.

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It only remains to check that $\sup_{n\geq 0}\mathbb{E}[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n] < +\infty$. Recall that $\Delta M_{n+1} = Y_{n+1} - \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n]$, where $Y_{n+1} = R(\xi_{n+1}) - SZ_n$. If we let $\|R\| := \max_{v\in\Sigma_m^{(d)}}\|R(v)\|$, then, for all $n \geq 0$, $\|Y_{n+1}\| \leq \|R\| + S$, implying that $\|\Delta M_{n+1}\| \leq 2(\|R\| + S)$ almost surely. In particular, we have $\sup_{n\geq 0}\mathbb{E}[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n] < +\infty$.

Finally, note that $f = h$ is uniformly bounded by $\|R\| + S$. We can thus apply Theorem 4, which concludes the proof. □

2.3. Central limit theorem (proof of Theorem 1(ii))

To prove Theorem 1(ii), the idea is again to apply standard theorems from the stochastic algorithms literature. We state here a weak version of a result by Zhang [22]. We also refer the reader to Laruelle and Pagès [14, Theorem A.2] (there are different versions of this paper and we refer the reader to [15] for the most recent publication).

Theorem 5. *Assume that θ_n satisfies recursion (2) with $\gamma_n = 1/n$ and that there exists $\theta \in \mathbb{R}^d$ a stable zero of f such that θ_n converges to θ with positive probability. Also assume that, for some $\delta > 0$,*

$$\sup_{n\geq 0}\mathbb{E}(\|\Delta\hat{M}_{n+1}\|^{2+\delta} \mid \mathcal{F}_n) < +\infty,$$
$$\mathbb{E}(\Delta\hat{M}_{n+1}\Delta\hat{M}_{n+1}^t \mid \mathcal{F}_n) \to \hat{\Gamma}, \qquad n \to \infty \text{ almost surely},$$

where $\hat{\Gamma}$ is a deterministic symmetric positive semi-definite matrix and, for some $\eta > 0$,

$$n^{3/2}\mathbb{E}[\|\hat{\varepsilon}_{n+1}\|^2\,\mathbf{1}_{\{\|\theta_n-\theta\|\leq\eta\}} \mid \mathcal{F}_n] \to 0, \qquad n \to \infty, \tag{3}$$

where $\mathbf{1}_A$ is the indicator function on the event A. Let $\hat{\Lambda}$ be the eigenvalue of $-\nabla f(\theta)$ with the largest real part. If we assume that θ_n converges almost surely to some deterministic limit θ, then we have the following.

- *If $\mathrm{Re}(\hat{\Lambda}) > \frac{1}{2}$ then $\sqrt{n}(\theta_n - \theta) \to \mathcal{N}(0, \hat{\Sigma})$ in distribution when $n \to \infty$, where*

$$\hat{\Sigma} = \int_0^{+\infty} (\mathrm{e}^{(\nabla f(\theta)+\mathrm{id}/2)u})^t\,\hat{\Gamma}\,\mathrm{e}^{(\nabla f(\theta)+\mathrm{id}/2)u}\,\mathrm{d}u.$$

Assume, additionally, that f is twice differentiable, and that all Jordan blocks of $\nabla f(\theta)$ associated to $\hat{\Lambda}$ have size 1. Then we have the following.

- *If $\mathrm{Re}(\hat{\Lambda}) = \frac{1}{2}$ then $\sqrt{n/\log n}(\theta_n - \theta) \to \mathcal{N}(0, \hat{\Sigma})$ in distribution when $n \to \infty$, where*

$$\hat{\Sigma} = \lim_{n\to\infty}\frac{1}{\log n}\int_0^{\log n} (\mathrm{e}^{(\nabla f(\theta)+\mathrm{id}/2)u})^t\,\hat{\Gamma}\,\mathrm{e}^{(\nabla f(\theta)+\mathrm{id}/2)u}\,\mathrm{d}u.$$

- *If $\mathrm{Re}(\hat{\Lambda}) < \frac{1}{2}$ then $n^{\mathrm{Re}(\hat{\Lambda})}(\theta_n - \theta)$ converges almost surely to a finite random variable.*

Remark 3. Note that assumption (3) is not stated as such in [14] and [22], in which the assumption depends on the values of $\mathrm{Re}(\hat{\Lambda})$. We leave it to the reader to check that (3) is stronger than the assumptions of [14] and [22], but since it holds in our particular case, we only state this weaker version.

Applying this result to our framework leads us to the proof of the second claim of Theorem 1.

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Proof of Theorem 1(ii). Recall that (see Lemma 2 for details)

$$Z_{n+1} - Z_n = \frac{1}{T_{n+1}}(h(Z_n) + \Delta M_{n+1} + \varepsilon_{n+1}).$$

We also have $T_n = T_0 + nS$ (by assumption (B)), which yields

$$\begin{aligned} Z_{n+1} - Z_n &= \frac{1}{(n+1)S}(h(Z_n) + \Delta M_{n+1} + \varepsilon_{n+1}) \\ &\quad + \frac{1}{(n+1)S}\left(\frac{(n+1)S}{T_0 + (n+1)S} - 1\right)(h(Z_n) + \Delta M_{n+1} + \varepsilon_{n+1}) \\ &= \frac{1}{n+1}(f(Z_n) + \Delta \hat{M}_{n+1} + \hat{\varepsilon}_{n+1}), \end{aligned}$$

where $f = h/S$ (note that this function is infinitely differentiable), $\Delta \hat{M}_{n+1} = \Delta M_{n+1}/S$, and

$$\hat{\varepsilon}_{n+1} = \frac{\varepsilon_{n+1}}{S} + \left(\frac{1}{1 + T_0/(n+1)S} - 1\right)\left(f(Z_n) + \Delta \hat{M}_{n+1} + \frac{\varepsilon_{n+1}}{S}\right).$$

This last equality implies that, almost surely when n tends to ∞,

$$\|\hat{\varepsilon}_{n+1}\| \leq \frac{\|\varepsilon_{n+1}\|}{S} + \mathcal{O}\left(\frac{1}{n}\right).$$

Recall that in the with-replacement case, $\varepsilon_n = 0$ for all integers n. In the without-replacement case, we have already proved that $\|\varepsilon_n\| = \mathcal{O}(1/T_n) = \mathcal{O}(1/n)$ (see (1)), implying that $\|\hat{\varepsilon}_n\| = \mathcal{O}(1/n)$, and

$$n^{3/2}\mathbb{E}[\|\hat{\varepsilon}_n\|^2 |\, \mathbf{1}_{\{|\theta_n - \theta| \leq \eta\}} \,|\mathcal{F}_{n-1}] = \mathcal{O}(n^{-1/2}) \to 0$$

almost surely when n tends to ∞ for all $\eta > 0$.

We have already mentioned that ΔM_{n+1} is almost surely bounded by $2(\|R\| + S)$, and, thus, $\sup_{n \geq 0} \mathbb{E}[\|\hat{\Delta} M_{n+1}\|^{2+\delta} \mid \mathcal{F}_n] < +\infty$ almost surely for all $\delta > 0$.

