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Practical use of a Lie algebra structure in non-dissipative fluid and plasma models

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Abstract. A class of Hamiltonian models is considered that both includes many important examples from fluid and plasma theory and admits an elementary and simple abstract formulation. The formalism is illustrated by explicit expressions for both incompressible and compressible ideal magnetohydrodynamics, including a Coriolis force term and, in the incompressible case, allowing for a sharp ideally conducting boundary. The small-amplitude expansion is studied for a general background state, and the abstract setting gives much control of the usually messy algebra. One result is a new general and surprisingly simple symmetric expression for the coupling strength in the resonant three-wave interaction process. It may provide a convenient starting point for wave coupling calculations and seems suitable for the use in conjunction with computer algebra when various concrete models are considered.

1. Introduction

The Hamiltonian structure underlying non-dissipative dynamics of fluids and plasmas is a deep and fundamental property of these models (see e.g. Holm et al. 1985; Salmon 1988; Shepherd 1990; Marsden and Ratiu 1994; Morrison 1998).

In a previous paper (Larsson 1998a), Hamiltonian dynamics on a general Hilbert space P was considered. The generalized Hamilton's equation

$$\dot{u} = J_u \frac{\delta H(u)}{\delta u} \equiv X(u) \tag{1.1}$$

was shown to possess a certain 'adjoint structure' that may be used to derive important properties of the small-amplitude expansion, i.e. of the Taylor series

$$X(u+\delta u) = X(u) + X_u^{(1)}\,\delta u + \frac{1}{2!} X_u^{(2)}\,(\delta u, \delta u) + \dots \quad . \tag{1.2}$$

Several implications for linearized theory may be obtained. For example, the linearized dynamics may be written as a Hermitian equation on a Hilbert space, just like the dynamical equation for the wave function in quantum mechanics, provided that the unperturbed equilibrium satisfy a certain stability criterion (Larsson 2000). In the particular case of ideal magnetohydrodynamics (MHD), this criterion is equivalent to the well-known stability result of Bernstein et al. (1958) (which concerns static equilibrium) and its recent generalization by

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Isichenko (1998) (which concerns general equilibrium). By including nonlinear terms, we may also consider nonlinear waves and wave interactions. In order to derive the interaction equations for the resonant three-wave interaction process, lengthy algebra is often needed – one reason for this is the need to solve also the adjoint linearized problem. The adjoint structure of the linear term in (1.2) saves us from this work, and, furthermore, the corresponding properties for the second-order term in (1.2) lead to a general proof of the Manley–Rowe relations (see Sec. 6.3 below and also Larsson 1996a). Applications of this general Hamiltonian theory for concrete calculations are given for example in Larsson (1998b) for drift waves, Larsson and Wiklund (1999) for Rossby waves, Wiklund (1998) for shallow-water equations and Axelsson (1999) for the ideal MHD modes.

Many important models may be written in the form (1.1) with a Poisson structure J_u that contains only terms that are independent of or linearly dependent of the u variable. In the present paper, we consider the following abstract setting for such a Hamiltonian system. We assume as given

(i) a real Hilbert space P with inner product $\langle\,,\rangle$;
(ii) a Lie algebra bracket $[\,,]$ on P;
(iii) a Hamiltonian observable $H\colon P\to\mathbb{R}$;
(iv) an antisymmetric linear operator $K\colon P\to P$ such that also the identity (2.7) below is valid; this is a condition that relates K to the Lie algebra bracket.

A generalized Hamilton's equation (1.1) is associated with (i)–(iv) by means of (2.8) and (2.9). This abstract formulation is essentially just a useful and self-contained part of the more advanced differential geometric setting of analytical mechanics as presented in, for example, Marsden and Ratiu (1994). In the present paper, we chose to focus on the Lie algebra structure and avoid detailed Lie group theory. One consequence of this is that we just consider Hamiltonian equations and not their derivation from Hamilton's principle. A particular feature of the formulation (i)–(iv) is the introduction of a Hilbert space structure. This is partly motivated from our consideration of the expansion (1.2) and the useful contact with the theory of self-adjoint linear operators on Hilbert space (Larsson 1998a, 2000).

Despite the simplicity of the formulation based on (i)–(iv), it is still possible to make constructive use of semidirect-product Lie algebras (for the full theory involving also the semidirect-product Lie groups, see Marsden et al. 1984). Dealing with a concrete model such as ideal MHD, for example, the first problem is to find the associated Lie algebra in accordance with (ii) above. This is not a difficult task if we know how to add a 'frozen-in' field by the semidirect product construction of a new Lie algebra using a Lie derivative representation of the old Lie algebra (Sec. 2.4).

The mathematical structure (i)–(iv) is given explicitly for the free rigid body in Sec. 2.3 and the heavy top in Sec. 2.5. These examples are included because of their simplicity and since they are considered in detail in Marsden and Ratiu (1994) and are thus useful for comparing formalisms. In Sec. 3, we consider incompressible ideal MHD with a Coriolis force, allowing for a sharp perfectly conducting boundary; in Sec. 4 we turn to compressible ideal MHD with a Coriolis force; and finally in Sec. 5, we consider the Vlasov–Poisson model. The derivations are not completely systematic, but the constructions are motivated

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and a lot of guidance is obtained merely by the fact that we are looking for a Lie algebra with some reasonable relevance to the model and, furthermore, by the semidirect product construction (Sec. 2.4).

In Sec. 6, the small-amplitude expansion (1.2) is considered. In Sec. 6.1, results for a general Hamiltonian equation (1.1) are presented for the operators $X_u^{(1)}$ and $X_u^{(2)}$ in the expansion (1.2). The linear result (Theorem 2) was derived and studied in Larsson (1998a, 2000), and is included for comparison with the corresponding second-order result (Theorem 3), which has not previously been presented in full generality. The proofs of the theorems are given in Appendix B. In Sec. 6.2, we then consider the same operators, but for the restricted class of models associated with the formulation (i)–(iv) above. In this case, more explicit expressions are obtained and presented in the Theorems 4 and 5, with proofs in Appendix C. The second-order result (Theorem 5) is particularly interesting, since the result is surprisingly simple and was obtained after considerable cancellations of terms. In Sec. 6.3, the results are applied to the three-wave interaction process. We note that the Manley–Rowe relations follow and that a nice expression for the coupling strength (6.23) is obtained. This result is then applied to the interaction of three Rossby waves, allowing for sheared flow, in Sec. 6.4, and the result of Larsson and Wiklund (1999) is verified.

2. An important class of Hamiltonian systems

2.1. A general Hamiltonian equation on Hilbert space

Non-dissipative fluid and plasma models are often Hamiltonian, but, since the usual physical variables are non-canonical, it is preferable to write the dynamical equations in the generalized non-canonical Hamiltonian form

$$\dot{u} = J_u H'_u \equiv X(u), \tag{2.1}$$

giving the dynamics in the space of states P, which we assume to have the structure of a real Hilbert space with inner product $\langle\,,\rangle$.† The *Hamiltonian* H is an observable $H\colon P\to\mathbb{R}$, and the *Poisson structure* J_u is a linear operator $J_u\colon P\to P$ that is antisymmetric and satisfies the Jacobi identity for Poisson structures given below. The functional derivative $H'_u \equiv \delta H(u)/\delta u$ is just the gradient on the Hilbert space P, defined by

$$\frac{d}{ds}H(u+sa)\bigg|_{s=0} = \langle a, H'_u\rangle \quad \text{for all } a\in P.$$

The Poisson structure is antisymmetric,

$$\langle a, J_u b\rangle = -\langle J_u a, b\rangle, \tag{2.2}$$

and satisfies the Jacobi identity for Poisson structures,‡

$$\left\langle J_u a, \frac{\delta}{\delta u}\langle J_u b, c\rangle\right\rangle + \left\langle J_u b, \frac{\delta}{\delta u}\langle J_u c, a\rangle\right\rangle + \left\langle J_u c, \frac{\delta}{\delta u}\langle J_u a, b\rangle\right\rangle = 0. \tag{2.3}$$

† It would formally be better to assume that the space of *vectors on state space* rather than the state space itself is a Hilbert space (see Larsson 1998a). However, we gain a simplification of statement with this slight abuse of terminology.

‡ The Jacobi identities for the Poisson structure (2.3) and for the Poisson bracket (2.4) are only equivalent in conjunction with the antisymmetry (cf. Larsson 2000).

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The Poisson bracket of two observables $F, G: P \to \mathbb{R}$ is defined by

$$\{F, G\}(u) = \langle F'_u, J_u G'_u \rangle. \tag{2.4}$$

The space of observables constitute a Lie algebra, with the Poisson bracket as algebra product. The time evolution of an observable is obtained as

$$\dot{F} = \{F, H\}. \tag{2.5}$$

2.2. An important class of Poisson structures

A small, but important for applications, class of Poisson structures is of the form $J_u = K + L_u$, where the operator K does not depend on the state u and the operator L_u is linear in u. It is easy to see that L_u must also be a Poisson structure and that a Lie algebra bracket $[\,,\,]$ is defined on P by

$$[a, b] = \frac{\delta}{\delta u} \langle L_u a, b \rangle. \tag{2.6}$$

In the following simple theorem, we proceed in the opposite direction and start from a Lie algebra structure on P, from which we construct a Poisson structure.

Theorem 1. *Let K be an antisymmetric linear operator and $[\,,\,]$ a Lie algebra bracket on the Hilbert space P. Assume that the following identity is valid*:

$$\langle Ka, [b, c] \rangle + \langle Kb, [c, a] \rangle + \langle Kc, [a, b] \rangle = 0 \quad \text{for all } a, b, c \in P, \tag{2.7}$$

and define a linear operator L_u by

$$\langle L_u a, b \rangle = \langle u, [a, b] \rangle \quad \text{for all } a, b \in P. \tag{2.8}$$

Then the operator J_u defined by

$$J_u = K + L_u \tag{2.9}$$

is a Poisson structure on P. The corresponding Poisson bracket is

$$\{F, G\} = \langle F'_u, KG'_u \rangle - \langle u, [F'_u, G'_u] \rangle. \tag{2.10}$$

Proof. The antisymmetric property of J_u is obvious. The Jacobi identity (2.3) follows easily from

$$\left\langle J_u a, \frac{\delta}{\delta u} \langle J_u b, c \rangle \right\rangle = \left\langle Ka + L_u a, \frac{\delta}{\delta u} \langle u, [b, c] \rangle \right\rangle = \langle Ka + L_u a, [b, c] \rangle$$

$$= \langle Ka, [b, c] \rangle + \langle u, [a, [b, c]] \rangle, \tag{2.11}$$

by use of (2.7) and the Jacobi identity for the Lie algebra bracket. □

2.3. Example: the free rigid body

We take the free rigid body as a simple example (cf. Marsden and Ratiu 1994, Sec. 1.2). All vectors are expressed in terms of the principal-axis boy frame, i.e. an orthonormal frame that is fixed in the body and in which the inertia tensor is diagonal. As state space, we take $P = \mathbb{R}^3$ with $u = \mathbf{\Pi} = (\Pi_1, \Pi_2, \Pi_3) \in P$ as the

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angular momentum of the rigid body. The relation to the body's angular velocity $\mathbf{\Omega}$ is given by $\mathbf{\Pi} = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3)$. As Lie algebra bracket on P, we choose the cross-product and as inner product the usual one in $\mathbb{R}^3$, i.e.

$$[\mathbf{a}, \mathbf{b}] = \mathbf{a} \times \mathbf{b}, \qquad \langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \mathbf{b} \tag{2.12a,b}$$

The Lie algebra thus defined is associated with the group of rotations and is thus the natural Lie algebra to appear in connection with rigid-body rotation. As Hamiltonian, we take the kinetic energy

$$H(\mathbf{\Pi}) = \tfrac{1}{2}\mathbf{\Pi} \cdot \mathbf{\Omega} = \tfrac{1}{2} \sum_{i=1}^{3} \Pi_i^2 / I_i. \tag{2.13}$$

From Theorem 1, we obtain the Poisson structure. First, we write (2.8) as

$$L_{\mathbf{\Pi}}\, \mathbf{a} \cdot \mathbf{b} = \mathbf{\Pi} \cdot \mathbf{a} \times \mathbf{b}, \tag{2.14}$$

and from (2.9), $K = 0$ and (2.14), we get

$$J_{\mathbf{\Pi}}\, \mathbf{a} = \mathbf{\Pi} \times \mathbf{a}. \tag{2.15}$$

Hamilton's generalized equation (2.1) for this example is

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}, \tag{2.16}$$

which we recognize as the dynamical equation for the free rigid body. The Poisson bracket is obtained from (2.10) as

$$\{F, G\}\,(\mathbf{\Pi}) = -\mathbf{\Pi} \cdot \frac{\partial F}{\partial \mathbf{\Pi}} \times \frac{\partial G}{\partial \mathbf{\Pi}}. \tag{2.17}$$

2.4. How to extend a system with 'frozen-in' fields

We want to extend the state space from P to $P \times Q$, where the dynamics on Q for the extended system has some kind of 'frozen-in' relation to the P-dynamics. The dynamics of the Q-variables is thus to be completely determined from the P-dynamics in the extended system. An example is ideal MHD, where Q above may include the specific entropy and the magnetic field.

A Hamiltonian system (2.1) is assumed to be given such that the Poisson structure is of the form (2.9). The space of 'new' fields Q is assumed to be a Hilbert space with inner product $\langle\, ,\rangle_Q$. The extended space of states $P \times Q$ is a Hilbert space with inner product

$$\left\langle \begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix} \right\rangle_{P \times Q} = \langle a, b \rangle_P + \langle \alpha, \beta \rangle_Q, \tag{2.18}$$

where we have denoted the inner product on the Hilbert space P by $\langle\, ,\rangle_P$.

The 'frozen-in' relation is expressed mathematically as a representation of the Lie algebra P on the space Q, i.e., for each $a \in P$, there is a linear operator $\Phi_a \colon Q \to Q$ such that

$$\Phi_a \Phi_b - \Phi_b \Phi_a = \Phi_{[a,b]}. \tag{2.19}$$

It is easy to prove that

$$\left[\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix} \right]_\Phi = \begin{pmatrix} [a, b] \\ \Phi_a \beta - \Phi_b \alpha \end{pmatrix}, \qquad a, b \in P \quad \text{and} \quad \alpha, \beta \in Q, \tag{2.20}$$

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defines a Lie algebra bracket on $P\times Q$. We observe that $a\to-\Phi_a^*$ is also a representation of P on Q. The asterisk denotes the (Hilbert space) adjoint of an operator. We thus obtain a second Lie algebra structure on $P\times Q$ defined by the bracket

$$\left[\begin{pmatrix}a\\ \alpha\end{pmatrix},\begin{pmatrix}b\\ \beta\end{pmatrix}\right]=\begin{pmatrix}[a,b]\\ -\Phi_a^*\beta+\Phi_b^*\alpha\end{pmatrix},\qquad a,b\in P\quad\text{and}\quad\alpha,\beta\in Q. \tag{2.21}$$

Mathematically, the constructions (2.20) and (2.21) are semidirect products for Lie algebras (cf. Marsden et al. 1984).

The antisymmetric operator $K\colon P\to P$ is trivially extended to an antisymmetric operator $K\colon P\times Q\to P\times Q$ by (we do not bother to give a new notation for the extended operator)

$$K\begin{pmatrix}a\\ \alpha\end{pmatrix}=\begin{pmatrix}Ka\\ 0\end{pmatrix}. \tag{2.22}$$

The choice of Hamiltonian for the extended system is in practice often the energy of the system, so here we are guided by our physical understanding of the system. The choice of representation of the Lie algebra P on Q will, in our fluid and plasma examples, also be quite natural and associated with the Lie derivative expressing the 'frozen-in' quality (see Appendix A). Then, as will be checked explicitly in the examples in the following sections, it turns out that the appropriate Lie bracket on $P\times Q$ is the one given by (2.21). Thus the Hamiltonian dynamics we thereby obtain is also the dynamics of the standard models we consider.

The Poisson bracket on $P\times Q$ corresponding to (2.21) is

$$\begin{aligned}\{F,G\}(u,\lambda)=&\langle F'_u,KG'_u\rangle_P-\langle u,[F'_u,G'_u]\rangle_P\\ &-\langle\lambda,-\Phi^*_{F'_u}G'_\lambda+\Phi^*_{G'_u}F'_\lambda\rangle_Q,\end{aligned} \tag{2.23}$$

which may also be written as

$$\begin{aligned}\{F,G\}(u,\lambda)=&\langle F'_u,KG'_u\rangle_P-\langle u,[F'_u,G'_u]\rangle_P\\ &+\langle G'_\lambda,\Phi_{F'_u}\lambda\rangle_Q-\langle F'_\lambda,\Phi_{G'_u}\lambda\rangle_Q.\end{aligned} \tag{2.24}$$

2.5. Example: the heavy top

An example of the construction above is an 'extension' of the free-body dynamics in Sec. 2.3 to heavy-top dynamics, where the top is spinning with one fixed point in an external gravitational field (cf. Marsden and Ratiu 1994, Sec. 1.9). We then take $Q=\mathbb{R}^3$ and let $\mathbf{\Gamma}=(\Gamma_1,\Gamma_2,\Gamma_3)\in Q$, expressed *in terms of the principal-axis body frame*, represent a vector of fixed length along the direction antiparallel to the gravitational force. The physical information obtained from $\mathbf{\Gamma}=(\Gamma_1,\Gamma_2,\Gamma_3)$ is the orientation of the top (apart from rotations with axis along the gravitational field). This information is needed to determine the potential energy of the top. The Hamiltonian is the total energy of the system

$$H(\mathbf{\Pi},\mathbf{\Gamma})=\tfrac{1}{2}\mathbf{\Pi}\cdot\mathbf{\Omega}+Mgl\,|\mathbf{\Gamma}|^{-1}\mathbf{\Gamma}\cdot\boldsymbol{\chi}, \tag{2.25}$$

where $l\boldsymbol{\chi}$ is the vector with length l from the fixed point to the centre of mass. Of course, this is a constant vector in any body frame. The total mass

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of the top is M, and g is the strength of the gravitational acceleration. The inner product on $P \times Q = \mathbb{R}^3 \times \mathbb{R}^3$ is

$$\left\langle \begin{pmatrix} \mathbf{a} \\ \boldsymbol{\alpha} \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ \boldsymbol{\beta} \end{pmatrix} \right\rangle = \mathbf{a} \cdot \mathbf{b} + \boldsymbol{\alpha} \cdot \boldsymbol{\beta}. \tag{2.26}$$

The body-frame 'motion' of $\boldsymbol{\Gamma}$ is, of course, only due to the true (inertial-frame) motion of the top. The associated natural representation $\Phi_{\mathbf{a}}: Q \to Q$ of the Lie algebra P in Sec. 2.3 is

$$\Phi_{\mathbf{a}} \boldsymbol{\alpha} = \mathbf{a} \times \boldsymbol{\alpha} \quad \text{for } \mathbf{a} \in P \quad \text{and } \boldsymbol{\alpha} \in Q. \tag{2.27}$$

In this case, $\Phi_{\mathbf{a}} = -\Phi_{\mathbf{a}}^*$, and we thus get from both (2.20) and (2.21) the bracket

$$\left[\begin{pmatrix} \mathbf{a} \\ \boldsymbol{\alpha} \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ \boldsymbol{\beta} \end{pmatrix} \right] = \begin{pmatrix} \mathbf{a} \times \mathbf{b} \\ \mathbf{a} \times \boldsymbol{\beta} - \mathbf{b} \times \boldsymbol{\alpha} \end{pmatrix}. \tag{2.28}$$

With $K = 0$, we now obtain, by use of (2.8) and (2.9), Hamilton's generalized equation (2.1) as

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\nabla}_{\boldsymbol{\Pi}} H + \boldsymbol{\Gamma} \times \boldsymbol{\nabla}_{\boldsymbol{\Gamma}} H, \tag{2.29a}$$

$$\dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\nabla}_{\boldsymbol{\Pi}} H. \tag{2.29b}$$

Substitution of the Hamiltonian (2.25) now results in the heavy-top equations

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega} + Mgl|\boldsymbol{\Gamma}|^{-1} \boldsymbol{\Gamma} \times \boldsymbol{\chi}, \tag{2.30a}$$

$$\dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega}. \tag{2.30b}$$

If we choose $|\boldsymbol{\Gamma}| = 1$ in (2.30) (note that $|\boldsymbol{\Gamma}|$ is conserved) then the usual form of the heavy-top equations is obtained (see e.g. Marsden and Ratiu 1994, p. 23)

3. Incompressible flow

We consider in this section the abstract formulation in Theorem 1 for incompressible flows on a domain $D \subset \mathbb{R}^3$ with boundary ∂D. The space of all (sufficiently nice) vector fields $\mathbf{w}: D \to \mathbb{R}^3$ is denoted by $\mathscr{H}(D)$. The set of divergence-free vector fields that are tangent to ∂D is denoted by $\mathscr{H}_{\mathrm{div}}(D)$, i.e.

$$\mathscr{H}_{\mathrm{div}}(D) = \{\mathbf{w} \in \mathscr{H}(D) | \boldsymbol{\nabla} \cdot \mathbf{w} = 0, \hat{\mathbf{n}} \cdot \mathbf{w} = 0 \text{ on } \partial D\}. \tag{3.1}$$

We use the Lie bracket and the standard inner product of vector fields

$$[\mathbf{a}, \mathbf{b}] = \mathbf{a} \cdot \boldsymbol{\nabla} \mathbf{b} - \mathbf{b} \cdot \boldsymbol{\nabla} \mathbf{a}, \qquad \langle \mathbf{a}, \mathbf{b} \rangle = \int_D \mathbf{a} \cdot \mathbf{b} \tag{3.2a,b}$$

The vector space $\mathscr{H}_{\mathrm{div}}(D)$ is the Lie algebra associated with the Lie group of volume-preserving diffeomorphisms, and thus does naturally appear in connection with incompressible flows. We give a direct verification that $\mathscr{H}_{\mathrm{div}}(D)$ is closed under the Lie bracket, i.e.

$$\mathbf{a}, \mathbf{b} \in \mathscr{H}_{\mathrm{div}}(D) \Rightarrow [\mathbf{a}, \mathbf{b}] \in \mathscr{H}_{\mathrm{div}}(D). \tag{3.3}$$

The following characterization of $\mathscr{H}_{\mathrm{div}}(D)$ is then useful.