Finally, note that

$$\begin{aligned} \mathbb{E}[\Delta M_{n+1} \Delta M_{n+1}^t \mid \mathcal{F}_n] &= \mathbb{E}[Y_{n+1} Y_{n+1}^t \mid \mathcal{F}_n] - \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n]\mathbb{E}[Y_{n+1}^t \mid \mathcal{F}_n] \\ &= \sum_{v \in \Sigma_m^{(d)}} P_n(v)(R(v) - SZ_n)(R(v) - SZ_n)^t \\ &\quad - \sum_{v,w \in \Sigma_m^{(d)}} P_n(v) P_n(w)(R(v) - SZ_n)(R(w) - SZ_n)^t. \end{aligned}$$

Since $Z_n \to \theta$ almost surely, we have, in both the with- and without-replacement cases, $P_n(v) \to \binom{m}{v_1, \ldots, v_d} \prod_{i=1}^d \theta_i^{v_i}$ almost surely, implying that

$$\mathbb{E}[\Delta \hat{M}_{n+1} \Delta \hat{M}_{n+1}^t \mid \mathcal{F}_n] = \frac{1}{S^2}\mathbb{E}[\Delta M_{n+1} \Delta M_{n+1}^t \mid \mathcal{F}_n] \xrightarrow{\text{A.S.}} \Gamma,$$

where

$$\begin{aligned} S^2\Gamma &= \sum_{v \in \Sigma_m^{(d)}} \binom{m}{v_1, \ldots, v_d}\left(\prod_{i=1}^d \theta_i^{v_i}\right)(R(v) - S\theta)(R(v) - S\theta)^t \\ &\quad - \sum_{v,w \in \Sigma_m^{(d)}} \binom{m}{v_1, \ldots, v_d}\binom{m}{w_1, \ldots, w_d}\left(\prod_{i=1}^d \theta_i^{v_i + w_i}\right)(R(v) - S\theta)(R(w) - S\theta)^t \end{aligned}$$

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$$= \sum_{v\in\Sigma_m^{(d)}} \binom{m}{v_1,\dots,v_d}\left(\prod_{i=1}^{d}\theta_i^{v_i}\right)(R(v)-S\theta)(R(v)-S\theta)^t - h(\theta)h(\theta)^t$$

$$= \sum_{v\in\Sigma_m^{(d)}} \binom{m}{v_1,\dots,v_d}\left(\prod_{i=1}^{d}\theta_i^{v_i}\right)(R(v)-S\theta)(R(v)-S\theta)^t,$$

since $h(\theta) = 0$. Note that Γ is the limit of a sequence of symmetric positive, semi-definite matrices, and, as such, it is also symmetric positive and semi-definite. Therefore, Theorem 5 applies, which concludes the proof. □

Remark 4. The result obtained in [22] also permits us to treat the cases of $\mathrm{Re}(\hat{\Lambda}) \leq S/2$ when the Jordan blocks associated $\hat{\Lambda}$ are not all of size 1. The asymptotic renormalisation then depends on the size of the largest blocks of $\nabla f(\theta)$ associated to $\hat{\Lambda}$. Applying this stronger version to our framework would result in a generalisation of Theorem 1 to these cases. Since this generalisation is quite technical to state and since most of the examples fall under Theorem 1 as it is, we do not provide more details.

2.4. The diagonal case (proof of Theorem 2)

Before proving Theorem 2, we provide a simple characterisation of the diagonal balanced urn schemes.

Lemma 3. *Under assumption (B), the function h is identically null on* $\Sigma^{(d)}$ *if and only if there exists an integer* σ *such that* $S = m\sigma$ *and*

$$R(v) = \sigma v \quad \textit{for all } v \in \Sigma_m^{(d)}. \tag{4}$$

Proof. First note that, straightforwardly, the fact that there exists an integer σ such that $S = m\sigma$ and such that (4) holds implies that $h \equiv 0$.

The reverse implication is less straightforward. First note that, for all $1 \leq i \leq d$, $h(\boldsymbol{e}_i) = R(m\boldsymbol{e}_i) - S\boldsymbol{e}_i = 0$, implying that $R(m\boldsymbol{e}_i) = S\boldsymbol{e}_i$ for all $1 \leq i \leq d$.

We reason by induction and prove that there exists σ such that $R(v) = \sigma v$ for all vectors $v \in \Sigma_m^{(d)}$ having at most k nonnull coordinates. Note that if this result holds, it implies that $\sigma = S/m$ and, thus, that m divides S (take $v = \boldsymbol{e}_1 + (m-1)\boldsymbol{e}_2$ and recall that $R_1(v) = \sigma$ is an integer).

Assume that this holds for some integer $k < m$, $R(v) = \sigma v$ for all vectors $v \in \Sigma_m^{(d)}$ having at most k nonnull coordinates. We prove that this property extends to vectors with at most $k+1$ nonzero coordinates. Without loss of generality, we can focus on vectors $v \in \Sigma_m^{(d)}$ such that the last $d-k-1$ coordinates are equal to 0, which is actually the set $\Sigma_m^{(k+1)}$ (with a slight abuse of notation since these vectors are still d-dimensional). Let $x = x_1\boldsymbol{e}_1 + \cdots + x_{k+1}\boldsymbol{e}_{k+1} \in \Sigma^{(d)}$, we have, by assumption,

$$\sum_{v\in\Sigma_m^{(k+1)}} \binom{m}{v_1,\dots,v_{k+1}}\left(\prod_{i=1}^{k+1}x_i^{v_i}\right)R(v) = Sx. \tag{5}$$

First note that, for all $k+2 \leq j \leq d$, we have

$$\sum_{v\in\Sigma_m^{(k+1)}} \binom{m}{v_1,\dots,v_{k+1}}\left(\prod_{i=1}^{k+1}x_i^{v_i}\right)R_j(v) = 0.$$

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Note that the tenability assumption implies that $R_j(v) \geq 0$ for all $v \notin \{m\boldsymbol{e}_1, \ldots, m\boldsymbol{e}_d\}$ (see Lemma 1). Moreover, we have proved that, for all $1 \leq i \leq d$, $R(m\boldsymbol{e}_i) = S\boldsymbol{e}_i$, implying, in particular, that $R_j(m\boldsymbol{e}_i) \geq 0$ for all $1 \leq j \leq d$. Therefore, the sum in the above display is a sum of nonnegative terms. The fact that it is equal to 0 therefore implies that all its terms are null, and, thus, that $R_j(v) = 0$ $(= \sigma v_j)$ for all $k+2 \leq j \leq d$. It remains to prove that the same equality, namely $R_j(v) = \sigma v_j$, holds for all $1 \leq j \leq k+1$.

Note that

$$\sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \right) (R(v) - \sigma v)$$

$$= \sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \mathbf{1}_{\{v_i \neq 0\}} \right) (R(v) - \sigma v),$$

since all $v \in \Sigma_m^{(k+1)}$ having at least one null coordinate satisfy $R(v) = \sigma v$ by the induction hypothesis. Therefore, (5) implies that

$$Sx = \sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \mathbf{1}_{\{v_i \neq 0\}} \right) (R(v) - \sigma v)$$

$$+ \sigma \sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \right) v. \tag{6}$$

Note that, for all $1 \leq j \leq k+1$,

$$x_j \frac{\partial}{\partial x_j} (x_1 + \cdots + x_{k+1})^m = x_j \frac{\partial}{\partial x_j} \left(\sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \right) \right)$$

$$= \sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \right) v_j,$$

implying that

$$mx = \sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \right) v_j.$$

Therefore, (6) implies that

$$Sx = \sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \mathbf{1}_{\{v_i \neq 0\}} \right) (R(v) - \sigma v) + \sigma m x,$$

and, thus, for all $x = x_1 \boldsymbol{e}_1 + \cdots + x_{k+1} \boldsymbol{e}_{k+1} \in \Sigma^{(d)}$,

$$\sum_{v \in \Sigma_m^{(k+1)}} \binom{m}{v_1, \ldots, v_{k+1}} \left(\prod_{i=1}^{k+1} x_i^{v_i} \mathbf{1}_{\{v_i \neq 0\}} \right) (R(v) - \sigma v) = 0.$$

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Take $x_p = t \in (0,1)$ and $x_i = (1-t)/k$ for all $i \in \{1,\dots,k+1\}\setminus\{p\}$. To all $v \in \Sigma_m^{(k+1)}$, we associate bijectively the couple (v_p, v_p^*), where v_p is the pth coordinate of v and $v_p^* \in \Sigma_{m-v_p}^{(k)}$ is the vector with k coordinates $v_1,\dots,v_{p-1},v_{p+1},\dots,v_{k+1}$. With this notation, we have

$$\begin{aligned}
0 &= \sum_{v\in\Sigma_m^{(k+1)}} \binom{m}{v_1,\dots,v_{k+1}} \left(\frac{t}{k}\right)^{v_p} (1-t)^{m-v_p} (R_p(v) - \sigma v_p) \\
&= \sum_{v_p=1}^{m} \sum_{v_p^*\in\Sigma_{m-v_p}^{(k)}} \binom{m}{v_1,\dots,v_{k+1}} \left(\frac{t}{k}\right)^{v_p} (R_p(v) - \sigma v_p) \sum_{\ell=0}^{m-v_p} \binom{m-v_p}{\ell} (-t)^{\ell} \\
&= \sum_{u=1}^{m} t^u \sum_{v_p=1}^{u} \sum_{v_p^*\in\Sigma_{m-v_p}^{(k)}} \binom{m}{v_1,\dots,v_{k+1}} \binom{m-v_p}{u-v_p} k^{-v_p} (R_p(v) - \sigma v_p)(-1)^{u-v_p},
\end{aligned}$$

implying that, for all $1 \le u \le m$,

$$\sum_{v_p=1}^{u} \binom{m-v_p}{u-v_p} k^{-v_p} (-1)^{u-v_p} \sum_{v_p^*\in\Sigma_{m-v_p}^{(k)}} \binom{m}{v_1,\dots,v_{k+1}} (R_p(v) - \sigma v_p) = 0.$$

For $u = 1$, this equation yields that $R_p(v) = \sigma$ for all vector v such that $v_p = 1$. Using the above equation for $u = 2$, one can then induce that $R_p(v) = 2\sigma$ for all vector v such that $v_p = 2$, and, inductively, prove that $R_p(v) = \sigma v_p$ for all $1 \le p \le k+1$.

Together, for all $v \in \Sigma_m^{(k+1)}$, we have $R(v) = \sigma v$, which concludes the proof. □

Proof of Theorem 2. We have (since we assume that the urn is balanced)

$$Z_{n+1} = Z_n + \frac{1}{T_0 + (n+1)S}(\Delta M_{n+1} + \varepsilon_{n+1}),$$

where we recall that ΔM_{n+1} is the increment of a martingale and $\|\varepsilon_{n+1}\| = \mathcal{O}(1/n)$ almost surely when n tends to ∞. We infer that, for all $1 \le k \le d$,

$$Z_{n,k} = Z_{0,k} + \sum_{i=0}^{n-1} \frac{\Delta M_{i+1,k}}{T_0 + (i+1)S} + \sum_{i=0}^{n-1} \frac{\varepsilon_{i+1,k}}{T_0 + (i+1)S}. \tag{7}$$

Note that the last sum of (7) converges almost surely since either $\varepsilon_i = 0$ for all i, or $|\varepsilon_{i,k}| \le \|\varepsilon_i\| = \mathcal{O}(1/i)$ almost surely. The first sum in (7) is a martingale and its quadratic variation is given by

$$\mathrm{H}_n := \sum_{i=0}^{n-1} \frac{\mathbb{E}[\Delta M_{i+1,k}^2 \mid \mathcal{F}_i]}{(T_0 + (i+1)S)^2}.$$

Recall that $\Delta M_{i+1,k}^2$ is almost surely bounded by $4(\|R\| + S)^2$ implying that H_n is almost surely convergent when n goes to ∞. Therefore, the martingale itself, i.e. the first sum in (7) converges almost surely to a finite random variable. In conclusion, Z_n converges almost surely to a random vector Z_∞. □

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3. The two-colour particular case

Proof of Corollary 1. (i) In the two-colour balanced case, the renormalised composition vector Z_n of the urn at time n is actually characterised by its first coordinate since $Z_{n,1} + Z_{n,2} = 1$. Thus, it is enough to study this first coordinate, and the problem becomes unidimensional and, thus, slightly easier. Although Corollary 1 could be deduced from Theorem 1, we provide a stand-alone and much simpler proof: we apply directly a result about one-dimensional stochastic algorithms (see [19, Corollary 2.7 and Theorem 2.9]). Let

$$g(x) := \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} (a_{m-k} - Sx).$$

Then

$$X_{n+1} - X_n = \frac{1}{n}\left(\frac{g(X_n)}{S} + \Delta M_{n+1} + \varepsilon_{n+1}\right),$$

where

$$\Delta M_{n+1} = \frac{1}{S}(Y_{n+1} - \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n]),$$

with $Y_{n+1} := a_{m-\zeta_{n+1}} - SZ_n$, and where $\varepsilon_{n+1} = 0$ in the with-replacement case and ε_{n+1} satisfies $|\varepsilon_n| = \mathcal{O}(1/n)$ almost surely when n tends to ∞ in the without-replacement case.

Since $g(x)$ is a polynomial, either it is identically null or it has isolated zeros. The $g \equiv 0$ case is called the *diagonal* case and is treated separately. If $g \not\equiv 0$ then g has at most m zeros on $[0, 1]$. In view of Corollary 2.7 and Theorem 2.9 of [19], we can conclude that X_n converges almost surely to one of the zeros of g where g' is nonpositive, depending on the initial composition of the urn. This proves Corollary 1(i).

(ii) If we assume that X_n converges almost surely to some stable zero $\theta_\star$ of g, then we can apply Theorem 5. We have

$$\mathbb{E}[(\Delta M_{n+1})^2 \mid \mathcal{F}_n] \to \Gamma := \frac{1}{S^2} \sum_{k=0}^{m} \binom{m}{k} \theta_\star^k (1-\theta_\star)^{m-k} (a_{m-k} - S\theta_\star)^2.$$

Finally, let $\Lambda = -g'(\theta_\star)/S$, and the following central limit theorem holds:

- if $\Lambda > \frac{1}{2}$ then $\sqrt{n}(X_n - \theta_\star) \to \mathcal{N}(0, \Gamma/(2\Lambda - 1))$ in distribution when $n \to \infty$;
- if $\Lambda = \frac{1}{2}$ then $\sqrt{n/\log n}(X_n - \theta_\star) \to \mathcal{N}(0, \Gamma)$ in distribution when $n \to \infty$;
- if $\Lambda < \frac{1}{2}$ then $n^\Lambda (X_n - \theta_\star)$ converges almost surely to a random variable.

This concludes the proof of Corollary 1. □

Diagonal case. In the two-colour case, assuming that both the urn is balanced and that $g \equiv 0$ implies that there exists an integer q such that

$$R = \begin{pmatrix} mq & 0 \\ (m-1)q & q \\ \vdots & \vdots \\ 0 & mq \end{pmatrix}.$$

This case was treated by Kuba and Mahmoud [11] and Kuba and Sulzbach [12].

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4. Examples

4.1. Two-colour examples

The first four examples are two-colour examples: we thus apply Corollary 1 to them. We choose to focus on nonaffine examples to which the results of Kuba and Mahmoud [11] do not apply; note that affine models correspond to the case when g is linear.