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Lemma 1. *Let a vector field* $\mathbf{w}: D \to \mathbb{R}^3$ *be given. Then* $\mathbf{w} \in \mathscr{H}_{\mathrm{div}}(D)$ *if and only if*

$$\int_D \mathbf{w} \cdot \boldsymbol{\nabla} \phi = 0 \quad \text{for all fields } \phi: D \to \mathbb{R}. \tag{3.4}$$

Proof. Consider the equality

$$\int_D \mathbf{w} \cdot \boldsymbol{\nabla} \phi = \int_D \boldsymbol{\nabla} \cdot (\phi \mathbf{w}) - \int_D \phi \boldsymbol{\nabla} \cdot \mathbf{w} = \int_{\partial D} \phi \hat{\mathbf{n}} \cdot \mathbf{w} - \int_D \phi \boldsymbol{\nabla} \cdot \mathbf{w}. \tag{3.5}$$

This is zero for *arbitrary* ϕ if and only if $\hat{\mathbf{n}} \cdot \mathbf{w} = 0$ on ∂D and $\boldsymbol{\nabla} \cdot \mathbf{w} = 0$ on D, and Lemma 1 thus follows. □

Now the implication (3.3) easily follows by use of Lemma 1 with $\mathbf{w} = [\mathbf{a}, \mathbf{b}]$. Just calculate

$$\begin{aligned} \int_D [\mathbf{a}, \mathbf{b}] \cdot \boldsymbol{\nabla} \phi &= \int_D \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \phi \times (\mathbf{a} \times \mathbf{b})) = \int_{\partial D} \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \phi \times (\mathbf{a} \times \mathbf{b}) \\ &= -\int_{\partial D} \boldsymbol{\nabla} \phi \cdot \hat{\mathbf{n}} \times (\mathbf{a} \times \mathbf{b}) = 0, \end{aligned} \tag{3.6}$$

and, by Lemma 1, we get $[\mathbf{a}, \mathbf{b}] \in \mathscr{H}_{\mathrm{div}}(D)$. The orthogonal projection operator $P_D: \mathscr{H}(D) \to \mathscr{H}_{\mathrm{div}}(D)$ may be obtained from the unique decomposition

$$\mathbf{w} = P_D \mathbf{w} + \boldsymbol{\nabla} \phi_{\mathbf{w}}, \qquad \text{where} \quad P_D \mathbf{w} \in \mathscr{H}_{\mathrm{div}}(D). \tag{3.7}$$

The scalar function $\phi_{\mathbf{w}}: D \to \mathbb{R}$ is then determined by

$$\boldsymbol{\nabla}^2 \phi_{\mathbf{w}} = \boldsymbol{\nabla} \cdot \mathbf{w} \quad \text{on } D, \tag{3.8a}$$

$$\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \phi_{\mathbf{w}} = \hat{\mathbf{n}} \cdot \mathbf{w} \quad \text{on } \partial D. \tag{3.8b}$$

3.1. Incompressible flow

The usual equations of incompressible flow on the domain D are

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\boldsymbol{\nabla} p, \qquad \boldsymbol{\nabla} \cdot \mathbf{v} = 0. \tag{3.9a,b}$$

These define a flow on $P = \mathscr{H}_{\mathrm{div}}(D)$, and may also be written in terms of the projection operator P_D as

$$\partial_t \mathbf{v} = -P_D(\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}). \tag{3.10}$$

Let us now apply Theorem 1 with the Lie algebra and Hilbert space $P = \mathscr{H}_{\mathrm{div}}(D)$, with operator $K = 0$ and with kinetic energy as Hamiltonian,

$$H(\mathbf{v}) = \tfrac{1}{2} \int_D \mathbf{v} \cdot \mathbf{v}. \tag{3.11}$$

Then $J_{\mathbf{v}} = L_{\mathbf{v}}$ and

$$\int_D L_{\mathbf{v}} \mathbf{a} \cdot \mathbf{b} \equiv \int_D J_{\mathbf{v}} \mathbf{a} \cdot \mathbf{b} = \int_D \mathbf{v} \cdot [\mathbf{a}, \mathbf{b}] \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathscr{H}_{\mathrm{div}}(D). \tag{3.12}$$

By use of the equality

$$\mathbf{v} \cdot [\mathbf{a}, \mathbf{b}] = \boldsymbol{\nabla} \cdot (\mathbf{b} \cdot \mathbf{v} \mathbf{a} - \mathbf{a} \cdot \mathbf{v} \mathbf{b}) - (\boldsymbol{\nabla} \times \mathbf{v}) \times \mathbf{a} \cdot \mathbf{b}, \tag{3.13}$$

we find

$$\int_D J_{\mathbf{v}} \mathbf{a} \cdot \mathbf{b} = -\int_D (\boldsymbol{\nabla} \times \mathbf{v}) \times \mathbf{a} \cdot \mathbf{b} \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathscr{H}_{\mathrm{div}}(D), \tag{3.14}$$

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which uniquely determine $J_{\mathbf{v}}\colon \mathscr{H}_{\mathrm{div}}(D) \to \mathscr{H}_{\mathrm{div}}(D)$ as

$$J_{\mathbf{v}}\mathbf{a} = -P_D((\boldsymbol{\nabla}\times\mathbf{v})\times\mathbf{a}). \tag{3.15}$$

The Hamiltonian equation (2.1) may now be written as (3.10), or equivalently as (3.9), with boundary conditions. The Poisson bracket (2.10) becomes

$$\{F, G\} = -\int_D \mathbf{v}\cdot[F'_{\mathbf{v}}, G'_{\mathbf{v}}]. \tag{3.16}$$

3.2. Incompressible flow with a Coriolis term

Let us add a force term $-2\boldsymbol{\Omega}\times\mathbf{v}$ to the dynamical equation (3.9a) for incompressible flow. This term may be used to model the Coriolis force. We assume that the field $\boldsymbol{\Omega}\colon D \to \mathbb{R}^3$ is given such that

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Omega} = 0. \tag{3.17}$$

The modified dynamical equation may be written as

$$\partial_t \mathbf{v} = -P_D(2\boldsymbol{\Omega}\times\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}). \tag{3.18}$$

Equation (3.18) may be obtained from Theorem 1 and (2.1) by choosing the operator K as (leaving the other choices in Sec. 3.1 unchanged)

$$K\mathbf{a} = -P_D(2\boldsymbol{\Omega}\times\mathbf{a}). \tag{3.19}$$

Thus we now have

$$J_{\mathbf{v}}\mathbf{a} = K\mathbf{a} + L_{\mathbf{v}}\mathbf{a} = -P_D(2\boldsymbol{\Omega}\times\mathbf{a}) - P_D((\boldsymbol{\nabla}\times\mathbf{v})\times\mathbf{a}) \tag{3.20}$$

In order to use Theorem 1, we must check that K is antisymmetric and that the identity (2.7) is satisfied. The antisymmetry follows by

$$\langle K\mathbf{a}, \mathbf{b}\rangle = \langle -P_D(2\boldsymbol{\Omega}\times\mathbf{a}), \mathbf{b}\rangle = -2\int_D \boldsymbol{\Omega}\times\mathbf{a}\cdot\mathbf{b} = -2\int_D \boldsymbol{\Omega}\cdot\mathbf{a}\times\mathbf{b}. \tag{3.21}$$

The identity (2.7) is proved by the calculation

$$\begin{aligned}
&\langle K\mathbf{a}, [\mathbf{b}, \mathbf{c}]\rangle + \langle K\mathbf{b}, [\mathbf{c}, \mathbf{a}]\rangle + \langle K\mathbf{c}, [\mathbf{a}, \mathbf{b}]\rangle \\
&\qquad \overset{(1)}{=} -2\int_D \boldsymbol{\Omega}\cdot(\mathbf{a}\times[\mathbf{b}, \mathbf{c}] + \mathbf{b}\times[\mathbf{c}, \mathbf{a}] + \mathbf{c}\times[\mathbf{a}, \mathbf{b}]) \\
&\qquad \overset{(2)}{=} 2\int_D \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}) \overset{(3)}{=} 2\int_D \boldsymbol{\nabla}\cdot[\boldsymbol{\Omega}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})] \\
&\qquad = 2\int_{\partial D} \hat{\mathbf{n}}\cdot[\boldsymbol{\Omega}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})] = 0.
\end{aligned} \tag{3.22}$$

The equality (1) in (3.22) is obtained by (3.19) and (3.3). For the equality (2), we use (3.23) below, and in the equality (3), we use $\boldsymbol{\nabla}\cdot\boldsymbol{\Omega} = 0$. For the last equality in (3.22), we observe that $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = 0$ on ∂D. We have used above

$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}) = -\mathbf{a}\times[\mathbf{b}, \mathbf{c}] - \mathbf{b}\times[\mathbf{c}, \mathbf{a}] - \mathbf{c}\times[\mathbf{a}, \mathbf{b}]. \tag{3.23}$$

A simple way of deriving (3.23) is to start from the dyadic identity

$$(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})\mathbf{I} = \mathbf{a}(\mathbf{b}\times\mathbf{c}) + \mathbf{b}(\mathbf{c}\times\mathbf{a}) + \mathbf{c}(\mathbf{a}\times\mathbf{b}) \tag{3.24}$$

(this identity can be seen on right scalar multiplication of both sides by $\mathbf{a}$, $\mathbf{b}$ and

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$\mathbf{c}$). If we now operate on (3.24) with $\boldsymbol{\nabla}\cdot$ and use $\boldsymbol{\nabla}\cdot\mathbf{a}=\boldsymbol{\nabla}\cdot\mathbf{b}=\boldsymbol{\nabla}\cdot\mathbf{c}=0$, we find (3.23) after a little algebra.