Example 1. Take

$$R = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

In this case, $g(x) = (1+x)(1-2x)$ with a single zero in $[0, 1]$ at $\frac{1}{2}$. Thus, almost surely when n goes to ∞, the proportion of white balls in the urn at time n converges to $\frac{1}{2}$. In this particular case, $S = 3$, and $\Lambda = -g'(\frac{1}{2})/S = 1 > \frac{1}{2}$. One can check that $\Gamma = \frac{1}{36}$, and since $2\Lambda - 1 = 1$, we have, by Corollary 1(ii),

$$\sqrt{n}\big(X_n - \tfrac{1}{2}\big) \to \mathcal{N}\big(1, \tfrac{1}{36}\big) \quad \text{in distribution when } n \to \infty.$$

Example 2. Take

$$R = \begin{pmatrix} 4 & 0 \\ 1 & 3 \\ 1 & 3 \end{pmatrix}.$$

In this case, $g(x) = (1-x)(1-3x)$, with two zeros in $[0, 1]$ at 1 and $\frac{1}{3}$. Note that $g'(1) = 2$ and $g'(\frac{1}{3}) = -2$, thus, almost surely when n tends to ∞, the proportion of white balls in the urn converges to $\frac{1}{3}$. Note that $\Lambda = -g'(\frac{1}{3})/S = \frac{1}{2}$. Also, one can check that $\Gamma = \frac{1}{18}$, which implies, by Corollary 1(ii), that

$$\sqrt{\frac{n}{\log n}}\left(X_n - \frac{1}{3}\right) \to \mathcal{N}\left(1, \frac{1}{18}\right) \quad \text{in distribution when } n \to \infty.$$

Example 3. Take

$$R = \begin{pmatrix} 7 & 1 \\ 3 & 5 \\ 1 & 7 \end{pmatrix}.$$

Then $g(x) = 2x^2 - 4x + 1$ and this polynomial has a unique root in $[0, 1]$, given by $\theta_\star = 1 - \sqrt{2}/2$. Applying Theorem 1(i), we find that the proportion of white balls X_n converges almost surely to $1 - \sqrt{2}/2$. Moreover, $\Lambda = -g'(\frac{1}{2})/8 = \sqrt{2}/4 < \frac{1}{2}$. Thus, by Corollary 1(ii), there exists a random variable Ψ such that

$$n^{\sqrt{2}/4}\big(X_n - \tfrac{1}{2}\big) \to \Psi \quad \text{almost surely when } n \to \infty.$$

Example 4. Take

$$R = \begin{pmatrix} 6 & 0 \\ 3 & 3 \\ 1 & 5 \end{pmatrix}.$$

In this case, $g(x) = (x-1)^2$, implying that 1 is the unique zero of g in $[0, 1]$. Thus, the proportion of white balls in the urn converges almost surely to 1. We have $g'(1) = 0$, in which case we are unable to apply Corollary 1(ii): we have no information about the speed of convergence of the proportion of white balls to its limit.

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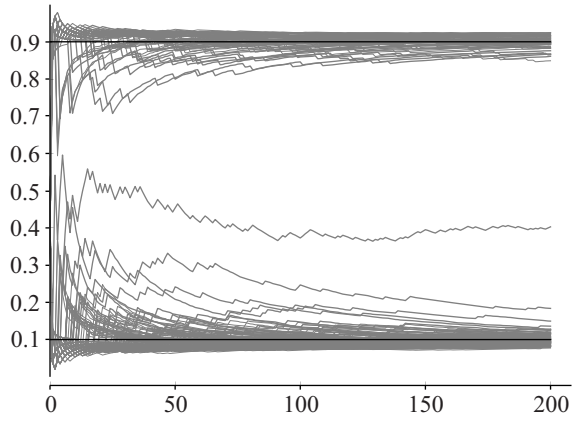


FIGURE 1: A hundred realisations of the urn process of Example 5, all starting at (4, 6) and running for 200 steps. We see that each trajectory converges to one of the two stable zeros of g, namely, $\frac{1}{10}$ and $\frac{9}{10}$.

Example 5. We consider an example for which $m = 3$:

$$R = \begin{pmatrix} 82 & 9 \\ 91 & 0 \\ 0 & 91 \\ 9 & 82 \end{pmatrix}.$$

In this case, $S = 91$, and $g(x) = -200(x - \frac{1}{10})(x - \frac{1}{2})(x - \frac{9}{10})$ admits three zeros on $[0, 1]$, namely $\frac{1}{10}$, $\frac{1}{2}$, and $\frac{9}{10}$. Note that $g'(\frac{1}{2}) > 0$ while $\frac{1}{10}$ and $t\,\frac{9}{10}$ are stable zeros of g. Therefore, the proportion of white balls in the urn converges to some random variable $\theta_\star$, and $\theta_\star \in \{\frac{1}{10}, \frac{9}{10}\}$ almost surely (see Figure 1). Note that $g'(\frac{1}{3}) = g'(\frac{2}{3}) = -64$ implying that $\Lambda = \frac{64}{91} > \frac{1}{2}$. Therefore, one can check that $\sqrt{n}(X_n - \theta_\star)$ converges in distribution to a centred normal distribution of variance $\frac{4131}{673\,40}$.

Note that although we know that X_n goes almost surely to a random variable X_∞ with support at $\{\frac{1}{10}, \frac{9}{10}\}$, it is an open problem to determine the distribution of X_∞, and it is expected that this distribution depends on the initial composition of the urn X_0.

4.2. Three-colour examples

For all three-colour examples, we use a computer algebra system to help with the calculations. Note that it is, in practice, more difficult to apply Theorem 1(i) than Theorem 1(ii); the strategy is to first find the zeros of the function h, then calculate the Jacobian of h at these zeros, extract their spectrum and isolate the zeros whose Jacobians only have negative eigenvalues. In all the examples below, there is at most one such zero and it is our candidate for the limit of the normalised composition vector Z_n; if there is no such zero, we can say no more about the considered urn (see Example 10) except that its limit set is a compact connected set stable by the flow of $\dot{x} = h(x)$.

Once we have a candidate θ to be the limit of Z_n, we need to check that $\langle h(Z_n),\ Z_n - \theta\rangle < 0$ for all n. The way we do this is by proving that $\langle h(x),\ x - \theta\rangle < 0$ for all $x \in \Sigma^{(d)}$ that is not a zero of h. This is the most intricate part. All the examples below are stated for $m = 2$, since it implies that $\langle h(x),\ x - \theta\rangle$ is a polynomial of order at most three and the set where it is negative can be calculated exactly (we do this with a computer algebra system).

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Finally, we prove that Z_n never reaches the other zeros of h, which proves convergence to θ by Theorem 1(i). Applying Theorem 1(ii) to obtain the fluctuations is then straightforward.

Example 6. We consider the three-colour urn scheme defined by the following replacement rule:

$$\begin{aligned} R(2,0,0) &= (1,0,0), & R(0,1,1) &= (1,0,0),\\ R(0,2,0) &= (0,1,0), & R(1,0,1) &= (0,1,0),\\ R(0,0,2) &= (0,0,1), & R(1,1,0) &= (0,0,1). \end{aligned}$$

In this case,

$$h(x) = \begin{pmatrix} x_1^2 + 2x_2x_3 - x_1 \\ x_2^2 + 2x_1x_3 - x_2 \\ x_3^2 + 2x_2x_3 - x_3 \end{pmatrix}.$$

The function $h(x)$ admits four zeros on the simplex $\Sigma_2^{(3)}$, given by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$.

We recall that the function h is a function from $\Sigma^{(d)}$ onto $\{(x_1,x_2,x_3)\in\mathbb{R}^d : x_1+x_2+x_3 = 0\}$, two spaces of dimension two, therefore, the Jacobian matrix of h at any point of $\Sigma^{(d)}$ is a 2×2 matrix. The eigenvalues of the Jacobian matrix of h at the first three zeros mentioned above are 1 and -3. The eigenvalues of $\nabla h(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ are -1 (with multiplicity 2), thus, this zero is stable and is our candidate to be the almost sure limit of Z_n. We actually have $\nabla h(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = -\mathrm{id}_2$.

One can check that the only solutions of $\langle h(x), x-\theta\rangle = 0$ on $\Sigma^{(3)}$, where $\theta = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$, are the four zeros of h. Since θ is a stable zero of h, $\langle h(x),\ x-\theta\rangle < 0$ in a neighbourhood of θ, and, thus, by continuity, $\langle h(x),\ x-\theta\rangle < 0$ for all x in $\Sigma^{(3)}$ such that $h(x)\neq 0$.