The dynamical equation (3.18) follows from Theorem 1 and (2.1). The Poisson bracket (2.10) may be written as

$$\{F,G\}=\int_D (2\boldsymbol{\Omega}\cdot F'_{\mathbf{v}}\times G'_{\mathbf{v}}-\mathbf{v}\cdot[F'_{\mathbf{v}},G'_{\mathbf{v}}]). \tag{3.25}$$

3.3. Incompressible ideal MHD with Coriolis term

We now add a 'frozen-in' magnetic field to the dynamics in Sec. 3.2. The wall ∂D of the domain D is assumed to be an ideal conductor, so the magnetic field satisfy the same boundary condition as the flow, i.e. we assume the magnetic field to be tangent to ∂D and choose $Q=\mathscr{H}_{\mathrm{div}}(D)$ (see Sec. 2.4). The space of states is now $P\times Q=\mathscr{H}_{\mathrm{div}}(D)\times\mathscr{H}_{\mathrm{div}}(D)$. The inner product is defined by

$$\left\langle\begin{pmatrix}\mathbf{a}\\ \boldsymbol{\alpha}\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ \boldsymbol{\beta}\end{pmatrix}\right\rangle_{P\times Q}=\int_D(\mathbf{a}\cdot\mathbf{b}+\boldsymbol{\alpha}\cdot\boldsymbol{\beta}). \tag{3.26}$$

As Hamiltonian function, we take

$$H\begin{pmatrix}\mathbf{v}\\ \mathbf{B}\end{pmatrix}=\int_D(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+\tfrac{1}{2}\mathbf{B}\cdot\mathbf{B}), \tag{3.27}$$

and as antisymmetric operator (see (2.22))

$$K\begin{pmatrix}\mathbf{a}\\ \boldsymbol{\alpha}\end{pmatrix}=\begin{pmatrix}K\mathbf{a}\\ \mathbf{0}\end{pmatrix}. \tag{3.28}$$

The magnetic field is most naturally represented by a 2-form, and the 'frozen-in' condition may then be simply expressed in terms of the Lie derivative. We choose to represent the magnetic field with the corresponding vector field (as is usually done in physics), and denote the corresponding Lie-derivative operator by $\mathscr{L}^{(2)}_{\mathbf{a}}$, where (see Appendix A)

$$\mathscr{L}^{(2)}_{\mathbf{a}}\colon\mathscr{H}_{\mathrm{div}}(D)\to\mathscr{H}_{\mathrm{div}}(D),\qquad \mathscr{L}^{(2)}_{\mathbf{a}}\mathbf{b}=[\mathbf{a},\mathbf{b}]. \tag{3.29a,b}$$

Note that we shall use the same notation for an operator also on the space $\mathscr{H}(D)$ including vector fields with non-vanishing divergence; in that case, an expression different from (3.29b) must be used (see (A 3) in Appendix A). A representation of the Lie algebra $P=\mathscr{H}_{\mathrm{div}}(D)$ of flows on the vector space $Q=\mathscr{H}_{\mathrm{div}}(D)$ of magnetic fields is defined by (see (A 4))

$$\Phi_{\mathbf{a}}=\mathscr{L}^{(2)}_{\mathbf{a}}. \tag{3.30}$$

The condition (2.19) is in this case just the Jacobi identity for the Lie bracket (3.2a). The Lie algebra bracket (2.21) is

$$\left[\begin{pmatrix}\mathbf{a}\\ \boldsymbol{\alpha}\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ \boldsymbol{\beta}\end{pmatrix}\right]=\begin{pmatrix}[\mathbf{a},\mathbf{b}]\\ -\mathscr{L}^{(2)*}_{\mathbf{a}}\boldsymbol{\beta}+\mathscr{L}^{(2)*}_{\mathbf{b}}\boldsymbol{\alpha}\end{pmatrix}. \tag{3.31}$$

We may write the Poisson bracket (2.24) as

$$\begin{aligned}\{F,G\}=&\int_D(2\boldsymbol{\Omega}\cdot F'_{\mathbf{v}}\times G'_{\mathbf{v}}-\mathbf{v}\cdot[F'_{\mathbf{v}},G'_{\mathbf{v}}])\\ &+\int_D(G'_{\mathbf{B}}\cdot[F'_{\mathbf{v}},\mathbf{B}]-F'_{\mathbf{B}}\cdot[G'_{\mathbf{v}},\mathbf{B}]).\end{aligned} \tag{3.32}$$

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We denote

$$u = \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix}, \qquad a = \begin{pmatrix} \mathbf{a} \\ \alpha \end{pmatrix} \tag{3.33}$$

and obtain from (3.31) and (2.8) and use of the identity (3.13) that

$$L_u a = \begin{pmatrix} P_D(-(\nabla \times \mathbf{v}) \times \mathbf{a} + (\nabla \times \alpha) \times \mathbf{b}) \\ -[\mathbf{a}, \mathbf{B}] \end{pmatrix}. \tag{3.34}$$

The generalized Hamilton's equation (2.1) may now be written as

$$\partial_t \mathbf{v} = -P_D(2\boldsymbol{\Omega} \times \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \mathbf{B} \cdot \nabla \mathbf{B}), \tag{3.35a}$$

$$\partial_t \mathbf{B} = -[\mathbf{v}, \mathbf{B}], \tag{3.35b}$$

which are the equations for incompressible ideal MHD including a Coriolis term.

4. Compressible flow

We consider in this section compressible flow on a domain $D \subset \mathbb{R}^3$. The fields are assumed to vanish at the boundary ∂D, so that partial integration may be performed without the appearance of surface terms. We allow for an external potential force field, which may be used to confine the flow within some finite volume without introducing hard walls. As field variable for the flow, we shall instead of the velocity use the momentum density $\mathbf{M} = \rho \mathbf{v}$, where $\mathbf{M} \in \mathscr{H}_0(D) \subset \mathscr{H}(D)$. The subscript zero indicates the vanishing of the vector fields at the boundary ∂D. The Lie bracket and the inner product are defined by

$$[\mathbf{a}, \mathbf{b}] = \mathbf{a} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{a}, \tag{4.1a}$$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_D \mathbf{a} \cdot \mathbf{b} \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathscr{H}_0(D). \tag{4.1b}$$

The bracket gives $\mathscr{H}_0(D)$ a Lie algebra structure.

The space of (sufficiently nice) real-valued functions on the domain D is denoted by $\mathscr{F}(D)$ and the subset of functions that vanish at the boundary ∂D is denoted by $\mathscr{F}_0(D)$. We shall use the semidirect product method (see Sec. 2.4) to include fields with some 'frozen-in' relation to the flow. In order to include the mass density ρ and specific entropy σ, we use representations of the Lie algebra $\mathscr{H}_0(D)$ on the vector space $\mathscr{F}_0(D)$ of real-valued functions on D. As inner product, we take

$$\langle \alpha, \beta \rangle_Q = \int_D \alpha \beta, \qquad \text{where} \quad Q = \mathscr{F}_0(D), \quad \alpha, \beta \in \mathscr{F}_0(D). \tag{4.2}$$

A magnetic field is included by use of a representation of the Lie algebra $\mathscr{H}_0(D)$ on the vector space $\mathscr{H}_0(D)$.

4.1. Barotropic flow

We consider the evolution equations for barotropic flow

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla h(\rho) - \nabla \phi_{\text{ext}}, \qquad \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{4.3}$$

where $h(\rho)$ is the specific enthalpy, which is related to the pressure $p(\rho)$ by $p'(\rho) = \rho h'(\rho)$. An external potential force is included, and may be used to

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confine the flow to some finite domain without introducing hard walls. We shall use the field variables $(\mathbf{M},\,\rho)=(\rho\mathbf{v},\rho)$, and rewrite (4.3) as

$$\partial_t\mathbf{M}+\boldsymbol{\nabla}\cdot(\rho^{-1}\mathbf{M}\mathbf{M})=-\rho\boldsymbol{\nabla}h(\rho)-\rho\boldsymbol{\nabla}\phi_{\text{ext}},\qquad \partial_t\rho+\boldsymbol{\nabla}\cdot\mathbf{M}=0. \tag{4.4}$$

The Lie algebra $\mathscr{H}_0(D)$ may be extended to a Lie algebra on the state space $P=\mathscr{H}_0(D)\times\mathscr{F}_0(D)$ by use of the representation $\boldsymbol{\Phi}_{\mathbf{a}}=\mathscr{L}^{(3)}_{\mathbf{a}}$ of the Lie algebra $\mathscr{H}_0(D)$ on the vector space $\mathscr{F}_0(D)$ (see Sec. 2.4 and Appendix A). This representation is naturally associated with the 'frozen-in' quality of the mass density ρ, which in this case amounts to mass conservation.

Mathematically, the mass density is best represented by a 3-form, and the corresponding Lie derivative is $\mathscr{L}^{(3)}_{\mathbf{a}}$ if we use the inner product to identify this 3-form with a scalar field (see Appendix A in Larsson 1996b). The construction (2.21) results in the Lie Algebra bracket

$$\left[\begin{pmatrix}\mathbf{a}\\ \alpha\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ \beta\end{pmatrix}\right]=\begin{pmatrix}[\mathbf{a},\mathbf{b}]\\ \mathscr{L}^{(0)}_{\mathbf{a}}\beta-\mathscr{L}^{(0)}_{\mathbf{b}}\alpha\end{pmatrix}=\begin{pmatrix}[\mathbf{a},\mathbf{b}]\\ \mathbf{a}\cdot\boldsymbol{\nabla}\beta-\mathbf{b}\cdot\boldsymbol{\nabla}\alpha\end{pmatrix}, \tag{4.5}$$

where we have used $\mathscr{L}^{(3)*}_{\mathbf{a}}=-\mathscr{L}^{(0)}_{\mathbf{a}}$ (see (4.17) below).

We take the operator $K=0$ and the Hamiltonian as the energy

$$H(\mathbf{M},\rho)=\int_D\left(\frac{1}{2\rho}\mathbf{M}\cdot\mathbf{M}+\rho U(\rho)+\rho\phi_{\text{ext}}\right), \tag{4.6}$$

where the specific internal energy $U(\rho)$ is related to the specific enthalpy by $h(\rho)=\partial_\rho[\rho U(\rho)]$. It follows from (2.8) and (2.9) that

$$J_u a=\begin{pmatrix}-\boldsymbol{\nabla}\cdot(\mathbf{a}\mathbf{M})-\boldsymbol{\nabla}\mathbf{a}\cdot\mathbf{M}-\rho\boldsymbol{\nabla}\alpha\\ -\boldsymbol{\nabla}\cdot(\rho\mathbf{a})\end{pmatrix},\quad \text{where}\quad u=\begin{pmatrix}\mathbf{M}\\ \rho\end{pmatrix},\quad a=\begin{pmatrix}\mathbf{a}\\ \alpha\end{pmatrix}. \tag{4.7}$$

The dynamics (4.4) is now obtained as Hamilton's generalized equation (2.1).