Finally, note that $Z_0 \neq (1,0,0)$ implies that $Z_n \neq (1,0,0)$ for all $n\geq 0$, and similarly for $(0,1,0)$ and $(0,0,1)$. Thus, Theorem 1(i) applies and we see that Z_n converges almost surely to $\theta = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$ (if the urn contained balls of more than one colour at time 0).

We now apply Theorem 1(ii). One can check that

$$\Gamma = \frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{8}$$

and

$$\Sigma = \int_0^{+\infty} (\mathrm{e}^{-u/2}\mathrm{id})^t\,\Gamma\,(\mathrm{e}^{-u/2}\mathrm{id})\,\mathrm{d}u = \Gamma.$$

Finally, note that the eigenvalue of $-\nabla h(\theta)$ with the smallest real part is $\Lambda = 1 > S/2 = \frac{1}{2}$. We have $2\,\mathrm{Re}(\Lambda)/S - 1 = 1$, implying that

$$\sqrt{n}\big(Z_n - \tfrac{\mathbf{1}}{3}\big) \to \mathcal{N}(0,\Gamma) \quad \text{in distribution when } n\to\infty,$$

where $\mathbf{1} = (1,1,1)$ and Γ given by (8).

Since Z_n and $\frac{\mathbf{1}}{3}$ are in the simplex $\Sigma^{(d)}$, we know that $\Sigma = \Gamma$ are of rank at most two, and that, in particular,

$$\Sigma(1,1,1)^t = \Gamma(1,1,1)^t = 0,$$

which can indeed be checked using (8).

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Example 7. See Figure 2. Consider the three-colour urn scheme defined by the following replacement rule:

$$\begin{aligned} R(2,0,0) &= (2,0,0), & R(0,1,1) &= (0,1,1),\\ R(0,2,0) &= (1,0,1), & R(1,0,1) &= (0,2,0),\\ R(0,0,2) &= (1,1,0), & R(1,1,0) &= (0,0,2). \end{aligned}$$

In this case,

$$h(x) = \begin{pmatrix} 2x_1^2 + x_2^2 + x_3^3 - 2x_1 \\ x_3(4x_1 + 2x_2 + x_3) - 2x_2 \\ x_2(4x_1 + x_2 + 2x_3) - 2x_3 \end{pmatrix}.$$

One can check that $h(x)$ admits two zeros on the simplex $\Sigma_2^{(3)}$, given by $(1,0,0)$ and $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$. The eigenvalues of the Jacobian matrix of h at $(1,0,0)$ are 2 and -6. The eigenvalues of $\nabla h(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$ are -2 and $-\frac{18}{5}$, thus, this zero is stable and is our candidate to be the almost sure limit of Z_n.

Using computer algebra software, we can also check that the only solutions of $\langle h(x),\ x-\theta\rangle = 0$ are $(1,0,0)$ and $\theta = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$. Since θ is a stable zero, $\langle h(x), x-\theta\rangle$ is negative in a neighbourhood of θ, and, by continuity, it is negative on $\Sigma^{(3)} \setminus \{(1,0,0), \theta\}$.

Note that if $Z_0 \neq (1,0,0)$ then $Z_n \neq (1,0,0)$ for all $n \geq 0$. We can thus apply Theorem 1(i) and conclude that if $Z_0 \neq (1,0,0)$ then Z_n converges almost surely to $\theta = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$.

We now apply Theorem 1(ii). We calculate that

$$\Gamma = \frac{1}{25}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix}, \qquad \Sigma = \frac{1}{25}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 19 & -6 \\ -1 & -6 & 19 \end{pmatrix}. \tag{9}$$

Finally, note that the eigenvalue of $-\nabla h(\theta)$ with the smallest real part is $\Lambda := 2 > S/2 = 1$, implying that, since $2\,\mathrm{Re}(\Lambda)/S - 1 = 1$,

$$\sqrt{n}\big(Z_n - \big(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}\big)^t\big) \to \mathcal{N}(0, \Sigma) \quad \text{in distribution when } n \to \infty,$$

where Σ is given by (9). (Again, one can check that, as expected, $\Sigma(1,1,1)^t = 0$.)

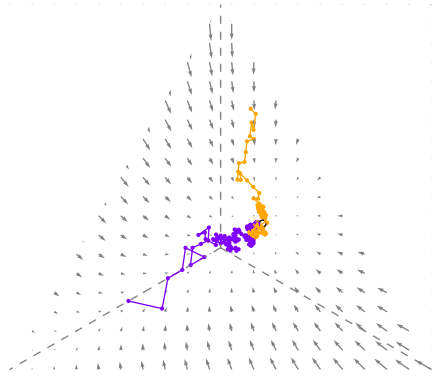


FIGURE 2: Two realisations of the urn of Example 7 starting respectively from $(10,3,3)$ and $(2,6,20)$, and running for 400 steps. The arrows are the vector fields associated to h on the simplex $\Sigma^{(3)}$; one can, in particular, see that $(1,0,0)$ is indeed an unstable zero while $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$ is stable.

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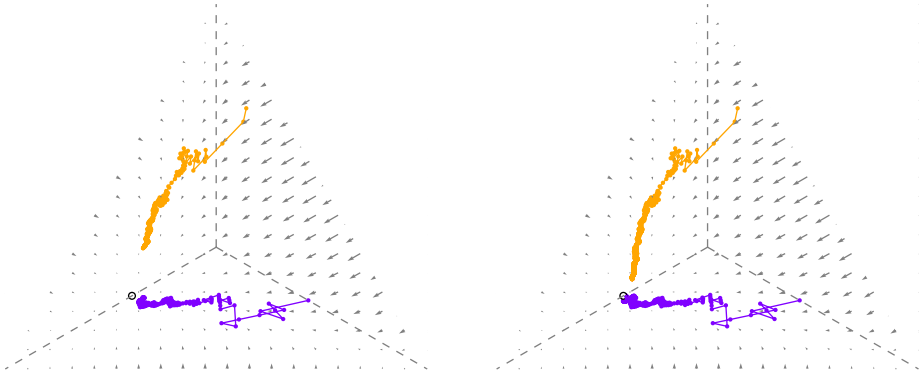


FIGURE 3: Two realisations of the urn of Example 8 starting respectively from (3, 10, 3) and (2, 5, 20), running for 400 steps (*left*) and 4000 steps (*right*). Both realisation converge to the stable zero $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$; one can compare the speed of convergence with the faster Example 7 shown in Figure 2.

Example 8. See Figure 3. Consider the three-colour urn scheme defined by the following replacement rule:

$$\begin{aligned} R(2,0,0) &= (3,0,0), & R(0,1,1) &= (3,0,0),\\ R(0,2,0) &= (0,3,0), & R(1,0,1) &= (1,1,1),\\ R(0,0,2) &= (0,0,3), & R(1,1,0) &= (1,1,1). \end{aligned}$$

In this case,

$$h(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3 - 3x_1 \\ 3x_2^2 + 2x_1x_2 + 2x_1x_3 - 3x_2 \\ 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 3x_3 \end{pmatrix}.$$

One can check that $h(x)$ admits four zeros on the simplex $\Sigma_2^{(3)}$, given by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$. The eigenvalues of ∇h at the first three zeros are $-2-\sqrt{13}$ and $-2+\sqrt{13}$. The eigenvalues of $\nabla h(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ are -1 and $-\frac{9}{5}$. Thus, $\theta = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ is a stable zero of h.

We also check that the only solutions of $\langle h(x),\ x-\theta\rangle = 0$ on $\Sigma^{(3)}$ are the zeros of h. Using the fact that $\langle h(x),\ x-\theta\rangle < 0$ on a neighbourhood of θ, thus, on the whole domain except at the zeros of h, and using the fact that $Z_0 \neq (1,0,0)$ implies $Z_n \neq (1,0,0)$ for all $n \geq 0$ and similarly for $(0,1,0)$ and $(0,0,1)$, we can apply Theorem 1(i) to find that the renormalised composition of the urn Z_n converges almost surely to $\theta = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ (if the urn contained balls of more than one colour at time 0).

We can now apply Theorem 1(ii). Note that, in this case, $\Lambda = 1 < S/2 = \frac{3}{2}$, and the Jordan block of $\nabla h(\theta)$ associated to 1 has size 1 (since ∇h has two distinct eigenvalues given by -1 and $-\frac{9}{5}$), implying that $n^{1/3}(Z_n - (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})^t)$ converges almost surely to a random variable Ψ.

Example 9. Consider the three-colour urn scheme defined by the following replacement rule:

$$\begin{aligned} R(2,0,0) &= (0,0,2), & R(0,1,1) &= (0,1,1),\\ R(0,2,0) &= (0,0,2), & R(1,0,1) &= (1,0,1),\\ R(0,0,2) &= (0,0,2), & R(1,1,0) &= (1,1,0). \end{aligned}$$

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We have

$$h(x) = 2\begin{pmatrix} x_1x_2 + x_1x_3 - x_1 \\ x_1x_2 + x_2x_3 - x_2 \\ x_1^2 + x_2^2 + x_3^2 + x_1x_3 + x_2x_3 - x_3 \end{pmatrix}.$$

The function h admits a unique zero on the simplex $\Sigma_2^{(3)}$, given by $(0, 0, 1)$. The eigenvalues of $\nabla h(0, 0, 1)$ are 0 (with multiplicity 2).

We can solve explicitly $\langle h(x),\ x - (0, 0, 1)\rangle$ on $\Sigma^{(3)}$ and check that $(0, 0, 1)$ is the only solution. Also, $\langle h(1, 0, 0), (1, 0, 0) - (0, 0, 1)\rangle = -4$. By continuity, $\langle h(x),\ x - (0, 0, 1)\rangle < 0$ for all $x \in \Sigma^{(3)} \setminus \{(0, 0, 1)\}$. Thus, Theorem 1(i) applies, and the renormalised composition vector Z_n converges almost surely to $(0, 0, 1)$. Nothing can be said about the rate of convergence since $\nabla h(0, 0, 1)$ admits zero as an eigenvalue (and Theorem 1(ii), thus, does not apply).

Example 10. (*Rock–paper–scissor game [16].*) Consider the three-colour urn scheme defined by the following replacement rule:

$$\begin{aligned} R(2, 0, 0) &= (1, 0, 0), & R(0, 1, 1) &= (0, 1, 0), \\ R(0, 2, 0) &= (0, 1, 0), & R(1, 0, 1) &= (0, 0, 1), \\ R(0, 0, 2) &= (0, 0, 1), & R(1, 1, 0) &= (1, 0, 0). \end{aligned}$$

This urn scheme was studied by Laslier and Laslier [16] as a model for the famous *rock–paper–scissors* game; the first colour represents *scissors*, the second *paper*, and the third *rock*, and the replacement rule encodes the standard rules of the game (scissors wins against paper, paper wins against rock, and rock wins against scissors). We have

$$h(x) = \begin{pmatrix} x_1(x_1 + 2x_2 - 1) \\ x_2(x_2 + 2x_3 - 1) \\ x_3(x_3 + 2x_1 - 1) \end{pmatrix}.$$

The function h admits four zeros, being $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, but none of them are stable.

In [16, Theorem 7] it was proved that the composition vector does not converge when n goes to ∞, but is concentrated on a cycle instead. Therefore, our Theorem 1 does not apply, further than saying that the limit set of $(Z_n)_{n\geq 0}$ is a compact connected set.

The authors also studied what they call a *three-alternative* scheme, in which a sample of three balls is sampled at every time step, and the replacement rule depends on the order of sampling of these balls: if one samples *(scissors, rock, paper)*, we obtain

(*scissors* against (*rock* against *paper*)) = *scissors* against *paper* = *scissors*,

while for the sample *(paper, scissors, rock)*, we have

(*paper* against (*scissors* against *rock*)) = *paper* against *rock* = *paper*.

In the rock–paper–scissor game, we see that an interesting extension of our model is to sample an ordered set of balls (or rather a sequence of balls) at each step and make the replacement rule depend on this sequence of m randomly sampled balls.

5. The two-colour nonbalanced case (proof of Theorem 3)

We have shown in this paper how stochastic algorithms can be used as a powerful tool to study multiple drawing Pólya urns. Our main achievement is to remove the *affinity* assumption that was previously made in the literature (see [11]) and, thus, considerably increase the number of urn schemes to which the strong law of large numbers and the central limit theorem apply.

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The main open question is how to generalise these results to the nonbalanced case: we show in this section how to prove Theorem 3.

There are two main difficulties occurring when treating the unbalanced case. First, there are very few results in the literature on stochastic algorithms that allow the sequence $(\gamma_n)_{n\geq 0}$ to be random (which is the case when the urn is nonbalanced since $\gamma_n = 1/T_n$). Second, we need to control the speed of convergence of the total number of balls divided by n, i.e. T_n/n to its limit.

To prove Theorem 3, we apply the work of Renlund [20]. In this work, a strong law of large numbers and a central limit theorem were proved, but only in the case when the stochastic algorithm $(\theta_n)_{n\geq 0}$ is one-dimensional, which is why this result can only be applied to two-colour Pólya urn schemes.

5.1. Estimates for the total number of balls in the urn

Proposition 1. *If X_n converges almost surely to a limit $\theta_\star$ then the sequence T_n/n converges almost surely to*

$$\omega := \sum_{k=0}^{m} \binom{m}{k} \theta_\star^k (1-\theta_\star)^{m-k} (a_{m-k} + b_{m-k}).$$

Proof. In this proof we use the notation $\mathbb{E}_i$ for the expectation conditionally on the canonical filtration $\mathcal{F}_i$. Write

$$\begin{aligned} \frac{T_{n+1}}{n+1} &= \frac{T_n}{n+1} + \frac{c_{m-\zeta_{n+1}}}{n+1} \\ &= \frac{T_0}{n+1} + \frac{1}{n+1}\sum_{i=0}^{n} (c_{m-\zeta_{i+1}} - \mathbb{E}_i[c_{m-\zeta_{i+1}}]) + \frac{1}{n+1}\sum_{i=0}^{n} \mathbb{E}_i[c_{m-\zeta_{i+1}}], \end{aligned} \tag{10}$$

where we recall that ζ_{i+1} is the number of white balls sampled at time $i+1$. Note that

$$\mathbb{E}_i[c_{m-\zeta_{i+1}}] = \sum_{k=0}^{m} \binom{k}{i} X_i^k (1-X_i)^{m-k} c_{m-k} \to \sum_{k=0}^{m} \binom{k}{i} \theta_\star^k (1-\theta_\star)^{m-k} c_{m-k} = \omega, \qquad i \to \infty,$$

and, thus, by Cesáro's lemma, almost surely,

$$\frac{1}{n+1}\sum_{i=0}^{n} \mathbb{E}_i[c_{m-\zeta_{i+1}}] \to \omega, \qquad n \to \infty.$$

Moreover, $\Delta D_{i+1} = c_{m-\zeta_{i+1}} - \mathbb{E}_i[c_{m-\zeta_{i+1}}]$ is a martingale difference sequence such that

$$\frac{\langle D\rangle_n}{n} = \frac{1}{n}\sum_{i=0}^{n} \mathbb{E}_i[(\Delta D_{i+1})^2] = \frac{1}{n}\sum_{i=0}^{n} [\mathbb{E}_i[c_{m-\zeta_{i+1}}^2] - \mathbb{E}_i[c_{m-\zeta_{i+1}}]^2].$$