4.2. Barotropic flow with a Coriolis force

We now consider the equations

$$\partial_t\mathbf{M}+\boldsymbol{\nabla}\cdot(\rho^{-1}\mathbf{M}\mathbf{M})+2\boldsymbol{\Omega}\times\mathbf{M}=-\rho\boldsymbol{\nabla}h(\rho)-\rho\boldsymbol{\nabla}\phi_{\text{ext}},\qquad \partial_t\rho+\boldsymbol{\nabla}\cdot\mathbf{M}=0, \tag{4.8}$$

where the Coriolis term involve a given divergence-free field $\boldsymbol{\Omega}$. With unchanged Hamiltonian, we obtain (4.8) as the Hamiltonian equation (2.1) if the Poisson structure is modified to

$$J_u a=\begin{pmatrix}-\boldsymbol{\nabla}\cdot(\mathbf{a}\mathbf{M})-\boldsymbol{\nabla}\mathbf{a}\cdot\mathbf{M}-2\rho\boldsymbol{\Omega}\times\mathbf{a}-\rho\boldsymbol{\nabla}\alpha\\ -\boldsymbol{\nabla}\cdot(\rho\mathbf{a})\end{pmatrix}. \tag{4.9}$$

This Poisson structure is linear in the u-variable, so the corresponding operator K is zero. Thus, while the Coriolis term for incompressible flow resulted in a non-vanishing operator K, leaving the Lie algebra unchanged (see Sec. 3.2), we now find (from (4.9), (2.8) and (2.9)) that the Lie algebra bracket must be modified to

$$\left[\begin{pmatrix}\mathbf{a}\\ \alpha\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ \beta\end{pmatrix}\right]=\begin{pmatrix}[\mathbf{a},\mathbf{b}]\\ \mathbf{a}\cdot\boldsymbol{\nabla}\beta-\mathbf{b}\cdot\boldsymbol{\nabla}\alpha-2\boldsymbol{\Omega}\cdot\mathbf{a}\times\mathbf{b}\end{pmatrix}. \tag{4.10}$$

It remains to prove that (4.10) satisfies all properties of a Lie algebra bracket. We know that this is true without the Coriolis term. Obviously (4.10) defines an

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antisymmetric bracket also in the presence of the Coriolis term. Thus only the Jacobi identity remains to be checked, and we then only need to consider terms involving $\boldsymbol{\Omega}$. From (4.10),

$$[a,[b,c]] = \left[\begin{pmatrix}\mathbf{a}\\ \alpha\end{pmatrix},\left[\begin{pmatrix}\mathbf{b}\\ \beta\end{pmatrix},\begin{pmatrix}\mathbf{c}\\ \gamma\end{pmatrix}\right]\right] = \begin{pmatrix}[\mathbf{a},[\mathbf{b},\mathbf{c}]]\\ A-2\boldsymbol{\Omega}\cdot\mathbf{C}-D\end{pmatrix}, \tag{4.11}$$

where $A = A(a,b,c)$ is independent of $\boldsymbol{\Omega}$, and

$$\mathbf{C}(a,b,c) = \mathbf{a}\cdot\boldsymbol{\nabla}(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\times[\mathbf{b},\mathbf{c}] \tag{4.12a}$$

and

$$D(a,b,c) = 2\boldsymbol{\nabla}\boldsymbol{\Omega}:\mathbf{a}(\mathbf{b}\times\mathbf{c}). \tag{4.12b}$$

We know from the Jacobi identity for vanishing Coriolis term that

$$A(a,b,c)+A(b,c,a)+A(c,a,b) = 0. \tag{4.13}$$

It is sufficient for us to show that

$$\mathbf{C}(a,b,c)+\mathbf{C}(b,c,a)+\mathbf{C}(c,a,b) = 0 \tag{4.14a}$$

and

$$D(a,b,c)+D(b,c,a)+D(c,a,b) = 0. \tag{4.14b}$$

The identity (4.14a) is obtained straightforwardly by use of (4.12a). The equality (4.14b) follows by use of by (3.23) and $\boldsymbol{\nabla}\cdot\boldsymbol{\Omega} = 0$. Thus the Jacobi identity for the bracket (4.10) is valid.

4.3. Ideal MHD

We now also include a 'frozen-in' magnetic field $\mathbf{B}$ and an advected specific entropy σ. The 'frozen-in' fields are $(\rho,\mathbf{B},\sigma)\in Q = \mathscr{F}_0(D)\times\mathscr{H}_0(D)\times\mathscr{F}_0(D)$. and, as in Sec. 4.1, representations of the Lie algebra $P = \mathscr{H}_0(D)$ are used to include these fields. For the density ρ, the representation $\mathscr{L}^{(3)}_{\mathbf{a}}$ on the vector space $\mathscr{F}_0(D)$ is used. Correspondingly, for the magnetic field, we use the representation $\mathscr{L}^{(2)}_{\mathbf{a}}$ on the vector space $\mathscr{H}_0(D)$, and, for the specific entropy, the representation $\mathscr{L}^{(0)}_{\mathbf{a}}$ on the vector space $\mathscr{F}_0(D)$. The choice of representation is associated with the fact that the density is naturally a 3-form, the magnetic field a 2-form and the specific entropy a 0-form. We denote elements in $P\times Q$ by

$$u = \begin{pmatrix}\mathbf{v}\\ \rho\\ \mathbf{B}\\ \sigma\end{pmatrix}, \qquad a = \begin{pmatrix}\mathbf{a}\\ \alpha_1\\ \boldsymbol{\alpha}\\ \alpha_5\end{pmatrix}, \qquad b = \begin{pmatrix}\mathbf{b}\\ \beta_1\\ \boldsymbol{\beta}\\ \beta_5\end{pmatrix}. \tag{4.15}$$

The Lie algebra bracket on $P\times Q$ obtained by the construction (2.21) is

$$\left[\begin{pmatrix}\mathbf{a}\\ \alpha_1\\ \boldsymbol{\alpha}\\ \alpha_5\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ \beta_1\\ \boldsymbol{\beta}\\ \beta_5\end{pmatrix}\right] = \begin{pmatrix}[\mathbf{a},\mathbf{b}]\\ \mathscr{L}^{(0)}_{\mathbf{a}}\beta_1-\mathscr{L}^{(0)}_{\mathbf{b}}\alpha_1-2\boldsymbol{\Omega}\cdot\mathbf{a}\times\mathbf{b}\\ \mathscr{L}^{(1)}_{\mathbf{a}}\boldsymbol{\beta}-\mathscr{L}^{(1)}_{\mathbf{b}}\boldsymbol{\alpha}\\ \mathscr{L}^{(3)}_{\mathbf{a}}\beta_5-\mathscr{L}^{(3)}_{\mathbf{b}}\alpha_5\end{pmatrix}, \tag{4.16}$$

where we have used (cf. (A 14a,b) in Larsson 1996b)

$$\mathscr{L}^{(k)}_{\mathbf{a}} = -\mathscr{L}^{(3-k)*}_{\mathbf{a}}, \qquad k = 0,1,2,3. \tag{4.17}$$

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With $K = 0$, we obtain from (2.8), (2.9) and (4.16) that

$$J_u a = \begin{pmatrix} -\nabla\cdot(\mathbf{a}\mathbf{M}) - \nabla\mathbf{a}\cdot\mathbf{M} - 2\rho\boldsymbol{\Omega}\times\mathbf{a} - \rho\nabla\alpha_1 - \mathbf{B}\times(\nabla\times\boldsymbol{\alpha}) + \alpha_5\nabla\sigma \\ -\nabla\cdot(\rho\mathbf{a}) \\ \nabla\times(\mathbf{a}\times\mathbf{B}) \\ -\mathbf{a}\cdot\nabla\sigma \end{pmatrix}. \tag{4.18}$$

The specific internal energy $U(\rho,\sigma)$ is a given function of density and specific entropy. With the system energy as Hamiltonian,

$$H(\mathbf{M},\rho) = \int_D \left(\frac{1}{2\rho}\mathbf{M}\cdot\mathbf{M} + \frac{1}{2}\mathbf{B}\cdot\mathbf{B} + \rho U(\rho,\sigma) + \rho\phi_{\text{ext}}\right), \tag{4.19}$$

the equations now obtained with $K = 0$ from Theorem 1 and (2.1) are

$$\dot{\mathbf{M}} = -\nabla\cdot(\rho^{-1}\mathbf{M}\mathbf{M}) - 2\boldsymbol{\Omega}\times\mathbf{M} + (\nabla\times\mathbf{B})\times\mathbf{B} - \nabla(\rho^2 U) - \rho\nabla\phi_{\text{ext}}, \tag{4.20}$$

$$\dot{\rho} = -\nabla\cdot\mathbf{M}, \qquad \dot{\mathbf{B}} = \nabla\times(\rho^{-1}\mathbf{M}\times\mathbf{B}), \qquad \dot{\sigma} = -\rho^{-1}\mathbf{M}\cdot\nabla\sigma. \tag{4.21a,b,c}$$

These are precisely the ideal MHD equations. The Poisson bracket is obtained from (2.23) as

$$\begin{aligned}\{F,G\} = &-\int_D (\mathbf{M}\cdot[F'_{\mathbf{M}}, G'_{\mathbf{M}}]) - \int_D \rho(\mathscr{L}^{(0)}_{F'_{\mathbf{M}}} G'_\rho - \mathscr{L}^{(0)}_{G'_{\mathbf{M}}} F'_\rho - 2\boldsymbol{\Omega}\cdot F'_{\mathbf{M}}\times G'_{\mathbf{M}}) \\ &- \int_D \mathbf{B}\cdot(\mathscr{L}^{(1)}_{F'_{\mathbf{M}}} G'_{\mathbf{B}} - \mathscr{L}^{(1)}_{G'_{\mathbf{M}}} F'_{\mathbf{B}}) - \int_D \sigma(\mathscr{L}^{(3)}_{F'_{\mathbf{M}}} G'_\sigma - \mathscr{L}^{(3)}_{G'_{\mathbf{M}}} F'_\sigma),\end{aligned} \tag{4.22}$$

which is the standard result (Morrison and Greene 1980).

5. The Vlasov–Poisson equations

Let us write the Vlasov–Poisson equations for a one-component plasma of particles with mass m and charge e and in the presence of an external magnetic field

$$\dot{f} = -\mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{r}} - \left(-\frac{e}{m}\frac{\partial\phi_f}{\partial\mathbf{r}} + \mathbf{v}\times\mathbf{B}_{\text{ext}}\right)\cdot\frac{\partial f}{\partial\mathbf{v}}, \qquad \nabla^2\phi_f = -e\int f\,d^3v. \tag{5.1a,b}$$

No neutralizing background has been included, in order to simplify the notations, so (5.1), as written here, are dynamical equations for a non-neutral plasma. For vanishing external magnetic field, these are formally the same as the Jeans equation of stellar dynamics. Important models for plasma physics are obtained by allowing for a second plasma component with opposite charge, with dynamics governed by some appropriate equations. For the Hamiltonian formulation of (5.1), we take $P = \mathscr{F}_0(\mathbb{R}^3\times\mathbb{R}^3)$.

As inner product on P, we define

$$\langle a,b\rangle = \int a(\mathbf{r},\mathbf{v})\,b(\mathbf{r},\mathbf{v})\,d^3r\,d^3v, \qquad a,b\in P, \tag{5.2}$$

and as Lie algebra bracket,

$$[a,b] = \frac{\partial a}{\partial\mathbf{r}}\cdot\frac{\partial b}{\partial\mathbf{v}} - \frac{\partial b}{\partial\mathbf{r}}\cdot\frac{\partial a}{\partial\mathbf{v}} + \frac{e}{m}\mathbf{B}_{\text{ext}}\cdot\frac{\partial a}{\partial\mathbf{v}}\times\frac{\partial b}{\partial\mathbf{v}}, \tag{5.3}$$

which is the Poisson bracket for single-particle motion in terms of the non-

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canonical variables $(\mathbf{r}, \mathbf{v})$, i.e. position and velocity. With $K = 0$ and with the system energy as Hamiltonian,

$$H(f) = \int \tfrac{1}{2} m\mathbf{v}\cdot\mathbf{v} f(\mathbf{r}, \mathbf{v}; t)\, d^3r\, d^3v + \int \tfrac{1}{2}|\boldsymbol{\nabla}\phi_f|^2 d^3r, \tag{5.4}$$

we get the dynamical equation (5.1a) as Hamilton's generalized equation (2.1).

6. The small-amplitude expansion

Consider a Hamiltonian dynamical system (2.1). We write Taylor expansions of the Hamiltonian

$$H(u+a) = H(u) + H_u^{(1)}(a) + \frac{1}{2!}H_u^{(2)}(a,a) + \frac{1}{3!}H_u^{(3)}(a,a,a) + \ldots, \tag{6.1}$$

the dynamical vector field

$$X(u+a) = X_u^{(1)}\, a + \frac{1}{2!}X_u^{(2)}(a,a) + \ldots \tag{6.2}$$

and the Poisson structure

$$J_{u+a} = J_u + J_u^{(1)}(a) + \frac{1}{2!}J_u^{(2)}(a,a) + \ldots \quad . \tag{6.3}$$

The terms in these expansions may be expressed in terms of directional derivatives in the Hilbert space P:

$$\left\langle a, \frac{\delta}{\delta u}\right\rangle H(u) = H_u^{(1)}(a), \qquad \left\langle a, \frac{\delta}{\delta u}\right\rangle \left\langle b, \frac{\delta}{\delta u}\right\rangle H(u) = H_u^{(2)}(a,b), \tag{6.4a,b}$$

$$\left\langle a, \frac{\delta}{\delta u}\right\rangle \left\langle b, \frac{\delta}{\delta u}\right\rangle \left\langle c, \frac{\delta}{\delta u}\right\rangle H(u) = H_u^{(3)}(a,b,c) \tag{6.4c}$$

and

$$\left\langle a, \frac{\delta}{\delta u}\right\rangle X(u) = X_u^{(1)}(a), \qquad \left\langle a, \frac{\delta}{\delta u}\right\rangle \left\langle b, \frac{\delta}{\delta u}\right\rangle X(u) = X_u^{(2)}(a,b), \tag{6.5a,b}$$

$$\left\langle a, \frac{\delta}{\delta u}\right\rangle J_u = J_u^{(1)}(a), \qquad \left\langle a, \frac{\delta}{\delta u}\right\rangle \left\langle b, \frac{\delta}{\delta u}\right\rangle J_u = J_u^{(2)}(a,b). \tag{6.5c,d}$$

The purpose of this section is now to study the expansion (6.2) from the Hamiltonian point of view.

6.1. General Hamiltonian vector fields

We first consider the linear term in the expansion (6.2). The theorem below has been proved and studied in Larsson (1998a, 2000). A reason to reconsider it here is to better understand the formulation and significance of the corresponding next-order result.

Theorem 2. *Consider a general Hamiltonian equation (2.1). Then*

$$(X_u^{(1)} J_u)^* = X_u^{(1)} J_u + J_u^{(1)}[X(u)]. \tag{6.6}$$

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Let us first reformulate (6.6) in a way that is suitable for stating the corresponding next-order result, and write

$$\langle a, X_u^{(1)} J_u b\rangle = \tfrac{1}{2}(\langle a, X_u^{(1)} J_u b\rangle + \langle b, X_u^{(1)} J_u a\rangle) + A_u(a,b), \tag{6.7a}$$

where

$$A_u(a,b) = \tfrac{1}{2}\langle a, J_u^{(1)}[X(u)]b\rangle. \tag{6.7b}$$

In (6.7a), we thus write $\langle a, X_1^{(1)} J_u b\rangle$ as the sum of the symmetric and some remaining part. The remaining part $A_u(a,b)$ must then satisfy

$$A_u(a,b) + A_u(b,a) = 0. \tag{6.8}$$

The corresponding result for the bilinear operator $X_u^{(2)}$ is formulated as follows.

Theorem 3. *Consider a general Hamiltonian equation (2.1). Then*

$$\begin{aligned}\langle a, X_u^{(2)}(J_u b, J_u c)\rangle = \tfrac{1}{3}(&\langle a, X_u^{(2)}(J_u b, J_u c)\rangle + \langle b, X_u^{(2)}(J_u c, J_u a)\rangle \\ &+ \langle c, X_u^{(2)}(J_u a, J_u b)\rangle) \\ &+ A_u(a,b,c),\end{aligned} \tag{6.9a}$$

where, in compact notation, we may formally write

$$A_u(a,b,c) = \frac{1}{3}\frac{d}{dt}(\langle a, J_u^{(1)}(J_u b)c\rangle + \langle a, J_u^{(1)}(J_u c)b\rangle). \tag{6.9b}$$

In order to get an explicit expression for $A_u(a,b,c)$, we must take the time derivative in (6.9b) assuming that a, b, c and u are time-dependent and obtain an expression that involves $\dot a, \dot b, \dot c$ and $\dot u$. Then, in the expression thus obtained, we substitute

$$\dot a = -X_{u_0}^{(1)*} a, \qquad \dot b = -X_{u_0}^{(1)*} b, \qquad \dot c = -X_{u_0}^{(1)*} c, \qquad \dot u = X(u) \tag{6.10}$$

The term $A_u(a,b,c)$ must, as is obvious from (6.9a), satisfy

$$A_u(a,b,c) + A_u(b,c,a) + A_u(c,a,b) = 0. \tag{6.11}$$

The equality (6.11) may also be verified directly from (6.9b) by use of the Jacobi identity (2.3) and the equality (6.5c) involving $J^{(1)}$.

The proof of Theorem 3 is given in Appendix B. Some of the significance of Theorem 3 is seen when it is applied to the resonant three-wave interaction process. Then the Manley–Rowe relations follow (see Sec. 6.3 below).

6.2. An important class of Hamiltonian vector fields

We consider now the class of Hamiltonian equations with a Poisson structure J_u that contain only terms that are constant or linear in u; that is the class considered in Theorem 1 and that is of main concern in the present paper. Then we get the following result for $X_u^{(1)}$.

Theorem 4. *Consider a Poisson structure of the form (2.9). Then*

$$\langle a, X_u^{(1)} J_u b\rangle = -H_u^{(2)}(J_u a, J_u b) - \tfrac{1}{2}\mathrm{perm}_2\langle J_u a, [b, H_u']\rangle + A_u(a,b), \tag{6.12a}$$

where

$$A_u(a,b) = -\tfrac{1}{2}\langle X(u), [a,b]\rangle \tag{6.12b}$$

and perm_2 *denotes the two terms obtained by permuting* (a,b).

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In the corresponding result for $X_u^{(2)}$, we use the notation H_u'' for the operator defined by $\langle a, H_u'' b\rangle = H_u^{(2)}(a,b)$ and the notation perm_6 for the six terms obtained by permuting (a,b,c).

Theorem 5. *Consider a Poisson structure of the form (2.9) and use notation as Sec. 2. Then*

$$\langle a, X_u^{(2)}(J_u b, J_u c)\rangle = -H_u^{(3)}(J_u a, J_u b, J_u c) - \tfrac{1}{3}\mathrm{perm}_6\langle J_u a, [b, H_u'' J_u c]\rangle + A_u(a,b,c), \tag{6.13a}$$

where formally

$$A_u(a,b,c) = \frac{1}{3}\frac{d}{dt}(\langle J_u b, [c,a]\rangle + \langle J_u c, [b,a]\rangle). \tag{6.13b}$$

In order to obtain an explicit expression for $A_u(a,b,c)$, we must take the time derivative in (6.13b) assuming that a, b, c and u are time-dependent. An expression that involves $\dot a, \dot b, \dot c$ and $\dot u$ is then obtained. Then we substitute (6.10) or, more explicitly,

$$\dot w = [w, H_u'] + H_u'' J_u w \quad \text{for } w = a, b, c \tag{6.14}$$

The proof of this result is given in Appendix C.

6.3. A note on resonant three-wave interaction

Let u_e be an equilibrium solution of (2.1) and let ζ_j, $j = 1,2,3$, denote three solutions of the adjoint linearized equation, i.e.

$$\dot\zeta_j = -X_e^{(1)*}\zeta_j \quad \text{for } j = 1,2,3, \tag{6.15}$$

that are time-harmonic with frequencies satisfying the resonance condition

$$\dot\zeta_j = -i\omega_j\zeta_j, \qquad \omega_1 + \omega_2 + \omega_3 = 0. \tag{6.16a,b}$$

In (6.15), as well as below, we write the subscript e in place of u_e. The linearized equation

$$\dot u_j = X_e^{(1)} u_j \tag{6.17}$$

is satisfied by $u_j = J_e\zeta_j$, as follows from Theorem 2 and (6.15). The three normal modes may interact resonantly, and an ansatz of the form

$$u = u_e + \left(\sum_{j=1}^{3} A_j u_j + \text{c.c.}\right), \tag{6.18}$$

where the amplitudes A_j are assumed to vary slowly with time, implies the coupled-mode equations

$$\frac{d\bar A_1}{dt} = -\frac{i\omega_1 V}{W_1}A_2 A_3, \qquad \frac{d\bar A_2}{dt} = -\frac{i\omega_2 V}{W_2}A_1 A_3, \qquad \frac{d\bar A_3}{dt} = -\frac{i\omega_3 V}{W_3}A_1 A_2, \tag{6.19}$$

where

$$V = \langle\zeta_1, X_e^{(2)}(u_2, u_3)\rangle = \langle\zeta_2, X_e^{(2)}(u_1, u_3)\rangle = \langle\zeta_3, X_e^{(2)}(u_1, u_2)\rangle \tag{6.20}$$

and

$$W_j = i\omega_j\langle\bar\zeta_j, u_j\rangle. \tag{6.21}$$

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The symmetry in (6.20) follows from Theorem 3. Note that the A_u terms in (6.9a) makes zero contribution to the coupling coefficient V. This is obtained directly from (6.9b) and the resonance condition (6.16b).