Note that

$$\begin{aligned} &\mathbb{E}_i[c_{m-\zeta_{i+1}}^2] - \mathbb{E}_i[c_{m-\zeta_{i+1}}]^2 \\ &\quad = \sum_{k=0}^{m} \binom{m}{k} c_{m-k}^2 X_i^k (1-X_i)^{m-k} - \left(\sum_{k=0}^{m} \binom{m}{k} c_{m-k} X_i^k (1-X_i)^{m-k}\right)^2 \\ &\quad \to \sum_{k=0}^{m} \binom{m}{k} c_{m-k}^2 \theta_\star^k (1-\theta_\star)^{m-k} - \left(\sum_{k=0}^{m} \binom{m}{k} c_{m-k} \theta_\star^k (1-\theta_\star)^{m-k}\right)^2, \qquad i \to \infty, \end{aligned}$$

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thus, by Cesáro's lemma, $\langle D\rangle_n/n$ converges to the same limit, and, by [6, Theorem 1.3.17], we find that $D_n/n \to 0$ almost surely when n tends to ∞. Therefore, by (10), T_n/n converges almost surely to ω. □

5.2. The associated stochastic algorithm

As already mentioned in Section 3, in the two-colour case, we can reduce the study to the first colour since $B_n + W_n = T_n$, and the associated stochastic algorithm is thus one-dimensional.

Lemma 4. *The proportion of white balls in the urn at time n, denoted by $X_n = W_n/T_n$, satisfies the following recursion:*

$$X_{n+1} = X_n + \frac{1}{T_{n+1}}(\tilde{g}(X_n) + \Delta M_{n+1} + \varepsilon_{n+1}),$$

where ε_{n+1} is $\mathcal{F}_n$-measurable and goes almost surely to 0 as n goes to ∞. Then

$$\tilde{g}(x) := (1-x)\sum_{k=0}^{m}\binom{m}{k}a_{m-k}x^k(1-x)^{m-k} - x\sum_{k=0}^{m}\binom{m}{k}b_{m-k}x^k(1-x)^{m-k},$$

and

$$\Delta M_{n+1} = Y_{n+1} - \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n]$$

with

$$Y_{n+1} := a_{m-\zeta_{n+1}} - Z_n(a_{m-\zeta_{n+1}} + b_{m-\zeta_{n+1}}).$$

The proof of this lemma is omitted since it is very similar to the proof of Lemma 2.

5.3. Law of large numbers

In view of Lemma 4, we can apply the following result of Renlund [20].

Theorem 6. *Assume that the sequence $(X_n)_{n\geq 0}$ satisfies*

$$X_{n+1} - X_n = \gamma_{n+1}(f(X_n) + V_{n+1}), \tag{11}$$

where $(\gamma_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ are two $\mathcal{F}_n$-measurable sequences of random variables and f is a continuous function from $[0,1]$ onto $\mathbb{R}$ such that $f(0) \geq 0$, $f(1) \leq 1$, and $f \not\equiv 0$. Assume that, almost surely,

$$\frac{c_1}{n} \leq \gamma_n \leq \frac{c_2}{n}, \qquad |V_n| \leq K_v, \qquad |f(X_n)| \leq K_f, \qquad |\mathbb{E}[\gamma_{n+1}V_{n+1} \mid \mathcal{F}_n]| \leq K_e\gamma_n^2,$$

where the constants c_1, c_2, K_v, K_f, and K_e are positive real numbers. Then $\theta_\star := \lim_{n\to+\infty} X_n$ exists almost surely, $f(\theta_\star) = 0$, and $f'(\theta_\star) \leq 0$.

This result allows us to prove Theorem 3(i).

Proof of Theorem 3(i). In view of Lemma 4, the proportion X_n of white balls in the urn satisfies

$$X_{n+1} = X_n + \frac{1}{T_{n+1}}(\tilde{g}(X_n) + \Delta M_{n+1} + \varepsilon_{n+1}),$$

which is the same as (11) with $f = \tilde{g}$, $V_{n+1} = \Delta M_{n+1} + \varepsilon_{n+1}$, and $\gamma_{n+1} = 1/T_{n+1}$. Note that, by the tenability assumption (see Lemma 1), $\tilde{g}(0) = a_m \geq 0$ and $\tilde{g}(1) = -b_0 \leq 0$ as requested in Lemma 4.

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By assumption, it follows that $\liminf_{n\to\infty} T_n/n > 0$, implying that there exists a constant c such that $T_n \geq cn$ for all $n \geq 0$. If we let $C = \max_{1\leq k\leq m} c_k$ then we have, almost surely, $cn \leq T_n \leq T_0 + Cn$, implying that there exist two constants c_1 and c_2 such that $c_1/n \leq \gamma_n \leq c_2/n$ almost surely for all $n \geq 0$.

Recall that $\Delta M_{n+1} = Y_{n+1} - \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n]$, where $Y_{n+1} = \tilde{g}(X_n)$. Note that the function $\tilde{g}$ is continuous and, thus, bounded on $[0, 1]$, implying that $|\Delta M_{n+1}| \leq 2\max_{[0,1]} \tilde{g}$. Also, $\varepsilon_{n+1} \to 0$ almost surely when n tend to ∞, implying that this sequence is bounded. Thus, there exists K_v such that $|V_n| = |\Delta M_n + \varepsilon_n| \leq K_v$ for all $n \geq 1$.

For all $n \geq 0$, we have $|\tilde{g}(X_n)| \leq \max_{[0,1]} \tilde{g}$, and, finally, since $T_{n+1} \geq c(n + 1)$ almost surely, we have

$$
\begin{aligned}
|\mathbb{E}[\gamma_{n+1} V_{n+1} \mid \mathcal{F}_n]| &= \left| \frac{1}{T_n} \mathbb{E}\left[\left(1 - \frac{c_{\xi_{n+1}}}{T_{n+1}} \right) V_{n+1} \;\middle|\; \mathcal{F}_n \right] \right| \\
&= \frac{\mathbb{E}[|\varepsilon_{n+1}| \mid \mathcal{F}_n]}{T_n} + \frac{1}{T_n} \mathbb{E}\left[\left| \frac{c_{\xi_{n+1}} V_{n+1}}{T_{n+1}} \right| \;\middle|\; \mathcal{F}_n \right],
\end{aligned}
$$

where we have used the fact that $\mathbb{E}[\Delta M_{n+1} \mid \mathcal{F}_n] = 0$. Recall that $\varepsilon_n = 0$ in the with-replacement case, and $|\varepsilon_n| = \mathcal{O}(1/T_n)$ in the without-replacement case. Also recall that, by assumption, $T_0 + cn \leq T_n \leq T_0 + Cn$, implying that there exists a constant K_v such that $|\mathbb{E}[\gamma_{n+1} V_{n+1} \mid \mathcal{F}_n]| \leq K_v \gamma_n^2$ as requested in Lemma 4.

We can thus apply Lemma 4, which proves Theorem 3(i). □

5.4. Central limit theorem

Our aim in this section is to apply Renlund's [20] central limit theorem for stochastic algorithms. This result is expressed as follows.

Theorem 7. (See [20].) *Let $(X_n)_{n\geq 0}$ satisfy (11) and $\theta_\star$ be a stable zero of f. Conditionally on the event $\{X_n \to \theta_\star\}$, let $\hat{\gamma}_n := n\gamma_n \hat{f}(X_{n-1})$, where $\hat{f}(x) = -f(x)/(x - \theta_\star)$, and assume that $\hat{\gamma}_n$ converges almost surely to some deterministic limit $\hat{\gamma}$.*

(i) *If $\hat{\gamma} > \frac{1}{2}$ and if $\mathbb{E}[(n\gamma_n V_n)^2 \mid \mathcal{F}_{n-1}] \to \sigma^2 > 0$, then $\sqrt{n}(X_n - \theta_\star) \to \mathcal{N}(0, \sigma^2/(2\hat{\gamma} - 1))$ in distribution when $n \to \infty$.*

(ii) *If $\hat{\gamma} = \frac{1}{2}$, $(\ln n)(\frac{1}{2} - \hat{\gamma}_n) \to 0$, and $\mathbb{E}[(n\gamma_n V_n)^2 \mid \mathcal{F}_{n-1}] \xrightarrow{\text{A.S.}} \sigma^2 > 0, n \to \infty$, then*

$$
\sqrt{\frac{n}{\ln n}} (X_n - \theta_\star) \to \mathcal{N}(0, \sigma^2) \quad \text{in distribution when } n \to \infty.
$$

Renlund also proved the following result.