For the class of models considered in Sec. 6.2, we obtain a useful more explicit expression for the coupling coefficient from Theorem 5. From (6.15), (6.16) and (C 19b), we have

$$-i\omega_j \zeta_j = [\zeta_j, H'_u] + H''_u \zeta_j, \tag{6.22}$$

which, when used in (6.13a), results in

$$V = -H_e^{(3)}(u_1, u_2, u_3) + \tfrac{1}{3}\mathrm{perm}_6(i\omega_3 \langle u_1, [\zeta_2, \zeta_3]\rangle + \langle u_1, [\zeta_2, [\zeta_3, H'_e]]\rangle) \tag{6.23}$$

The above result is useful but rather formal as it now stands, since the validity of the ansatz (6.18) must also be confirmed. In practice, certain modifications may be needed, depending on the situation. We may, for example, include a spectrum of interacting waves. The coupling strength (6.23) is still a quantity of importance.

6.4. Example: interaction between Rossby waves

The resonant interaction between three Rossby waves in a periodic channel allowing for shear flow was considered by Larsson and Wiklund (1999) (hereinafter referred to as [LW]) by use of a Hamiltonian formulation (2.1) of the dynamics and the interaction equations (6.15)–(6.21). We shall now reconsider this problem and use the new result (6.23). The results obtained agree, but starting from the result (6.23) saves us from much algebra.

The presentation below is very brief and some details omitted here can be found in [LW]. We consider two-dimensional flow in a channel $D = [0, L] \times [0, l]$ in (x, y) space, and use the stream function as dynamical variable. The space P consists of functions $\Psi\colon D \to \mathbb{R}$ such that the associated flow $\mathbf{u} = (-\partial_y \Psi, \partial_x \Psi)$ is tangent to the boundary of D at $y = 0$ and $y = l$. In the x direction, we choose periodic boundary conditions. On the space P, we define an inner product and Lie Algebra bracket as

$$\langle a, b\rangle = \int_D \boldsymbol{\nabla} a \cdot \boldsymbol{\nabla} b, \qquad [a, b] = a'_x b'_y - a'_y b'_x \qquad \text{for } a, b \in P. \tag{6.24}$$

The Hamiltonian and the K operator are

$$H(\Psi) = \tfrac{1}{2}\langle \Psi, \Psi\rangle, \qquad K\Psi = \Delta^{-1}[f, \Psi]. \tag{6.25}$$

The validity of (2.7) in Theorem 1 is easily checked. The dynamical equation is then obtained as

$$\partial_t \Psi = \Delta^{-1}[\Delta \Psi + f, \Psi], \tag{6.26}$$

where the inverse Laplace operator is defined in [LW].

Any function $\Psi_e = \Psi_e(y)$ is an equilibrium solution of (6.26), and represents a shear flow $U_e(y) = -\Psi'_e(y)$ along the channel. The Coriolis parameter is assumed to depend only on the y variable: $f = f(y)$. We consider such an equilibrium and the ansatz

$$\Psi = \Psi_e + \left(\sum_{j+1}^{3} A_j(t)\, \tilde{\psi}_j(y)\, e^{-i\omega_j t + ik_j x} + \text{c.c.} \right), \tag{6.27}$$

where $\psi_j = \tilde{\psi}_j(y) \exp(i\omega_j t + ik_j x)$, $\psi_j = J_e \zeta_j$, and ζ_j solves the corresponding

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adjoint linearized equation (6.15). Then ψ_j solves the linearized equation. We also assume the resonance conditions

$$\omega_1+\omega_2+\omega_3=0, \qquad k_1+k_2+k_3=0. \tag{6.28}$$

We now like to use (6.23) and calculate

$$\langle\psi_1,[\zeta_2,\zeta_3]\rangle=-Lk_1k_2k_3\int_0^l Q_e'(k_3^{-1}\tilde\xi_1\tilde\xi_2\tilde\xi_3'-k_2^{-1}\tilde\xi_1\tilde\xi_2'\tilde\xi_3)\,dy \tag{6.29}$$

and

$$\begin{aligned}\langle\psi_1,[\zeta_2,[\zeta_3,\Psi_e]]\rangle &= ik_1k_2k_3L\int_0^l Q_e'U_e'\zeta_1\zeta_2\zeta_3\,dy\\ &\quad+ik_1k_2k_3L\int_0^l Q_e'U_e\left(\zeta_1\zeta_2\zeta_3'-\frac{k_3}{k_2}\zeta_1\zeta_2'\zeta_3\right)dy,\end{aligned} \tag{6.30}$$

where $Q_e'=\Psi_e'''+f'$. Substitution of (6.29) and (6.30) into (6.23) yields

$$V=iLk_1k_2k_3\int_0^l Q_e'\left(2U_e'\zeta_1\zeta_2\zeta_3-\mathrm{perm}_3\left(\frac{\omega_1-k_1U_e}{k_1}\zeta_1'\zeta_2\zeta_3\right)\right)dy, \tag{6.31}$$

which agrees with (41) in [LW].

7. Conclusions

A class of Hamiltonian system has been studied that both includes many important examples and admits a simple abstract formulation. A main new result is the expression (6.13) for the second-order term in the small-amplitude expansion (1.2). When applied to the problem of resonant three-wave interaction, the result (6.23) is obtained. This latter expression is valid for an arbitrary equilibrium state. A simple example, involving the interaction of Rossby waves in the presence of shear flow, has been studied in detail (see Sec. 6.4 above and compare it with the recent paper by Larsson and Wiklund 1999).

The focus in the present paper is more on the space of states as a Lie algebra than on the Poisson structure or the Poisson bracket. This is practical both for stating and deriving results. Note that the Lie algebra bracket appears a number of times in the results (6.13) and (6.23). The result (6.23) looks simple when expressed in terms of the Lie algebra bracket. However, when applied to ideal MHD, for example, the bracket (4.16) must be used, and we easily get very long and complicated expressions. Computer algebra may be useful for the lengthy algebra, and the formalism of the present paper provide a highly structured and interesting starting point for such studies.

Analytical mechanics for fluid and plasma dynamics is naturally expressed in terms of differential geometry, and many difficult and abstract concepts are used (see e.g. Marsden and Ratiu 1994). The Lagrangian–Hamiltonian formalism may then, in a deep and beautiful way, be generalized by use of Lie group theory. However, in practice, most of this formalism is inaccessible to many physicists who are working with fluid and plasma theory. There is strong motivation for an easier formalism that makes the Hamiltonian structure more available. Of course, the level of formalism needed depends on the problems addressed. In the present paper, it has been shown how essential general results

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for the small-amplitude expansion may be obtained within an comparatively simple and elementary abstract formulation. This formalism also promotes simple but complete and rigorous proofs of the Hamiltonian structure for various concrete models. Take, for example, ideal MHD. With the Lie algebra structure obtained in Sec. 4.3 it takes only a little work to show that the correct dynamical equations are obtained. Also the verification that (4.16) indeed constitutes a Lie algebra bracket is quite easy thanks to the semidirect product construction. The main part of the corresponding problem when we study a given Poisson bracket for ideal MHD would be to check the Jacobi identity by direct algebra. This may easily become a rather time-consuming exercise. In the more advanced differential geometric considerations, the semidirect product for Lie groups is essential; as is seen in the present paper, the corresponding semidirect product for Lie algebras may still play a constructive role within a much easier formalism.

The full set of Maxwell's equations are not included in any of the models considered in Sections 3–5 above. It is, however, not difficult to consider such models within the formalism of Theorem 1. One must then use the canonical momentum, which involves the magnetic vector potential, and accordingly the fields of electrodynamics are chosen to be this vector potential together with the electric field.

In order to apply the method of the present paper and obtain the interaction equations for the three-wave coupling process, we need to know explicitly the structure in Theorem 1 for the model of interest. For many models, we may use published results for the non-canonical Poisson bracket. It may also be possible to proceed as in Secs 3 and 4 in order to find the structure. However, there is no universal method to find the Hamiltonian structure of a given model. Once the Theorem 1 structure has been found, the present paper provides results for the mode-coupling process as explained in Sec. 6 and with (6.23) as a main result.

Appendix A. Lie derivatives

In this appendix, we denote by A and B scalar fields $A, B\colon D \to \mathbb{R}$, and by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ vector fields $\mathbf{a}, \mathbf{b}, \mathbf{c}\colon D \to \mathbb{R}^3$, where D is some domain in $\mathbb{R}^3$. That is, in notation introduced in Secs 3 and 4, $A, B \in \mathscr{F}(D)$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathscr{H}(D)$. We define operators $\mathscr{L}^{(0)}_{\mathbf{a}}$ and $\mathscr{L}^{(3)}_{\mathbf{a}}$ on $\mathscr{F}(D)$ by

$$\mathscr{L}^{(0)}_{\mathbf{a}}(B) = \mathbf{a}\cdot\boldsymbol{\nabla}B, \qquad \mathscr{L}^{(3)}_{\mathbf{a}}(B) = \boldsymbol{\nabla}\cdot(B\mathbf{a}), \tag{A 1a,b}$$

and operators $\mathscr{L}^{(1)}_{\mathbf{a}}$ and $\mathscr{L}^{(2)}_{\mathbf{a}}$ on $\mathscr{H}(D)$ by

$$\mathscr{L}^{(1)}_{\mathbf{a}}\,\mathbf{b} = \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) - \mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) \equiv \mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{b} + \boldsymbol{\nabla}\mathbf{a}\cdot\mathbf{b} \tag{A 2}$$

and

$$\mathscr{L}^{(2)}_{\mathbf{a}}\,\mathbf{b} = \mathbf{a}\boldsymbol{\nabla}\cdot\mathbf{b} - \boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) \equiv \mathbf{b}\boldsymbol{\nabla}\cdot\mathbf{a} + [\mathbf{a},\mathbf{b}]. \tag{A 3}$$

These operators satisfy

$$\mathscr{L}^{(k)}_{\mathbf{a}}\,\mathscr{L}^{(k)}_{\mathbf{b}} - \mathscr{L}^{(k)}_{\mathbf{b}}\,\mathscr{L}^{(k)}_{\mathbf{a}} = \mathscr{L}^{(k)}_{[\mathbf{a},\mathbf{b}]} \quad \text{for } k = 0,1,2,3, \tag{A 4}$$

and they thus give representations of the Lie algebra $\mathscr{H}(D)$. The operator $\mathscr{L}^{(k)}_{\mathbf{a}}$ is associated with the Lie derivative of a k-form. More details are given in Larsson (1996b, Appendix A).

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Appendix B. Proofs of Theorems 2 and 3

Consider the following set of equations associated with (2.1):

$$\dot{u} = X(u), \qquad \dot{\zeta} = -X_u^{(1)*}\zeta, \qquad \partial_\varepsilon u = J_u \zeta \tag{B 1a,b,c}$$

A general Hamiltonian theorem now states that the equation (B 1c) is propagated in time by (B 1a,b) (see Larsson 1998a). In other words, if (B 1c) is satisfied at some initial time $t = t_i$ for two ε-parametrized curves $u(\varepsilon)$ and $\zeta(\varepsilon)$ in P, and we let these two curves evolve in time in accordance with the dynamics (B 1a,b), then (B 1c) is valid at each fixed time. We now substitute power series

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots, \qquad \zeta = \zeta_1 + \varepsilon\zeta_2 + \varepsilon^2\zeta_3 + \ldots \tag{B 2a,b}$$

into (B 1), Taylor-expand around u_0, and find by equating terms of equal order in ε the set of equations

$$\dot{u}_0 = X(u_0), \tag{B 3}$$

$$\dot{u}_1 = X^{(1)}_{u_0} u_1, \qquad \dot{u}_2 = X^{(1)}_{u_0} u_2 + \tfrac{1}{2} X^{(2)}_{u_0}(u_1, u_1), \tag{B 4a,b}$$

$$\dot{\zeta}_1 = X^{(1)*}_{u_0}\zeta_1, \qquad \dot{\zeta}_2 = -X^{(1)*}_{u_0}\zeta_2 - X^{(2)*}_{u_0, u_1}\zeta_1, \tag{B 5a,b}$$

$$u_1 = J_{u_0}\zeta_1, \qquad 2u_2 = J_{u_0}\zeta_2 + J^{(1)}_{u_0}[u_1]\zeta_1. \tag{B 6a,b}$$

In (B 5b), we have used a new notation in the last term, where the adjoint of the linear operator defined by

$$X^{(2)}_{u_0, u_1} a = X^{(2)}_{u_0}(u_1, a) \tag{B 7}$$

appears. These equations are, like (B 1), consistent and by use of (B 6) in (B 5) we should get (B 4). Theorem 2 follows easily by use of (B 3), (B 4a), (B 5a) and (B 6a). To see this, we may simply express $\dot{u}_1$ in terms of ζ_1 in two different ways: (1) we substitute (B 6a) on the right-hand side of (B 4a); (2) we take the time derivative of (B 6a) and substitute (B 5a) on the right-hand side. The results obtained must agree, and Theorem 2 follows □

To prove Theorem 3, we may proceed in a similar way. It is, however, easier to give a short proof by calculating backwards, so let us start by proving the intermediate result

$$\langle a, X^{(2)}_{u_0}(u_1, u_1)\rangle - \langle X^{(2)}_{u_0}(J_{u_0} a, u_1), \zeta_1\rangle = \partial_t \langle a, J^{(1)}_{u_0}(u_1)\zeta_1\rangle, \tag{B 8}$$

where we have used a compact notation as in Theorem 3, so that we assume (6.10a). We use (B 6b) on the right-hand side to eliminate the $J^{(1)}$ term and then calculate the time derivative. By use of (6.10a) and (B 4b), we get rid of all u_2 terms, and the remaining part of (B 8) may be written as

$$-\langle X^{(2)}_{u_0}(J_{u_0} a, u_1), \zeta_1\rangle = \langle a, X^{(1)}_{u_0} J_{u_0}\zeta_2 - J^{(1)}_{u_0}[\dot{u}_0]\zeta_2 - J_{u_0}\dot{\zeta}_2\rangle. \tag{B 9}$$

We use (B 7) and manipulate the left-hand side so that a becomes free in the inner product; on the right-hand side, we use (B 3) and Theorem 2. From (B 5b), we then get $0 = 0$, so the validity of (B 8), from which we started, has now been checked. The identity (B 8) with (B 6a) defining u_1 may now quite easily be written as Theorem 3. □

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Appendix C. Proofs of Theorems 4 and 5

We assume that a Poisson structure of the form (2.9) is given. The useful identity

$$\langle J_u^{(1)}[a]\,b,c\rangle = \langle a,[b,c]\rangle \tag{C 1}$$

is obtained by operating on the equality

$$\langle J_u\,b,c\rangle = \langle Kb,c\rangle + \langle u,[b,c]\rangle \tag{C 2}$$

with the direction derivative $\langle a,\delta/\delta u\rangle$ and using (6.5). We shall now derive (6.12), and start with the calculation

$$\begin{aligned}\langle a,X_u^{(1)}\,b\rangle &= \left\langle b,\frac{\delta}{\delta u}\right\rangle\langle a,X(u)\rangle = \left\langle b,\frac{\delta}{\delta u}\right\rangle\langle a,J_uH_u'\rangle\\ &= \langle a,J_u^{(1)}[b]\,H_u'+J_u\,H_u''\,b\rangle = \langle a,J_u\,H_u''\,b\rangle-\langle b,[a,H_u']\rangle.\end{aligned} \tag{C 3}$$

We make the replacement $b\to J_u\,b$ in this identity:

$$\begin{aligned}\langle a,X_u^{(1)}\,J_u\,b\rangle &= \langle a,J_u\,H_u''\,J_u\,b\rangle-\langle J_u\,b,[a,H_u']\rangle\\ &= -H_u^{(2)}(J_u\,a,J_u\,b)-\langle J_u\,b,[a,H_u']\rangle.\end{aligned} \tag{C 4}$$

Now (6.12) and thus Theorem 4 follow by use of (2.7)–(2.9). □

The calculation of $X_u^{(2)}$ proceeds in a similar way. For a general Hamiltonian system (2.1),

$$\begin{aligned}\langle a,X_u^{(2)}(b,c)\rangle &= \left\langle b,\frac{\delta}{\delta u}\right\rangle\left\langle c,\frac{\delta}{\delta u}\right\rangle\langle a,J_u\,H_u'\rangle\\ &= \left\langle b,\frac{\delta}{\delta u}\right\rangle\langle a,J_u^{(1)}[c]\,H_u'+J_u\,H_u''\,c\rangle\\ &= \langle a,J_u^{(2)}[b,c]\,H_u'+J_u^{(1)}[c]\,H_u''\,b+J_u^{(1)}[b]\,H_u''\,c+J_u\,H_u'''(b,c)\rangle,\end{aligned} \tag{C 5}$$

where the notation $H_u'''(b,c)$ is defined by $\langle a,H_u'''(b,c)\rangle = H_u^{(3)}(a,b,c)$. In the particular case (2.9) that we now consider, this reduces to

$$\langle a,X_u^{(2)}(b,c)\rangle = -\langle c,[a,H_u''\,b]\rangle-\langle b,[a,H_u''\,c]\rangle-H_u^{(3)}(J_u\,a,b,c), \tag{C 6}$$

by use of (C 1) and $J_u^{(2)}=0$. By substituting $b\to J_u\,b$ and $c\to J_u\,c$ in (C 6) and defining

$$B_u(a,b,c)\equiv\langle J_u\,a,[b,c]\rangle, \tag{C 7}$$

we get

$$\begin{aligned}\langle a,X_u^{(2)}(J_u\,b,J_u\,c)\rangle &= -B_u(c,a,H_u''\,J_u\,b)-B_u(b,a,H_u''\,J_u\,c)\\ &\quad -H_u^{(3)}(J_u\,a,J_u\,b,J_u\,c).\end{aligned} \tag{C 8}$$

In order to prove that (C 8) is equivalent to (6.13), we need the equality

$$\begin{aligned}&B_u(c,a,H_u''\,J_u\,b)+B_u(b,a,H_u''\,J_u\,c)+\frac{1}{3}\frac{d}{dt}(B_u(b,c,a)+B_u(c,b,a))\\ &\qquad = \frac{1}{3}\operatorname{perm}_6 B_u(a,b,H_u''\,J_u\,c),\end{aligned} \tag{C 9}$$

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where the time-derivative term is to be interpreted in a formal compact notation as explained in Theorem 5.

The remaining part of this appendix consists of a proof of (C 9). The following identities are then useful:

$$B_u(a,b,c)+B_u(b,c,a)+B_u(c,a,b)=0, \tag{C 10}$$

$$B_u(a,b,c)=-B_u(a,c,b), \tag{C 11}$$

$$B_u(a,b,[c,d])+B_u(a,c,[d,b])+B_u(a,d,[b,c])=0, \tag{C 12}$$

$$B_u([a,b],c,d)=-B_u([c,d],a,b). \tag{C 13}$$

The identity (C 10) follows by first rewriting it using (C 7), (2.9), (2.8) and the Jacobi identity for the Lie algebra and the identity (2.7). The identity (C 11) is evident from (C 7). The identity (C 12) follow from (C 7) and the Jacobi identity for the bracket. The identity (C 13) is obvious from (C 7). Another identity is

$$\frac{d}{dt}B_u(a,b,c)=B_u(H'_u,a,[b,c])+B_u(\dot{a},b,c)+B_u(a,\dot{b},c)+B_u(a,b,\dot{c}), \tag{C 14}$$

which is obtained by use of (C 7), (C 1) and (2.1).

We now proceed by rewriting the left-hand side of (C 9). First we use (C 14) and then (6.14). After this, we obtain two kind of terms, those that involve H'_u and those that involves H''_u, and the strategy is now to get these into one of the respective forms (allowing also for permutations of a, b and c)

$$B_u(a,b,[c,H'_u]) \quad \text{or} \quad B_u(a,b,H''_u J_u c). \tag{C 15a,b}$$

This is always possible by use of (C 10)–(C 13). For example,

$$\begin{aligned} B_u(H'_u,a,[b,c]) &= -B_u(a,b,[c,H'_u])-B_u(c,b,[a,H'_u]) \\ &\quad +B_u(a,c,[b,H'_u])+B_u(b,c,[a,H'_u]), \end{aligned} \tag{C 16}$$

$$B_u([a,H'_u],b,c)=B_u(c,b,[a,H'_u])-B_u(b,c,[a,H'_u]), \tag{C 17}$$

$$B_u(H''_u J_u a,b,c)=B_u(c,b,H''_u J_u a)-B_u(b,c,H''_u J_u a). \tag{C 18}$$

It is quite straightforward to get these expressions by use of (C 10)–(C 13). We thus rewrite (C 9) in a systematic way so that all terms are of the form (C 15a) or (C 15b). The equality (C 9) is now seen to be valid.

We also obtain from (C 3) that

$$X^{(1)}_u b=J^{(1)}_u[b]H'_u+J_u H''_u b, \qquad X^{(1)*}_u a=-[a,H'_u]-H''_u J_u a. \tag{C 19a,b}$$

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