Theorem 8. (See [20].) *Let $(X_n)_{n\geq 0}$ be a sequence of random variables that satisfies (11), and $\theta_\star$ be a stable zero of f. Conditionally on $\{X_n \to \theta_\star\}$, assume that there exists $\hat{\gamma} \in (0, \frac{1}{2})$ such that, almost surely when n tends to ∞,*

$$
\hat{\gamma}_n - \hat{\gamma} = \mathcal{O}\left(|X_n - \theta_\star| + \frac{1}{n} \right).
$$

Then $n^{\hat{\gamma}}(X_n - \theta_\star)$ converges almost surely to a finite random variable when n tends to ∞.

Theorem 6(i) applies to our framework and allows us to prove Theorem 3(ii).

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Proof of Theorem 3(ii). The idea of the proof is to apply Renlund's result to the stochastic algorithm of Lemma 4. In this case, $\hat{\gamma}_n = (n/T_n)\tilde{g}(X_{n-1})/(\theta_\star - X_{n-1})$, $f = \tilde{g}$, and $V_n = \Delta M_n + \varepsilon_n$. First note that, since $\tilde{g}$ is a polynomial, it is, in particular, differentiable in $\theta_\star$, and, thus, conditionally on $\{X_n \to \theta_\star\}$, we have

$$\frac{\tilde{g}(X_{n-1})}{\theta_\star - X_{n-1}} \to -\tilde{g}'(\theta_\star).$$

Moreover, Lemma 1 yields that $\hat{\gamma}_n \to -\tilde{g}'(\theta_\star)/\omega =: \lambda$. We now turn our attention to

$$\mathbb{E}[(n\gamma_n V_n)^2 \mid \mathcal{F}_{n-1}] = \mathbb{E}\left[\left(\frac{n(\Delta M_n + \varepsilon_n)}{T_n}\right)^2 \,\middle|\, \mathcal{F}_{n-1}\right].$$

Note that $T_n/n \to \omega$ almost surely, $\varepsilon_n \to 0$ almost surely, and $\Delta M_n = Y_{n+1} - \mathbb{E}[Y_{n+1} \mid \mathcal{F}_{n-1}]$. A simple calculation (using $\tilde{g}(\theta_\star) = 0$) then leads to

$$\mathbb{E}\left[\left(\frac{n(\Delta M_n + \varepsilon_n)}{T_n}\right)^2 \,\middle|\, \mathcal{F}_{n-1}\right] \to \frac{1}{\omega^2} H(\theta_\star) \quad \text{almost surely when } n \to \infty.$$

We can thus apply Theorem 7, which concludes the proof of Theorem 3(ii). □

As we now see, we need information about the speed of convergence of T_n/n to its limit in order to solve the two remaining cases: $\lambda = \frac{1}{2}$ and $\lambda < \frac{1}{2}$. And, in view of Theorem 8, it would be enough to prove that

$$\left|\frac{T_n}{n} - \omega\right| = \mathcal{O}\left(|Z_n - \theta_\star| + \frac{1}{n}\right).$$

Unfortunately, we are not yet able to prove this statement.

5.5. Examples

Theorem 3 already permits us to treat some particular cases of nonbalanced Pólya urn processes. In this section we provide examples to which our result applies, but also some examples that do not fall into our framework, either because $\lambda \leq \frac{1}{2}$ or $\sigma^2 = 0$.

Note that in all examples given below, $\min_{0\leq k\leq m} c_k \geq 1$ (recall that $c_k = a_k + b_k$), implying, in particular, that $\liminf_{n\to\infty} T_n/n > 0$, as required in Theorem 3.

Example 11. Take

$$R = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

In this case, $\tilde{g}(x) = (1-x)^3 - x^3$, and this function admits a unique zero at $\theta_\star = \frac{1}{2}$. This particular case thus falls under Theorem 3 and X_n converges almost surely to $\frac{1}{2}$. We also have $\omega = \frac{5}{2}$ and $\tilde{g}'(\frac{1}{2}) = -\frac{3}{2}$, implying that $\lambda = \frac{3}{5} > \frac{1}{2}$. Finally, note that $H(\frac{1}{2}) = \frac{1}{8}$, $\sigma^2 = \frac{1}{50}$, and $2\lambda - 1 = \frac{1}{5}$ meaning that

$$\sqrt{n}\big(X_n - \tfrac{1}{2}\big) \to \mathcal{N}\big(0, \tfrac{1}{10}\big) \quad \text{in distribution as } n \to \infty.$$

Example 12. Let a and b be two distinct integers, and let $a_{m-k} = a(m-k)$ and $b_{m-k} = bk$ for all $0 \leq k \leq m$. This example was studied in [1], where it was proved that X_n converges almost surely to $\sqrt{a}/(\sqrt{a}+\sqrt{b})$, and that

$$\sqrt{n}\left(X_n - \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) \to \mathcal{N}\left(0, \frac{\sqrt{ab}}{3m(\sqrt{a}+\sqrt{b})^2}\right) \quad \text{almost surely as } n \to \infty. \tag{12}$$

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This result can be reproved as a corollary of our main results. One can check that

$$\tilde{g}(x) = m[a(1-x)^2 - bx^2],$$

with a unique root in [0, 1] at $\theta_\star := \sqrt{a}/(\sqrt{a}+\sqrt{b})$. Thus, Theorem 3(i) yields the almost sure convergence of Z_∞ to $\theta_\star$. Moreover, $\tilde{g}'(\theta_\star) = -2m\sqrt{ab}$ and $\omega = m\sqrt{ab}$, implying that $\lambda = 2$. Finally,

$$H(\theta_\star) = \frac{m(ab)^{3/2}}{(\sqrt{a}+\sqrt{b})^2}, \qquad \sigma^2 = \frac{\sqrt{ab}}{m(\sqrt{a}+\sqrt{b})^2}, \qquad 2\lambda - 1 = 3,$$

which, applying Theorem 3(ii), yields (12).

Example 13. Let a and b be two distinct integers, and let $a_{m-k} = ak$ and $b_{m-k} = b(m-k)$ for all $0 \le k \le m$. This example was studied in [1], where it was proved that, if $a < b$ (respectively, $a > b$), X_n converges almost surely to 0 (respectively, 1) when n tends to ∞, and

$$n^{-a/b} X_n \to \Psi \quad \text{almost surely,}$$

where Ψ is an absolutely continuous random variable (in which the first and second moments are calculated).

We now attempt to apply our results. We see that

$$\tilde{g}(x) = m(a-b)x(1-x).$$

This polynomial admits two zeros, 0 and 1. Note that $\tilde{g}'(0) = -\tilde{g}'(1) = m(a-b)$. Thus, by Theorem 3(i), if $a < b$ then X_n converges almost surely to 0, and if $a > b$, X_n converges almost surely to 1. In both cases,

$$\omega = \max(a,b)m,$$

and, thus, $\lambda = |a-b|/\max(a,b)$. Also note that in both cases, $H(\theta_\star) = 0$, which means that $\sigma^2 = 0$. Even in the cases when $\lambda > \frac{1}{2}$, this example does not fall into our framework since $\sigma^2 = 0$.

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